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A General Qualitative Spatio-Temporal Model Based on Intervals

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Abstract: Many real-world problems involve qualitative reasoning about space and/or time. Actually, it is an adequate tool for dealing with situations in which information is not sufficiently precise. However, despite its numerous applications, it is difficult for people from outside the field to incorporate the required reasoning techniques into their methods. In this paper, we present a general, easy-to-use framework that integrates and solves the reasoning process of all qualitative models based on intervals. This framework has been divided into: (1) a representation magnitude and (2) the resolution of the reasoning process. Mainly, the developed method for solving the reasoning process is based on the definition of two algorithms: the qualitative sum and the qualitative difference. In addition, here, different instances of the model as well as some practical applications of them are presented.

Key Words: qualitative models, comonsense reasoning, spatial reasoning
Category: I.2

1 Introduction

Humans have a remarkable capability to perform a wide variety of physical and mental tasks without any measurements and any computations. Familiar examples are parking a car, cooking a meal, or summarizing a story. In performing such tasks, humans make decisions based on information that is mostly perception, rather than accurate measurement [Zadeh, 2001]. So, qualitative reasoning is concerned with representation formalisms that are considered close to conceptual schemata used by humans for reasoning about their physical environment in particular, about processes or events and about the spatial environment in which they are situated [Westphal and Wölfl, 2009] [Renz and Nebel, 2007].

Thus, a qualitative representation is the result of an abstraction process, which can be defined as that representation which *makes only as many distinctions as necessary to identify objects, events, situations, etc. in a given context* [Renz and Nebel, 2007]

[Hernández, 1994]. The way to define those distinctions depends on two different aspects. The first one is the level of *granularity*. In this context, granularity refers to a matter of precision in the sense of the amount of information which is included in the representation. Therefore, a fine level of granularity will provide a more detailed information than a coarse level.

The second aspect corresponds to the distinction between *comparing* magnitudes and *naming* magnitudes [Clementini et al., 1997]. This distinction refers to the usual comparison between *absolute* and *relative*. From a spatial point of view, this controversy corresponds to the way of representing the relationships among objects. As [Levinson, 2003] pointed out, *absolute* defines an object's location in terms of arbitrary bearings such as cardinal directions (e.g. North, South, East, West), by resulting in binary relationships. Instead, *relative* leads to ternary relationships. Consequently, for *comparing* magnitudes, an object b is any compared relationship to another object a from the same Point of View (PV). It is worth noting that the comparison depends on the orientation of both objects *with respect to* (wrt) the PV, since objects a and b can be at any orientation wrt the PV. An example is the qualitative treatment of compared distances [Escrig and Toledo, 2001] (see Figure 1). In this case, only two extreme orientations are considered: (1) both objects a and b are at the same orientation wrt the PV, represented by $b[Rel]^{S}_{PV}a$; and, (2) objects a and b are in the opposite orientation wrt the PV ($b[Rel]^{O}_{PV}a$).

Figure 1: An example of the *compared* distances as represented in [Escrig and Toledo, 2001]

On the other hand, *naming* magnitudes divides the magnitude of any concept into intervals (sharply or overlapped separated, depending on the context (see Figure 2)) such that qualitative labels are assigned to each interval. Note that the result of reasoning with regions of this kind can provide imprecision. This imprecision will be solved by providing disjunction in the result. That is, if an object can be found in several qualitative regions, q_i or q_{i+1} or $\ldots$ or q_n, then all possibilities are listed as follows $\{q_i, q_{i+1}, \ldots, q_n\}$ by indicating this situation.

Although qualitative models based on comparing magnitudes and qualitative naming models based on intervals have been studied, some models have not been solved up to now. Table 1 presents some of the qualitative models developed for dealing with

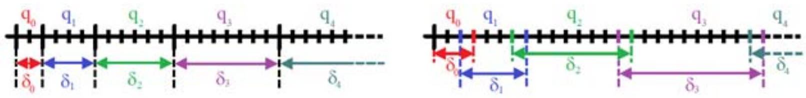


Figure 2: An example of structure relations where: acceptance areas are sharply separated (left) and acceptance areas are overlapped (right)

certain spatial concepts. Note that only some of the developed models are illustrated, since it is not possible to depict all of them by lack of space. Moreover, *acceleration* has been also included (in shady cells) despite it is firstly described in a qualitative way in this paper.

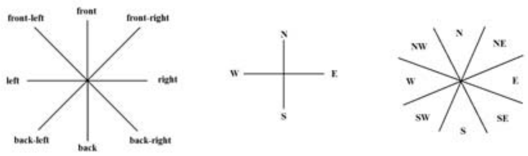
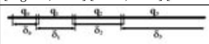
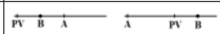
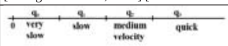
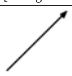
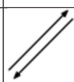
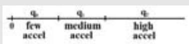
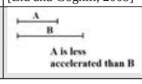
Magnitude	**Naming models (based on intervals)**	**Comparing models**
Orientation	 [Pacheco et al., 2006] [Skiadopoulos and Koubarakis, 2005] [Renz and Mitra, 2004] [Ligozat, 1998] [Frank, 1996] [Hernández, 1994]	
Distance	 [Escrig and Toledo, 2000] [Clementini et al., 1997] [Jong, 1994] [Zimmermann, 1993]	 [Escrig and Toledo, 2001]
Velocity	 [Escrig and Toledo, 2002]	
Trajectories	 [Gottfried, 2008] [de Weghe et al., 2005a] [de Weghe et al., 2005b]	 [Liu and Goghill, 2005]
Acceleration		

Table 1: Qualitative naming models versus qualitative comparing models

However, although qualitative reasoning is an established field of pursued by investigators from many disciplines including geography [van de Weghe et al., 2006], psychology [Knauff et al., 2004], ecology [Cioaca et al., 2009] [Salles and Bredeweg, 2006], biology [King et al., 2005] [Guerrin and Dumas, 2001], robotics [Liu et al., 2008] [Liu, 2008] [Holzmann, 2007] [Moratz and Wallgrün, 2003] and Artificial Intelligence [Cohn and Hazarika, 2001], the number of practical applications that make use of it is comparatively small. One reason for this can be seen in the difficulty for people from outside the field to incorporate the required reasoning techniques into their methods. So, the aim of this paper is to design a general, easy-to-use framework that overcomes that problem. With that propose we present a general systemic algorithm that integrates and solves the reasoning process of all qualitative models based on intervals. From the starting point that the development of any qualitative model consists of a representation of the magnitude at hand and the reasoning process, the structure of this paper is as follows: Section 2 describes the designed representation of a magnitude; the reasoning process is introduced in Section 3, while the experimental results, that is, different instances of the general model and some practical applications of them, are presented in Section 4, and discussed in Section 5.

2 Magnitude Representation

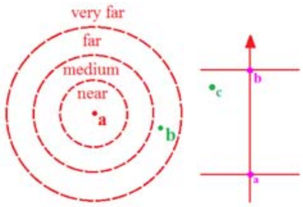
In qualitative spatial reasoning, it is common to consider a particular aspect of the physical world, that is, a magnitude such as topology or distance, and to develop a system of qualitative relationships between entities which cover that aspect of the world to some degree. Therefore, the first issue to be solved refers to the way to represent the magnitude to be modelled.

Focusing on qualitative naming models based on intervals, any magnitude is represented by the following three elements:

1. The **number of objects** implied in each relation (i.e. *arity*). A relationship is *binary* when there are only two objects implied. So, an object acts as reference (a) and the other one is referred (b). For instance, how far an object is wrt another object is a binary relationship as defined in [Jong, 1994] (see Figure 3a). In this example, the two-dimensional space is divided into several tracks centred in the reference object a. Each track is associated to a unique qualitative value (e.g. *near, medium, far, very far*). So, the relationship between objects a and b will be determined by the track of the interval-based system in which the object b is. Therefore, in the shown example, *b wrt a* is *far*, in other words, b is far from a.

   On the contrary, a relationship is *ternary* when three objects are implied (*c wrt ab*) such that two objects form the reference system (ab) and the other object (c) is referred wrt such reference system. For example, [Freksa and Zimmermann, 1992] represented the qualitative orientation information by an *orientation grid*. This grid is aligned to the orientation determined by two points in space, a and b, unlike the previous case. Thus, the space is divided into nine qualitative regions (i.e. *front-left, front, front-right, left, identical-front, right, back-left, back and back-right*). In this way, the orientation of the object *c wrt ab* will correspond to any of these qualitative regions. In particular, in this example, *c wrt ab* is *back-left*.

