THE CHARACTER OF TOPOLOGICAL GROUPS, VIA PONTRYAGIN-VAN KAMPEN DUALITY

CRISTINA CHIS, M. VINCENTA FERRER, SALVADOR HERNÁNDEZ, AND BOAZ TSABAN

Abstract. The Birkhoff-Kakutani Theorem asserts that a topological group is metrizable if and only if it has countable character. We develop and apply tools for the estimation of the character for a wide class of nonmetrizable topological groups.

We consider abelian groups whose topology is determined by a countable cofinal family of compact sets. These are precisely the closed subgroups of Pontryagin-van Kampen duals of *metrizable* abelian groups, or equivalently, complete abelian groups whose dual is metrizable. By investigating these connections, we show that also in these cases, the character can be estimated, and that it is determined by the weights of the *compact* subsets of the group, or of quotients of the group by compact subgroups. It follows, for example, that the density and the local density of an abelian metrizable group determine the character of its dual group. Our main result applies to the more general case of closed subgroups of Pontryagin-van Kampen duals of abelian Čech-complete groups.

Even in the special case of free abelian topological groups, our results extend a number of results of Nickolas and Tkachenko, which were proved using laborious elementary methods.

In order to obtain concrete estimations, we establish a natural bridge between the studied concepts and pcf theory, which allows the direct application of several major results from that theory. We include an introduction to these results, their use, and their limitations.

1. Overview and main results

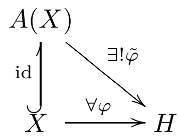
The topological structure of a topological group is completely determined by its local structure at an element. The most fundamental invariant of the local structure is the *character*, the minimal cardinality of a local basis. Metrizable groups have countable character, and the celebrated Birkhoff-Kakutani Theorem asserts that this is the only case where the character is countable.

The computation of the character of nonmetrizable groups may be a hard task. For example, even the character of free abelian topological groups is only known in very special cases. The *free abelian topological group* $A(X)$ over a Tychonoff space X is the abelian topological group with the universal property, that each continuous function φ from X into any abelian topological group H has a unique extension to

2010 *Mathematics Subject Classification.* Primary: 22A05, 22D35, 54H11, 03E04, 54A25; Secondary: 22B05, 43A40, 03E17, 03E35, 03E75.

Key words and phrases. Character of a topological group, dual group, Pontryagin van Kampen duality, compact-open topology, metrizable group, locally quasi-convex group, bounded sets, free topological group, cofinality, pcf theory.

a continuous homomorphism $\tilde{\varphi} : A(X) \to H$.



As a set, $A(X)$ is the family of all formal linear combinations of elements of X over the integers. But the topology of $A(X)$ is very complex, and in general, it is not known how to determine the character of $A(X)$ from the properties of X.

In this paper, we make use of the fact that groups from an important class of topological groups, whose character estimation was intractable for earlier methods, contain open subgroups whose Pontryagin-van Kampen duals are *metrizable*. An introduction to the pertinent part of this duality theory will be given in Section 5.

A subset C of a partially ordered set P is *cofinal* (in P) if for each $p \in P$, there is $c \in C$ such that $p \leq c$. In this paper, families of sets are always ordered by $\subseteq$.

All groups considered in this overview are assumed, without further notice, to be locally quasiconvex. This is a mild restriction, meaning that the group admits reasonably many continuous homomorphisms into the circle group.

The complete abelian groups whose dual is metrizable are exactly the ones whose topology is determined by a countable cofinal family of compact subsets.[1] The class of abelian groups containing open subgroups of this type includes, in addition to all locally compact abelian groups:

- all free abelian groups on a compact space, indeed on any space whose topology is determined by a countable cofinal family of compact subsets;
- all dual groups of countable projective limits of metrizable, or more generally Čech complete, abelian groups;
- all dual groups of abelian pro-Lie groups defined by countable systems [22, 26]; and
- all countable direct sums, closed subgroups, and finite products of groups from this class [22].

Consider $\mathbb{N}^\mathbb{N}$ with the partial order $f \leq g$ if $f(n) \leq g(n)$ for all n. The *cofinality* of a partially ordered set P, denoted $\mathrm{cof}(P)$, is the minimal cardinality of a cofinal subset of P. $\mathfrak{d}$ is the cofinality of $\mathbb{N}^\mathbb{N}$ with respect to $\leq$. This cardinal was extensively studied [13, 7], and for the present purposes it may be thought of as some constant cardinal between $\aleph_1$ and the continuum (inclusive).

For a cardinal κ (thought of as a set of cardinality κ), $[\kappa]^{\aleph_0}$ is the family of all countable subsets of κ. The *weight* of a topological space X is the minimal cardinality of a basis of open sets for the topology of X. For brevity, define the *compact weight* of X to be the supremum of the weights of compact subsets of X. For nondiscrete (locally) compact groups, the character is equal to the (compact) weight. The main theorem of this paper, stated in an inner language, is the following.

Theorem 1. *Assume that the group G has an open subgroup H such that H is abelian non-locally compact, and the topology of H is determined by a countable*

[1] I.e., there are compact sets $K_1, K_2, \ldots \subseteq G$ such that each compact $K \subseteq G$ is contained in some K_n, and for each $U \subseteq G$ with all $U \cap K_n$ open in K_n, U is open in G. Groups satisfying the first condition are often named *hemicompact*. Groups satisfying both conditions are often named k_ω.

cofinal family of compact subsets. Let κ be the compact weight of H, and λ be the minimum among the compact weights of the quotients of H by compact subgroups. Then: the character of G is the maximum of $\mathfrak{d}$, κ, and the cofinality of $[\lambda]^{\aleph_0}$.

In particular, if G has no proper compact subgroups (this is the case, e.g., for $A(X)$), or more generally, if quotients by compact subgroups do not decrease the compact weight of G, then the character of G is the maximum of $\mathfrak{d}$ and $\operatorname{cof}([\kappa]^{\aleph_0})$.

Theorem 1 reduces the computation of the character of G to the purely combinatorial task of estimating the cofinality of $[\lambda]^{\aleph_0}$. This is a central task in Shelah's pcf theory. The last sections of this paper are dedicated to an introduction of this theory and its applications in our context. In contrast to cardinal exponentiation, $\operatorname{cof}([\lambda]^{\aleph_0})$ is very tame. For example, if there are no large cardinals (in a certain canonical model of set theory)[2], then $\operatorname{cof}([\lambda]^{\aleph_0})$ is simply λ if λ has uncountable cofinality, and λ^+ (the successor of λ) otherwise. Thus, the axiom *SSH*, asserting that $\operatorname{cof}([\lambda]^{\aleph_0}) \leq \lambda^+$, is extremely weak. Moreover, without any special hypotheses, $\operatorname{cof}([\lambda]^{\aleph_0})$ can be estimated, and in many cases computed exactly.

For brevity, denote the character of a topological group G by $\chi(G)$. Following is a summary of consequences of the main theorem.

Theorem 2. *In the notation of Theorem 1:*

1. $\chi(G) \leq \kappa^{\aleph_0}$.
2. *If* $\kappa = \kappa^{\aleph_0}$, *then* $\chi(G) = \kappa$.
3. *If* $\lambda = \aleph_n$ *for some* n, *then* $\chi(G) = \max(\mathfrak{d}, \kappa)$.
4. *If* $\lambda = \aleph_\mu$, *for a limit cardinal* μ *below the first fixed point of the* $\aleph$ *function, and* μ *has uncountable cofinality, then* $\chi(G) = \max(\mathfrak{d}, \kappa)$.
5. *If* $\lambda = \aleph_\alpha$ *is smaller than the first fixed point of the* $\aleph$ *function, then* $\chi(G)$ *is smaller than* $\max(\mathfrak{d}^+, \kappa^+, \aleph_{|\alpha|^{+4}})$.
6. *If SSH holds, then:*
   (a) *If* $\lambda < \kappa$ *or* $\operatorname{cof}(\lambda) > \aleph_0$, *then* $\chi(G) = \max(\mathfrak{d}, \kappa)$.
   (b) *If* $\lambda = \kappa$ *and* $\operatorname{cof}(\lambda) = \aleph_0$, *then* $\chi(G) = \max(\mathfrak{d}, \kappa^+)$.

The proof of these theorems spans throughout the entire paper, but the paper is designed so that each reader can read the sections accessible to him or her, and take as granted the other ones, using the index at the end of the paper in case of need for a definition.

In Section 2, we set up a general framework for studying bounded sets in topological groups. The level of generality is just the one needed to capture available methods from the context of topological vector spaces, and import them to the seemingly different context of separable topological groups with translations by elements of a dense subset. This is done in Section 3, which concludes by showing that in metrizable groups, precompact subsets of dense subgroups determine the precompact subsets of the full group, and consequently, the precompact sets in the group and in its dense subgroup have the same cofinal structure. These are, essentially, the only two results from the first two sections which are needed for the remaining sections. In a first reading of Sections 2 and 3, the reader may wish to consider only the special case of topological groups with translations by elements of a dense subset, since this is the case needed in the concluding results of these sections.

---

[2] It is not even possible to prove, using the standard axioms of set theory, that the existence of such cardinals is *consistent*.

In Section 4, the approach of Section 3 is generalized from separable to arbitrary metrizable groups. The *density* of a topological group G, $\mathrm{d}(G)$, is the minimal cardinality of a dense subset of that space. We define the *local density* of G, $\mathrm{ld}(G)$, to be the minimal density of a neighborhood of the identity element of G. Let $\mathrm{PK}(G)$ denote the family of all precompact subsets of G. The main result of this section is the following.[3]

Theorem 3. *Let G be metrizable non-locally precompact group. The cofinality of $\mathrm{PK}(G)$ is equal to the maximum of $\mathfrak{d}$, $\mathrm{d}(G)$, and $\mathrm{cof}([\mathrm{ld}(G)]^{\aleph_0})$.*

In Section 5 we use Theorem 3 and methods of Pontryagin-van Kampen duality to prove the following theorem.

Theorem 4. *Let G be a complete abelian group whose dual group is a metrizable non-locally precompact group Γ. Then $\chi(G)$ is the maximum of $\mathfrak{d}$, $\mathrm{d}(\Gamma)$, and $\mathrm{cof}([\mathrm{ld}(\Gamma)]^{\aleph_0})$.*

This already puts us in a position to prove, in Section 6, the following result.[4]

Theorem 5. *Let X be a space whose topology is determined by a countable cofinal family of compact subsets. Let κ be the compact weight of X. Then the character of $A(X)$ is the maximum of $\mathfrak{d}$ and $\mathrm{cof}([\kappa]^{\aleph_0})$. In particular:*

1. *$\chi(A(X)) \leq \kappa^{\aleph_0}$, and if $\kappa = \kappa^{\aleph_0}$, then $\chi(A(X)) = \kappa$.*
2. *If $\kappa = \aleph_n$ for some $n \in \mathbb{N}$, then $\chi(A(X)) = \max(\mathfrak{d}, \aleph_n)$.*
3. *If $\kappa = \aleph_\mu$, for μ smaller than the first fixed point of the $\aleph$ function, and μ is a limit cardinal of uncountable cofinality, then $\chi(A(X)) = \max(\mathfrak{d}, \aleph_\mu)$*
4. *If $\kappa = \aleph_\alpha$ is smaller than the first fixed point of the $\aleph$ function, then $\chi(A(X))$ is smaller than $\max(\mathfrak{d}^+, \aleph_{|\alpha|^{+4}})$.*
5. *If SSH holds, then:*
   (a) *If $\mathrm{cof}(\kappa) > \aleph_0$, then $\chi(A(X)) = \max(\mathfrak{d}, \kappa)$.*
   (b) *If $\mathrm{cof}(\kappa) = \aleph_0$, then $\chi(A(X)) = \max(\mathfrak{d}, \kappa^+)$.*

Moreover, Nickolas and Tkachenko proved that for Lindelöf spaces X, the characters of the free abelian and free *nonabelian* topological groups over X are equal [31]. Thus, Theorem 5 also holds for the free nonabelian topological group $F(X)$.

