DISJOINTNESS PRESERVING MAPPINGS BETWEEN BSE DITKIN ALGEBRAS *

JUAN J. FONT†

Abstract

Let A and B be regular Banach function algebras. A linear map T defined from A into B is said to be *disjointness preserving or separating* if $f \cdot g \equiv 0$ implies $T(f) \cdot T(g) \equiv 0$ for all $f, g \in A$. We prove that if there exists a disjointness preserving bijection between two BSE Ditkin algebras with a BAI, then they are isomorphic as algebras. As a corollary we can deduce that two of these algebras are algebraically isomorphic if there exists a surjective isometry between them for the supremum norm.

1.- Introduction.

Since the 40's, when disjointness preserving mappings began to be used, many authors have studied them on several contexts. Among others, on Banach lattices (see e.g. [1], [2] or [6]), on spaces of continuous functions (see e.g. [14], [3], [7], [15] or [12]), on group algebras of locally compact Abelian groups ([8]), on Fourier algebras ([10] and [20]) and on some others (see e.g. [16], [17] or [5]).

In [9], we extended the definition of disjointness preserving mappings to the class of regular Banach function algebras. Let us recall that a linear map T defined from a regular Banach function algebra A into such an algebra B is said to be *disjointness preserving or separating* if $f \cdot g \equiv 0$ implies $T(f) \cdot T(g) \equiv 0$ for all $f, g \in A$.

In [8] we proved that the existence of a disjointness separating bijection between the group algebras of two locally compact Abelian groups implies that these algebras are algebraically isomorphic. A similar result was obtained

---

* *2000 Mathematics Subject Classification:* 43A15, 46J10, 47B48.

†Research partially supported by the Spanish Ministry of Science and Education (Grant number MTM2008-04599), and by Bancaixa (Projecte P1-1B2008-26).

in [10] (resp. [20]) for Fourier algebras (resp. generalized Fourier algebras) of amenable locally compact groups.

In this paper we extend the above results to a wider class of regular Banach function algebras which includes group algebras and Fourier algebras: the class of BSE Ditkin algebras with a BAI (bounded approximate identity). Let us recall that BSE algebras were introduced in [21] (see the definition in section 3) motivated by the Bochner-Schoenberg-Eberlein characterization of the Fourier-Stieltjes transforms of measures on a locally compact abelian group. BSE Ditkin algebras with a BAI has recently attracted the attention of some authors. For example, one of the main results in [22] consists of an abstract analog of Cohen's Idempotent Theorem for such type of Banach algebras.

We prove here that if there exists a disjointness preserving bijection between two BSE Ditkin algebras with a BAI, then they are isomorphic as algebras. As a corollary we can deduce that two BSE Ditkin algebras with a BAI are algebraically isomorphic if there exists a surjective supremum norm isometry between them.

2.- Background.

Let $(A, \|\cdot\|)$ be a commutative Banach algebra which may or may not have an identity element. Let Φ_A be the (locally compact) *structure space* of A. The *Gelfand transform* of $f \in A$ is denoted by $\hat{f}$. $\widehat{A}$ will stand for the point-separating subalgebra of $C_0(\Phi_A)$ consisting of all $\hat{f}$, $f \in A$.

Next we gather the main results concerning disjointness preserving maps between regular Banach function algebras, which can be found in [9]:

In the sequel, let A and B be regular semisimple commutative Banach algebras, which is to say, regular Banach function algebras. Associated with a disjointness preserving map $T : A \longrightarrow B$, we can define a linear mapping $\hat{T} : \widehat{A} \longrightarrow \widehat{B}$ as $\hat{T}(\hat{f}) := \widehat{T(f)}$ for every $f \in A$. Since A and B are semisimple, it is easy to check that T is disjointness preserving if and only if $\hat{T}$ is disjointness preserving. In like manner, T is injective (resp. surjective) if and only if $\hat{T}$ is injective (resp. surjective).

If $\gamma \in \Phi_B$, let $\delta_\gamma \circ \hat{T} : \widehat{A} \to \mathbf{C}$ be the functional defined as $(\delta_\gamma \circ \hat{T})(\hat{f}) := \hat{T}(\hat{f})(\gamma)$ for all $f \in A$.