2. The **set of relations between objects**. It depends on the considered level of granularity. In a formal way, this set of relations between objects is expressed by means of the definition of a *Reference System (RS)*. A RS will contain, at least, a couple of components:



(a) **(b)**

Figure 3: Relations between objects: (a) Binary: object *a* is the reference and object *b* is referred. The space is divided into a four-distance system (in red) centred in the reference object *a* [Jong, 1994] (b) Ternary: objects *a* and *b* define a reference system (in red) such that object *c* is referenced wrt such system. In this case, the space is divided into nine qualitative regions determined by the oriented path from object *a* to object *b* [Freksa and Zimmermann, 1992]

– A *set of qualitative symbols* in increasing order represented by $Q = \{q_0, q_1, ..., q_n\}$, where q_0 is the qualitative symbol closest to the *Reference Object (RO)* and q_n is the one furthest away, going to infinity. Here, by cognitive considerations, the acceptance areas have been chosen in increasing size. Note that this set defines the different areas in which the workspace is divided and the number of them will depend on the *granularity* of the task, as abovementioned

– The *structure relations*, $\Delta r = \{\delta_0, \delta_1, ..., \delta_n\}$, describe the acceptance areas for each qualitative symbol q_i. So, δ_0 corresponds to the acceptance area of qualitative symbol q_0; δ_1 to the acceptance area of symbol q_1 and so on. These acceptance areas are quantitatively defined by means of a set of close or open intervals delimited by two extreme points: the initial point of the interval j, δ_j^i, and the ending point of the interval j, δ_j^e. Thus, the structure relations are rewritten by:

$$\begin{cases} \Delta r = \left\{\left[\delta_0^i, \delta_0^e\right[, \left[\delta_1^i, \delta_1^e\right[, \ldots, \left[\delta_n^i, \delta_n^e\right[\right\} & \text{if open intervals are considered} \\ \Delta r = \left\{\left[\delta_0^i, \delta_0^e\right], \left[\delta_1^i, \delta_1^e\right], \ldots, \left[\delta_n^i, \delta_n^e\right]\right\} & \text{otherwise} \end{cases}$$

As a consequence, the acceptance area of a particular magnitude entity, $AcAr(entity)$, is δ_j if its value is between the initial and ending points of δ_j, that is, $\delta_j^i \leq value\,(entity) \leq \delta_j^e$

3. The **operations**. The number of operations associated to a representation corresponds to the possible change in the PV. For instance, if the relationship is binary (*b wrt a*), only one operation can be defined: inverse (*a wrt b*). Nevertheless, it is

possible to define five different operations when the relationship between objects is ternary (*c wrt ab*) [Freksa and Zimmermann, 1992]: inverse (*c wrt ba*), homing (*a wrt bc*), homing-inverse (*a wrt cb*), shortcut (*b wrt ac*) and shortcut-inverse (*b wrt ca*). An iconic representation of the obtained relationships from these operations is depicted in Figure 4.

original relation inverse homing homing-inverse shortcut shortcut-inverse

Figure 4: Iconic representation of the relationship *c wrt ab* and the result of applying the five operations to the original relationship

3 The Reasoning Process

The *reasoning process* is divided into two parts:

– **The Basic Step of the Inference Process (BSIP)**. It can be defined as: "given two relationships, (1) the object *b wrt a reference system*, *RS1*, and (2) the object *c wrt another reference system, RS2*, such that the object *b* is included into the second reference system, the BSIP obtains the relationship *c wrt RS1*". Figure 5 shows the general BSIP for orientation and positional models not based on projections (Figure 5a) as well as two particular examples of the BSIP: (Figure 5b) when binary relationships are considered and (Figure 5c) when ternary relationships are used. Note that in spatial reasoning, the BSIP is usually represented by *composition tables*. These tables encode semantic, i.e., domain-specific information about (spatial or temporal) configuration between two entities if information is available about how these entities are related to some third entity. Their content can be obtained either by hand or automatically by means of algorithms, if they exist.

– **The Complete Inference Process (CIP)**. It is necessary when more than two objects (in binary relationships) or three objects (in ternary relations) are involved in the reasoning mechanism. Mainly, it consists of repeating the BSIP as many times as possible with the initial information and the information provided by some BSIP until no more information can be inferred

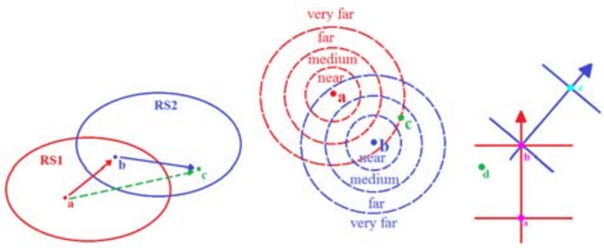


Figure 5: The general BSIP for qualitative models not based on projections (a); (b) a BSIP example for binary relationships based on the named distances system [Jong, 1994]; (c) shows a BSIP example when ternary relationships are used [Freksa and Zimmermann, 1992]

3.1 The Basic Step of the Inference Process

Basically, the BSIP is defined as the process of inferring the relationship between two (or three) entities of a magnitude from the knowledge of two other relationships such that there is an object in common in both relationships. The way to infer the new relationship depends on the considered magnitude. However, all qualitative models based on intervals define the magnitude in the same way, as abovementioned. For that reason, an abstraction can be done by resulting in a general algorithm. Here, we propose a general algorithm based on qualitative sums and differences that solves the inference process for all models based on intervals.

3.1.1 The General Algorithm

As previously introduced, magnitudes are represented by three different elements: the number of objects implied in each relationship, the set of relationships between entities and the operations that can be defined. Nevertheless, it is worthy noting that there is a difference between concepts of commonsense knowledge. So, for example, *time* is a scalar magnitude, while *space* is much more complex mainly due to its inherent multi-dimensionality. This inherent feature leads to a higher degree of freedom and an increased possibility of describing entities and relationships between entities. Because of the richness of space and its multi-dimensionality, most work in qualitative reasoning has focused on single aspects of space such as, for example, topology, orientation or distance. Nevertheless, as pointed out in [Freksa, 1992], relationships between entities can be seen as movements in the space or spatial deformations in physical space. As a result, when the relationships between entities are considered as directed vectors and using as reference orientation *ab*, three different situations can take place (see Figure 6):

– relationships between entities are in the same orientation

– relationships between entities are in the opposite orientation

– relationships between entities are at any orientation

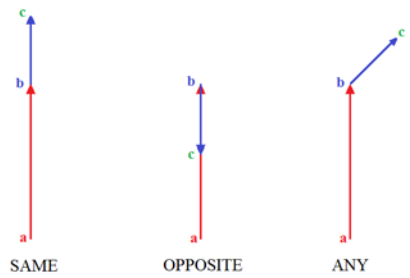


Figure 6: Representation of relationships between entities in terms of orientation by using as reference orientation *ab*

Therefore, the inferred relationship will be composed of all possible relationships between the entities by considering the three possible orientations. According to a deeper analysis of the possible orientations, it is clear that the extreme cases are obtained when the implied objects are in the same orientation and when they are in the opposite one. Consequently, if both extreme cases are solved, the result will be built as a disjunction of qualitative symbols from the inferred area closest to the RO to the furthest one. With the aim of automatically solving these extreme cases, we have defined the qualitative sum of intervals and the qualitative difference of intervals.

3.1.2 The Qualitative Sum

Let q_i be the qualitative symbol which represents a relationship *b wrt a reference system RS1*, and let q_j be the qualitative symbol referred to the relationship *c wrt another reference system RS2*, such that *b* is included into the second reference system.

Supposing that the two relationships are binary, we would have a situation similar to the one illustrated in Figure 7. In this example, from the knowledge *b wrt a* = q_3 and *c wrt b* = q_2, *c wrt a* will be inferred. Graphically, after locating both entities b and c at any place in their corresponding qualitative areas, q_i and q_j respectively (extreme cases are depicted in Figure 8), it is clear that the possible resulting relationships are $\{q_3, q_4\}$. However, it is possible to achieve the same solution from a mathematical point of view. The development of such a method has several advantages. It does not require to represent the relationships for any composition. This is important specially when the dimensionality of the magnitude is high. Moreover, it can be applied to all the models based on intervals since the reasoning mechanism is the same in all of them.

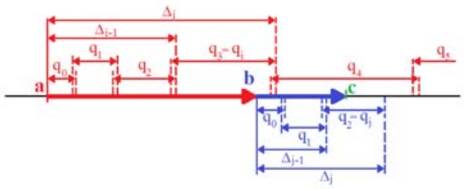


Figure 7: Example of qualitative sum when structure relations with overlapped acceptance areas are used

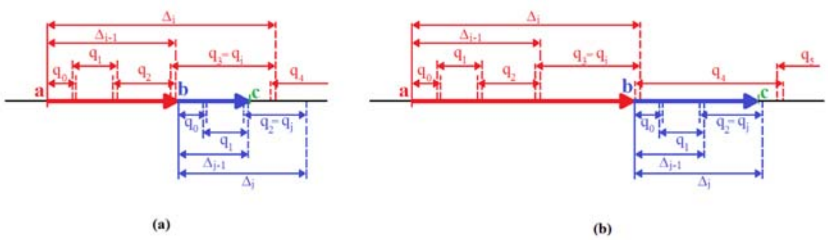


Figure 8: Extreme positions at which entities b and c can be located by keeping the relationships *b wrt a reference system RS1* and *c wrt another reference system RS2*. So, (a) refers the case when both entities are located at the initial points of their acceptance areas (leading to LB); while (b) represents the case when both entities are located at the ending points of their respective acceptance areas (leading to UB)