The result in Theorem 5 that the character of $A(X)$ is the maximum of $\mathfrak{d}$ and $\mathrm{cof}([\kappa]^{\aleph_0})$ was previously known only in few, very special cases, for example when X is compact, or when, in addition to the premise in our theorem, all compact subsets of X are metrizable [31]. Even in these special cases, their proof (which used elementary methods) was considerably more difficult than our proof for the more general theorem.

In Section 7 we develop the remaining Pontryagin-van Kampen theory required to deduce Theorem 1 from Theorem 4.

Section 8 introduces and applies pcf theory, to obtain the concrete estimations in Theorems 2 and 5, and Section 9 proves some freedom in these estimations, answering a problem of Bonanzinga and Matveev raised in a different context.

We note that all estimations in Theorem 2 apply to Theorem 4 as well, which may be viewed by some readers as the main result of this paper.

[3] In Theorem 3, which is of independent interest, we do not require that G is locally quasiconvex or abelian.

[4] We state Theorem 5 in full because the estimations are slightly simpler than those in Theorem 2.

2. Bounded sets in topological groups

The unifying concept of this paper is that of boundedness in topological groups. This concept plays a central role in a number of studies in functional analysis and topology. In its most abstracted form, a *boundedness* (or *bornology* [6]) on a topological space X is a family of subsets of X which is closed under taking subsets and unions of finitely many elements, and contains all finite subsets of X.[5] The abstract approach has found applications in several areas of mathematics – see the introduction and references in [6]. In particular, Vilenkin [37] applied this approach in the realm of topological groups. Here, we focus on *well-behaved* boundedness notions in topological groups, which make it possible to simultaneously extend some earlier studies in locally convex topological vector spaces as well as seemingly unrelated studies of general topological groups.

We use the following notational conventions throughout the paper: For a set X, $P(X)$ denotes the family of all subsets of X, and $\mathrm{Fin}(X)$ denotes the family of all *finite* subsets of X. An *operator* t on $P(X)$ is a function $t : P(X) \to P(X)$. Throughout, G is an infinite Hausdorff topological group with identity element e (or 0 if G is restricted to be abelian), and T is a set of operators on $P(G)$.

Definition 2.1. For an operator t on $P(G)$, write $t * A$ for $t(A)$, $A \subseteq G$. Let T be a set of operators on $P(G)$.

(1) For $F \subseteq T$, $F * A$ denotes $\bigcup_{t\in F} t * A$.
(2) A set $B \subseteq G$ is T-*bounded* (*bounded*, when T is clear from the context) if for each neighborhood U of e, there is a finite $F \subseteq T$ such that $B \subseteq F * U$.

The following axioms guarantee that the family of T-bounded sets is a boundedness notion.

Definition 2.2. A *boundedness system* is a pair (G, T) such that G is a topological group, T is a set of operators on $P(G)$, and the following conditions hold:

(B1) For each open U and each $t \in T$, $t * U$ is open;
(B2) For each neighborhood U of e, $T * U = G$;
(B3) For each T-bounded $A \subseteq G$ and each $t \in T$, $t * A$ is T-bounded;
(B4) For all $A \subseteq B \subseteq G$ and each $t \in T$, $t * A \subseteq t * B$;
(B5) For each $S \subseteq T$ with $|S| < |T|$, there is a neighborhood U of e such that $S * U \neq G$;
(B6) For each n, there is a neighborhood U of e such that for all $F \subseteq T$ with $|F| \leq n$, $F * U \neq G$.

A boundedness system (G, T) is said to be *metrizable* if G is metrizable.

Axiom (B5) is assumed since one can restrict attention to a set $T' \subseteq T$ of minimal cardinality such that $T' * U = G$ for each neighborhood U of e. Axiom (B6) is added to avoid trivialities. By moving to the semigroup of operators generated by T, we may assume that T is a semigroup. We will, however, not make use of this fact.

Precompact sets need not be bounded when G is not complete, but we have the following.

Lemma 2.3. *For each boundedness system* (G, T)*:*

(1) *Every compact* $K \subseteq G$ *is bounded.*

[5] In set theoretic terms, this defines a (not necessarily proper) *ideal* on X containing all singletons.

(2) *The family of bounded subsets of G is a boundedness.* □

The following two examples of boundedness systems are well known. In these examples, we identify T with some set of parameters defining the elements of T. In general, we may identify T with any set S of the same cardinality, by modifying the definition of $*$ appropriately.

Example 2.4 (Standard boundedness on topological vector spaces). Let E be a topological vector space. Take $T = \mathbb{N}$, and define $n * A = \{nv : v \in A\}$ for each $A \subseteq V$. For example, (B2) holds since $\lim_n \frac{1}{n}v = \vec{0}$ for each $v \in E$. The $\mathbb{N}$-bounded sets are those bounded in the ordinary sense.

In Example 2.4, if E is a locally convex topological vector space, we may alternatively define $n*A = nA = \{v_1+\dots+v_n : v_1, \dots, v_n \in A\}$ for each $A \subseteq V$, and obtain the same bounded sets. More generally, for any connected multiplicative topological group G, we can take $T = \mathbb{N}$ and $n*A = A^n = \{a_1a_2\cdots a_n : a_1, a_2, \dots, a_n \in A\}$. Let U be an open and symmetric neighborhood of e. Then $\mathbb{N} * U$ is an open, and therefore also closed, subgroup of G. Thus, $\mathbb{N} * U = G$.

Example 2.5 (Standard boundedness on Topological groups). Fix a dense subset T of G of minimal cardinality. For our purposes, it does not matter which dense subset we take. Define $t * A = tA = \{ta : a \in A\}$ for all $t \in T, A \subseteq G$. The T-bounded sets are the precompact subsets of G. Axiom (B6) holds because our groups are assumed to be infinite Hausdorff.

When a topological group also happens to be a topological vector space, the term *standard boundedness system on G* has two contradictory interpretations. When we wish to use the one of topological vector spaces, we will say so explicitly.

The two canonical examples were combined by Hejcman [24], who considered the case $T = D \times \mathbb{N}$, where D is a dense subset of G, and $(d, n) * A = dA^n$. The T-bounded sets are the standard bounded sets when G is a topological vector space, and the precompact sets when G is a locally compact group.

Definition 2.6. Let (G, T) be a boundedness system. A set $A \subseteq G$ is *κ-bounded* (with respect to T) if, for each neighborhood U of e, there is $S \subseteq T$ such that $|S| \leq \kappa$, and $A \subseteq S * U$. The *boundedness number* of A in (G, T), denoted $\mathrm{b}_T(A)$, is the minimal κ such that A is κ-bounded.

Axiom (B6) asserts that $\mathrm{b}_T(G) \geq \aleph_0$.

For the standard boundedness system (G, T) on a topological group G (Example 2.5), $\mathrm{b}_T(G)$ does not depend on the choice of the dense subset T. Indeed, we have the following.

Definition 2.7. For a topological group G and a set $A \subseteq G$, $\mathrm{b}(A)$ is the minimal cardinal κ such that for each neighborhood U of e, there is $S \subseteq A$ such that $|S| \leq \kappa$, and $A \subseteq SU$.

Lemma 2.8 (folklore)**.** *Let (G, T) be a standard boundedness system on G. Then:*

(1) $\mathrm{b}_T(A) = \mathrm{b}(A)$ *for all* $A \subseteq G$.
(2) *If* $A \subseteq B \subseteq G$, *then* $\mathrm{b}(A) \leq \mathrm{b}(B)$.

Proof. (2) Clearly, $\mathrm{b}_T(A) \leq \mathrm{b}_T(B)$. Thus, it suffices to prove (1).

($\geq$) Fix a neighborhood U of e in G. Let V be a neighborhood of e in G, such that $V = V^{-1}$ and $V^2 \subseteq U$. Let $S \subseteq T$ be such that $|S| \leq \mathrm{b}_T(A)$, and $A \subseteq SV$. By thinning out S if needed, we may assume that for each $s \in S$, $sV \cap A \neq \emptyset$. For each $s \in S$, pick an element $a_s \in sV \cap A$. Then $s \in a_s V$, and thus $sV \subseteq a_s V^2 \subseteq a_s U$. Let $S' = \{a_s : s \in S\}$. Then $S' \subseteq A$, $|S'| \leq |S| \leq \mathrm{b}_T(A)$, and $A \subseteq SV \subseteq S'U$.

($\leq$) Similar, using that T is dense in G. □

Lemma 2.9. *For a standard boundedness system* (G, T) *on a topological group,* $|T| = \mathrm{d}(G)$. □

Thus, if (G, T) is a boundedness system with G a σ-compact group, then $\mathrm{b}_T(G) = \aleph_0$. But if G is (nonmetrizable and) not separable, then for the standard boundedness system on G, $|T| = \mathrm{d}(G) > \aleph_0$. That is, for each neighborhood U of e there is a countable $S \subseteq T$ such that $S * U = G$, but there is no such S independent on U.

Recall that for infinite cardinals κ and λ, $\kappa \cdot \lambda = \max(\kappa, \lambda)$.

Proposition 2.10. *Let* (G, T) *be a boundedness system. Then*

$$\mathrm{b}_T(G) \leq |T| \leq \chi(G) \cdot \mathrm{b}_T(G).$$

In particular:

(1) *For metrizable* G, $|T| = \mathrm{b}_T(G)$.
(2) $\mathrm{b}(G) \leq \mathrm{d}(G) \leq \chi(G) \cdot \mathrm{b}(G)$.
(3) *For metrizable* G, $\mathrm{b}(G) = \mathrm{d}(G)$.

Proof. $|T| \leq \chi(G) \cdot \mathrm{b}_T(G)$: Let $\{U_\alpha : \alpha < \chi(G)\}$ be a neighborhood base of G at e. For each $\alpha < \chi(G)$, let $S_\alpha \subseteq T$ be such that $|S_\alpha| \leq \mathrm{b}_T(G)$, and $S_\alpha * U_\alpha = G$. Let $S = \bigcup_{\alpha < \chi(G)} S_\alpha$. For each neighborhood U of e, $S * U = G$. If follows that $|T| = |S| \leq \chi(G) \cdot \mathrm{b}_T(G)$.

For (2) and (3), consider the standard boundedness system on G. □

Thus, when considering metrizable groups, we may replace $\mathrm{b}_T(G)$ by $|T|$, or by $\mathrm{d}(G)$ when the standard boundedness system is considered.

We give some examples, using the (multiplicative) *torus group* $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Example 2.11. The inequalities in Proposition 2.10 cannot be improved, not even for the standard boundedness system (Item 3 of the proposition) on powers of the torus: For compact groups G of cardinality 2^κ, $\mathrm{b}(G) = \aleph_0$, and $\mathrm{d}(G) = \log(\kappa)$, where $\log(\kappa)$ is defined as $\min\{\lambda : \kappa \leq 2^\lambda\}$ [11, Theorem 3.1].

Thus, for infinite κ, $\mathrm{b}(\mathbb{T}^\kappa) = \aleph_0$, $\mathrm{d}(\mathbb{T}^\kappa) = \log(\kappa)$, and $\chi(\mathbb{T}^\kappa) = \kappa$. The inequality $\aleph_0 \leq \log(\kappa) \leq \kappa$ cannot be improved: Let $\mathfrak{c} = 2^{\aleph_0}$.

(1) $\kappa = \aleph_0$ gives $\mathrm{b}(G) = \mathrm{d}(G) = \chi(G) = \aleph_0$.
(2) $\kappa = \mathfrak{c}$ gives $\mathrm{b}(G) = \mathrm{d}(G) = \aleph_0 < \chi(G) = \mathfrak{c}$.
(3) $\kappa = \mathfrak{c}^+$ gives $\mathrm{b}(G) = \aleph_0 < \mathrm{d}(G) = \log(\mathfrak{c}^+) < \chi(G) = \mathfrak{c}^+$.
(4) $\kappa = \beth_\omega$ gives $\mathrm{b}(G) = \aleph_0 < \mathrm{d}(G) = \chi(G) = \beth_\omega$.[6]

3. When T is countable

Methods and ideas from the context of topological vector spaces, developed by Saxon and Sánchez-Ruiz [34], and by Burke and Todorcevic [9], generalize in a straightforward manner to general boundedness systems (G, T) with T countable.

