COMPARISON OF EXIT MOMENT SPECTRA FOR EXTRINSIC METRIC BALLS

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ABSTRACT. We prove explicit upper and lower bounds for the L^1-moment spectra for the Brownian motion exit time from extrinsic metric balls of submanifolds P^m in ambient Riemannian spaces N^n. We assume that P and N both have controlled radial curvatures (mean curvature and sectional curvature, respectively) as viewed from a pole in N. The bounds for the exit moment spectra are given in terms of the corresponding spectra for geodesic metric balls in suitably warped product model spaces. The bounds are sharp in the sense that equalities are obtained in characteristic cases. As a corollary we also obtain new intrinsic comparison results for the exit time spectra for metric balls in the ambient manifolds N^n themselves.

1. INTRODUCTION

We consider a complete Riemannian manifold (M^n, g) and the induced Brownian motion X_t defined on M. The L^p-moments of the exit time of X_t from smooth precompact domains D in the manifold are given by the following integrals (see [H, KD, KDM, Mc, Dy]):

$$\mathcal{A}_{p,k}(D) = \left(\int_D (u_k(x))^p \, dV \right)^{1/p} \quad , \tag{1.1}$$

where the functions u_k are defined inductively as the sequence of solutions to the following hierarchy of boundary value problems

$$\begin{aligned} \Delta u_1 + 1 &= 0 \ \text{ on } \ D \\ u_1|_{\partial D} &= 0 \quad , \end{aligned} \tag{1.2}$$

and for $k \geqslant 2$,

$$\begin{aligned} \Delta u_k + k\, u_{k-1} &= 0 \ \text{ on } \ D \\ u_k|_{\partial D} &= 0 \quad . \end{aligned} \tag{1.3}$$

Here Δ denotes the Laplace-Beltrami operator on (M^n, g). The first solution $u_1(x)$ is the mean time of first exit from D for the Brownian motion starting at the point x in D, see [Dy, Ma1].

The quantity $\mathcal{A}_{1,1}(D)$ is known as the *torsional rigidity* of D. This name stems from the fact that if $D \subseteq \mathbb{R}^2$, then $\mathcal{A}_{1,1}(D)$ represents the torque required per unit angle of twist

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and per unit beam length when twisting an elastic beam of uniform cross section D, see [Ba] and [PS]. The torsional rigidity plays a role in the exit moment spectrum similar to the role played by the first positive Dirichlet eigenvalue in the Dirichlet spectrum. See also [Ch1, Ch2] and [BBC, BG].

Perhaps the most relevant example and token of interest in these problems is given by the St. Venant torsion problem. It is a precise analog of the Rayleigh conjecture about the fundamental tone of a membrane. In 1856 Saint-Venant conjectured that among all cross sections with a given area, the circular disk has maximum torsional rigidity. The first proof of this conjecture was given by G. Polya in 1948, see [Po] and [PS].

In view of the isoperimetric inequality for domains in $\mathbb{R}^2$ and in view of the domain monotonicity of $\mathcal{A}_{1,1}(D)$ it thence follows, that among all cross sections with a given *circumference*, the circular disk has maximum torsional rigidity. In other words, in $\mathbb{R}^2$ the boundary-relative torsional rigidity is maximized by the circular disks.

Since we shall similarly only be concerned with $p = 1$, and since our results for the higher moments in the exit time moment spectrum are also in this sense isoperimetric type inequalities we define:

Definition 1.1. The *isoperimetric exit moment spectrum of* D is defined by $\{\widehat{\mathcal{A}}_1(D), \widehat{\mathcal{A}}_2(D), \cdots\}$, where

$$\widehat{\mathcal{A}}_k(D) = \frac{\mathcal{A}_{1,k}(D)}{\mathrm{Vol}(\partial D)} = \frac{1}{\mathrm{Vol}(\partial D)} \int_D u_k(x)\, dV \quad . \tag{1.4}$$

If we formally define $u_0(x) = 1$ for all $x \in D$, then all the solutions u_k – including $u_1(x)$ – are uniformly generated by induction from (1.3). With this natural extension of the u_k sequence we thence have from Definition 1.1:

$$\widehat{\mathcal{A}}_0(D) = \frac{1}{\mathrm{Vol}(\partial D)} \int_D u_0(x)\, dV = \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\partial D)} \quad , \tag{1.5}$$

which is precisely the isoperimetric quotient for D.

We will henceforth refer to the list $\{\widehat{\mathcal{A}}_0(D), \widehat{\mathcal{A}}_1(D), \widehat{\mathcal{A}}_2(D), \cdots\}$ as the *extended* isoperimetric exit moment spectrum of D.

Here we restrict our study to be concerned with the exit moment spectra of a specific kind of domains, the so-called extrinsic R-balls D_R defined in submanifolds P^m which are properly immersed into ambient Riemannian manifolds N^n with controlled sectional curvatures.

Suppose p is a pole in N, see [S]. An extrinsic p-centered R-ball D_R of the submanifold P is then, roughly speaking, the intersection between the submanifold and the ambient metric R-ball centered at p in the ambient space N.

The isoperimetric relations satisfied by these extrinsic balls have been studied and applied in a number of contexts, see e.g. [Pa2, MP1, MP4, HMP, MP5]. In these works we use R-balls and R-spheres in tailor made rotationally symmetric (warped product) model spaces M^m_w as comparison objects.

The simplest settings considered are given by the minimal submanifolds P^m in real space forms $\mathbb{K}^n(b)$ of constant sectional curvature $b \leqslant 0$. In these specific cases we have the following isoperimetric inequalities, see [CLY, Ma1, Ma2, Pa2, MP1]:

$$\frac{\operatorname{Vol}(D_R)}{\operatorname{Vol}(\partial D_R)} \leqslant \frac{\operatorname{Vol}(B_R^{b,m})}{\operatorname{Vol}(S_R^{b,m-1})} \quad , \tag{1.6}$$

where $B_R^{b,m}$ and $S_R^{b,m-1} = \partial B_R^{b,m}$ denote, respectively, the geodesic R-ball and the geodesic R-sphere in the real space form $\mathbb{K}^n(b)$.