Therefore, the qualitative sum of the two corresponding intervals δ_i and δ_j, results in a range of qualitative symbols given by:

$$AcAr\,(\varDelta_{i-1} + \varDelta_{j-1}) \ldots AcAr\,(\varDelta_i + \varDelta_j) \tag{1}$$

where $\varDelta_k$ represents the distance from the origin to δ_j, i.e. the sum of consecutive intervals from the origin to δ_k. This concept can be mathematically defined by assuming that δ_+ is the origin of positive values and δ_- is the corresponding origin of negative values, as follows:

$$\forall k = 0, 1, \ldots, n \ \varDelta_k = \begin{cases} \delta_k^e - \delta_+^i \text{ if } \delta_k^e \geq 0 \\ |\delta_k^i - \delta_-^e| \text{ otherwise} \end{cases} \tag{2}$$

Algorithm 1 Qualitative sum

Input: q_i : relationship *b wrt a RS, RS1*
q_j : relationship *c wrt another RS, RS2* (*b* is included into the RS2)
Δr : structure relations
Output: $Result$: disjunction of qualitative symbols for the relationship *c wrt RS1*
BEGIN
if $\Delta_j \ll \delta_i$ **then**
$UB \leftarrow q_i$;
else if $i == max$ **then**
$UB \leftarrow q_i$;
else
$Find\_UB\_qualitative\_sum\,(\Delta_j, \delta_{i+1}, \Delta r, i+1, UB)\,;$
end if
$Find\_LB\_qualitative\_sum\,(\Delta_{j-1}, \delta_i, \Delta r, i, LB)\,;$
$Build\_Result\,(LB, UB, Result)\,;$
END

The developed method proposed to solve the qualitative sum of intervals, sketched in Algorithm 1, is divided into three steps:

1. **Obtaining the Upper Bound (UB) of the result** (see Algorithm 2). It corresponds to the case in which entities b and c are equivalent to the ending points of their respective acceptance areas (see Figure 8b). Under this hypothesis, three different cases can occur:
   - The distance from the origin of qualitative areas to δ_j, Δ_j, is much lower than the interval δ_i. In this case, the absorption rule is applied. This rule, stated in [Clementini et al., 1995], means that whether an interval δ_i is k times greater than another, δ_j, then it can be assumed, without loss of information, that the sum or difference of them is δ_i. Mathematically, this generality is expressed as follows:

$$(\delta_i \gg \delta_j) \Leftrightarrow \delta_i \geq k * \delta_j \Rightarrow \delta_i \pm \delta_j \approx \delta_i \tag{3}$$

     where k is a constant which depends on the context. Thus, if that rule is applied, the interval δ_j will be disregarded *wrt* δ_i. In our algorithm, Δ_j is disregarded *wrt* δ_i such that the UB corresponds to q_i
   - δ_i corresponds to the last defined qualitative area. This fact leads q_i to be the UB of the result
   - Otherwise, an iterative procedure has been defined to recursively search for the minimum qualitative area, δ_k, which satisfies:

$$\Delta_j \leq \delta_{i+1} + \delta_{i+2} + \ldots + \delta_k \Leftrightarrow \Delta_j \leq \Delta_{(i+1)..k} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} \Delta_j \leq \left(\delta_k^e - \delta_{i+1}^i\right) \text{ if } \delta_{i+1}^i \geq 0 \\ \Delta_j \leq \left(|\delta_k^i - \delta_{i+1}^e|\right) \text{ otherwise} \end{cases} \tag{4}$$

It stops when it comes to the last qualitative defined region or when the sum of acceptance areas from the origin to δ_j, i.e. Δ_j, is less than or equal to the sum of acceptance areas starting from δ_{i+1} to δ_k with $k > i$.

Going back to the example shown in Figure 7, suppose that the acceptance areas have been defined such as $\Delta r = \{[0,4], [3,8], [7,15], [13,25], [22,37], [34,\infty[\}$. So, $\Delta_j = \Delta_2 = \delta_2^e - \delta_+^i = 15 - 0 = 15$. Therefore, the algorithm searches for that δ_k that satisfies Equation 4. In this case, $k = 4$ since $\Delta_{(i+1)..(i+1)} = \left(\delta_{i+1}^e - \delta_{i+1}^i\right) = \left(\delta_4^e - \delta_4^i\right) = 37 - 22 = 15$ which is equal to Δ_2. As a result, the UB of the qualitative sum is q_4.

Algorithm 2 Find UB qualitative sum

Input: $\Delta_j : \left(\delta_j^e - \delta_+^i\right)$ or $|\delta_j^i - \delta_-^e|$ if $\delta_j^i \geq 0$ or not, respectively
$\Delta_{inc} : \delta_{i+1} + \delta_{i+2} + \ldots + \delta_k$
Δr : structure relations
k : index of the qualitative area under study (initially $i+1$)
Output: $Result$: upper bound of the disjunction of qualitative symbols for the relationship *c wrt RS1*
BEGIN
if $k == max$ **then**
  $UB \leftarrow q_k$;
else if $\Delta_j \leq \Delta_{inc}$ **then**
  $UB \leftarrow q_k$;
else
  $Find\_UB\_qualitative\_sum\,(\Delta_j, \Delta_{inc} + \delta_{k+1}, \Delta r, k+1, UB)\,;$
end if
END

2. **Obtaining the Lower Bound (LB) of the result**. With this purpose, a recursive function has been implemented (see Algorithm 3). Note that, unlike the previous case, values of entities b and c are supposed to be equivalent to the initial points of their respective acceptance areas (see Figure 8a). Thus, the expression to be satisfied in this case is:

$$\Delta_{j-1} \leq \delta_i + \delta_{i+1} + \ldots + \delta_k \tag{5}$$

It will stop when it comes to the last qualitative region of the structure relations or when the distance from the origin to the qualitative area previous to δ_j, that is, Δ_{j-1}, is less than or equal to the sum of acceptance areas starting from δ_j to δ_k with $k \leq i$. Again, consider the example depicted in Figure 7. Now, $\Delta_{j-1} = \Delta_1 = \delta_1^e - \delta_+^i = 8 - 0 = 8$ is required. And the searched qualitative symbol q_k is provided by the Equation 5. In this case, it is $k = 3$ given that $\Delta_{i..i} = \left(\delta_i^e - \delta_i^i\right) = \left(\delta_3^e - \delta_3^i\right) = 25 - 13 = 12$, that is greater than Δ_1. Consequently, the LB of the qualitative sum for this example is q_3

Algorithm 3 Find LB qualitative sum

Input: $\Delta_{j-1} : \left(\delta_{j-1}^e - \delta_+^i\right)$ or $|\delta_{j-1}^i - \delta_-^e|$ if $\delta_{j-1}^i \geq 0$ or not, respectively
$\Delta_{inc} : \delta_i + \delta_{i+1} + \ldots + \delta_k$
Δr : structure relations
k : index of the qualitative area under study (initially i)

Output: $Result$: lower bound of the disjunction of qualitative symbols for the relationship *c wrt RS1*

BEGIN
if $k == max$ **then**
  $LB \leftarrow q_k$;
else if $\Delta_{j-1} \leq \Delta_{inc}$ **then**
  $LB \leftarrow q_k$;
else
  $Find\_LB\_qualitative\_sum\left(\Delta_{j-1}, \Delta_{inc} + \delta_{k+1}, \Delta r, k+1, LB\right)$;
end if
END

3. **Building the result** (see Algorithm 4). Basically, the implemented procedure provides the list of qualitative regions from the LB to the UB. Again, based on the illustrated example, the resulting disjunct of qualitative symbols expressing the relationship *c wrt RS1* would be $\{q_3, q_4\}$

3.1.3 The Qualitative Difference

When the given relationships are opposite directed, as the example shown in Figure 9, the qualitative difference of intervals must be solved. With this aim, a new method, outlined in Algorithm 5, has been designed. With a similar reasoning mechanism to the qualitative sum, the qualitative difference of two intervals δ_i and δ_j is given by:

$$AcAr\left(\Delta_i - \Delta_j\right) \ldots AcAr\left(\Delta_{i-1} - \Delta_{j-1}\right) \tag{6}$$

Algorithm 4 Build Result

Input: LB : qualitative symbol of the lower bound of the result
UB : qualitative symbol of the upper bound of the result
Output: $Result$: disjunction of qualitative symbols for the relationship *c wrt RS1*
BEGIN
$Result \leftarrow \{\}$;
for $q_k = LB$ TO UB **do**
  $Result \leftarrow Result \cup q_k$;
end for
END

Algorithm 5 Qualitative difference

Input: q_i : relationship *b wrt a RS, RS1*
q_j : relationship *c wrt another RS, RS2* (*b* is included into the RS2)
Δr : structure relations
Output: $Result$: disjunction of qualitative symbols for relationship *c wrt RS1*
BEGIN
if $\Delta_i \geq \Delta_j$ **then**
  if $\Delta_j \ll \delta_i$ **then**
    $LB \leftarrow q_i$;
  else if $i == 0$ **then**
    $LB \leftarrow q_i$;
  else
    $Find\_LB\_qualitative\_difference\,(\Delta_j, \delta_{i-1}, \Delta r, i-1, LB)$;
  end if
  $Find\_UB\_qualitative\_difference\,(\Delta_{j-1}, \delta_i, \Delta r, i, UB)$;
else
  if $\Delta_i \ll \delta_j$ **then**
    $LB \leftarrow q_j$;
  else if $j == 0$ **then**
    $LB \leftarrow q_j$;
  else
    $Find\_LB\_qualitative\_difference\,(\Delta_i, \delta_{j-1}, \Delta r, j-1, LB)$;
  end if
  $Find\_UB\_qualitative\_difference\,(\Delta_{i-1}, \delta_j, \Delta r, j, UB)$;
end if
$Build\_Result\,(LB, UB, Result)$
END