[6] The cardinal $\beth_\omega$ is defined as the supremum of all cardinals $\beth_n$, $n \in \mathbb{N}$, where $\beth_1 = 2^{\aleph_0}$ and for each $n > 1$, $\beth_n = 2^{\beth_{n-1}}$.

Even for the standard boundedness systems on topological groups, some of the obtained results were apparently not observed earlier.

Definition 3.1. (G,T) is *locally bounded* if there is in G a neighborhood base at e, consisting of bounded sets.

Definition 3.2. Let P,Q be partially ordered sets. $P \preceq Q$ if there is an order-preserving $f : P \to Q$ with image cofinal in Q. P is *cofinally equivalent* to Q if $P \preceq Q$ and $Q \preceq P$.

If $P \preceq Q$, then $\mathrm{cof}(Q) \leq \mathrm{cof}(P)$.

Definition 3.3. Let (G,T) be a boundedness system. $\mathrm{Bdd}_T(G)$ is the family of T-bounded subsets of G. $\mathrm{Bdd}_T(G)$ is considered with the partial order $\subseteq$. When (G,T) is a standard boundedness system, $\mathrm{Bdd}_T(G)$ is the family of precompact subsets of G, which we denote for simplicity by $\mathrm{PK}(G)$.

Remark 3.4. If G is T-bounded, then $\mathrm{Bdd}_T(G)$ is cofinally equivalent to the singleton $\{1\}$.

For a function $f : X \to Y$ and $A \subseteq X, B \subseteq Y$, we use the notation $f[A] = \{f(a) : a \in A\}$, and $f^{-1}[B] = \{x \in X : f(x) \in B\}$.

For locally convex topological vector spaces with the standard boundedness structure, the following is pointed out in [9, Theorem 2.5]. Recall that when T is countable, we may identify T with $\mathbb{N}$.

Proposition 3.5. *If a boundedness system* $(G,\mathbb{N})$ *is locally bounded and* G *is unbounded, then* $\mathrm{Bdd}_{\mathbb{N}}(G)$ *is cofinally equivalent to* $\mathbb{N}$.

Proof. Fix a bounded neighborhood U of e, such that for each finite $F \subseteq \mathbb{N}$, $F*U \neq G$. Define $\varphi : G \to \mathbb{N}$ by

$$\varphi(g) = \min\{n : g \in n * U\}.$$

The functions $K \mapsto \max\varphi[K]$ and $n \mapsto \varphi^{-1}[\{1,\dots,n\}]$ establish the required cofinal equivalence. □

Systems which are *not* locally bounded are more interesting in this respect. Assume that G is metrizable, and let U_n, $n \in \mathbb{N}$, be a neighborhood base at e.

Definition 3.6. $\Psi : G \to \mathbb{N}^{\mathbb{N}}$ is defined by

$$x \mapsto \varphi_x(n) = \min\{m : x \in m * U_n\}.$$

For a bounded set $B \subseteq \mathbb{N}^{\mathbb{N}}$, $f = \max B \in \mathbb{N}^{\mathbb{N}}$ is defined by $f(n) = \max\{g(n) : g \in B\}$. Define functions $\mathrm{Bdd}_{\mathbb{N}}(G) \to \mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}} \to \mathrm{Bdd}_{\mathbb{N}}(G)$, respectively, by

$$\begin{aligned} K &\mapsto \max\Psi[K]; \\ f &\mapsto \Psi^{-1}[\{g \in \mathbb{N}^{\mathbb{N}} : g \leq f\}]. \end{aligned}$$

Both functions are monotone, and the image of the latter is cofinal in $\mathrm{Bdd}_{\mathbb{N}}(G)$.

For locally convex topological vector spaces with the standard boundedness structure, the following is proved in [34, Proposition 1] and in [9, Theorem 2.5].

Theorem 3.7. *Let* $(G,\mathbb{N})$ *be a metrizable non-locally bounded boundedness system. Then* $\mathrm{Bdd}_{\mathbb{N}}(G)$ *is cofinally equivalent to* $\mathbb{N}^{\mathbb{N}}$.

Proof. As compact sets are bounded, it suffices to show that there is a neighborhood base U_n, $n \in \mathbb{N}$, at e, and for each $f \in \mathbb{N}^{\mathbb{N}}$, there is a compact $K \subseteq G$, such that $f \leq \max \Psi[K]$.

Let U_n, $n \in \mathbb{N}$, be a descending neighborhood base at e. As U_1 is not bounded, we may assume (by shrinking U_2 if needed) that there is no m such that $U_1 \subseteq \{1, \dots, m\} * U_2$. Continuing in the same manner, we may assume that for each n, there is no m such that $U_n \subseteq \{1, \dots, m\} * U_{n+1}$.

Given $f \in \mathbb{N}^{\mathbb{N}}$, choose for each n an element $x_n \in U_n \setminus \{1, \dots, f(n)\} * U_{n+1}$. As the original sequence U_n was descending to e, x_n converges to e, and thus $K = \{x_n : n \in \mathbb{N}\} \cup \{e\}$ is a compact set as required. □

Corollary 3.8. *Let G be a separable metrizable non-locally precompact group. Then* $\mathrm{PK}(G)$ *is cofinally equivalent to* $\mathbb{N}^{\mathbb{N}}$.

Definition 3.9. For a topological space X, $C(X, \mathbb{T})$ is the group of all continuous functions from X to $\mathbb{T}$ with pointwise multiplication, endowed with the *compact-open topology*. That is, a neighborhood base at the constant function 1 is given by the sets

$$\{f \in C(X, \mathbb{T}) : (\forall x \in K)\ |f(x) - 1| < \epsilon\},$$

where K is a compact subset of X, and ϵ is a positive real number.

A *Polish group* is a complete, separable, metrizable group. We give two well known examples of non-locally compact Polish groups, and where it is not immediately clear (without Corollary 3.8) that $\mathrm{PK}(G)$ is cofinally equivalent to $\mathbb{N}^{\mathbb{N}}$.

Example 3.10. Let L be a Lie group, for example $\mathbb{T}$ or the group of unitary $n \times n$ complex matrices. Let K be a compact metric space. $C(K, L)$ is a Polish group, with the metric given by the supremum norm. $C(K, L)$ is not locally compact (unless K is finite). By Lemma 3.7, the family of compact subsets of $C(K, L)$ is cofinally equivalent to $\mathbb{N}^{\mathbb{N}}$.

Example 3.11. Consider the group $S_{\mathbb{N}}$ of permutations on $\mathbb{N}$, where for each finite $F \subseteq \mathbb{N}$, the set U_F of all permutations fixing F is a neighborhood of the identity. This defines a neighborhood base at the identity permutation, and thus a topology on $S_{\mathbb{N}}$. $S_{\mathbb{N}}$ is a (nonabelian) Polish group, and it is not locally compact. Thus, its compact subsets are cofinally equivalent to $\mathbb{N}^{\mathbb{N}}$.

For $f, g \in \mathbb{N}^{\mathbb{N}}$, $f \leq^* g$ means: $f(n) \leq g(n)$ for all but finitely many n. $\mathfrak{b}$ is the minimal cardinality of a $\leq^*$-unbounded subset of $\mathbb{N}^{\mathbb{N}}$. $\mathfrak{b}$ is uncountable, and can consistently be any regular uncountable cardinal (details are available in [13, 7]).

For locally convex topological vector spaces with the standard boundedness structure, the following is Corollary 2.6 of [9].

Corollary 3.12. *Let $(G, \mathbb{N})$ be a metrizable boundedness system.*

1. *For each $\mathcal{F} \subseteq \mathrm{Bdd}_{\mathbb{N}}(G)$ with $|\mathcal{F}| < \mathfrak{b}$, there is a countable $\mathcal{S} \subseteq \mathrm{Bdd}_{\mathbb{N}}(G)$ such that each member of $\mathcal{F}$ is contained in a member of $\mathcal{S}$.*
2. *Each union of less than $\mathfrak{b}$ bounded subsets of G is a union of countably many bounded subsets of G.*

Proof. The statements are immediate when G is locally bounded. Thus, assume it is not. Then (1) follows from the cofinal equivalence of $\mathrm{Bdd}_{\mathbb{N}}(G)$ and $\mathbb{N}^{\mathbb{N}}$, and (2) follows from (1). □

Definition 3.13. A group G is *metrizable modulo precompact* if there is a precompact subgroup K of G, such that the coset space G/K is metrizable.

Example 3.14. All Čech-complete groups, and all almost-metrizable groups, are metrizable modulo precompact.

For nonabelian G, the coset space G/K need not be a group since we do not require K to be a *normal* subgroup. However, the concept of boundedness extends naturally to the coset space G/K, and we have the following.

Lemma 3.15. *Let K be a precompact subgroup of G, and $\pi : G \to G/K$ be the canonical quotient map.*

1. *If $P \in \mathrm{PK}(G)$, then $\pi[P] \in \mathrm{PK}(G/K)$.*
2. *If $Q \in \mathrm{PK}(G/K)$, then $\pi^{-1}[Q] \in \mathrm{PK}(G)$.*
3. *$\mathrm{PK}(G)$ is cofinally equivalent to $\mathrm{PK}(G/K)$.*

Proof. (1) Precompactness of K is not needed here: Let U be a neighborhood of eK in G/K. As $\pi^{-1}[U]$ is a neighborhood of e in G, there is a finite $F \subseteq G$ such that $P \subseteq F\pi^{-1}[U]$. Then $\pi[P] \subseteq \pi[F\pi^{-1}[U]] = FU$.

(2) Let U be a neighborhood of e in G. Take a neighborhood W of e such that $W^2 \subseteq U$. As K is precompact, there is a neighborhood V of e such that $VK \subseteq KW$.[7] As K is precompact, there is a finite $I \subseteq G$ such that $K \subseteq IW$.

$\pi[V]$ is a neighborhood of eK in G/K. Take a finite subset F of G such that $Q \subseteq \pi[F]\pi[V]$. Then $\pi^{-1}[Q] \subseteq \pi^{-1}[\pi[F]\pi[V]] = FKVK \subseteq FK^2W = FKW \subseteq FIW^2 \subseteq FIU$, and FI is finite.

(3) If $P \in \mathrm{PK}(G)$, then $Q = \pi[P] \in \mathrm{PK}(G/K)$, and $\pi^{-1}[Q] \in \mathrm{PK}(G)$, and contains P. Thus, the map $Q \mapsto \pi^{-1}[Q]$ shows that $\mathrm{PK}(G/K) \preceq \mathrm{PK}(G)$. Similarly, if $Q \in \mathrm{PK}(G/K)$, then $P = \pi^{-1}[Q] \in \mathrm{PK}(G)$, and $Q = \pi[P] \in \mathrm{PK}(G/K)$, and thus the map $P \mapsto \pi[P]$ gives $\mathrm{PK}(G) \preceq \mathrm{PK}(G/K)$. □

Corollary 3.16. *Let G be a separable, metrizable modulo precompact, Baire group. If G is a union of fewer than $\mathfrak{b}$ precompact sets, then G is locally precompact.*

Proof. By Lemma 3.15, we may assume that G is metrizable. By Corollary 3.12, G is a union of countably many precompact sets. As the closure of precompact sets is precompact, we may assume that these sets are closed. As G is Baire, one of these sets has nonempty interior. It follows that there is a precompact neighborhood of e. □

If every bounded subset of a normed space is separable, then the space is separable. Dieudonné [12] asked to what extent this can be generalized to locally convex topological vector spaces. Burke and Todorcevic answered this question completely, by showing that the same assertion holds in all locally convex topological vector spaces if, and only if, $\aleph_1 < \mathfrak{b}$ [9]. One direction of this assertion is generalized as follows.[8]

Theorem 3.17. *Let $(G, \mathbb{N})$ be a metrizable boundedness system, and $\mathrm{d}(G) < \mathfrak{b}$. If all bounded subsets of G are separable, then G is separable.*

[7] This is standard: Take a neighborhood W_0 of e with $W_0^2 \subseteq W$, and then take a finite $F \subseteq K$ such that $K \subseteq FW_0$. For each $g \in F$, $e{\cdot}g = g \in FW_0$, and thus there is a neighborhood V_g of e with $V_g \cdot g \subseteq FW_0$. Take $V = \bigcap_{g\in F} V_g$. Then $VF \subseteq FW_0$, and thus $VK \subseteq VFW_0 \subseteq FW_0W_0 \subseteq FW$.