With the notation introduced above we may state this result as follows:

$$\widehat{\mathcal{A}}_0(D_R) \leqslant \widehat{\mathcal{A}}_0(B_R^{b,m}) \quad . \tag{1.7}$$

In passing we note that when equality is attained in (1.7) for some fixed radius R, and when the ambient space N^n is the hyperbolic space $\mathbb{H}^n(b)$, $b < 0$, then the minimal submanifold itself is a totally geodesic hyperbolic subspace $\mathbb{H}^m(b)$ of $\mathbb{H}^n(b)$, see [Pa2]. Thus, in analogy with the St. Venant torsion problem – and in analogy with the classical isoperimetric problem itself – we also obtain strong rigidity conclusions from equalities in these isoperimetric estimates.

1.1. **A first glimpse of the main results.** In the present paper we extend the inequalities (1.7) and prove isoperimetric inequalities of this type for *every* element $\widehat{\mathcal{A}}_k(D_R)$, $k \geqslant 0$, in the extended isoperimetric exit moment spectrum for extrinsic metric balls.

Before stating this extension for *minimal submanifolds* in constant curvature ambient spaces below we note, that this is but a shadow of our main results, Theorem 4.1 and Theorem 4.2 in section 4, where we prove both upper *and lower* bounds for the isoperimetric exit moment spectrum under more relaxed curvature conditions. The main condition for the lower bounds is a lower bound on the sectional curvatures of the ambient space and the upper bounds for the spectrum stem similarly from an upper bound on the ambient sectional curvatures. Moreover, in our *general results* the submanifolds are not assumed beforehand to be minimal.

Theorem 1.2. *Let P^m be a minimal submanifold properly immersed in the real space form $\mathbb{K}^n(b)$ with constant sectional curvature $b \leqslant 0$. Let D_R be an extrinsic R-ball in P^m, with center at a point $p \in P$. Then we have for the extended isoperimetric exit moment spectrum of D_R, i.e. for all $k \geqslant 0$:*

$$\widehat{\mathcal{A}}_k(D_R) \leqslant \widehat{\mathcal{A}}_k(B_R^{b,m}) \quad , \tag{1.8}$$

where $B_R^{b,m}$ is the geodesic ball of radius R in $\mathbb{K}^m(b)$.

When the ambient space is hyperbolic space $\mathbb{H}^n(b)$, $b < 0$, then equality in (1.8) for some radius R and for some value of $k \geqslant 0$ implies that D_R – and in fact all of P^m – is totally geodesic in $\mathbb{H}^n(b)$, so that equality is attained for all k and for every smaller p-centered extrinsic ball in P^m.

In order to illustrate our use of the upper and lower bounds on the ambient space sectional curvatures in the more general setting alluded to above – and since we believe that the following result is also in itself of independent interest – we extract here a purely *intrinsic* consequence from the proofs of Theorems 4.1 and 4.2. The notion of radial sectional curvatures and the geometric analytic notions associated with the model spaces are defined precisely in section 2 below.

Theorem 1.3. *Let B_R^N be a geodesic ball of a complete Riemannian manifold N^n with a pole p and suppose that the p-radial sectional curvatures of N^n are bounded from below (respectively from above) by the p_w-radial sectional curvatures of a w-warped model space M_w^n. Then the extended* *isoperimetric exit moment spectrum of B^N_R satisfies for all $k \geqslant 0$ the following respective inequalities:*

$$\widehat{\mathcal{A}}_k(B^N_R) \geqslant (\leqslant) \widehat{\mathcal{A}}_k(B^w_R) \quad , \tag{1.9}$$

where B^w_R is the geodesic ball in the model space M^n_w.

Equality in (1.9) for some $k \geqslant 0$ implies that B^N_R is isometric to the warped product model ball B^w_R and hence again that equality is attained for all $k \geqslant 0$ and for every smaller p-centered extrinsic ball in P^m.

The proofs of these results, Theorem 1.2 and 1.3 are given in section 5 at the end of this paper.

2. Preliminaries and Comparison Setting

We first consider a few conditions and concepts that will be instrumental for establishing our results.

2.1. Extrinsic metric balls.

We consider a properly immersed m-dimensional submanifold P^m in a complete Riemannian manifold N^n. Let p denote a point in P and assume that p is a pole of the ambient manifold N. We denote the distance function from p in N^n by $r(x) = \mathrm{dist}_N(p, x)$ for all $x \in N$. Since p is a pole there is - by definition - a unique geodesic from x to p which realizes the distance $r(x)$. We also denote by r the restriction $r|_P : P \longrightarrow \mathbb{R}_+ \cup \{0\}$. This restriction is then called the extrinsic distance function from p in P^m. The corresponding extrinsic metric balls of (sufficiently large) radius R and center p are denoted by $D_R(p) \subseteq P$ and defined as any connected component which contains p of the set:

$$D_R(p) = B_R(p) \cap P = \{x \in P \,|\, r(x) < R\} \quad ,$$

where $B_R(p)$ denotes the geodesic R-ball around the pole p in N^n. The extrinsic ball $D_R(p)$ is a connected domain in P^m, with boundary $\partial D_R(p)$. Since P^m is assumed to be unbounded and properly immersed into N, we have for every R that $B_R(p) \cap P \neq P$.

2.2. The curvature bounds.

We now present the curvature restrictions which constitute the geometric framework of our investigations.

Definition 2.1. Let p be a point in a Riemannian manifold M and let $x \in M - \{p\}$. The sectional curvature $K_M(\sigma_x)$ of the two-plane $\sigma_x \in T_xM$ is then called a *p-radial sectional curvature* of M at x if σ_x contains the tangent vector to a minimal geodesic from p to x. We denote these curvatures by $K_{p,M}(\sigma_x)$.