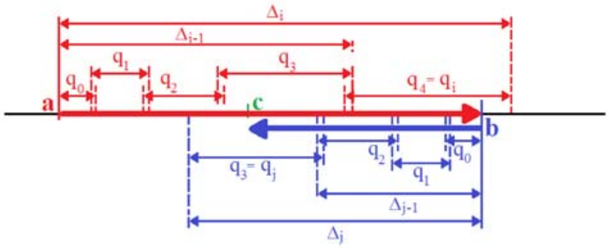


Figure 9: Example of qualitative difference when structure relations with overlapped acceptance areas are used

Nevertheless, with that definition, a bigger amount can be subtracted of a lower one (i.e. obtaining $\Delta_i - \Delta_j$ when $\Delta_j > \Delta_i$). For solving that, the advantage of the commutative property [Clementini et al., 1995] is used. Therefore, two definitions of the resulting range of acceptance areas are distinguished by depending on the relationship between the two amounts Δ_i and Δ_j:

$$\begin{cases} AcAr\left(\Delta_i - \Delta_j\right) \ldots AcAr\left(\Delta_{i-1} - \Delta_{j-1}\right) \text{ when } \Delta_i \geq \Delta_j \\ AcAr\left(\Delta_j - \Delta_i\right) \ldots AcAr\left(\Delta_{j-1} - \Delta_{i-1}\right) \text{ otherwise} \end{cases} \quad (7)$$

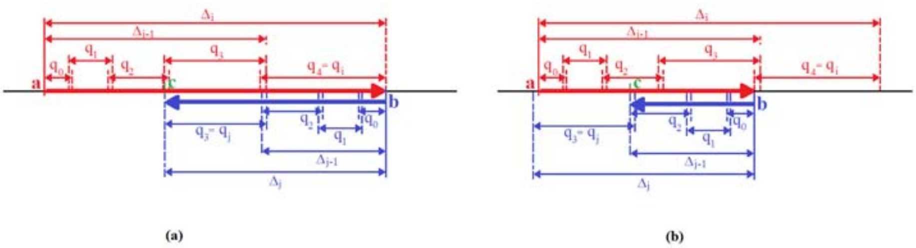


Figure 10: Extreme positions where entities b and c can be located by keeping the relationships *b wrt a reference system* and *c wrt another reference system*. So, (a) refers the case when both entities are located at the ending points of their acceptance areas (leading to LB); while (b) represents the case when both entities are situated at the initial points of their respective acceptance areas (leading to UB)

From that definition, the process to obtain the qualitative difference consists of the following three steps:

1. **Obtaining the upper bound** (Algorithm 6). The UB of the range of acceptance areas is computed considering that the entity values are equivalent to the initial points of their acceptance areas (see Figure 10b). Under this hypothesis, a recursively function that searches for the minimum acceptance area, δ_k, that satisfies the comparison $\Delta_{j-1} \leq \delta_i + \delta_{i-1} + \ldots + \delta_k$, has been implemented. This process will stop when it comes to consider the first region of the relation structure or when the sum of acceptance areas from the origin to δ_j (without including δ_j), i.e. Δ_{j-1}, is less than or equal to the sum of acceptance areas starting from δ_i to δ_j with $k \leq i$.

   As an example, suppose that the acceptance areas have been defined such as $\Delta r = \{[0,4], [3,8], [7,15], [13,25], [22,37], [34,\infty[\}$. So, from the knowledge *b wrt a* = q_4 and *c wrt b* = q_3, *c wrt a* must be inferred. We know $\Delta_{j-1} = \Delta_2 = 15 - 0 = 15$. Thus, the algorithm searches for that δ_k that satisfies $\Delta_{j-1} \leq \delta_i + \delta_{i-1} + \ldots + \delta_k$. In this example, $k = 3$ since $\Delta_{i-1..i} = \Delta_{3..4} = 37 - 13 = 24$ is greater than Δ_2, whereas $\Delta_{i..i} = \Delta_{4..4} = 37 - 22 = 15$ is less than that amount. Consequently, the upper bound of the resulting disjunct of relationships corresponding to *c wrt a* is q_3. Graphically, it can be observed in Figure 10b that the entity c is in the area where the acceptance areas δ_2 and δ_3 overlap. Consequently, as we are searching for the upper bound, the resulting acceptance area for this case is δ_3.

Algorithm 6 Find UB qualitative difference

Input: $\Delta_{j-1} : \left(\delta^e_{j-1} - \delta^i_+\right)$ or $|\delta^i_{j-1} - \delta^e_-|$ if $\delta^i_{j-1} \geq 0$ or not, respectively
$\Delta_{inc} : \delta_i + \delta_{i-1} + \ldots + \delta_k$
Δr : structure relations
k : index of the qualitative area under study (initially i)
Output: $Result$: upper bound of the disjunction of qualitative symbols for the relationship *c wrt RS1*

BEGIN
if $k == 0$ **then**
  $UB \leftarrow q_k;$
else if $\Delta_{j-1} \leq \Delta_{inc}$ **then**
  $UB \leftarrow q_k;$
else
  $Find\_UB\_qualitative\_difference\left(\Delta_{j-1}, \Delta_{inc} + \delta_{k-1}, \Delta r, k-1, UB\right);$
end if
END

2. **Obtaining the lower bound** (Algorithm 7). The LB is computed supposing that the entity values are equivalent to the ending points of their acceptance areas (see Figure 10a). So, as in the case of the upper bound of the qualitative sum, three cases can occur:

   – Whether the absorption rule is satisfied, the LB will be δ_i or δ_j by depending on $\Delta_i \geq \Delta_j$ or $\Delta_j > \Delta_i$ respectively

   – If δ_i (or δ_j when $\Delta_j > \Delta_i$) is the first defined acceptance area, then δ_i (or δ_j) is the LB because there is no any previous area to be considered

   – Otherwise, a recursive backward search among the defined qualitative areas is applied. Its aim is to find the qualitative area δ_k that satisfies the relationship $\Delta_j \leq \delta_{i-1} + \delta_{i-2} + \ldots + \delta_k$ ($\Delta_i \leq \delta_{j-1} + \delta_{j-2} + \ldots + \delta_k$).

   Considering again the illustrated example in Figure 9, we have that $\Delta_j = \Delta_3 = 25 - 0 = 25$ and the searched k is equal to 2, as depicted in Figure 10a, given that $\Delta_{i-1..i-1} = \Delta_{3..3} = 25 - 13 = 12 < \Delta_3$ and $\delta_{i-1} + \delta_{i-2} = \Delta_{i-2..i-1} = \Delta_{2..3} = 25 - 7 = 18$ which is greater than Δ_3. As a result, the lower bound for our example is q_2

Algorithm 7 Find LB qualitative difference

Input: Δ_j : $\left(\delta^e_j - \delta^i_+\right)$ or $|\delta^i_j - \delta^e_-|$ if $\delta^i_j \geq 0$ or not, respectively
Δ_{inc} : $\delta_{i-1} + \delta_{i-2} + \ldots + \delta_k$
Δr : structure relations
k : index of the qualitative area under study (initially $i + 1$)
Output: $Result$: lower bound of the disjunction of qualitative symbols for the relationship *c wrt RS1*

BEGIN
if $k == 0$ **then**
  $LB \leftarrow q_k$;
else if $\Delta_j \leq \Delta_{inc}$ **then**
  $LB \leftarrow q_k$;
else
  $Find\_LB\_qualitative\_difference\left(\Delta_j, \Delta_{inc} + \delta_{k-1}, \Delta r, k-1, LB\right)$;
end if
END

3. **Building the result** (Algorithm 4). The same procedure used for the qualitative sum is applied to obtain the desired result for this operation. In the shown example, the output of this procedure would be $\{q_2, q_3\}$

Take into account that the operation (i.e. qualitative sum or qualitative difference) that solves the reasoning process of any magnitude, mainly depends on two different aspects:

- the sign of the magnitude values (i.e. positive and/or negative)
- the *physical* definition of the magnitude

Hence, as it will be shown with some instances, different magnitudes will require different operations to solve their reasoning process.

3.2 The Complete Inference Process

The Complete Inference Process (CIP) consists of repeating the BSIP as many times as possible with the initial information and the information provided by any previous BSIP until no more information can be inferred. It is necessary when more than two objects (in binary relationships) or three objects (in ternary relationships) are involved in the reasoning mechanism.

As knowledge about relationships between entities is often given in the form of *constraints*, the CIP can be formalized as a *Constraint Satisfaction Problem (CSP)* (see [Westphal and Wölfl, 2009] [Rossi et al., 2006] [Dimopoulos and Stergiou, 2006] [Barták, 2005] [Kumar, 1992] for a survey). Note that a CSP is *consistent* if it has a *solution*. Moreover, a CSP can be represented by a *constraint network* where each node is labelled by a variable X_i or by the variable index i, and each directed edge is labelled by the relationship between the variables it links. Consequently, a path consistency algorithm can be used as a heuristic test for whether the defined constraint network is *consistent* [Allen, 1983], and, therefore, if the CSP has a *solution*. Thus, a number of algorithms for path consistency has been developed from its definition: a constraint graph is *path consistent* if for pairs of nodes (i, j) and all paths $i - i_1 - i_2 - ... - i_n - j$ between them, the direct constraint $c_{i,j}$ is tighter than the indirect constraint along the path, i.e. the composition of constraint $c_{i,i_1} \otimes ... \otimes c_{i_n,j}$ [Frühwirth, 1994a] [Frühwirth, 1994b].