In general, a disjointness preserving map $T : A \longrightarrow B$ induces a continuous mapping h of Φ_B into $\Phi_A \cup \{\infty\}$, which may make no sense if A and B are not regular. We call h the *support map* of T. If T is continuous, then it is a *weighted composition map*; i.e., $(\delta_\gamma \circ \hat{T})(\hat{f}) = \hat{T}(\hat{f})(\gamma) = \kappa(\gamma)\hat{f}(h(\gamma))$ for all $\gamma \in \Phi_B$ and all $f \in A$, where the weight function $\kappa : \Phi_B \to \mathbf{C}$ is continuous, and the range of h is contained in Φ_A. If, in addition, T is surjective, then the point-separating property of $\widehat{B}$ easily implies that κ is nonvanishing on Φ_B.

The main result in [9] is the following:

Theorem 1 *Let $T : A \longrightarrow B$ be a disjointness preserving bijection. If A satisfies Ditkin's condition (i.e., if A is a Ditkin algebra), then*

1. *T is continuous*
2. *T^{-1} is disjointness preserving.*
3. *If also B satisfies Ditkin's condition, then the support map of T, h, is a homeomorphism of Φ_A onto Φ_B.*

As a consequence of this theorem and the above paragraphs, if there exists a disjointness preserving bijection T of A onto B, then $\hat{T}(\hat{f})(\gamma) = \kappa(\gamma)\hat{f}(h(\gamma))$ for all $f \in A$ and all $\gamma \in \Phi_B$. Since T^{-1} is also disjointness preserving and, consequently, continuous, we can write $\hat{T}^{-1}(\hat{g})(\zeta) = \Psi(\zeta)\hat{g}(h^{-1}(\zeta))$ for all $g \in B$ and all $\zeta \in \Phi_A$, where h^{-1} can be proved to be the inverse of the homeomorphism h. We will call $\kappa \in C(\Phi_B)$ and $\Psi \in C(\Phi_A)$ the *weight functions associated to T*.

3.- The results.

Let $\mathcal{A}$ be a semisimple commutative Banach algebra. A multiplier T on $\mathcal{A}$ is a bounded linear operator on $\mathcal{A}$ into itself which satisfies $T(f \cdot g) = f \cdot T(g) = T(f) \cdot g$ for all $f, g \in \mathcal{A}$. $M(\mathcal{A})$ denotes the commutative Banach algebra consisting of all multipliers on $\mathcal{A}$. By [18, Corollary 1.2.1], we may identify $M(\mathcal{A})$ with the normed algebra of all bounded continuous functions ϕ on $\Phi_{\mathcal{A}}$ such that $\phi\hat{\mathcal{A}} \subset \hat{\mathcal{A}}$. It is then apparent that multipliers are examples of disjointness preserving mappings.

Theorem 2 *Let A and B regular semisimple commutative Banach algebras. Then A and B are (algebra) isomorphic if and only if there exists a continuous disjointness preserving bijection between them whose (associated) weight functions are multipliers.*

Proof. Let us suppose that there exists a continuous disjointness preserving bijection T of A onto B. First we claim that $(\hat{g} \circ h^{-1}) \in \widehat{A}$ for all $g \in B$. To prove this, let $\zeta \in \Phi_A$ and $f \in A$ such that $\hat{f}(\zeta) = 1$. Hence

$$\begin{aligned} 1 = \hat{f}(\zeta) &= \hat{T}^{-1}(\hat{T}(\hat{f}))(\zeta) \\ &= \Psi(\zeta) \cdot \hat{T}(\hat{f})(h^{-1}(\zeta)) \\ &= \Psi(\zeta) \cdot \kappa(h^{-1}(\zeta)) \cdot \hat{f}(h(h^{-1}(\zeta))) \\ &= \Psi(\zeta) \cdot \kappa(h^{-1}(\zeta)); \end{aligned}$$

that is, $\Psi(\zeta) \cdot \kappa(h^{-1}(\zeta)) = 1$ for all $\zeta \in \Phi_A$. On the other hand, from the fact that $\widehat{B}$ is an ideal in $M(B)$ (see [18]) and since, by hypothesis, $\kappa : \Phi_B \to \mathbf{C}$ belongs to $M(B)$, we infer that $\kappa \cdot \kappa \cdot (\hat{f} \circ h)$ belongs to $\widehat{B}$ for every $f \in A$. Consequently,