In order to control the mean curvatures $H_P(x)$ of P^m at distance r from p in N^n we introduce the following definition:

Definition 2.2. The p-radial mean curvature function for P in N is defined in terms of the inner product of H_P with the N-gradient of the distance function $r(x)$ as follows:

$$\mathcal{C}(x) = -\langle \nabla r(x), H_P(x) \rangle \quad \text{for all} \quad x \in P \ .$$

In the following definition, we are going to generalize the notion of *radial mean convexity condition* introduced in [MP5], [HMP].

Definition 2.3. (see [MP5]) We say that the submanifold P satisfies a *radial mean convexity condition from below* controlled by a smooth radial function $h_1(r)$ (respectively, *from above* controlled by a smooth radial function $h_2(r)$) from the point $p \in P$ such that

$$\begin{aligned} \mathcal{C}(x) &\geqslant h_1(r(x)) \text{ for all } x \in P \quad (h_1(r) \text{ bounds } \textit{from below}) \\ \mathcal{C}(x) &\leqslant h_2(r(x)) \text{ for all } x \in P \quad (h_2(r) \text{ bounds } \textit{from above}) \end{aligned} \tag{2.1}$$

The radial bounding functions $h_1(r)$ and $h_2(r)$ are related to the global extrinsic geometry of the submanifold. For example, it is obvious that minimal submanifolds satisfy a radial mean convexity condition from above and from below, with bounding functions $h_2 = 0$ and $h_1 = 0$. On the other hand, it can be proved, see the works [Sp, DCW, Pa1, MP5], that when the submanifold is a convex hypersurface, then the constant function $h_1(r) = 0$ is a radial bounding function from below.

The final notion needed to describe our comparison setting is the idea of *radial tangency*. If we denote by ∇r and $\nabla^P r$ the gradients of r in N and P respectively, then we have the following basic relation:

$$\nabla r = \nabla^P r + (\nabla r)^{\perp} \quad , \tag{2.2}$$

where $(\nabla r)^{\perp}(q)$ is perpendicular to T_qP for all $q \in P$.

When the submanifold P is totally geodesic, then $\nabla r = \nabla^P r$ in all points, and, hence, $\|\nabla^P r\| = 1$. On the other hand, and given the starting point $p \in P$, from which we are measuring the distance r, we know that $\nabla r(p) = \nabla^P r(p)$, so $\|\nabla^P r(p)\| = 1$. Therefore, the difference $1 - \|\nabla^P r\|$ quantifies the radial *detour* of the submanifold with respect the ambient manifold as seen from the pole p. To control this detour locally, we apply the following

Definition 2.4. We say that the submanifold P satisfies a *radial tangency condition* at $p \in P$ when we have a smooth positive function $g(r)$ so that

$$\mathcal{T}(x) \,=\, \|\nabla^P r(x)\| \geqslant g(r(x)) \,>\, 0 \quad \text{for all} \quad x \in P\,. \tag{2.3}$$

Remark 2.5. Of course, we always have

$$\mathcal{T}(x) \,=\, \|\nabla^P r(x)\| \leqslant 1 \quad \text{for all} \quad x \in P\,. \tag{2.4}$$

Remark 2.6. We observe, that the assumption $\|\nabla^P r(x)\| > 0$ implies that the properly immersed extrinsic ball D_R in P can have only trivial topology. It follows directly from Theorem 3.1 in [Mi], since $r(x)$ is a smooth function on $P - \{p\}$ without critical points, that D_R is diffeomorphic to the standard unit ball in $\mathbf{R}^m$.

2.3. **Model Spaces.** As mentioned previously, the model spaces M^m_w serve first and foremost as comparison controller objects for the radial sectional curvatures of N^n.

Definition 2.7 (See [Gri], [GreW]). A $w-$model M^m_w is a smooth warped product with base $B^1 = [\,0, R[\, \subset \mathbb{R}$ (where $0 < R \leqslant \infty$), fiber $F^{m-1} = S^{m-1}_1$ (i.e. the unit $(m-1)-$sphere with standard metric), and warping function $w : [\,0, R[\, \to \mathbb{R}_+ \cup \{0\}$ with $w(0) = 0$, $w'(0) = 1$, and $w(r) > 0$ for all $r > 0$. The point $p_w = \pi^{-1}(0)$, where π denotes the projection onto B^1, is called the *center point* of the model space. If $R = \infty$, then p_w is a pole of M^m_w.

Remark 2.8. The simply connected space forms $\mathbb{K}^m(b)$ of constant curvature b can be constructed as $w-$models with any given point as center point using the warping functions

$$w(r) = Q_b(r) = \begin{cases} \frac{1}{\sqrt{b}} \sin(\sqrt{b}\, r) & \text{if } b > 0 \\ r & \text{if } b = 0 \\ \frac{1}{\sqrt{-b}} \sinh(\sqrt{-b}\, r) & \text{if } b < 0 \end{cases} \quad . \tag{2.5}$$

Note that for $b > 0$ the function $Q_b(r)$ admits a smooth extension to $r = \pi/\sqrt{b}$. For $b \leqslant 0$ any center point is a pole.