A straight-forward way to enforce path-consistency on a CSP is to strengthen relationships by successively applying the following operation until a fixed point is reached:

$$c_{ij} \; := \; c_{ij} \oplus c_{ik} \otimes c_{kj} \tag{8}$$

where the part ($c_{ik} \otimes c_{kj}$) of the formula computes composition and it obtains the constraint c_{ij}. This result is intersected ($\oplus$) with the preceding computed or user-defined constraints (if they exist). The complexity of such an algorithm is $O\left(n^3\right)$ where n is the number of nodes in the constraint graph [Bessière, 1996] [Mackworth and Freuder, 1985].

It is worth noting that, as pointed out by [Condotta et al., 2006], any CSP solver that uses generalized arc consistency (GAC) as constraint propagation achieves the same pruning as a typical qualitative solver because the GAC is equivalent to path consistency in Qualitative Reasoning (QR), when applied to the finite CSP encoding of a qualitative CSP instance. In addition, given that the efficiency of the constraint propagation is crucial for any constraint solver, [Westphal and Wölfl, 2009] compared the path consistency algorithm from QR to established GAC-variants by concluding that a state-of-the-art implementation of QR methods gives the most efficient performance for classical, small-sized qualitative calculi. Moreover, although the SAT approach provides robust results for specific hard instances, particularly where the path consistency algorithm is known to give bad estimates of satisfiability, it is generally much more resource consuming and highly dependent on the used encoding scheme. For that reason, we have decided to use a CSP solver.

However, although path consistency eliminate some values of variable domains that will never appear in a solution, a search algorithm is still needed to solve the CSP. One way of solving this kind of problems is by means of *Constraint Logic Programming (CLP)* extended with *Constraint Handling Rules* (CHRs). So, on the one hand, CLP is a paradigm based on *First Order Predicate Logic* that combines the declarative of logic programming. Moreover, it provides a means to separate *competence* of a program (also called *logic* or *what*) from *performance* (*control* or *how*) with the efficiency of constraint solving [Smolka, 1994]. The main idea is to replace unification of terms -the heart of a logic programming system- by constraint handling in a constraint domain such that a constraint (or a set of constraints) is satisfied. The scheme is called CLP(X) [Jaffar and Lassez, 1987], where the argument X represents a computational domain such as, for example, reals ($CLP(\Re)$ [Jaffar and Maher, 1992]), rationals ($CLP(Q)$), Boolean constraints including all truth functions (solved by a resolution-based method), temporal intervals ($CLP(Temp)$ [Ibánez, 1994]), integers or finite domains ($CLP(FD)$ [Hentenryck and Deville, 1991]).

Thus, a CLP program is defined as a finite set of clauses, while CHRs are logical formulas which basically define *simplification* and *propagation* over user-defined constraints [Frühwirth, 1994a]. In such way, *simplification* replaces constraints by a simpler constraints while preserving logical equivalence; and *propagation* adds new constraints logically redundant but being able to cause further simplification. So, repeatedly applying CHRs, the constraints are incrementally solved as in a built-in constraint solver. Consequently, CHRs allow the system to faster achieve an answer without backtracking.

In this context, a *Constraint Solver* (CS) is a CLP+CHRs program composed of a finite set of clauses from the CLP language and from the language of CHRs. Given that the BSIP is different from each instance of the general qualitative model and the CIP is the repetition of the BSIP, a different CS will be defined for solving each CIP, although the structure of the program is kept for a complete CLP implementation.

4 Instances of the General Model

With the purpose of checking the performance of the proposed algorithm, three different instances of the general model have been used: qualitative model on naming distance, qualitative velocity and qualitative acceleration. For simplicity, the complete CIP is only outlined, since its complete implementation has been done by means of CLP+CHRs (available in `http://www.robot.uji.es/lab/plone/Members/emartine`).

4.1 Qualitative Model on Naming Distance

Distance is a physical magnitude that expresses the length of the path (the line or the curve) described by an object moving through space. Again, the relationships between entities of this magnitude can be considered as directed vectors. This is specially important for the inference process. To properly cope with that, we start from the definition of the BSIP [Escrig and Toledo, 2001]: given two distances between three spatial objects *a*, *b* and *c*, we want to find the distance between the two objects which is not initially given. It is worth noting that spatial orientation information, or more specifically, directional information about the environment is crucial for establishing spatial location and for path planning. Actually, localization tasks are very fundamental for almost all animals and human beings. As a result, motivated by cognitive considerations, qualitative orientation information [Freksa and Zimmermann, 1992] [Freksa, 1992] and distance have been integrated.

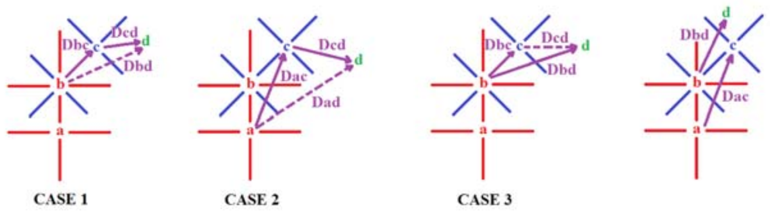


Figure 11: Four different cases in the BSIP for qualitative orientation integrated with qualitative naming distance model

This integration leads to ternary relationships such that four points are required to infer new knowledge. The distance can be measured from the first point of the front/back dichotomy of the RS or from the second point of the front/back dichotomy of the RS. Therefore, it is possible to define four different cases for the reasoning process (see Figure 11 where dashed lines correspond to the distance relationship to be inferred):

1. **CASE 1**: given the distance relationships *c,ab from 1st* (represented in the Figure 11 by *Dbc*) and *d,bc from 1st* (*Dcd*) -i.e. both distances are measured from the first point of the front/back dichotomy of the RS-, the relationship *d,ab from 1st* (*Dab*) is obtained

2. **CASE 2**: given the distance relationships *c,ab from 2nd* (*Dac*) and *d,bc from 1st* (*Dcd*) -i.e. the first distance is measured from the second point of the front/back dichotomy of the RS and the second distance relationship is obtained from the first point of the front/back dichotomy of the RS-, the relationship *d,ab from 2nd* (*Dad*) is found

3. **CASE 3**: given the relationships *c,ab from 1st* (*Dbc*) and *d,bc from 2nd* (*Dbd*) -i.e. the first distance is measured from the first point of the front/back dichotomy of the RS whereas the second relationship is obtained from the second point of the front/back dichotomy of the RS- the relationship *d,bc from 1st* (*Dcd*) is inferred

4. The fourth case occurs when both distances (*Dac* and *Dbd*) are measured from the second point of the front/back dichotomy of the RS. In this case, the distance relationships are independent, therefore it is not possible to derive any further information unless the distance relationship between the entities *a* and *b* is known by means of another relationship

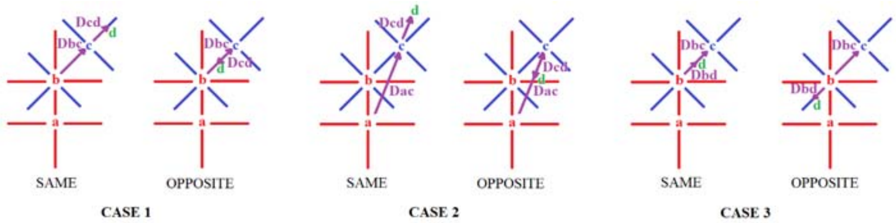


Figure 12: Same and opposite directions for CASE 1, CASE 2 and CASE 3

Therefore, the integration of orientation information about the distance relationships involved in the inference process, influences the distance relationships in the following way (see Figure 12): for *CASE 1* the distance relationship inferred is the qualitative sum of the qualitative distances when the orientations of *Dbc* and *Dcd* are the same, and the qualitative difference of the qualitative distances when these orientations are the opposite. By means of a similar reasoning process, *CASE 2* is solved. In this case, the distances involved are *Dbc* and *Dcd*, although the concepts of *same* and *opposite* orientations have changed. In both cases, the same and opposite orientations will determine the upper and lower bounds for the composition of heterogeneous distance ranges in any

Dbc orientation	Dcd orientation to obtain the SAME direction	Dcd orientation to get the OPPOSITE direction
*	sf	sm, ib, sb

Table 2: Orientation relationships of Dbc and Dcd necessary to obtain the SAME and the OPPOSITE direction for *CASE 1*. Note that (*) refers to all the orientation regions in which the space is divided into

orientation. For *CASE 3*, however, the resulting distance *Dcd* when the distances *Dbc* and *Dbd* are in the same orientation is obtained by solving the qualitative difference between the qualitative distances, whereas when these distances are in the opposite orientation, the qualitative sum of those qualitative distances will be required. In this case, the same orientation will determine the LB and the opposite orientation will determine the UB when the distances are measured from any orientation. This knowledge has been taken into account in the inference algorithms.