[8] Theorem 3.17 is trivial when applied to standard boundedness systems on topological groups, but is nontrivial in general.

Proof. Assume otherwise, and let D be a discrete subset of G of cardinality $\aleph_1$. As $\aleph_1 < \mathfrak{b}$, we have by Corollary 3.12 that D is a union of countably many bounded sets. Thus, D has a (discrete, of course) bounded subset of cardinality $\aleph_1$. □

Lemma 3.18. *Using the notation of Definition 3.6: For all $m_1, \dots, m_n \in \mathbb{N}$, $\Psi^{-1}[\{f \in \mathbb{N}^\mathbb{N} : f(k) \le m_k, k = 1, \dots, n\}]$ is open.* □

Proposition 3.19. *For each sequence $x_n \to x$ in G, there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that φ_{y_n} converges to a function $f \le \varphi_x$.*

Proof. By Lemma 3.18, $\varphi_{x_n}(1) \le \varphi_x(1)$ for all but finitely many n. Thus, there is $m_1 \le \varphi_x(1)$ such that $I_1 = \{n : \varphi_{x_n}(1) = m_1\}$ is infinite. Inductively, given the infinite $I_{k-1} \subseteq \mathbb{N}$, we have by Lemma 3.18 that $\varphi_{x_n}(k) \le \varphi_x(k)$ for all but finitely many $n \in I_{k-1}$, and thus there is $m_k \le \varphi_x(k)$ such that $I_k = \{n \in I_{k-1} : \varphi_{x_n}(k) = m_k\}$ is infinite.

For each k, pick $i_k \in I_k$ with $i_k > i_{k-1}$. Then $\varphi_{x_{i_k}} \to f$, where $f(k) = m_k \le \varphi_x(k)$ for all k. □

The next result tells that if the group has a small dense subset, then the bounded subsets of its completion are determined by the bounded subsets of any dense subgroup of G. A special case of it was proved by Grothendieck [23], and extended in [9, Theorem 2.1], for G a separable metrizable locally convex topological vector space.

Theorem 3.20. *Let $(G, \mathbb{N})$ be a metrizable boundedness system, and $\mathrm{d}(G) < \mathfrak{b}$. Let D be a dense subset of G. For each bounded $K \subseteq G$, there is a bounded $J \subseteq D$ such that $K \subseteq \overline{J}$.*

Proof. Assume that G is locally compact, and let U be a compact neighborhood of e. Take a finite $F \subseteq \mathbb{N}$ such that $K \subseteq F * U$, and let $J = D \cap (F * U)$. Then $K \subseteq \overline{J}$.

Next, assume that G is not locally compact. As $\mathrm{d}(G) < \mathfrak{b}$, there is $K' \subseteq K$ such that $|K'| < \mathfrak{b}$ and $K \subseteq \overline{K'}$. For each $x \in K'$, let $\{x_n\}$ be a sequence in D converging to x. By Proposition 3.19, we may assume that $\{\varphi_{x_n}\}$ converges to a function $\varphi'_x \le \varphi_x$. $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact and thus bounded. Take g_x such that $\varphi_{x_n} \le g_x$ for all n.

As $|K'| < \mathfrak{b}$, there is $h \in \mathbb{N}^\mathbb{N}$ such that $g_x \le^* h$ for all $x \in K'$. We require also that all elements of $\Psi[K]$ are $\le h$. For each $x \in K'$, $\varphi_{x_n} \le h$ for all but finitely many n: Indeed, let N be such that $g_x(k) \le h(k)$ for all $k > N$. For all but finitely many n,

$$\varphi_{x_n}(1) = \varphi'_x(1) \le \varphi_x(1) \le h(1), \dots, \varphi_{x_n}(N) = \varphi'_x(N) \le \varphi_x(N) \le h(N),$$

as $x \in K$, and for $k > N$, $\varphi_{x_n}(k) \le g_x(k) \le h(k)$. Thus, for $J = D \cap \Psi^{-1}[\{f \in \mathbb{N}^\mathbb{N} : f \le h\}]$, we have that $K' \subseteq \overline{J}$, and therefore also $K \subseteq \overline{J}$. □

It seems that the following special case of Theorem 3.20 was not noticed before.

Corollary 3.21. *Let G be metrizable, and H be a dense subgroup of G. For each precompact $K \subseteq G$, there is a precompact $J \subseteq H$ such that $K \subseteq \overline{J}$.*

Proof. As K is precompact and G is metrizable, K is separable. As H is dense in G and K is separable, there is a countable $D \subseteq H$ such that $K \subseteq \overline{D}$. We may assume that D is a group. Let $G' = \overline{D}$, and apply Theorem 3.20 to G' and D to obtain a bounded $J \subseteq D$ such that $K \subseteq \overline{J}$. □

Example 3.22. Consider the permutation group $S_{\mathbb{N}}$ from Example 3.11. By Corollary 3.21, each compact subset of $S_{\mathbb{N}}$ is contained in the closure of some precompact set of finitely supported permutations.

Remark 3.23. There is no assumption on the density of G in corollary 3.21. However, metrizability is needed: A *P-group* is a group where every G_δ set is open. For each complete P-group G with a proper dense subgroup H, and each $g \in G$, $\{g\}$ is not contained in the closure of any precompact subset of H. Indeed, if $B \subseteq H$ is precompact, then $\overline{B}$ is a compact subset of G, and thus finite (countably infinite subsets of P-spaces are closed and discrete), and thus $\overline{B} \subseteq H$.

For a concrete example, let $\mathbb{Z}_2$ be the two element group, and take $G = (\mathbb{Z}_2)^\kappa$ for some $\kappa > \aleph_0$, with the countable box topology, and let H be the group of all $g \in (\mathbb{Z}_2)^\kappa$ which are supported on a countable set.

Corollary 3.21 implies the following.

Corollary 3.24. *Let G be metrizable, and H be a dense subgroup of G. Then* $\mathrm{PK}(H)$ *is cofinally equivalent to* $\mathrm{PK}(G)$. □

4. The cofinality of the family of bounded sets

For locally convex topological vector spaces with the standard boundedness structure, the following is proved in [34, Theorem 1] and in [9, Theorem 2.5]. In its general form, it follows from Proposition 3.5 and Theorem 3.7.

Corollary 4.1. *Let $(G, \mathbb{N})$ be a boundedness system.*

1. *If G is bounded, then* $\mathrm{cof}(\mathrm{Bdd}_{\mathbb{N}}(G)) = 1$.
2. *If G is locally bounded and unbounded, then* $\mathrm{cof}(\mathrm{Bdd}_{\mathbb{N}}(G)) = \aleph_0$.
3. *If G is metrizable non-locally bounded, then* $\mathrm{cof}(\mathrm{Bdd}_{\mathbb{N}}(G)) = \mathfrak{d}$. □

Lemma 4.2. *Let (G, T) be a boundedness system.*

1. *If G is bounded, then* $\mathrm{cof}(\mathrm{Bdd}_T(G)) = 1$.
2. *If G is unbounded, then:*
   - (a) $\aleph_0 \leq \mathrm{cof}(\mathrm{Bdd}_T(G))$.
   - (b) $\mathrm{b}_T(G) \leq \mathrm{cof}(\mathrm{Bdd}_T(G))$.
   - (c) *If $\chi(G) \leq |T|$ (in particular, for metrizable G), then* $|T| \leq \mathrm{cof}(\mathrm{Bdd}_T(G))$.

Proof of (2). (a) Otherwise, G is the union of finitely many bounded sets, and thus bounded.

(b) Let $\kappa = \mathrm{cof}(\mathrm{Bdd}_T(G))$. By (a), $\kappa \geq \aleph_0$. Let $\{K_\alpha : \alpha < \kappa\}$ be cofinal in $\mathrm{Bdd}_T(G)$. For each neighborhood U of e, there are finite $F_\alpha \subseteq T$, $\alpha < \kappa$, such that $K_\alpha \subseteq F_\alpha * U$. Let $S = \bigcup_{\alpha<\kappa} F_\alpha$. Then $|S| = \kappa$, and $S * U$ contains $\bigcup_{\alpha<\kappa} K_\alpha = G$.

(c) Apply (b) and Proposition 2.10. □

Lemma 4.3.

1. *Let (G, T) be an unbounded locally bounded metrizable boundedness system. Then* $\mathrm{cof}(\mathrm{Bdd}_T(G)) = |T|$.
2. *For each metrizable nonprecompact locally precompact group G,* $\mathrm{cof}(\mathrm{PK}(G)) = \mathrm{d}(G)$.

Proof of (1). Let U be a bounded neighborhood of e. Then $\{F * U : F \in \mathrm{Fin}(T)\}$ is cofinal in $\mathrm{Bdd}_T(G)$, and thus $\mathrm{cof}(\mathrm{Bdd}_T(G)) \leq |\mathrm{Fin}(T)| = |T|$. Apply Lemma 4.2. □

Definition 4.4. For a set X, $\mathrm{Fin}(X)^{\mathbb{N}}$ is the set of all functions $f : \mathbb{N} \to \mathrm{Fin}(X)$. This set is partially ordered by defining $f \subseteq g$ as $f(n) \subseteq g(n)$ for all n.

$\mathrm{cof}(\mathrm{Fin}(X)^{\mathbb{N}})$ depends only on $|X|$.

Lemma 4.5. *Let (G, T) be a metrizable boundedness system, and let $\kappa = |T|$. Then:*

1. $\mathrm{Fin}(\kappa)^{\mathbb{N}} \preceq \mathrm{Bdd}_T(G)$.
2. $\mathrm{cof}(\mathrm{Bdd}_T(G)) \leq \mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}})$.
3. $\mathrm{cof}(\mathrm{PK}(G)) \leq \mathrm{cof}(\mathrm{Fin}(\mathrm{d}(G))^{\mathbb{N}})$.

Proof of (1). Fix a neighborhood base U_n, $n \in \mathbb{N}$, at e. For each $f \in \mathrm{Fin}(\kappa)^{\mathbb{N}}$, define

$$K_f = \bigcap_{n \in \mathbb{N}} f(n) * U_n.$$

Then each $K_f \in \mathrm{Bdd}_T(G)$, and $\{K_f : f \in \mathrm{Fin}(\kappa)^{\mathbb{N}}\}$ is cofinal in $\mathrm{Bdd}_T(G)$. □

The following concept is central for the main results of this section.

Definition 4.6. The *local density* of a group G is the cardinal

$$\mathrm{ld}(G) = \min\{\mathrm{d}(U) : U \text{ is a neighborhood of } e \text{ in } G\}.$$

G has *stable density* if $\mathrm{ld}(G) = \mathrm{d}(G)$.

G has local density κ if, and only if, G has a local base at e, consisting of elements of density κ.

Lemma 4.7. $\mathrm{ld}(G)$ *is the minimal density of a clopen subgroup H of G. Thus, G has stable density if, and only if,* $\mathrm{d}(H) = \mathrm{d}(G)$ *for all clopen $H \leq G$.*

Proof. Let $U \subseteq G$ be an open neighborhood of e, with $\mathrm{d}(U) = \mathrm{ld}(G)$. Take $H = \langle U \rangle$. H is an open subgroup of G, and is thus also closed. □

Example 4.8. If G is connected, then G has stable density.