In the papers [O'N, GreW, Gri, MP3, MP4], we have a complete description of these model spaces and their key properties. In particular the sectional curvatures K_{p_w, M_w} in the radial directions from the center point p_w are determined by the radial function

$$K_{p_w, M_w}(\sigma_x) \;=\; K_w(r) \;=\; -\frac{w''(r)}{w(r)} \quad , \tag{2.6}$$

and the mean curvature of the distance sphere of radius r from the center point is

$$\eta_w(r) = \frac{w'(r)}{w(r)} = \frac{d}{dr} \ln(w(r)) \quad . \tag{2.7}$$

2.4. **The isoperimetric comparison spaces.** Given the bounding functions $g(r), h(r)$ (when in the following no specific index is given, then h represents any one of the bounding functions $h_1(r)$ or $h_2(r)$), and the ambient curvature controller function $w(r)$ described is subsections 2.2 and 2.3, as in [MP5, HMP] we construct new model spaces $C^m_{w,g,h}$. For completeness, we recall their construction:

Definition 2.9. Given a smooth positive function $g(r(x)) > 0$ satisfying $g(0) = 1$ and $g(r(x)) \leqslant 1$ for all $x \in P$, a *stretching function* s is defined as follows

$$s(r) \;=\; \int_0^r \frac{1}{g(t)}\, dt \quad . \tag{2.8}$$

It has a well-defined inverse $r(s)$ for $s \in [\,0, s(R)\,]$ with derivative $r'(s) = g(r(s))$. In particular $r'(0) = g(0) = 1$.

Definition 2.10 ([MP5])**.** The *isoperimetric comparison space* $C^m_{w,g,h}$ is defined as the $W-$model space M^m_W which has base interval $B = [\,0, s(R)\,]$ and warping function $W(s)$ defined by

$$W(s) \;=\; \Lambda^{\frac{1}{m-1}}(r(s)) \quad , \tag{2.9}$$

where the auxiliary function $\Lambda(r)$ satisfies the following differential equation:

$$\begin{aligned} \frac{d}{dr}\{\Lambda(r)w(r)g(r)\} \;&=\; \Lambda(r)w(r)g(r)\left(\frac{m}{g^2(r)}\left(\eta_w(r) - h(r)\right)\right) \\ &=\; m\,\frac{\Lambda(r)}{g(r)}\left(w'(r) - h(r)w(r)\right) \quad , \end{aligned} \tag{2.10}$$

and the following boundary condition:

$$\frac{d}{dr}_{|_{r=0}}\left(\Lambda^{\frac{1}{m-1}}(r)\right) = 1 \quad . \tag{2.11}$$

In spite of its relatively complicated construction, $C^m_{w,g,h}$ is indeed a model space M^m_W with a well defined pole p_W at $s = 0$: $W(s) \geqslant 0$ for all s and $W(s)$ is only 0 at $s = 0$, where also, because of the explicit construction in definition 2.10 and because of equation (2.11):

$W'(0) = 1$.

Note that, when $g(r) = 1$ for all r and $h(r) = 0$ for all r, then the stretching function $s(r) = r$ and $W(s(r)) = w(r)$ for all r. In this case $C^m_{w,g,h}$ simply reduces to the w warped model space M^m_w.

The spaces $M^m_W = C^m_{w,g,h}$ will be applied as those spaces, where our bounds on the exit moment spectrum are attained.

2.5. **Balance conditions.** In the paper [HMP] we considered and applied a balance condition on the general model spaces M^m_W, that we shall also need in the sequel:

Definition 2.11. The model space $M^m_W = C^m_{w,g,h}$ is *$w-$balanced* (respectively *strictly $w-$balanced*) if the following holds for all $s \in [0, s(R)]$:

$$q_W(s)\left(\eta_w(r(s)) - h(r(s))\right) \geqslant (>)\, g(r(s))/m \quad . \tag{2.12}$$

Here $q_W(s)$ is the isoperimetric quotient function

$$\begin{aligned} q_W(s) &= \frac{\mathrm{Vol}(B^W_s)}{\mathrm{Vol}(S^W_s)} \\ &= \frac{\int_0^s W^{m-1}(t)\,dt}{W^{m-1}(s)} \\ &= \frac{\int_0^{r(s)} \frac{\Lambda(u)}{g(u)}\,du}{\Lambda(r(s))} \quad . \end{aligned} \tag{2.13}$$

Remark 2.12. In particular the w-balance condition for $M^m_W = C^m_{w,g,h}$ implies that

$$\eta_w(r) - h(r) > 0 \tag{2.14}$$

wherever $g(r) > 0$.

Remark 2.13. The above definition of a (strict) $w-$balance condition for M^m_W is clearly an extension of the balance condition (from below) as defined in [MP4, Definition 2.12]. The condition in that paper is obtained precisely when $g(r) = 1$ and $h(r) = 0$ for all $r \in [0, R]$ so that $r(s) = s$, $W(s) = w(r)$, and

$$q_w(r)\eta_w(r) \geqslant 1/m \quad . \tag{2.15}$$

This particular condition is of instrumental importance for the respective proofs of Theorem 1.2 and Theorem 4.2. For these settings it is easy to verify that every warping function $w(r)$ which gives a negatively curved model space M^m_w satisfies the strict version of (2.15) for all r – using (2.13) for the functions $q_w(r)$, see also [MP4, Observation 3.12 and Examples 3.13]. In particular, the hyperbolic constant curvature spaces $M^m_w = \mathbb{H}^m(b)$, $b < 0$, all satisfy:

$$q_w(r)\eta_w(r) > 1/m \quad . \tag{2.16}$$

2.6. **Comparison Constellations.** We now present the precise settings where our main results take place, introducing the notion of *comparison constellations* as they were previously defined in [HMP]. For that purpose we shall bound the previously introduced notions of radial curvature and tangency by the corresponding quantities attained in the special model spaces, the *isoperimetric comparison spaces* defined above.