As a consequence, the procedure to obtain the resulting distance relationship depends on the orientation relationships. An analysis of the different orientation relationships reveals that, for *CASE 1*, the distance relationships will be in the same direction when the orientation relationship for Dcd is *straight-forward (sf)*, regardless the orientation relationship for Dbc (see Table 2). On the other hand, the opposite direction is obtained when the orientation relationship for Dcd is *straight-middle (sm)*, *identical-back (ib)* or *straight-back (sb)*, whatever is the orientation relationship for Dbc. With a similar reasoning, Table 3 and Table 4 for *CASE 2* and *CASE 3* have been obtained.

Dac orientation	Dcd orientation to obtain the SAME direction	Dcd orientation to get the OPPOSITE direction
*	sf, idf, sm	sb

Table 3: Orientation relationships of Dac and Dcd necessary to obtain the SAME and the OPPOSITE direction for *CASE 2*. Note that (*) refers to all the orientation regions in which the space is divided into

From all this knowledge, the procedure to solve the BSIP is sketched in Algorithm 8. So, the algorithm obtains the inferred qualitative distance information (d_k) with its corresponding orientation (o_k) receiving as input two single qualitative distance symbols (d_i, d_j) with their corresponding single qualitative orientation relationships (o_i, o_j). For that, the procedure firstly determines whether these qualitative distances have the same, opposite or any comparing orientation by checking Tables 2, 3 or 4, depending on the case under study. With the aim of determining the case under study, two parameters are introduced: $FirstOrSecond_i$ and $FirstOrSecond_j$ whose values will be $1st$ or $2nd$, depending on the point of the RS from which the qualitative distances have been measured, that is, the $1st$ or $2nd$ point of the front/back dichotomy.

Dbc orientation	Dbd orientation to obtain the SAME direction	Dbd orientation to get the OPPOSITE direction
lf, l, lm, ibl, bl	rf	lm, ibl, bl
rf, r, rm, ibr, br	lf	rm, ibr, br
sf, sb	sf	sm, ib, sb
sm	sm, ib, sb	sf

Table 4: Orientation relationships of Dbc and Dbd necessary to obtain the SAME and the OPPOSITE direction for *CASE 3*

This task is accomplished by the function $comparing\_orientation$. Then, the disjunction of relationships corresponding to the inferred orientation is obtained by looking up the composition table for the Freksa and Zimmermann's approach [Freksa, 1992] developed by [Escrig and Toledo, 1998]. Note that the composition table for orientation is directly used. The reason lies on the fact that orientation relationships are not defined as intervals and, therefore, the defined algorithms (i.e. *qualitative_sum* and *qualitative_difference*) cannot be used to solve its BSIP.

Afterwards, different algorithms are performed, based on the relative orientation. When the orientations related to the two initial qualitative distances are the same and the opposite, Algorithm 1 or Algorithm 5 will be called by depending on the $Type$. Thus, $Type == 1$ corresponds to *CASE 1* and *CASE 2*, while $Type == 2$ refers to *CASE 3* (see Figure 11). On the contrary, the general case is when the initial qualitative distances are in any other orientation. In this situation, the fact that compositions of distance relationships which have the same and opposite orientations correspond to the extremes of the disjunction of the resulting inferred distance is exploited. So, based on the conditions to obtain the upper and lower bound for each case (i.e. *CASE 1*, *CASE 2* or *CASE 3*), four different methods have been implemented (see Algorithms 9, 10, 11 and 12) such that they will be invoked depending on the case under study.

The performance of the proposed method has been evaluated by means of two different qualitative representations of the distance magnitude defined in [Escrig and Toledo, 1998]. In that way, results automatically obtained were compared with those handwritten in [Escrig and Toledo, 1998], by being exactly the same.

As previously introduced, the computation of the full inference process for positional information is viewed as an instance of the CSP, where the constraints are *special* ternary constraints: $c_{c,ab \text{ from } 1st}$ (which represents the relationship *c, ab from 1st*) and $c_{c,ab \text{ from } 2nd}$ (which represents the relationship *c, ab from 2nd*). They are *special* in the sense that the orientation information, which is included in the positional information, corresponds to ternary constraints ($c_{c,ab}$).

Algorithm 8 BSIP for Qualitative Distance integrated with Qualitative orientation

Input: d_i first distance relationship (Dbc, Dac or Dbc)
o_i : orientation relationship for d_i
$FirstOrSecond_i$: the point of the RS from which d_i has been measured
d_j : second distance relationship (Dcd or Dbd)
o_j : orientation relationship for d_j
$FirstOrSecond_j$: the point of the RS from which d_j has been measured ($1st$ or $2nd$)
Δr : structure relations
$Type$: type of case under study: 1 corresponds to *CASE 1* and *CASE 2,* while a value of 2 refers to *CASE 3*

Output: d_k : the inferred distance relationship (Dbd, Dad or Dbd)
o_k the inferred orientation relationship

BEGIN
$comparing\_orientation(FirstOrSecond_i, o_i, FirstOrSecond_j, o_j, Orientation)$;
$o_k \leftarrow orientation\_composition\_table(o_i, o_j)$;
if $Orientation == same$ **then**
  if $Type == 1$ **then**
    $d_k \leftarrow qualitative\_sum(d_i, d_j, \Delta r)$;
  else
    $d_k \leftarrow qualitative\_difference(d_i, d_j, \Delta r)$;
  end if
else if $Orientation == opposite$ **then**
  if $Type == 1$ **then**
    $d_k \leftarrow qualitative\_difference(d_i, d_j, \Delta r)$;
  else
    $d_k \leftarrow qualitative\_sum(d_i, d_j, \Delta r)$;
  end if
else
  if $Type == 1$ **then**
    $composition\_same\_direction\_UB(d_i, d_j, \Delta r, UB)$;
    $composition\_opposite\_direction\_LB(d_i, d_j, \Delta r, LB)$;
  else
    $composition\_opposite\_direction\_UB(d_i, d_j, \Delta r, UB)$;
    $composition\_same\_direction\_LB(d_i, d_j, \Delta r, LB)$;
  end if
  $Build\_Result(LB, UB, d_k)$;
end if
END

Algorithm 9 composition same direction UB

Input: q_i : first relationship
q_j : the second relationship
Δr : structure relations
Output: $Result$: upper bound of the disjunction of qualitative symbols for the inferred relationship

BEGIN
if $\Delta_j \ll \delta_i$ **then**
  $UB \leftarrow q_i$
else if $i == max$ **then**
  $UB \leftarrow q_i$
else
  $Find\_UB\_qualitative\_sum\,(\Delta_j, \delta_{i+1}, \Delta r, i+1, UB)$
end if
END

Algorithm 10 composition opposite direction LB

Input: q_i : the first relationship
d_j : the second relationship
Δr : structure relations
Output: $Result$: lower bound of the disjunction of qualitative symbols for the inferred relationship

BEGIN
if $\Delta_i \geq \Delta_j$ **then**
  if $\Delta_j \ll \delta_i$ **then**
    $LB \leftarrow q_i$
  else if $i == 0$ **then**
    $LB \leftarrow q_i$
  else
    $Find\_LB\_qualitative\_difference\,(\Delta_j, \delta_{i-1}, \Delta r, i-1, LB)$
  end if
else
  if $\Delta_i \ll \delta_j$ **then**
    $LB \leftarrow q_j$
  else if $j == 0$ **then**
    $LB \leftarrow q_j$
  else
    $Find\_LB\_qualitative\_difference\,(\Delta_i, \delta_{j-1}, \Delta r, j-1, LB)$
  end if
end if
END

Algorithm 11 composition opposite direction UB

Input: q_i : first relationship
q_j : the second relationship
Δr : structure relations
Output: $Result$: upper bound of the disjunction of qualitative symbols for the inferred relationship
BEGIN
if $\Delta_i \geq \Delta_j$ **then**
$Find\_UB\_qualitative\_difference\,(\Delta_{j-1}, \delta_i, \Delta r, i, UB)\,;$
else
$Find\_UB\_qualitative\_difference\,(\Delta_{i-1}, \delta_j, \Delta r, j, UB)\,;$
end if
END

Algorithm 12 composition same direction LB

Input: q_i : the first relationship
d_j : the second relationship
Δr : structure relations
Output: $Result$: lower bound of the disjunction of qualitative symbols for the inferred relationship
BEGIN
$Find\_LB\_qualitative\_sum\,(\Delta_{j-1}, \delta_i, \Delta r, i, LB)\,;$
END

Particularly, the operation to compute path-consistency for positional information together with orientation information is defined in the following way:

$$\begin{cases} c_{d,ab \text{ from } 1st} := c_{d,ab \text{ from } 1st} \oplus c_{c,ab \text{ from } 1st} \otimes c_{d,bc \text{ from } 1st} \\ c_{d,ab \text{ from } 2nd} := c_{d,ab \text{ from } 2nd} \oplus c_{c,ab \text{ from } 2nd} \otimes c_{d,bc \text{ from } 1st} \\ c_{d,bc \text{ from } 1st} := c_{d,bc \text{ from } 1st} \oplus c_{c,ab \text{ from } 1st} \otimes c_{d,bc \text{ from } 2nd} \end{cases} \tag{9}$$

4.1.1 A Practical Application

With the purpose of validating the performance of the proposed method, we have implemented it on a real robot. The goal is developing a perceptual system capable of achieving a full $3D$ awareness for interaction control/planning in the surrounding space by using the interplay between vision and motion.