Definition 4.9. Let V be a neighborhood of e in G. A set $A \subseteq G$ is a V*-grid* if the sets aV, $a \in A$, are pairwise disjoint. A is a *grid* if it is a V-grid for some neighborhood V of e.

The intersection of a precompact set and a grid must be finite.

Lemma 4.10. *Let G be a metrizable group with stable density. Let $\kappa = \mathrm{d}(G)$, and U be a neighborhood of e.*

1. *For each $\lambda < \kappa$, U contains a grid of cardinality greater than λ.*
2. *If* $\mathrm{cof}(\kappa) > \aleph_0$*, then U contains a grid of cardinality κ.*

Proof. (1) Let $V \subseteq U$ be a symmetric neighborhood of e, such that for each $S \subseteq G$ with $|S| = \lambda < \kappa$, SV^2 does not contain U.

By Zorn's Lemma, there is a maximal V-grid A in U. As V is symmetric, $U \subseteq AV^2$. It follows that $|A| > \lambda$.

(2) Let $\{V_n : n \in \mathbb{N}\}$ be a symmetric local base at e, and for each n let A_n be a maximal V_n-grid in U. The previous argument shows that for each $\lambda < \kappa$, there is n such that $|A_n| > \lambda$. Thus, $\sup_n |A_n| = \kappa$. As $\mathrm{cof}(\kappa) > \aleph_0$, there is n with $|A_n| = \kappa$. □

We are now ready for the main results of this section. Given partially ordered sets $P_1,\dots,P_k$, define the *coordinate-wise partial order* on $P_1\times\ldots\times P_k$ by $(a_1,\dots,a_k)\le(b_1,\dots,b_k)$ if $a_1\le b_1,\dots,a_k\le b_k$.

Definition 4.11. For cardinals κ,λ, the family

$$[\kappa]^\lambda=\{A\subseteq\kappa: |A|=\lambda\}$$

is partially ordered by $\subseteq$.

Theorem 4.12. *Let G be a metrizable non-locally precompact group of stable density κ. Then:*

(1) $\mathrm{PK}(G)$ *is cofinally equivalent to* $\mathbb{N}^\mathbb{N}\times[\kappa]^{\aleph_0}$.
(2) $\mathrm{cof}(\mathrm{PK}(G))=\mathfrak{d}\cdot\mathrm{cof}([\kappa]^{\aleph_0})$.

Theorem 4.12 follows from the following two propositions.

Proposition 4.13. *Let G be a metrizable non-locally precompact group of stable density κ. Then:*

(1) $\mathrm{PK}(G)$ *is cofinally equivalent to* $\mathrm{Fin}(\kappa)^\mathbb{N}$.
(2) $\mathrm{cof}(\mathrm{PK}(G))=\mathrm{cof}(\mathrm{Fin}(\kappa)^\mathbb{N})$.

Proof. If $\mathrm{cof}(\kappa)>\aleph_0$, let $\kappa_n=\kappa$ for all n. Otherwise, κ_n, $n\in\mathbb{N}$, be such that $\kappa_n<\kappa_{n+1}$ for all n, and $\sup_n\kappa_n=\kappa$.

Let $\{U_n:n\in\mathbb{N}\}$ be a decreasing local base at e. For each n, there is by Lemma 4.10 a grid $A_n\subseteq U_n$ with $|A_n|=\kappa_n$.

Let $P\in\mathrm{PK}(G)$. Then $P\cap A_n$ is finite for all n. Thus, we can define $\Psi:\mathrm{PK}(G)\to\prod_n\mathrm{Fin}(A_n)$ by

$$P\mapsto f \text{ with } f(n)=P\cap A_n$$

for all n.

Ψ is cofinal: For each $f\in\prod_n\mathrm{Fin}(A_n)$, $P=\bigcup_n f(n)\cup\{e\}$ is a countable set converging to e, and thus compact, and for each n, $f(n)\subseteq\Psi(P)(n)$.

As Ψ is monotone and cofinal, $\mathrm{PK}(G)\preceq\prod_n\mathrm{Fin}(A_n)$.

Lemma 4.14. *If $\kappa_n\le\kappa_{n+1}$ for all n, and $\sup_n\kappa_n=\kappa$, then*

$$\prod_n\mathrm{Fin}(\kappa_n)\preceq\mathbb{N}^\mathbb{N}\times\prod_n\mathrm{Fin}(\kappa_n)\preceq\mathrm{Fin}(\kappa)^\mathbb{N}.$$

To prove the first assertion, map f to the pair (h,f), where $h(n)=\max f(n)\cap\omega$ (or 0 if $f(n)\cap\omega$ is empty).

For the second assertion, map (h,g) to the function

$$f(n)=\bigcup_{m\le h(n)}g(m).\quad\square$$

Now, apply Lemma 4.5.

Proposition 4.15. *For each infinite cardinal κ:*

(1) $\mathrm{Fin}(\kappa)^\mathbb{N}$ *is cofinally equivalent to* $\mathbb{N}^\mathbb{N}\times[\kappa]^{\aleph_0}$.
(2) $\mathrm{cof}(\mathrm{Fin}(\kappa)^\mathbb{N})=\mathfrak{d}\cdot\mathrm{cof}([\kappa]^{\aleph_0})$.

Proof of (1). $\mathrm{Fin}(\kappa)^{\mathbb{N}} \preceq \mathbb{N}^{\mathbb{N}} \times [\kappa]^{\aleph_0}$: Given $f \in \mathrm{Fin}(\kappa)^{\mathbb{N}}$, define $g_f \in \mathbb{N}^{\mathbb{N}}$ by $g_f(n) = \max(f(n) \cap \omega) \cup \{0\}$, and $s_f = \bigcup_n f(n)$. The map $f \mapsto (g_f, s_f)$ is monotone and cofinal.

$\mathbb{N}^{\mathbb{N}} \times [\kappa]^{\aleph_0} \preceq \mathrm{Fin}(\kappa)^{\mathbb{N}}$: For each $s \in [\kappa]^{\aleph_0}$, fix a surjection $r_s : \mathbb{N} \to s$. The mapping of $(f, s) \in \mathbb{N}^{\mathbb{N}} \times [\kappa]^{\aleph_0}$ to $g \in \mathrm{Fin}(\kappa)^{\mathbb{N}}$, defined by

$$g(n) = \{r_s(1), r_s(2), \ldots, r_s(f(n))\}$$

for all n, is monotone and cofinal. □

We now treat the general case, using the following observation: If H is a clopen subgroup of G of density $\mathrm{ld}(G)$, then H has stable density, G/H is discrete, and $\mathrm{d}(G) = |G/H| \cdot \mathrm{ld}(G)$.

Theorem 4.16. *Let G be a metrizable non-locally precompact group.*

1. *Let H be a clopen subgroup of G, of density $\mathrm{ld}(G)$. Then $\mathrm{PK}(G)$ is cofinally equivalent to $\mathrm{Fin}(G/H) \times \mathbb{N}^{\mathbb{N}} \times [\mathrm{ld}(G)]^{\aleph_0}$.*
2. $\mathrm{cof}(\mathrm{PK}(G)) = \mathfrak{d} \cdot \mathrm{d}(G) \cdot \mathrm{cof}([\mathrm{ld}(G)]^{\aleph_0})$.

Proof. (1) $\mathrm{d}(H) = \mathrm{ld}(G) = \mathrm{ld}(H)$.

Lemma 4.17. *For each clopen subgroup H of G, $\mathrm{PK}(G)$ is cofinally equivalent to $\mathrm{Fin}(G/H) \times \mathrm{PK}(H)$.*

Proof. Fix a set $S \subseteq G$ of coset representatives, that is such that for each $g \in G$, $|S \cap gH| = 1$. We need to show that $\mathrm{PK}(G)$ is cofinally equivalent to $\mathrm{Fin}(S) \times \mathrm{PK}(H)$.

For $A \subseteq G$ let $S(A) = \{s \in S : sH \cap A \neq \emptyset\}$.

$$P \mapsto \left(S(P), H \cap \bigcup_{s \in S(P)} s^{-1}P \right)$$

is a monotone and cofinal map from $\mathrm{PK}(G)$ to $\mathrm{Fin}(S) \times \mathrm{PK}(H)$.

For the other direction, we can map each $(F, P) \in \mathrm{Fin}(S) \times \mathrm{PK}(H)$ to FP. □

This, together with Theorem 4.12, proves (1).

(2) By (1),

$$\mathrm{cof}(\mathrm{PK}(G)) = |G/H| \cdot \mathfrak{d} \cdot \mathrm{cof}([\mathrm{ld}(G)]^{\aleph_0}).$$

The statement follows, using that $|G/H| \leq \mathrm{d}(G) \leq \mathrm{cof}(\mathrm{PK}(G))$ (Lemma 4.2). □

Example 4.18. For all cardinals $\lambda \leq \kappa$, there are metrizable groups G with $\mathrm{ld}(G) = \lambda$ and $\mathrm{d}(G) = \kappa$. For example, a product of a discrete group of cardinality κ and $C(\mathbb{T}^\lambda, \mathbb{T})$.

An extreme example is where G is discrete: We obtain $\mathrm{ld}(G) = 1$, and $\mathrm{d}(G) = |G|$, and indeed $\mathrm{PK}(G) = \mathrm{Fin}(G/\{e\})$.

$\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}})$ also appears, in a different context, in a recent work of Bonanzinga and Matveev [8]. We will return to this towards the end of this paper.

5. Abelian groups and Pontryagin-van Kampen duality

In the remainder of the paper, all considered groups are assumed to be abelian, and we use the additive notation and 0 for the trivial element. In particular, we identify $\mathbb{T}$ with the additive group $[-1/2, 1/2)$, having addition defined by identifying $\pm 1/2$.

A character on a topological group G is a continuous group homomorphism from G to the torus group $\mathbb{T}$. This is a collision in terminology, which may be solved as follows: Characters on G are its continuous homomorphisms into $\mathbb{T}$, whereas *the* character of G is the minimal cardinality of a local base of G at e. The set of all characters on G, with pointwise addition, is a group.

Let $\mathrm{K}(G)$ denote the family of all compact subsets of G. For a set $A \subseteq G$ and a positive real ϵ, define

$$[A, \epsilon] = \{\chi \in \widehat{G} : (\forall a \in A)\ |\chi(a)| \leq \epsilon\}.$$

The sets $[K, \epsilon] \subseteq \widehat{G}$ ($K \in \mathrm{K}(G), \epsilon > 0$) form a neighborhood base at the trivial character, defining the compact-open topology. We write $\widehat{G}$ for the topological group obtained in this manner.

G is *reflexive* if the evaluation map

$$E : G \to \widehat{\widehat{G}}$$

defined by $E(g)(\chi) = \chi(g)$ for all $g \in G, \chi \in \widehat{G}$, is a topological isomorphism. The *Pontryagin-van Kampen Theorem* asserts that every locally compact abelian group is reflexive.

Let K be a compact subset of G. For each n, the set $K_n = K \cup 2K \cup \cdots \cup nK$ is compact, and $[K_n, 1/4] \subseteq [K, 1/4n]$. Thus, the sets $[K, 1/4]$, $K \in \mathrm{K}(G)$, also form a neighborhood base of $\widehat{G}$ at the trivial character.

Definition 5.1. For $A \subseteq G$, $A^{\triangleright} = [A, 1/4]$. Similarly, for $X \subseteq \widehat{G}$, $X^{\triangleleft} = \{g \in G : (\forall \chi \in X)\ |\chi(g)| \leq \frac{1}{4}\}$.

Lemma 5.2 ([4, Proposition 1.5])**.** *For each neighborhood U of 0 in G, $U^{\triangleright} \in \mathrm{K}(\widehat{G})$.*

Definition 5.3 (Vilenkin [37])**.** A set $A \subseteq G$ is *quasiconvex* if $A^{\triangleright\triangleleft} = A$. G is *locally quasiconvex* if it has a neighborhood base at its identity, consisting of quasiconvex sets.