Definition 2.14. Let N^n denote a complete Riemannian manifold with a pole p and distance function $r = r(x) = \mathrm{dist}_N(p, x)$. Let P^m denote an unbounded complete and properly immersed submanifold in N^n. Suppose $p \in P^m$ and suppose that the following conditions are satisfied for all $x \in P^m$ with $r(x) \in [\,0, R]$:

(1) The p-radial sectional curvatures of N are bounded from below by the p_w-radial sectional curvatures of the $w-$model space M^m_w:

$$\mathcal{K}(\sigma_x) \geqslant -\frac{w''(r(x))}{w(r(x))} \quad .$$

(2) The p-radial mean curvature of P is bounded from below by a smooth radial function $h_1(r)$:

$$\mathcal{C}(x) \geqslant h_1(r(x)) \quad .$$

(3) The submanifold P satisfies a *radial tangency condition* at $p \in P$, with smooth positive radial function $g(r)$ such that

$$\mathcal{T}(x) = \|\nabla^P r(x)\| \geqslant g(r(x)) > 0 \quad \text{for all} \quad x \in P \,. \tag{2.17}$$

Let C^m_{w,g,h_1} denote the W-model with the specific warping function $W : \pi(C^m_{w,g,h_1}) \to \mathbb{R}_+$ constructed in Definition 2.10, (subsection 2.4), via w, g, and $h = h_1$. Then the triple $\{N^n, P^m, C^m_{w,g,h_1}\}$ is called an *isoperimetric comparison constellation bounded from below* on the interval $[\,0, R]$.

A *constellation bounded from above* is given by the following dual setting defining the special W-model spaces $C^m_{w,1,h_2}$ with the uniform choice $g = 1$:

Definition 2.15. Let N^n denote a Riemannian manifold with a pole p and distance function $r = r(x) = \mathrm{dist}_N(p, x)$. Let P^m denote an unbounded complete and properly immersed submanifold in N^n. Suppose the following conditions are satisfied for all $x \in P^m$ with $r(x) \in [\,0, R]$:

(1) The p-radial sectional curvatures of N are bounded from above by the p_w-radial sectional curvatures of the $w-$model space M^m_w:

$$\mathcal{K}(\sigma_x) \leqslant -\frac{w''(r(x))}{w(r(x))} \quad .$$

(2) The p-radial mean curvature of P is bounded from above by a smooth radial function $h_2(r)$:

$$\mathcal{C}(x) \leqslant h_2(r(x)) \quad .$$

Let $C^m_{w,1,h_2}$ denote the W-model with the specific warping function $W : \pi(C^m_{w,1,h_2}) \to \mathbb{R}_+$ constructed in Definition 2.10 via w, $g = 1$, and $h = h_2$. Then the triple $\{N^n, P^m, C^m_{w,1,h_2}\}$ is called an *isoperimetric comparison constellation bounded from above* on the interval $[\,0, R]$.

2.7. **Laplacian Comparison.** We begin this section recalling the following Laplacian comparison Theorem for manifolds with a pole (see [GreW, JK, Ma1, Ma2, MP3, MP4, MP5, MM] for more details and previous applications).

Theorem 2.16. *Let N^n be a manifold with a pole p, let M^m_w denote a $w-$model space with center p_w. Let us consider a smooth function $f : \mathbb{R}_+ \to \mathbb{R}$ and the restricted distance function from the pole $r : P \to \mathbb{R}$.*

Then we have the following dual Laplacian inequalities for the modified distance functions

$$f \circ r : P \to \mathbb{R};\; f \circ r(x) := f(r(x)) \;\forall x \in P$$

(i) Suppose that every p-radial sectional curvature at $x \in N - \{p\}$ is bounded by the p_w-radial sectional curvatures in M^m_w as follows:

$$\mathcal{K}(\sigma(x)) \,=\, K_{p,N}(\sigma_x) \geqslant -\frac{w''(r)}{w(r)} \quad . \tag{2.18}$$

Then we have for every smooth function $f(r)$ with $f'(r) \leqslant 0$ for all r, (respectively $f'(r) \geqslant 0$ for all r):

$$\begin{aligned}\Delta^P(f\circ r) \,\geqslant (\leqslant)\, &\big(\, f''(r) - f'(r)\eta_w(r)\,\big)\,\|\nabla^P r\|^2 \\ &+ mf'(r)\,\Big(\eta_w(r) + \langle\, \nabla^N r,\, H_P\,\rangle\Big) \quad ,\end{aligned} \tag{2.19}$$

where H_P denotes the mean curvature vector of P in N.

(ii) Suppose that every p-radial sectional curvature at $x \in N - \{p\}$ is bounded by the p_w-radial sectional curvatures in M^m_w as follows:

$$\mathcal{K}(\sigma(x)) \,=\, K_{p,N}(\sigma_x) \leqslant -\frac{w''(r)}{w(r)} \quad . \tag{2.20}$$

Then we have for every smooth function $f(r)$ with $f'(r) \leqslant 0$ for all r, (respectively $f'(r) \geqslant 0$ for all r):

$$\begin{aligned}\Delta^P(f\circ r) \,\leqslant (\geqslant)\, &\big(\, f''(r) - f'(r)\eta_w(r)\,\big)\,\|\nabla^P r\|^2 \\ &+ mf'(r)\,\Big(\eta_w(r) + \langle\, \nabla^N r,\, H_P\,\rangle\Big) \quad ,\end{aligned} \tag{2.21}$$

where H_P denotes the mean curvature vector of P in N.