$$\begin{aligned} \hat{T}^{-1}(\kappa \cdot \kappa \cdot (\hat{f} \circ h))(\zeta) &= \Psi(\zeta) \cdot \kappa(h^{-1}(\zeta)) \cdot \kappa(h^{-1}(\zeta)) \cdot \hat{f}(h(h^{-1}(\zeta))) \\ &= \kappa(h^{-1}(\zeta)) \cdot \hat{f}(\zeta) \end{aligned}$$

for all $\zeta \in \Phi_A$. This implies that the function $(\kappa \circ h^{-1}) \cdot \hat{f}$ belongs to $\widehat{A}$ for all $f \in A$, which is to say that $(\kappa \circ h^{-1})$ belongs to $M(A)$. Hence, since $\widehat{A}$ is an ideal in $M(A)$ and the function $\Psi \cdot (\hat{g} \circ h^{-1})$ belongs to $\widehat{A}$, we have that $(\kappa \circ h^{-1}) \cdot \Psi \cdot (\hat{g} \circ h^{-1}) = (\hat{g} \circ h^{-1})$ belongs to $\widehat{A}$ for all $g \in B$.

In like manner, we can prove that $\hat{f} \circ h$ belongs to $\widehat{B}$ for all $f \in A$. Hence, it is now clear, since $h : \Phi_B \longrightarrow \Phi_A$ is a homeomorphism, that the mapping $\hat{T}_h : \widehat{A} \longrightarrow \widehat{B}$, defined as $\hat{T}_h(\hat{f}) := \hat{f} \circ h$, is a surjective algebra isomorphism, which, by semisimplicity, provides the desired algebra isomorphism of A onto B.

The converse is clear. □

Theorem 3 *Let A and B be Ditkin algebras. Then A and B are (algebra) isomorphic if and only if there exists a disjointness preserving bijection between them whose weight functions are multipliers.*

Proof. Combine Theorems 1 and 2. □

Next we show that Ditkin algebras with a BAI have local units thanks to the Cohen Factorization Theorem ([13]).

Proposition 1 *Let A be a Ditkin algebra which has an approximate identity of bound b. Then for each compact $K \subset \Phi_A$ and each $\epsilon > 0$ there exists $k \in A$ such that $\hat{k}$ has compact support, $\hat{k} \equiv 1$ on K and $\|k\| < b + \epsilon$.*

Proof. Since A is regular, we can find $f \in A$ such that $\hat{f} \equiv 1$ on K. By Cohen Factorization Theorem, given $\delta > 0$, we can write $f = f_1 f_2$, where $f_1, f_2 \in A$, $\|f_1\| \leq b$ and $\|f - f_2\| < \delta$. Hence, if we define $g_1 := f_1 - f_1(f - f_2)$, then $\hat{g}_1 \equiv 1$ on K and $\|g_1\| < b(1+\delta)$. By [19, p. 205], we know that there exists $g_2 \in A$ such that $\hat{g_2}$ has compact support and $||g_1 - g_2|| < \delta$. Hence we can now define the following function in A:

$$k = g_2 \sum_{n=0}^{\infty} (g_1 - g_2)^n.$$

It is apparent that $\hat{k}$ has compact support and that, if $x \in K$, then

$$\hat{k}(x) = \hat{g}_2(x) \frac{1}{1 - \hat{g}_1(x) + \hat{g}_2(x)} = 1.$$

Furthermore, by choosing an appropiate δ,

$$\|k\| \leq \frac{b(1+2\delta)}{1-\delta} < b + \epsilon$$

as was to be proved.

□

Let A be a commutative Banach algebra. A complex-valued function κ on $\Phi_{\mathcal{A}}$ is said to satisfy the BSE-condition if there exists $C > 0$ such that, for every finite collection $c_1, ..., c_n$ of complex numbers and $\alpha_1, ..., \alpha_n$ in $\Phi_{\mathcal{A}}$,

$$|\sum_{j=1}^{n} c_j \kappa(\alpha_j)| \leq C \left\| \sum_{j=1}^{n} c_j \alpha_j \right\|_{A^*}$$

where A^* denotes the dual space of A. This condition is motivated by the Bochner-Schoenberg-Eberlein theorem, which characterizes the Fourier-Stieltjes transforms of measures on a locally compact abelian group. A group algebra A is called a BSE-algebra ([21]) if the continuous functions on $\Phi_{\mathcal{A}}$ satisfying the BSE-condition are precisely the functions of the form $\hat{w}$ where $w \in M(A)$.