For each $A \subseteq G$, $A^{\triangleright}$ is a quasiconvex subset of $\widehat{G}$. Thus, $\widehat{G}$ is locally quasiconvex for all topological groups G. Moreover, local quasiconvexity is hereditary for arbitrary subgroups.

$A^{\triangleright\triangleleft}$ is the smallest quasiconvex subset of G containing A, and is closed.

In the case where G is a topological vector space G is locally quasiconvex in the present sense if, and only if, G is a locally convex topological vector space in the ordinary sense [4].

If G is locally quasiconvex, its characters separate points of G, and thus the evaluation map $E : G \to G^{\wedge\wedge}$ is injective. For each quasiconvex neighborhood U of 0 in G, $U^{\triangleright}$ is a compact subset of $\widehat{G}$ (Lemma 5.2), and thus $U^{\triangleright\triangleright}$ is a neighborhood of 0 in $G^{\wedge\wedge}$. As $E[G] \cap U^{\triangleright\triangleright} = E[U^{\triangleright\triangleleft}] = E[U]$, we have that E is open [4, Lemma 14.3].

Lemma 5.4. *Let G be a complete locally quasiconvex group. Let $\widehat{\mathcal{N}}$ be the family of all neighborhoods of 0 in $\widehat{G}$. Then:*

(1) $(\widehat{\mathcal{N}}, \supseteq)$ *is cofinally equivalent to* $(\mathrm{K}(G), \subseteq)$.
(2) $\chi(\widehat{G}) = \operatorname{cof}(\mathrm{K}(G))$.

Proof of (1). We have seen above that the monotone map $\triangleright : \mathrm{K}(G) \to \widehat{\mathcal{N}}$ is cofinal.

Consider the other direction. Let $K \in \mathrm{K}(G)$, and take $U = K^{\triangleright} \in \widehat{\mathcal{N}}$. By Lemma 5.2, $U^{\triangleright} \in \mathrm{K}(G^{\wedge\wedge})$. Now,

$$K \subseteq K^{\triangleright\triangleleft} = U^{\triangleleft} = E^{-1}[U^{\triangleright} \cap E[G]].$$

As G is complete, $U^{\triangleright} \cap E[G]$ is compact. As G is locally quasiconvex, E is open, and therefore $E^{-1}[U^{\triangleright} \cap E[G]]$ is compact. Thus, the monotone map $\triangleleft : \widehat{\mathcal{N}} \to \mathrm{K}(G)$ is also cofinal. □

Remark 5.5. As can be seen from the proof of Lemma 5.4, the assumption that G is complete can be wakened to the so-called *quasiconvex compactness property*: That for each $K \in \mathrm{K}(G)$, $K^{\triangleright\triangleleft} \in \mathrm{K}(G)$.

We obtain the following result, which extends to topological abelian groups a result of Saxon and Sanchez-Ruiz for the strong dual of a metrizable space [34, Corollary 2].[9]

A topological space X is a *k-space* if the topology of X is determined by its compact subsets, that is, $F \subseteq X$ is closed if (and only if) $F \cap K$ is closed in K for all $K \in \mathrm{K}(G)$. Every metrizable space is a k-space. A *k-group* is a topological group which is a k-space.

Let G be the dual of a metrizable group Γ. If Γ is (pre)compact, then by Pontryagin's Theorem, G is discrete, that is $\chi(G) = 1$. Item (1) of the following proposition is known [11, Theorem 3.12(ii)].

Proposition 5.6. *Let G be the dual of a metrizable, nonprecompact group Γ.*
(1) *If Γ is locally precompact, then $\chi(G) = \mathrm{d}(\Gamma)$.*
(2) *If Γ is non-locally precompact, then $\chi(G)$ is the maximum of $\mathfrak{d}$, $\mathrm{d}(\Gamma)$, and $\operatorname{cof}([\mathrm{ld}(\Gamma)]^{\aleph_0})$.*

Proof. Außenhofer [3] and independently Chasco [10] proved that a metrizable group and its completion have the same (topological) dual group. Since the density and local density of a metrizable group are equal to those of its completion, we may assume that Γ is complete.

Since Γ is metrizable, it is a k-space, and therefore $G = \widehat{\Gamma}$ is complete [4, Proposition 1.11]. By Lemma 5.4 and the completeness of Γ,

$$\chi(G) = \chi(\widehat{\Gamma}) = \operatorname{cof}(\mathrm{K}(\Gamma)) = \operatorname{cof}(\mathrm{PK}(\Gamma)).$$

(1) By Lemma 4.3, $\operatorname{cof}(\mathrm{PK}(\Gamma)) = \mathrm{d}(\Gamma)$.
(2) By Theorem 4.16 and Theorem 4.15,

$$\operatorname{cof}(\mathrm{PK}(\Gamma)) = \mathrm{d}(\Gamma) \cdot \operatorname{cof}(\mathrm{Fin}(\mathrm{ld}(\Gamma))^{\mathbb{N}}) = \mathfrak{d} \cdot \mathrm{d}(\Gamma) \cdot \operatorname{cof}([\mathrm{ld}(\Gamma)]^{\aleph_0}). \quad \square$$

Even for locally quasiconvex G, the evaluation map E need not be continuous. If it is, then G is isomorphic to its image $E[G]$ in $G^{\wedge\wedge}$.

[9] As every locally convex topological vector space is connected, it has stable density and therefore the concept of local density is not required in [34]. As stated here, our theorem does not generalize that of Saxon and Sanchez-Ruiz. There is a natural extension of our approach which implies their result as well, by replacing $\mathrm{K}(G)$ with more general boundedness notions on G. For concreteness, we do not present our results in full generality.

Definition 5.7. A topological group G is *subreflexive* if the evaluation map $E : G \to E[G]$ is a topological isomorphism. In this case, we identify G with its image $E[G] \leq G\hat{}\hat{}$.

Remark 5.8. If G is subreflexive, then G is locally quasiconvex. Indeed, $G\hat{}\hat{}$ is locally quasiconvex, being a dual group, and therefore so is its subgroups $E[G]$, which is isomorphic to G.

Lemma 5.9. *Let G be subreflexive. Then $\{K^{\lhd} : K \in \mathrm{K}(\widehat{G})\}$ is a neighborhood base at e in G.*

Proof. Let $K \in \mathrm{K}(\widehat{G})$. $K^{\rhd}$ is a neighborhood of 0 in $G\hat{}\hat{}$. As G is subreflexive, $K^{\lhd}$ is a neighborhood of 0 in G.

Let U be a neighborhood of e in G. As G is locally quasiconvex, we may assume that U is quasiconvex. Then $K = U^{\rhd}$ is a compact subset of $\widehat{G}$ (Lemma 5.2), and $K^{\lhd} = U^{\rhd\lhd} = U$. □

Proposition 5.10. *Let G be subreflexive, and $\mathcal{N}$ be the family of all neighborhoods of 0 in G. Then:*

(1) *$(\mathcal{N}, \supseteq)$ is cofinally equivalent to $(\mathrm{K}(\widehat{G}), \subseteq)$.*
(2) *$\chi(G) = \mathrm{cof}(\mathrm{K}(\widehat{G}))$.*

Proof of (1). By Lemma 5.9, the monotone map $\lhd : \mathrm{K}(\widehat{G}) \to \mathcal{N}$ is cofinal. The monotone map $\rhd : \mathcal{N} \to \mathrm{K}(\widehat{G})$ is also cofinal: Let $K \in \mathrm{K}(\widehat{G})$. By Lemma 5.9, $K^{\lhd} \in \mathcal{N}$, and $(K^{\lhd})^{\rhd} \supseteq K$. □

Even complete subreflexive groups G need not be reflexive. The following corollary tells that, however, $G\hat{}\hat{}$ is not much larger than G. (See also Theorem 7.6 and Corollary 7.7 below.) Außenhofer made related observations in [3, 5.22]. Question 5.23 in [3] asks whether the character group of an abelian metrizable group is reflexive.

Corollary 5.11.

(1) *For subreflexive G with $\widehat{G}$ complete, $\chi(G\hat{}\hat{}) = \chi(G)$.*
(2) *If G is a locally quasiconvex k-group, then $\chi(G\hat{}\hat{}) = \chi(G)$.*

Proof. (1) $\widehat{G}$ is locally quasiconvex. By Lemma 5.4 and Proposition 5.10, $\chi(G\hat{}\hat{}) = \mathrm{cof}(\mathrm{K}(\widehat{G})) = \chi(G)$.

(2) By Corollary 7.4 below, G is subreflexive. As G is a k-group, $\widehat{G}$ is complete. Apply (1). □

The first two items in the following theorem are well known.

Theorem 5.12. *Let G be a subreflexive group, such that the group $\Gamma = \widehat{G}$ is metrizable. Then $\chi(G) = \mathrm{cof}(\mathrm{PK}(\Gamma))$. Thus,*

(1) *If Γ is precompact, then $\chi(G) = 1$, that is, G is discrete.*
(2) *If Γ is nonprecompact locally precompact, then $\chi(G) = \mathrm{d}(\Gamma)$.*
(3) *If Γ is non-locally precompact, then $\chi(G) = \mathfrak{d} \cdot \mathrm{d}(\Gamma) \cdot \mathrm{cof}([\mathrm{ld}(\Gamma)]^{\aleph_0})$.*

Proof. By Proposition 5.10, $\chi(G) = \mathrm{cof}(\mathrm{K}(\widehat{G})) = \mathrm{cof}(\mathrm{K}(\Gamma))$. Let Δ be the completion of Γ. Δ is locally quasiconvex too, and metrizable, and thus subreflexive. By Corollary 3.24, $\mathrm{cof}(\mathrm{K}(\Delta)) = \mathrm{cof}(\mathrm{PK}(\Gamma))$.

It remains to prove that $\mathrm{K}(\Gamma)$ is cofinally equivalent to $\mathrm{K}(\Delta)$. By the Außenhofer-Chasco Theorem, we may identify $\widehat{\Delta}$ with $\widehat{\Gamma}$. As G is subreflexive, we also identify G with its image in $G^{\wedge\wedge} = \widehat{\Gamma}$, and similarly for Δ.

$\mathrm{K}(\Delta) \preceq \mathrm{K}(\Gamma)$: Let $K \in \mathrm{K}(\Delta)$. Then $K^{\triangleright}$ is a neighborhood of 0 in $\widehat{\Delta} = \widehat{\Gamma} = G^{\wedge\wedge}$. As G is subreflexive, $K^{\triangleright} \cap G$ is a neighborhood of 0 in G, and thus $(K^{\triangleright} \cap G)^{\triangleright} \in \mathrm{K}(\widehat{G}) = \mathrm{K}(\Gamma)$. Define $\Phi(K) = (K^{\triangleright} \cap G)^{\triangleright}$. For each $K \in \mathrm{K}(\Gamma)$, $K \in \mathrm{K}(\Delta)$ and $\Phi(K) \supseteq K$. Thus, Φ is cofinal.

$\mathrm{K}(\Gamma) \preceq \mathrm{K}(\Delta)$: Let $K \in \mathrm{K}(\Gamma)$. Then $K^{\triangleright}$ is a neighborhood of 0 in $\widehat{\Gamma} = \widehat{\Delta}$. Thus, $K^{\triangleright\triangleright} \in \mathrm{K}(\Delta^{\wedge\wedge})$, and as Δ is complete, $K^{\triangleright\triangleright} \cap \Delta \in \mathrm{K}(\Delta)$. Define $\Psi : \mathrm{K}(\Gamma) \to \mathrm{K}(\Delta)$ by $\Psi(K) = K^{\triangleright\triangleright} \cap \Delta$. For each $C \in \mathrm{K}(\Delta)$, $C^{\triangleright}$ is a neighborhood of 0 in $\widehat{\Delta} = \widehat{\Gamma}$, and thus there is $K \in \mathrm{K}(\Gamma)$ such that $K^{\triangleright} \subseteq C^{\triangleright}$. Then $K^{\triangleright\triangleright} \supseteq C^{\triangleright\triangleright} \supseteq C$, and therefore $\Psi(K) = K^{\triangleright\triangleright} \cap \Delta \supseteq C$. This shows that Ψ is cofinal.