3. Exit moment spectra of R-balls in model spaces

We have the following result concerning the exit moment spectrum of a geodesic R-ball $B^w_R \subseteq M^m_w$:

Proposition 3.1. *Let $\tilde{u}_k$ be the solution of the boundary value problems* (1.3)*, defined on the geodesic R-ball B^w_R in a warped model space M^m_w.*

Then

$$\tilde{u}_1(r) \,=\, \int_r^R \frac{\int_0^t w^{m-1}(s)\,ds}{w^{m-1}(t)}\,dt, \tag{3.1}$$

and

$$\tilde{u}_k'(r) \,=\, -k\,\frac{\int_0^r w^{m-1}(s)\tilde{u}_{k-1}(s)\,ds}{w^{m-1}(r)}. \tag{3.2}$$

Therefore,

$$\widehat{\mathcal{A}}_k(B^w_R) = -\frac{1}{k+1}\,\tilde{u}'_{k+1}(R) \quad . \tag{3.3}$$

Proof. A straightforward computation gives

$$\Delta\tilde{u}_k = \frac{\left(\tilde{u}_k' w^{m-1}\right)'}{w^{m-1}} = -k\tilde{u}_{k-1}, \tag{3.4}$$

which gives (3.1) and (3.2). So, if

$$\tilde{u}_k(r) = k \int_r^R \frac{\int_0^t w^{m-1}(s)\tilde{u}_{k-1}(s)\, ds}{w^{m-1}(t)}\, dt,$$

the boundary condition $\tilde{u}_k(R) = 0$ is satisfied and as a consequence of the Maximum Principle for elliptic operators, the functions $\tilde{u}_k$ are the only solutions to the boundary value problems defined on B^w_R and given by (1.3).

Therefore, applying the Divergence Theorem, we obtain

$$\begin{aligned} \widehat{\mathcal{A}}_k(B^w_R) \cdot \mathrm{Vol}(S^w_R) &= \int_{B^w_R} \tilde{u}_k\, dV = -\frac{1}{k+1} \int_{B^w_R} \Delta \tilde{u}_{k+1}\, dV \\ &= -\frac{1}{k+1} \int_{S^w_R} \langle \nabla \tilde{u}_{k+1}, \nabla r \rangle\, dA = -\frac{1}{k+1}\, \tilde{u}'_{k+1}(R) \cdot \mathrm{Vol}(S^w_R) \quad , \end{aligned} \tag{3.5}$$

where S^w_R is the geodesic R-sphere in M^m_w, and the claim is proved. □

3.1. **A key lemma.** Let us consider now an isoperimetric comparison model space M^m_W and let $\tilde{u}^W_k$ be the radial functions given by (3.2), which are the solutions of the problems (1.3) defined on the geodesic ball $B^W_{s(R)}$. We define the functions $f_k : [0, R] \to \mathbb{R}$ as $f_k = \tilde{u}^W_k \circ s$, where s is the stretching function given by (2.8).

Then we have the following lemma, which will be of instrumental importance for the proofs of the main results below:

Lemma 3.2. *Let M^m_W be an isoperimetric comparison model space that is w-balanced in the sense of Definition 2.11 with $h = h_1$ or $h = h_2$. Then for all $k \geqslant 1$,*

$$f''_k(r) - f'_k(r)\eta_w(r) \geqslant 0 \quad .$$

If $k \geqslant 2$ or if M^m_W is strictly *balanced, then the inequality is in fact a strict inequality:*

$$f''_k(r) - f'_k(r)\eta_w(r) > 0 \quad .$$

Proof. By equation (2.8),

$$\begin{aligned} f''_k(r) &= \tilde{u}^{W''}_k(s(r))(s'(r))^2 + \tilde{u}^{W'}_k(s(r))s''(r) \\ &= \frac{1}{g^2(r)}(\tilde{u}^{W''}_k(s(r)) - \tilde{u}^{W'}_k(s(r))g'(r)). \end{aligned} \tag{3.6}$$

Since the functions $\tilde{u}^W_k$ are the solution of the problems (1.3) on $B^W_{s(R)}$, using equation (3.4),

$$\tilde{u}^{W''}_k(s(r)) = -k\, \tilde{u}^W_{k-1}(s(r)) - (m-1)\frac{W'(s(r))}{W(s(r))}\, \tilde{u}^{W'}_k(s(r)).$$

Taking into account the explicit construction of M^m_W, i.e. equations (2.9) and (2.10), a straightforward computation shows that

$$(m-1)\frac{W'(s(r))}{W(s(r))} = \frac{m}{g(r)}(\eta_w(r) - h(r)) - g(r)\eta_w(r) - g'(r),$$

and consequently,

$$\tilde{u}^{W''}_k(s(r)) = -k\, \tilde{u}^W_{k-1}(s(r)) - \frac{m}{g(r)}(\eta_w(r) - h(r))\, \tilde{u}^{W'}_k(s(r)) + (\eta_w(r)g(r) + g'(r))\, \tilde{u}^{W'}_k(s(r)).$$

Replacing the expression of $\tilde{u}^{W''}_k(s(r))$ in equation (3.6) we obtain that

$$g^2(r)\, f''_k(r) = -kf_{k-1}(r) + (g^2(r)\eta_w(r) - m(\eta_w(r) - h(r)))f'_k(r),$$

and

$$g^2(r)(f_k''(r) - f_k'(r)\eta_w(r)) = -kf_{k-1}(r) - m(\eta_w(r) - h(r))f_k'(r). \tag{3.7}$$

Since $f_k'(r) = \tilde{u}_k^{W\prime}(s(r))/g(r) < 0$, the functions f_k are strictly decreasing in $]\,0, R\,]$ for all $k \geqslant 1$ and consequently by (3.2)

$$f_k'(r) \quad = \quad -k\,\frac{\int_0^{s(r)} W^{m-1}(s)\tilde{u}_{k-1}(s)\,ds}{W^{m-1}(s(r))g(r)} = -k\,\frac{\int_0^r \frac{\Lambda(t)}{g(t)} f_{k-1}(t)\,dt}{\Lambda(r)g(r)} \tag{3.8}$$

$$\leqslant (<) \quad -kf_{k-1}(r)\frac{\int_0^r \frac{\Lambda(t)}{g(t)}\,dt}{\Lambda(r)g(r)} = -kf_{k-1}(r)q_W(s(r))/g(r), \tag{3.9}$$

where the last equality is obtained using equation (2.13). Note that we can assume that $\tilde{u}_0 \equiv 1$ and therefore $f_0 \equiv 1$ too, so that only in the case $k = 1$ can we have equality in (3.9).