Lemma 1 *Let A be a Ditkin algebra with BAI and B a BSE Ditkin algebra. Let $T : A \longrightarrow B$ be a disjointness preserving bijection. Then the weight function κ belongs to $M(B)$.*

Proof. Let $\{\alpha_1, ..., \alpha_n\}$ be a subset of $\Phi_{\mathcal{B}}$ and $\epsilon > 0$. By Proposition 1, there exists $f \in A$ such that $\|f\| < b + \epsilon$ and $\hat{f}(h(\alpha_i)) = 1$ for $i = 1, ..., n$.

Let $\{c_1, ..., c_n\} \subset \mathbf{C}$. Then, since $\hat{T}$ is continuous (Theorem 1 (1)), we have

$$\begin{aligned}\left|\sum_{i=1}^{n} c_i \cdot \kappa(\alpha_i)\right| &= \left|\sum_{i=1}^{n} c_i \cdot \hat{T}(\hat{f})(\alpha_i)\right| \\ &\leq \left\|\hat{T}(\hat{f})\right\| \left\|\sum_{i=1}^{n} c_i \delta_{\alpha_i}\right\|_{A^*} \\ &\leq \left\|\hat{T}\right\| (b+\epsilon) \left\|\sum_{i=1}^{n} c_i \delta_{\alpha_i}\right\|_{A^*}\end{aligned}$$

Consequently, κ satisfies the BSE-condition and, as B is a BSE algebra, $\kappa \in M(B)$.

□

Theorem 4 *Let A and B be BSE Ditkin algebras with BAI. Then A and B are algebra isomorphic if and only if there exists a disjointness preserving bijection between them.*

Proof. It is a straightforward consequence of Lemma 1 and Theorem 3.

□

Corollary 1 *Let A and B be BSE Ditkin algebras with BAI. Then A and B are algebra isomorphic if and only if $\hat{A}$ and $\hat{B}$ are $\|\cdot\|_\infty$-isometric; i.e., there exists a linear bijection T of A onto B such that $\left\|\hat{f}\right\|_\infty = \left\|\hat{T}(\hat{f})\right\|_\infty$ for all $f \in A$.*

Proof. By [4, Theorem 4.1 and Lemma 2.1]) we know that

$$\partial B = \bigcup_{\zeta \in \partial A} \{\gamma \in \Phi_B : \left|\hat{f}(\zeta)\right| = \left|\hat{T}(\hat{f})(\gamma)\right| \text{ for all } f \in A\},$$

where ∂A and ∂B stand for the Shilov boundaries of $\hat{A}$ and $\hat{B}$ respectively. But, since $\hat{A}$ is a regular subalgebra of $C_0(\Phi_A)$, it is well known that the Shilov boundary of $\hat{A}$ coincides with Φ_A. Hence, we indeed have

$$\Phi_B = \bigcup_{\zeta \in \Phi_A} \{\gamma \in \Phi_B : \left|\hat{f}(\zeta)\right| = \left|\hat{T}(\hat{f})(\gamma)\right| \text{ for all } f \in A\}.$$

The remainder of the proof consists of checking that T is disjointness preserving and applying Theorem 4. Assume, contrary to what we claim, that there are $\hat{f}, \hat{g} \in A$ with disjoint cozero sets such that $\hat{T}(\hat{f}) \cdot \hat{T}(\hat{g}) \not\equiv 0$. Let us choose $\gamma_0 \in \Phi_B$ such that $\left|\hat{T}(\hat{f})(\gamma_0)\right| > 0$ and $\left|\hat{T}(\hat{g})(\gamma_0)\right| > 0$. In virtue of the paragraph above, there exists $\zeta_0 \in \Phi_A$ such that $\left|\hat{f}(\zeta_0)\right| = \left|\hat{T}(\hat{f})(\gamma_0)\right|$ for all $f \in A$. Since the cozero sets of $\hat{f}$ and $\hat{g}$ are disjoint, we have that either $\hat{f}(\zeta_0) = 0$ or $\hat{g}(\zeta_0) = 0$, which yields that either $\hat{T}(\hat{f})(\gamma_0) = 0$ or $\hat{T}(\hat{f})(\gamma_0) = 0$. This contradiction proves that T is disjointness preserving.