(1) and (2) follow, using Lemma 4.3 and Theorem 4.16. □

Theorem 5.12 is stronger than Proposition 5.6: Duals of metrizable groups are subreflexive, and have a metrizable dual.

6. Application to the free topological groups

A topological space X is *hemicompact* if $\mathrm{cof}(\mathrm{K}(X)) \leq \aleph_0$. X is a k_ω *space* if it is a hemicompact k-space. Denote the weight of a topological space X by $\mathrm{w}(X)$.

The following theorem extends several results of Nickolas and Tkachenko [30, 31].[10] For example, they proved that if X is *compact*, then

$$\chi(A(X)) = \mathfrak{d} \cdot \mathrm{cof}([\mathrm{w}(X)]^{\aleph_0}),$$

and that if X is a k_ω space such that all compact subsets of X are metrizable, then $\chi(A(X)) = \mathfrak{d}$. Nickolas and Tkachenko's results were proved by direct methods. Even in these special cases, their arguments are sophisticated and technically very involved.

Theorem 6.1. *Let X be a k_ω space of compact weight κ. Then*

$$\chi(A(X)) = \mathfrak{d} \cdot \mathrm{cof}([\kappa]^{\aleph_0}).$$

Proof. Außenhofer [3] and independently Galindo-Hernández [17] proved that for a class of spaces X containing k-spaces (namely, Ascoli μ-spaces), $A(X)$ is subreflexive. Pestov [32] proved that for a class of spaces X containing k_ω spaces (namely, μ-spaces), $\widehat{A(X)} = C(X, \mathbb{T})$. As X is k_ω, $C(X, \mathbb{T})$ has a countable local base at 0 (namely, the sets $[K_n, 1/n]$ where $\{K_n : n \in \mathbb{N}\}$ is cofinal in $\mathrm{K}(X)$). Thus, $C(X, \mathbb{T})$ is metrizable.

Moreover, $C(X, \mathbb{T})$ is non-locally precompact. Thus, Theorem 5.12 applies.

Lemma 6.2 (Engelking [14, 3.4.16])**.** *If X is locally compact and* $\mathrm{w}(X)$ *is infinite, then* $\mathrm{w}(C(X, \mathbb{T})) \leq \mathrm{w}(X)$.

Lemma 6.3. *Let X be a Tychonoff space of compact weight κ. Then:*

(1) $\mathrm{b}(C(X, \mathbb{T})) = \mathrm{b}(C(X, \mathbb{R})) = \kappa$.

[10] See, e.g., the results numbered 2.12, 2.18, 2.22 in [30], and those numbered 2.9, 3.5, 3.7 in [31].

(2) *If X is hemicompact (or just* $\operatorname{cof}(K(X)) \leq \kappa$*), then*

$$\mathrm{b}(C(X,\mathbb{T})) = \mathrm{d}(C(X,\mathbb{T})) = \mathrm{ld}(C(X,\mathbb{T})) = \mathrm{w}(C(X,\mathbb{T})) = \kappa.$$

In particular, $C(X,\mathbb{T})$ *has stable density.*

Proof. For each cofinal family $\mathcal{K} \subseteq \mathrm{K}(X)$, and for $Y = \mathbb{T}$ or $\mathbb{R}$, the mapping $f \mapsto (f|_K : K \in \mathcal{K})$ is an embedding of $C(X,Y)$ in $\prod_{K\in\mathcal{K}} C(K,Y)$.

(1) In the case $\mathcal{K} = \mathrm{K}(X)$, we have by Lemma 6.2 that

$$\begin{aligned} \mathrm{b}(C(X,Y)) &\leq \mathrm{b}\left(\prod_{K\in\mathrm{K}(X)} C(K,Y)\right) = \sup_{K\in\mathrm{K}(X)} \mathrm{w}(C(K,Y)) \leq \\ &\leq \sup_{K\in\mathrm{K}(X)} \mathrm{w}(K), \end{aligned}$$

Let $K \in \mathrm{K}(X)$. Take $S \subseteq C(X,Y)$ with $|S| = \mathrm{b}(C(X,Y))$, such that $S+[K,1/16] = C(X,Y)$. Then $\{f^{-1}(-1/16,1/16) \cap K : f \in S\}$ is a base of K: Let $p \in U \cap K$, U open in X. As X is Tychonoff, there is $g \in C(X,Y)$ such that g is $1/4$ on $X \setminus U$ and $g(p) = 0$. As $S+[K,1/16] = C(X,Y)$, there is $f \in S$ such that $|f(x)-g(x)| \leq 1/16$ for each $x \in K$. It follows that $p \in g^{-1}(-1/16,1/16) \cap K \subseteq U \cap K$. Thus, $\mathrm{w}(K) \leq \mathrm{b}(C(X,Y))$ for each $K \in \mathrm{K}(X)$.

(2) By (1), $\kappa = \mathrm{b}(C(X,\mathbb{R})) \leq \mathrm{d}(C(X,\mathbb{R}))$. As $C(X,\mathbb{R})$ is connected, $\mathrm{d}(C(X,\mathbb{R})) = \mathrm{ld}(C(X,\mathbb{R}))$. For each $\epsilon < 1/2$ and each compact $K \subseteq X$, $[K,\epsilon]$ is the same in $C(X,\mathbb{R})$ and in $C(X,\mathbb{T})$. Thus,

$$\kappa \leq \mathrm{ld}(C(X,\mathbb{R})) \leq \mathrm{ld}(C(X,\mathbb{T})) \leq \mathrm{d}(C(X,\mathbb{T})) \leq \mathrm{w}(C(X,\mathbb{T})).$$

In the case where $|\mathcal{K}| = \operatorname{cof}(K(X))$,

$$\begin{aligned} \mathrm{w}(C(X,\mathbb{T})) &\leq \mathrm{w}\left(\prod_{K\in\mathcal{K}} C(K,\mathbb{T})\right) = |\mathcal{K}| \cdot \sup_{K\in\mathcal{K}} \mathrm{w}(C(K,\mathbb{T})) \leq \\ &\leq \operatorname{cof}(\mathrm{K}(X)) \cdot \sup_{K\in\mathrm{K}(X)} \mathrm{w}(K) \leq \kappa \cdot \kappa = \kappa. \quad \square \end{aligned}$$

We therefore have, by Theorem 5.12, that $\chi(A(X))$ is the maximum of $\mathfrak{d}$ and $\operatorname{cof}([\kappa]^{\aleph_0})$, where $\kappa = \mathrm{d}(C(X,\mathbb{T})) = \sup\{\mathrm{w}(K) : K \in \mathrm{K}(X)\}$.

This completes the proof of Theorem 6.1. $\square$

Example 6.4. If X is compact, or locally compact σ-compact, then X is a k_ω space, and thus Theorem 6.1 applies.

Together with the results of Sections 8 and 9, we obtain Theorem 5 and the following.

Corollary 6.5. *Let* $\bigsqcup_n K_n$ *be the direct union of compact sets* K_n *with* $\mathrm{w}(K_n) = \aleph_n$ *(e.g.,* $K_n = \mathbb{T}^{\aleph_n}$*). Fix* γ *with* $1 \leq \gamma < \aleph_1$*. It is consistent (relative to the consistency of ZFC with appropriate large cardinal hypotheses) that*

$$\aleph_\omega < \mathfrak{b} = \mathfrak{d} = \aleph_{\omega+1} < \chi\left(A\left(\bigsqcup_{n\in\mathbb{N}} K_n\right)\right) = \aleph_{\omega+\gamma+1} = \mathfrak{c}. \quad \square$$

The following deep theorem implies that our results also apply to free nonabelian groups. Let $F(X)$ be the free (nonabelian) group over X.

Theorem 6.6 (Nickolas-Tkachenko [31])**.** *If X is Lindelöf, then* $\chi(A(X)) = \chi(F(X))$*.*

7. The inner theorem

We begin with an inner characterization of subreflexivity.

Definition 7.1. $V \subseteq G$ is a *k-neighborhood* of 0 if for each $K \in \mathrm{K}(G)$ with $0 \in K$, $V \cap K$ is a neighborhood of 0 in K.

Lemma 7.2 (Hernández-Trigos-Arrieta [25])**.**

(1) *Let G be a k-group. Every quasiconvex k-neighborhood of* 0 *is a neighborhood of* 0.

(2) *Let U be a quasiconvex subset of a locally quasiconvex group G. U is a k-neighborhood of* 0 *if, and only if,* $U^{\triangleright} \in \mathrm{K}(\widehat{G})$.

We obtain the following.

Theorem 7.3. *A group G is subreflexive if, and only if, G is locally quasiconvex, and each quasiconvex k-neighborhood of the identity in G is a neighborhood of the identity.*

Proof. ($\Leftarrow$) Let $F \in \mathrm{K}(\widehat{G})$, and $K \in \mathrm{K}(G)$. By Ascoli's Theorem, the restrictions of the elements of F to K form an equicontinuous subset of $C(K, \mathbb{T})$. Hence, if K contains 0, then $F^{\triangleright} \cap K$ is a neighborhood of 0 in K. Again, taking intersections, we have that $F^{\triangleleft} \cap K$ is a neighborhood of 0 in K. Thus, $F^{\triangleleft}$ is a neighborhood of 0.

($\Rightarrow$) Let W be a quasiconvex k-neighborhood of 0. Then $W^{\triangleright}$ is compact in $\widehat{G}$. As G is subreflexive, $W = W^{\triangleright\triangleleft}$ is a neighborhood of 0 in G. □

Lemma 7.2 and Theorem 7.3 imply the following.

Corollary 7.4 (folklore)**.** *Every locally quasiconvex k-group is subreflexive.* □

For locally convex topological vector spaces and countable weight, the following result was proved by Ferrando, Kąkol, and M. López Pellicer [16].

Theorem 7.5. *Let G be a locally quasiconvex group.*

(1) $\mathrm{b}(\widehat{G})$ *is equal to the compact weight of G.*

(2) *If $\widehat{G}$ is metrizable, then* $\mathrm{d}(\widehat{G})$ *equal to the compact weight of G.*

Proof of (1). ($\leq$) As $\widehat{G} \leq C(G, \mathbb{T})$, we have by Lemmata 2.8 and 6.3 that $\mathrm{b}(\widehat{G}) \leq \mathrm{b}(C(G, \mathbb{T})) = \sup\{\mathrm{w}(K) : K \in \mathrm{K}(G)\}$.

($\geq$) Let $K \in \mathrm{K}(G)$. Since $[K, 1/8]$ is a neighborhood of the identity of $\widehat{G}$, there is $S \subseteq \widehat{G}$ with $|S| \leq \mathrm{b}(\widehat{G})$, such that $S + [K, 1/8] = \widehat{G}$.

S separates the points of K: Let a_1, a_2 be distinct elements of K. As G is locally quasiconvex, there is $\chi \in \widehat{G}$ such that $|\chi(a_1 - a_2)| > 1/4$. As $\chi \in \widehat{G} = S + [K, 1/8]$, there are $\alpha \in S$ and $\beta \in [K, 1/8]$ such that $\chi = \alpha + \beta$. Then $|\beta(a_1 - a_2)| \leq |\beta(a_1)| + |\beta(a_2)| \leq 2/8 = 1/4$, and thus $|\alpha(a_1 - a_2)| \geq |\chi(a_1 - a_2)| - 1/4 > 0$.

Thus, the minimal topology on K which makes all elements of S continuous is Hausdorff, and as K is compact, its topology (which is minimal Hausdorff) coincides with it. Thus, $\mathrm{w}(K) \leq |S| \leq \mathrm{b}(\widehat{G})$. □

An unpublished result of Außenhofer asserts that, if G is a separable metrizable group, then all higher character groups of G are separable. This is in accordance with item (3) of the following theorem.