In a first step to this end, based on the importance of relative disparity between objects for accurate hand-eye coordination, we developed a virtual reality environment that implements robotic reaching tasks from stereo visual cues [Martínez-Martín et al., 2011]. Going furher away, we propose here a qualitative classification of the objects observed in a scene with the purpose of obtaining positional and orientational information of an object to be grasped.

For that, a humanoid torso endowed with a pan-tilt-vergence stereo head and two multi-joint arms (see Figure 13). The head mounts two cameras with a resolution of 1024x768 pixels that can acquire colour images at 30 Hz. The baseline between cameras is 270 mm and the motor positions are provided by high resolution optical encoders.

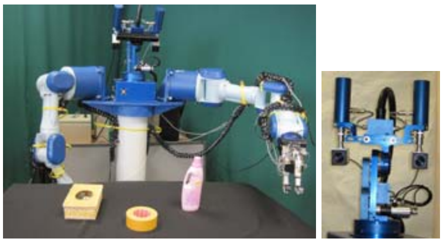


Figure 13: Experimental set-up: external view of the used humanoid robot (left) and a detail view of pan/tilt/vergence head (right)

So, from the disparity map obtained by using the disparity approach proposed in [Martínez-Martín et al., 2011], the system can determine the different distance and orientation relationships between the objects in the scene and, with that information, being able to achieve the task at hand. So, the system is initially focused on a reference object whose location in the scene is known (see Figure 14). Then, the system is visually focused on its arm such that the reference object is also visible in the image. In that way, the system has qualitative naming distance relationships from its arm to the reference object and from the reference object to the target object. Using the reasoning mechanism implemented in the previous section, the system is able to know the distance and orientation relationship between its body and the target object. That knowledge is finally used to estimate the system's motion in order to approximate to the target object.

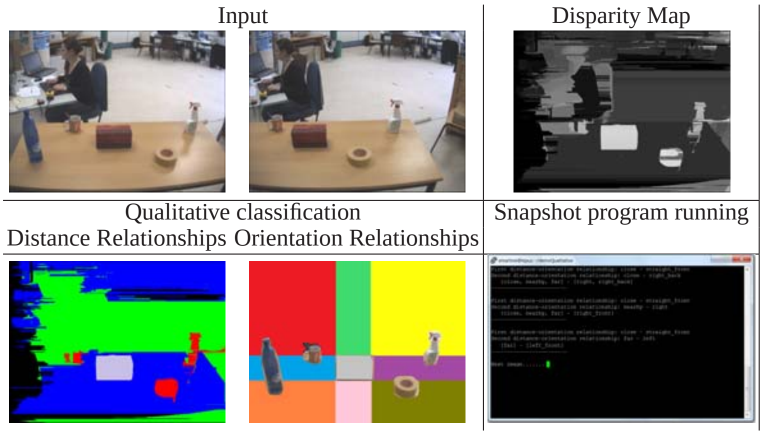


Figure 14: Results obtained with the real robot when the qualitative naming distance model proposed in the previous section has been used. In this case, distance relationships are determined by $\Delta r = \{[0, 60[\, , [60, 200[\, , [200, 255]\}$ and $Q = \{$closer (c), nearby (n), further (f)$\}$ coded in the image by red, green and blue respectively, while the reference object is coloured by gray. On the other hand, orientation relationships correspond to Freksa and Zimmermann's approach [Freksa, 1992] such that lf is coded by red, sf by green, rf by yellow, l by blue, idef by gray, r by purple, lm by orange, sm by rose and rm by olive

4.2 Qualitative Velocity

The velocity is the physical concept that measures the distance travelled by an object per unit of time. From a physical point of view, this concept is defined as:

$$Velocity = \frac{Space}{Time} \tag{10}$$

So, the BSIP for the concept of velocity can be defined as: given two velocity relationships between three spatio-temporal entities *a*, *b* and *c*, we want to find the velocity relationship between the two entities which is not initially given. However, it is important to take into account that the relative movement of the implied objects can be in any direction. For that reason, the BSIP is studied integrating the velocity concept with a qualitative orientation model. Note that, for this case, the qualitative orientational model of Freksa and Zimmerman [Freksa and Zimmermann, 1992, Freksa, 1992] has been redefined as shown in Figure 15 since the reference object is always in the b point. In that way, it is possible to reason with the extreme angles which define the INT part of the orientation reference system (ORS).

From that definition, we have developed the algorithm sketched in Algorithm 13 such that the specific cases are graphically shown in Figure 16. So, it is clear that when any velocity relationship is zero, both velocity and orientation will be equal to the other involved relationship. When the two velocity relationships have the same orientation, the resulting relationship has the same orientation and its value corresponds to the qua-

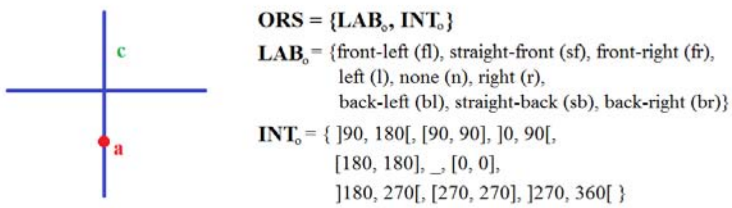


Figure 15: Redefinition of the qualitative orientational model of Freksa and Zimmerman [Freksa and Zimmermann, 1992] [Freksa, 1992]

litative sum of both relationships. On the contrary, if the relationships has an opposite orientation, the resulting relationship will be obtained as their qualitative difference and its orientation will be equal to that of higher velocity value. On the other hand, in the case both relationships have the same orientation but it corresponds to an open interval, the resulting relationship has the same orientation, although its value will be a disjunction of velocity relationships from the result of applying the pythagorean theorem to the UB of the qualitative sum. When the orientation relationships corresponds to an open and a close interval such that one extreme of an interval matches up with an extreme of the other interval, then the resulting relationship will have the orientation of the open interval, while its value will be obtained from the pythagorean theorem and the qualitative sum. The last special case refers to the case two orientation relationships are perpendicular. In that situation, the resulting relationship results of the pythagorean theorem, whereas its orientation is the orientation between the orientations of the initial relationships. Finally, the remaining situations are solved by means of qualitative difference and the pythagorean theorem. With regard to its orientation, it corresponds to all the possible orientation relationships.

Again, the performance of the proposed method has been tested by comparing the results with those obtained by hand. The results obtained for the same orientation have been compared to the handwritten ones [Escrig and Toledo, 2002] by being the same.

Algorithm 13 BSIP for Qualitative Velocity integrated with Qualitative orientation

Input: v_i first velocity relationship; o_i : orientation relationship for v_i
v_j : second velocity relationship; o_j : orientation relationship for v_j
Δr_v : velocity structure relations; Δr_o : orientation structure relations
Output: v_k, o_k : the inferred velocity and orientation relationships, respectively

```
BEGIN
if v_i == zero_velocity then v_k ← v_j;  o_k ← o_j;
else if v_j == zero_velocity then v_k ← v_i;  o_k ← o_i;
else if same_orientation(o_i, o_j) then
  v_k ← qualitative_sum(v_i, v_j, Δr_v);  o_k ← o_i;
else if opposite_orientation(o_i, o_j) then
  v_k ← qualitative_difference(v_i, v_j, Δr_v);
  if Δ_{o_i} > Δ_{o_j} then o_k ← o_i;
  else if Δ_{o_i} < Δ_{o_j} then o_k ← o_j;
  else o_k ← none;
else if same_qualitative_orientation(o_i, o_j) then
  if Δ_{v_j} ≪ δ_{v_i} then UB_{v_k} ← v_i;
  else if i == max then UB_{v_k} ← v_i;
  else Find_UB_qualitative_sum (Δ_{v_j}, δ_{v_{i+1}}, Δr_v, i + 1, UB_{v_k}) ;
  LB_{v_k} ← pythagorean_theorem_LB(v_i, v_j);
  Build_Result(LB_{v_k}, UB_{v_k}, v_k);  o_k ← o_i;
else if extreme_coincidence(o_i, o_j) then
  if Δ_j ≪ δ_i then UB_{v_k} ← v_i;
  else if i == max then UB_{v_k} ← v_i;
  else Find_UB_qualitative_sum (Δ_{v_j}, δ_{v_{i+1}}, Δr, i + 1, UB_{v_k}) ;
  LB_{v_k} ← pythagorean_theorem_LB(v_i, v_j);
  Build_Result(LB_{v_k}, UB_{v_k}, v_k);  o_k ← open_interval(o_i, o_j);
else if perpendicular_orientation(o_i, o_j) then
  v_k ← pythagorean_theorem(v_i, v_j); o_k ← intermediate_orientation(o_i, o_j);
else
  if Δ_{v_i} ≥ Δ_{v_j} then
    if Δ_{v_j} ≪ δ_{v_i} then LB ← v_i;
    else if i == 0 then LB ← v_i;
    else Find_LB_qualitative_difference (Δ_{v_j}, δ_{v_{i-1}}, Δr_v, i − 1, LB_{v_k}) ;
  else
    if Δ_{v_i} ≪ δ_{v_j} then LB ← q_j
    else if j == 0 then LB ← q_j
    else Find_LB_qualitative_difference (Δ_{v_i}, δ_{v_{j-1}}, Δr_v, j − 1, LB) ;
  UB_{v_k} ← pythagorean_theorem_UB(v_i, v_j);
  Build_Result(LB_{v_k}, UB_{v_k}, v_k);  o_k ← all_orientation_relationships();
END
```