$\square$

Remark 1 The above corollary is not true for general Banach function algebras. Indeed, H^∞, the Banach algebra of bounded analytic functions on the open unit disk, and H_0^∞, the subalgebra of all elements in H^∞ which vanish at the origin, are isometric but are not algebraically isomorphic.

A similar situation can be found in [11], where the authors provide two isometric semisimple commutative Banach algebras which are not isomorphic as Banach algebras.

References

[1] Y. Abramovich, *Multiplicative representation of disjointness preserving operators.* Indag. Math. **45** (1983), 265-279.

[2] Y. Abramovich and A.K. Kitover, *Inverses of disjointness preserving operators.* Mem. Amer. Math. Soc. **n. 679** (2000).

[3] J. Araujo, E. Beckenstein and L. Narici, *Biseparating maps and homeomorphic realcompactifications.* J. Math. Anal. Appl. **192** (1995), 258-265.

[4] J. Araujo and J.J. Font, *Linear isometries between subspaces of continuous functions.* Trans. Amer. Math. Soc. **349** (1997), 413-428.

[5] J. Araujo and L. Dubarbie, *Biseparating maps between Lipschitz function spaces.* Preprint.

[6] W. Arendt and D.R. Hart, *The spectrum of quasi-invertible disjointness preserving operators.* J. Funct. Anal. **68** (1986), 149-167.

[7] J.J. Font and S. Hernández, *Separating maps between locally compact spaces.* Arch. Math. (Basel), **63** (1994), 158-165.

[8] J.J. Font and S. Hernández, *Automatic continuity and representation of certain linear isomorphisms between group algebras.* Indag. Math., **6** (4) (1995), 397-409.

[9] J.J. Font, *Automatic continuity of certain linear isomorphisms between regular Banach function algebras.* Glasgow Math., **77** (1998), 333-343.

[10] J.J. Font, *Disjointness preserving mappings between Fourier algebras.* Colloq. Math., **39** (1997), 179-187.

[11] O. Hatori, T. Miura and H. Oka, *An example of multiplicatively spectrum-preserving maps between non-isomorphic semi-simple commutative Banach algebras.* Nihonkai Math. J., **18** (2007), 11-15.

[12] S. Hernández, E. Beckenstein and L. Narici, *Banach-Stone theorems and separating maps.* Manuscripta Math., **86** (1995), 409-416.

[13] C. Herz, *Harmonic analysis for subgroups.* Ann. Inst. Fourier, **23** (1973), 91-123.

[14] K. Jarosz, *Automatic continuity of separating linear isomorphisms.* Canad. Math. Bull., **33** (2) (1990), 139-144.

[15] J.S. Jeang and N. C. Wong, *Weighted composition operators of $C_0(X)$'s.* J. Math. Anal. Appl., **201** (3) (1996), 981-993.

[16] A. Jiménez-Vargas, *Disjointness preserving operators between little Lipschitz algebras.* J. Math. Anal. Appl., **337** (2) (2008), 984-993.

[17] A. Jiménez-Vargas and M. Villegas-Vallecillos, *Lipschitz algebras and peripherally-multiplicative maps.* Acta Math. Sinica (Eng. Ser.), **24** (8) (2008), 1233–1242.

[18] R. Larsen, *An introduction to the theory of multipliers.* Springer-Verlag, Berlin (1971).

[19] R. Larsen, *Banach algebras: An introduction.* Marcel Dekker (1973).

[20] M.S. Monfared, *Extensions and isomorphisms for the generalized Fourier algebras of a locally compact group.* J. Funct. Anal., **198** (2003), 413444.

[21] S.E. Takahasi and O. Hatori, *Commutative Banach algebras which satisfy a Bochner-Schoenberg-Eberlein type theorem.* Proc. Amer. Math. Soc., **110** (1990), 149-158.

[22] A. Ulger, *Multipliers with closed range on commutative semisimple Banach algebras.* Studia Math., **153** (2002), 59-80.

Departamento de Matemáticas, Universidad Jaume I, Campus Riu Sec,
Castellón, Spain.
E-mail address: font@mat.uji.es