Finally, combining the above inequality with equation (3.7) we get:

$$g^3(r)(f_k''(r) - f_k'(r)\eta_w(r)) \;\geqslant (>)\; kf_{k-1}(r)\,(-g(r) + m\,q_W(s(r))(\eta_w(r) - h(r))) \geqslant (>)\, 0$$

by the balance condition (2.12) – respectively the strict balance condition – and the fact that g and f_{k-1} are positive functions. □

4. Lower and Upper bounds for the isoperimetric exit moments

We are now ready to prove the first of our main results.

Theorem 4.1. *Let $\{N^n, P^m, C^m_{w,g,h_1}\}$ denote a comparison constellation bounded from below in the sense of Definition 2.14. Assume that $M^m_W = C^m_{w,g,h_1}$ is w-balanced in the sense of Definition 2.11. Let D_R be an extrinsic R-ball in P^m, with center at a point $p \in P$ which also serves as a pole in N. According to remark 2.6, our assumption $g(r(x)) > 0$ implies trivial topology of the extrinsic ball D_R. For all $k \geqslant 0$, i.e. for the extended exit moment spectrum, we also have:*

$$\widehat{\mathcal{A}}_k(D_R) \geqslant \widehat{\mathcal{A}}_k(B^W_{s(R)}) \quad , \tag{4.1}$$

where $B^W_{s(R)}$ is the geodesic $s(R)$-ball in C^m_{w,g,h_1}.

Proof. Consider the functions $f_k = \tilde{u}_k^W \circ s$ of Lemma 3.2. Let r denote the smooth distance to the pole p on M. We define $v_k : D_R \to \mathbb{R}$ by $v_k(q) = f_k(r(q))$.

Using Theorem 2.16, Lemma 3.2, equation (3.7) and the fact that $f_k'(r) \leqslant 0$, we have that

$$\Delta^P v_k = \Delta^P(f_k \circ r) \;\geqslant\; (f_k''(r) - f_k'(r)\eta_w(r))\|\nabla^P r\|^2 + m\,f_k'(r)(\eta_w(r) - h_1(r)) \tag{4.2}$$

$$\geqslant\; (f_k''(r) - f_k'(r)\eta_w(r)) \cdot g^2(r) + m\,f_k'(r)(\eta_w(r) - h_1(r)) \tag{4.3}$$

$$=\; -kf_{k-1}(r) = -k\,v_{k-1}, \quad \text{on } D_R$$

Now, we are going to prove *inductively* that if we denote by u_k the solutions of the hierarchy of boundary value problems on D_R given by (1.3), then $v_k \leqslant u_k$ on D_R.

For $k = 1$, since f_0 is assumed to be identically 1, inequality (4.2) gives us that

$$\Delta^P v_1 \geqslant -1 = \Delta^P u_1,$$

so $\Delta^P(v_1 - u_1) \geqslant 0$ on D_R and $(v_1 - u_1) = 0$ on ∂D_R. Applying the Maximum Principle we conclude that $v_1 \leqslant u_1$ on D_R.

Suppose now that $v_k \leqslant u_k$ on D_R, then as a consequence of inequality (4.2) we get

$$\Delta^P v_{k+1} \geqslant -(k+1)\,v_k \geqslant -(k+1)\,u_k = \Delta^P u_{k+1},$$

and $(v_{k+1} - u_{k+1}) = 0$ on ∂D_R, so applying again the Maximum Principle we have $v_{k+1} \leqslant u_{k+1}$.

Summarizing we have so far: $v_k \leqslant u_k$ and $\Delta^P v_k \geqslant \Delta^P u_k$ on D_R for all $k \geqslant 1$. Taking these inequalities into account and applying Divergence theorem we then get

$$\begin{aligned}
\widehat{\mathcal{A}}_k(D_R)\cdot \operatorname{Vol}(\partial D_R) &= \int_{D_R} u_k d\,V = -\frac{1}{k+1}\int_{D_R} \Delta^P u_{k+1} d\,V \\
&\geqslant -\frac{1}{k+1}\int_{D_R} \Delta^P v_{k+1} d\,V = -\frac{1}{k+1}\int_{\partial D_R} \langle \nabla^P v_{k+1}, \frac{\nabla^P r}{\|\nabla^P r\|}\rangle d\,A \\
&= -\frac{1}{k+1} f'_{k+1}(R)\int_{\partial D_R} \|\nabla^P r\| d\,A.
\end{aligned}$$

Since $f'_{k+1}(R) = \tilde{u}^{W\prime}_{k+1}(s(R))/g(R) \leqslant 0$ and $\|\nabla^P r\| \geqslant g(r)$, we conclude that

$$\widehat{\mathcal{A}}_k(D_R) \geqslant -\frac{1}{k+1}\,\frac{\tilde{u}^{W\prime}_{k+1}(s(R))}{g(R)}\,g(R) \;= \widehat{\mathcal{A}}_k(B^W_{s(R)}),$$

by equation (3.3). And this proves the claim in (4.1). □

Theorem 4.2. *Let $\{N^n, P^m, C^m_{w,1,h_2}\}$ denote a comparison constellation bounded from above. Assume that $M^m_W = C^m_{w,1,h_2}$ is w-balanced in the sense of Definition 2.11. Let D_R be a smooth precompact extrinsic R-ball in P^m with center at a point $p \in P$ which also serves as a pole in N. Then, for all $k \geqslant 0$, i.e. for the extended isoperimetric exit moment spectrum we have:*

$$\widehat{\mathcal{A}}_k(D_R) \leqslant \widehat{\mathcal{A}}_k(B^W_R) \quad , \tag{4.4}$$

where B^W_R is the geodesic ball in $C^m_{w,1,h_2}$.

If M^m_W is strictly balanced then equality in (4.4) for some fixed radius R and some fixed $k \geqslant 0$ implies that D_R is a geodesic cone in N and that the equality is in fact attained for all $k \geqslant 0$ and for every smaller p-centered extrinsic ball in P^m.