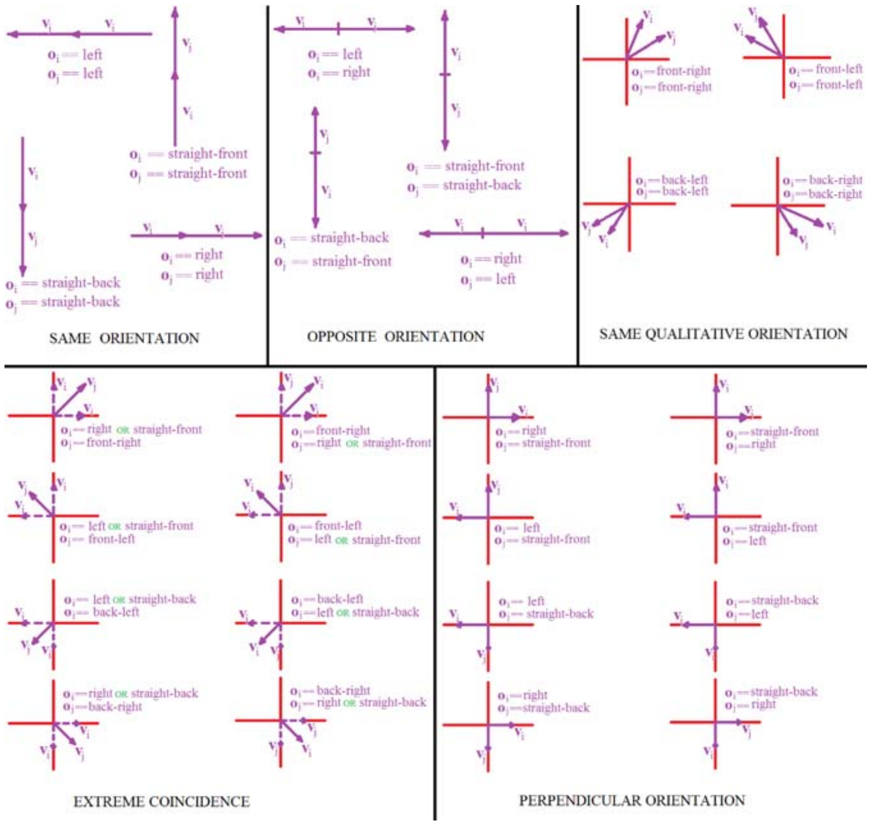


Figure 16: Orientation distinctions done to properly solve the BSIP (named in the same way that they appear in the Algorithm 13)

From the BSIP definition, the CIP can be defined. Anagously to the previous case, the computation of the full inference process for qualitative velocity can be viewed as an instance of the CSP. So, in order to determine whether a graph is complete we repeatedly compute the following operation:

$$c_{x,y} := c_{x,y} \oplus c_{x,z} \otimes c_{z,y} \quad (11)$$

until a fixed point is reached.

4.2.1 A Practical Application

Again, a real application of the proposed method is presented. In this case, the qualitative velocity model has been implemented on a mobile robot. The aim of this system is to assist human beings in performing a variety of tasks such as carrying person's tools or delivering parts. One of the major requirements of such robotic assistants is the ability to track and follow a moving person through a non-predetermined, unstructured environment. To achieve this goal, two different tasks have to be carried out: person recognition and segmentation from the surrounding environment, and motion control to follow the person using the recognition results. In particular, in this section, we proposed a qualitative reasoning method to achieve the second task to be performed.

For that, an indoor pan-tilt-zoom (PTZ) camera was mounted on a Pioneer 3-DX mobile platform [Adept-Technology, 2004] without restricting its autonomy and flexibility as depicted in Figure 17. The core of the PTZ system is a Canon VC-C4 analog colour camera [Canon, 2001] with a resolution of $320x240$ pixels, which is integrated with the mobile platform hardware.

Figure 17: Experimental set-up: external view of the used mobile platform (left) and a more detailed view of the camera (right)

So, on the one hand, the system knows both its velocity and its orientation through the information obtained from its motors. On the other hand, an image processing based on optical flow provides an estimation of the velocity and orientation relationships corresponding to the person to be followed. Therefore, from these two relationships, the system is able to determine the required velocity-orientation relationship in order to properly follow and assist that person. An example of the obtained results can be seen in Figure 18.

4.3 Qualitative Acceleration

Finally, the acceleration is another physical concept that measures how an object's speed or direction changes over time, in other terms, acceleration is the rate of change of velocity as a function of time. Physically, it can be defined as:

$$Acceleration = \frac{Velocity}{Time} = \frac{Space}{Time^2} \tag{12}$$

This definition is very similar to the velocity one. The main difference is the fact that the values for the intervals in which the workspace is divided into, can be both positives

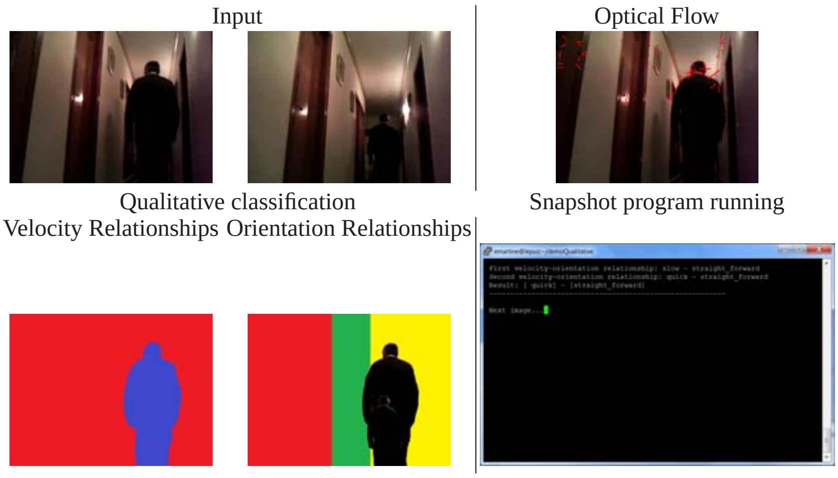


Figure 18: Results obtained with the real robot when the qualitative velocity model proposed in the previous section has been used. In this case, velocity relationships are labelled as $Q = \{$zero, slow, normal, quick$\}$ coded in the image by purple, red, green and blue respectively. On the other hand, orientation relationships correspond to the modified Freksa and Zimmermann's approach such that fl is coded by red, sf by green, fr by yellow, l by blue, r by purple, bl by orange, sb by rose and br by olive

and negatives, whereas only positive values are possible for the velocity magnitude. And, another thing to consider is the existing relationship between acceleration and time which is different to the defined by the velocity model.

So, from the definition of the BSIP: given two acceleration relationships between three spatio-temporal entities *a*, *b* and *c*, we want to find the acceleration relationship between the two points which is not initially given. Again, the relative movement of the implied entities can be in any direction. Therefore, it is necessary to integrate a qualitative orientation model. As in the previous qualitative model, we have used the modified Freksa and Zimmerman's approach. An algorithm very similar to that sketched in Algorithm 13 has been developed to solve the acceleration BSIP.

The next step is the computation of the full inference process. As foregoing introduced, it can be viewed as an instance of the CSP such that the determination of a graph completeness is obtained by repeatedly computing the following operation:

$$c_{x,y} := c_{x,y} \oplus c_{x,z} \otimes c_{z,y} \tag{13}$$

until a fixed point is reached.

5 Conclusions and Future Work

In this paper, we have presented a general framework that allows investigators from other disciplines to easily incorporate the required reasoning techniques into their methods. For that, we have developed a general systemic algorithm that integrates and solves the reasoning process of all qualitative models based on intervals. Note that the development of such a method has several advantages such as it does not require to represent the relationships for any composition.

So, from the starting point that the development of any qualitative model consists of a representation of the magnitude at hand and the reasoning process, we have designed a general, abstract magnitude representation and a general method for solving the reasoning process based on the definition of two algorithms: the qualitative sum and the qualitative difference.

In addition, focused on assessing the method's performance, we have used three different magnitudes: (1) naming distance, (2) qualitative velocity and (3) qualitative acceleration, by obtaining the same results as those handwritten when they existed [Escrig and Toledo, 1998, Escrig and Toledo, 2002]. Furthermore, we have presented two different real robotic applications. In that way, robots have been provided with intelligent abilities to solve service robotics problems such as grasping or navigation.

Note that, in this paper, we have deeply analysed the Basic Step of the Inference Process (BSIP) of the different instances. Moreover, we have given the corresponding definitions to compute the path-consistency since these problems have been formalized as a Constraint Satisfaction Problem (CSP). Although, due to lack of space, the complete implementation has not been included in this paper, it can be found in `http://www.robot.uji.es/lab/plone/Members/emartine`.

As a future work we will investigate the development of new qualitative models based on intervals of aspects such as: time, weight, body sensations (such as hunger, sleepiness, tiredness, love, etc.), etc.

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