Proof. The proof of this theorem follows closely the lines of the proof of Theorem 4.1. Since there are, however, some crucial and obvious differences we take this space to point them out explicitly. In the present case we have $s(r) = r$ because $g(r) \equiv 1$ (see equation (2.8)). Therefore $f_{k+1} = \tilde{u}^W_{k+1}$ so that $v_{k+1} = \tilde{u}^W_{k+1} \circ r$. Thence v_{k+1} is the solution of the boundary value problems (1.3) on B^W_R transplanted to D_R.

The new geometric setting given by the *comparison constellation bounded from above* gives us now:

$$\begin{aligned}
\Delta^P v_k = \Delta^P(f_k \circ r) &\leqslant (f''_k(r) - f'_k(r)\eta_w(r))\|\nabla^P r\|^2 + m\,f'_k(r)(\eta_w(r) - h_2(r)) && (4.5)\\
&\leqslant (f''_k(r) - f'_k(r)\eta_w(r)) + m\,f'_k(r)(\eta_w(r) - h_2(r)) && (4.6)\\
&= -kf_{k-1}(r) = -k\,v_{k-1}, \quad \text{on } D_R \quad .
\end{aligned}$$

Again we prove *inductively* that if u_k denotes the family of solutions of the hierarchy of boundary value problems on D_R given by (1.3), then $v_k \geqslant u_k$ on D_R.

For $k = 1$, since f_0 is still assumed to be identically 1, inequalities (4.6) and (4.5) give us that

$$\Delta^P v_1 \leqslant -1 = \Delta^P u_1,$$

so $\Delta^P(v_1 - u_1) \leqslant 0$ on D_R and $(v_1 - u_1) = 0$ on ∂D_R. Applying the Maximum Principle we conclude that $v_1 \geqslant u_1$ on D_R.

Suppose now that $v_k \geqslant u_k$ on D_R, then again as a consequence of inequalities (4.5) and (4.6) we get

$$\Delta^P v_{k+1} \leqslant -(k+1)\, v_k \leqslant -(k+1)\, u_k = \Delta^P u_{k+1},$$

and $(v_{k+1} - u_{k+1}) = 0$ on ∂D_R, so applying again the Maximum Principle we have $v_{k+1} \geqslant u_{k+1}$.

We have: $v_k \geqslant u_k$ and $\Delta^P v_k \leqslant \Delta^P u_k$ on D_R for all $k \geqslant 1$. The Divergence theorem gives the claim in (4.4):

$$\begin{aligned}
\widehat{\mathcal{A}}_k(D_R)\cdot \mathrm{Vol}(\partial D_R) &= \int_{D_R} u_k d\,V = -\frac{1}{k+1}\int_{D_R} \Delta^P u_{k+1} d\,V \\
&\leqslant -\frac{1}{k+1}\int_{D_R} \Delta^P v_{k+1} d\,V \qquad (4.7)\\
&= -\frac{1}{k+1} f'_{k+1}(R) \int_{\partial D_R} \|\nabla^P r\| d\,A \\
&\leqslant \widehat{\mathcal{A}}_k(B^W_R)\cdot \mathrm{Vol}(\partial D_R) \quad . \qquad (4.8)
\end{aligned}$$

Suppose that M^m_W is strictly balanced and that we have equality in (4.4). Then we must have equalities in (4.8), (4.7), and (4.6) as well. In particular the last mentioned equality gives $\|\nabla^P r\| \equiv 1$ because we have from (3.2) that $(f''_k(r) - f'_k(r)\eta_w(r)) > 0$. Therefore $\nabla^P r = \nabla^N r$ and D_R is a geodesic cone swept out by the radial geodesics from p. □

5. Intrinsic and constant curvature results

In this short section we finally show how to obtain the results stated in the introduction from Theorem 4.1 and Theorem 4.2.

Proof of Theorem 1.2. This theorem follows immediately from Theorem 4.2 once we show that the comparison space M^m_W is strictly w-balanced. But we have $g = 1$ and $h_2 = 0$ so that M^m_W is $M^m_w = \mathbb{H}^m(b)$, $b < 0$, which is strictly w-balanced according to remark 2.13. The equality case gives even more significant rigidity: Since D_R is here a *minimal* geodesic cone, then by analytic continuation D_R and in fact all of P^m is totally geodesic in the hyperbolic space $\mathbb{H}^n(b)$, see [Ma1]. □

Proof of Theorem 1.3. We consider the intrinsic versions of (the proofs of) Theorem 4.1 and Theorem 4.2 assuming that $P^m = N^n$. In this case, the extrinsic distance to the pole p becomes the intrinsic distance in N, so, the extrinsic domains D_R become the geodesic balls B^N_R of the ambient manifold N and for all $x \in P$ we have:

$$\begin{aligned}
\nabla^P r(x) &= \nabla r(x), \\
H_P(x) &= 0.
\end{aligned}$$

As a consequence, $\|\nabla^P r\| \equiv 1$, so $g(r(x)) = 1$ and $\mathcal{C}(x) = h_1(r(x)) = h_2(r(x)) = 0$. The stretching function becomes the identity $s(r) = r$, $W(s(r)) = w(r)$, and the isoperimetric comparison spaces C^m_{w,g,h_1} and $C^m_{w,1,h_2}$ reduce to the same auxiliary model space M^m_w. Since $\|\nabla r\| \equiv 1$, we do not need to control the sign of $(f''_k(r) - f'_k(r)\eta_w(r))$ in equations (2.19) and (2.21). For this reason it is not necessary to assume any w-balance conditions in these cases. The theorem and the two-sided bounds in (1.9) then follow directly from the inequalities in Theorem 4.1 and Theorem 4.2. If equality is satisfied, then B^N_R has all its radial curvatures equal to the radial curvatures of M^m_w, hence they are isometric, see [MP4]. □

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