Theorem 7.6. *Let G be a topological group, and let κ be the compact weight of $\widehat{G}$.*

1. *If G is subreflexive, then* $\mathrm{b}(G) = \mathrm{b}(G^{\wedge\wedge}) = \kappa$.
2. *If G is a locally quasiconvex k-group, then* $\mathrm{b}(G) = \mathrm{b}(G^{\wedge\wedge}) = \kappa$.
3. *If G is locally quasiconvex and metrizable, then* $\mathrm{d}(G) = \mathrm{d}(G^{\wedge\wedge}) = \kappa$.

Proof. (1) As $G \leq G^{\wedge\wedge}$, $\mathrm{b}(G) \leq \mathrm{b}(G^{\wedge\wedge})$. By Theorem 7.5, $\mathrm{b}(G^{\wedge\wedge}) = \kappa$. We prove that $\kappa \leq \mathrm{b}(G)$.

Let K be a compact subset of $\widehat{G}$. As G is subreflexive, the set

$$U = (K \cup 2K)^{\lhd} = \{g \in G : (\forall \chi \in K)\ |\chi(g)| \leq 1/8\}$$

is a neighborhood of 0 in G. Let $S \subseteq G$ be such that $|S| \leq \mathrm{b}(G)$, and $S + U = G$.

S separates points of K: Let $\chi, \psi \in K$ be distinct. As $G^{\rhd} = \{0\}$, there is $g \in G$ such that $|(\chi - \psi)(g)| > 1/4$. Take $s \in S, u \in U$, such that $g = s + u$. Then

$$|(\chi - \psi)(s)| \geq |(\chi - \psi)(g)| - |(\chi - \psi)(u)| > 1/8.$$

It follows that $\mathrm{w}(K) \leq |S| \leq \mathrm{b}(G)$.

(2) Locally quasiconvex metrizable groups are subreflexive, being locally quasiconvex k-groups (Corollary 7.4). □

Mikhail Tkachenko pointed out to us that our results imply the following.

Corollary 7.7. *For all subreflexive G with $\widehat{G}$ complete,* $\mathrm{w}(G^{\wedge\wedge}) = \mathrm{w}(G)$.

Proof. This follows from Corollary 5.11 and Theorem 7.6, using the fact $\mathrm{w}(G) = \mathrm{b}(G) \cdot \chi(G)$ for all topological groups [2]. □

We now turn to characterizing the local density of $\widehat{G}$ in terms of inner properties of G.

A mapping is *compact covering* if each compact subset of the range space is covered by the image of a compact subset of the domain.

Lemma 7.8. *Let H be a compact subgroup of G. Then the canonical projection $\pi : G \to G/H$ is compact covering.*

Proof. For each compact $K \subseteq G/H$, $\pi^{-1}[K]$ is compact. □

Lemma 7.9. *Let G be a topological group, and H be a compact subgroup of G. Then $\widehat{G/H}$ is topologically isomorphic to $H^{\rhd}$.*

Proof. The homeomorphism $\varphi : \widehat{G/H} \to \widehat{G}$ defined by $\varphi(\chi) = \chi \circ \pi$ is continuous and injective, and its image is $\{\chi \in \widehat{G} : \chi|_H = 0\} = H^{\rhd}$.

φ is open: Let U be a neighborhood of the identity of $\widehat{G/H}$. We may assume that $U = K^{\rhd}$ for some compact $K \subseteq G/H$. By Lemma 7.8, we may assume that $K = \pi[K']$ for some compact $K' \in \mathrm{K}(G)$. We may also assume that $K' \supseteq H$. Then $K'^{\rhd} \subseteq H^{\rhd}$, and therefore

$$\varphi[U] = \varphi[\pi[K']^{\rhd}] = \{\varphi(\chi) : \chi \in \pi[K']^{\rhd}\} = \{\chi \circ \pi : \chi \circ \pi \in K'^{\rhd}\} = K'^{\rhd}$$

is open. □

Lemma 7.10. *Let H be an open subgroup of G. Then the topological groups $\widehat{G/H}$ and $H^{\rhd}$ are isomorphic.*

Proof. As G/H is discrete, the topology on $\widehat{G/H}$ is the finite-open, and π is finite-covering. φ is open because $\widehat{G/H}$ is compact. □

For brevity, denote the compact weight of a group G by $kw(G)$.

Proposition 7.11. *Let G be a locally quasiconvex k_ω group. Then*

$$\mathrm{ld}(\widehat{G}) = \min\left\{kw(G/H) : H \leq G \text{ compact}\right\}.$$

Proof. ($\geq$) Let Γ be an open subgroup of G such that $\mathrm{d}(\Gamma) = \mathrm{ld}(\widehat{G})$. As G is k_ω, $\widehat{G}$ is first countable and thus metrizable. By Corollary 7.4 below, G is subreflexive. As k_ω groups are complete, $\Gamma^\triangleleft = \Gamma^\triangleright \cap G$ is an intersection of a compact group and a complete group, and is thus compact.

By Lemma 7.9, $\widehat{G/\Gamma^\triangleleft}$ is isomorphic to $\Gamma^{\triangleleft\triangleright}$, which contains Γ. By definition, Γ separates the points of $G/\Gamma^\triangleleft$, and therefore so does every dense subset of Γ. Thus, $\mathrm{w}(K) \leq \mathrm{d}(\Gamma)$ for all compact sets $K \subseteq G/\Gamma^\triangleleft$.

($\leq$) Let H be a compact subgroup of G. By Lemma 7.9, $\widehat{G/H}$ is isomorphic to $H^\triangleright$. As $H^\triangleright \leq \widehat{G}$, it is metrizable, and thus by Corollary 7.5,

$$\mathrm{d}(H^\triangleright) = \mathrm{d}(\widehat{G/H}) = kw(G/H).$$

As $H^\triangleright$ is open, $\mathrm{ld}(\widehat{G}) \leq \mathrm{d}(H^\triangleright)$. □

G is *locally hemicompact* (respectively, *locally k_ω*) if G contains an *open* hemicompact (respectively, k_ω) subgroup. The first item of the following theorem is an immediate consequence of the Pontryagin-van Kampen Theorem. The second item is new.

Theorem 7.12. *Let G be a locally quasiconvex, locally k_ω group. Let H be an open k_ω subgroup of G, of compact weight κ. Let $\lambda = \min\{kw(H/K) : K \leq H \text{ compact}\}$. Then:*

1. *If H is nondiscrete and locally compact, then $\chi(G) = \kappa$.*
2. *If H is non-locally compact, then $\chi(G)$ is the maximum of $\mathfrak{d}$, κ, and $\mathrm{cof}([\lambda]^{\aleph_0})$.*

Proof of (2). As H is open in G, $\chi(G) = \chi(H)$. G is locally quasiconvex, and therefore so is H. By Lemma 7.4, H is subreflexive. By hemicompactness, $\Gamma := \widehat{H}$ is metrizable. By Theorem 5.12,

$$\chi(H) = \mathfrak{d} \cdot \mathrm{d}(\Gamma) \cdot \mathrm{cof}([\mathrm{ld}(\Gamma)]^{\aleph_0}).$$

By Theorem 7.5(2), $\mathrm{d}(\Gamma) = \kappa$. By Proposition 7.11, $\mathrm{ld}(\Gamma) = \lambda$. □

Concrete estimations are given in the overview (Section 1). The remaining sections provide proofs for these estimations and additional details.

8. Shelah's theory of possible cofinalities

In this section, we provide estimations for $\mathrm{cof}([\kappa]^{\aleph_0})$, and in the next one, we establish some freedom in its determination. The estimations given here either appear explicitly in works of Shelah, or are easy consequences thereof. For the reader's convenience, we provide proofs.

Lemma 8.1. *For each $\kappa > \aleph_0$, $\kappa \leq \mathrm{cof}([\kappa]^{\aleph_0}) \leq \kappa^{\aleph_0}$.*

Proof. $\mathrm{cof}([\kappa]^{\aleph_0}) \leq |[\kappa]^{\aleph_0}| = \kappa^{\aleph_0}$.

For the other inequality, note that if $A \subseteq [\kappa]^{\aleph_0}$ and $|A| < \kappa$, then $|\bigcup A| \leq |A| \cdot \aleph_0 < \kappa$, and thus $\bigcup A \neq \kappa$. In particular, A is not cofinal in $[\kappa]^{\aleph_0}$. □

For each cardinal λ, $\kappa = \lambda^{\aleph_0}$ has the property $\kappa^{\aleph_0} = \kappa$. The most well-known cases are where $\kappa = 2^\lambda$ for some λ, but there are many more. E.g., if $\kappa^{\aleph_0} = \kappa$, then the same is true for the subsequent cardinal κ^+, and therefore also for κ^{++}, etc. This is also the case when κ is inaccessible. If GCH holds, this is the case for all cardinals, except for those of cofinality $\aleph_0$.

Corollary 8.2. *For each infinite κ with $\kappa^{\aleph_0} = \kappa$, $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \operatorname{cof}([\kappa]^{\aleph_0}) = \kappa$.*

Proof. If $\kappa^{\aleph_0} = \kappa$, then $\kappa \geq \mathfrak{c} \geq \mathfrak{d}$. Apply Theorem 4.15 and Lemma 8.1. □

Lemma 8.3. *For each $\kappa > \aleph_0$, $\operatorname{cof}([\kappa]^{\aleph_0}) = \kappa \cdot \sup\{\operatorname{cof}([\lambda]^{\aleph_0}) : \lambda \leq \kappa, \operatorname{cof}(\lambda) = \aleph_0\}$.*

Proof. ($\geq$) Monotonicity and Lemma 8.1.

($\leq$) If $\operatorname{cof}(\kappa) = \aleph_0$, this follows from the fact that $\kappa \leq \operatorname{cof}([\kappa]^{\aleph_0})$ (Lemma 8.1).

If $\operatorname{cof}(\kappa) > \aleph_0$, then each countable subset of κ is bounded in κ. Thus,

$$[\kappa]^{\aleph_0} = \bigcup_{\alpha<\kappa} [\alpha]^{\aleph_0},$$

and therefore $\operatorname{cof}([\kappa]^{\aleph_0}) \leq \kappa \cdot \sup\{\operatorname{cof}([\lambda]^{\aleph_0}) : \lambda < \kappa\}$. The statement for $\kappa = \aleph_1$ follows, and by induction, for each $\lambda < \kappa$ with $\lambda > \aleph_1$,

$$\begin{aligned} \operatorname{cof}([\lambda]^{\aleph_0}) &= \lambda \cdot \sup\{\operatorname{cof}([\mu]^{\aleph_0}) : \mu \leq \lambda, \operatorname{cof}(\mu) = \aleph_0\} \leq \\ &\leq \kappa \cdot \sup\{\operatorname{cof}([\mu]^{\aleph_0}) : \mu \leq \kappa, \operatorname{cof}(\mu) = \aleph_0\}. \quad \square \end{aligned}$$

Corollary 8.4. *For each κ, if $\operatorname{cof}([\kappa]^{\aleph_0}) = \kappa$, then $\operatorname{cof}([\kappa^+]^{\aleph_0}) = \kappa^+$.* □

Item (1) of the following corollary is well known [1], and Item (2) was proved, independently, by Bonanzinga and Matveev [8].

Corollary 8.5.

(1) $\operatorname{cof}([\aleph_0]^{\aleph_0}) = 1$*, and for each* $n \geq 1$, $\operatorname{cof}([\aleph_n]^{\aleph_0}) = \aleph_n$.

(2) $\operatorname{cof}(\operatorname{Fin}(\aleph_0)^{\mathbb{N}}) = \mathfrak{d}$*, and for each* $n \geq 1$, $\operatorname{cof}(\operatorname{Fin}(\aleph_n)^{\mathbb{N}}) = \mathfrak{d} \cdot \aleph_n$. □

Already for $\kappa = \aleph_\omega$, the situation is different. A diagonalization argument as in König's Lemma gives the following.

Lemma 8.6 (folklore)**.** *For singular κ, $\operatorname{cof}([\kappa]^{\operatorname{cof}(\kappa)}) > \kappa$.*

Corollary 8.7. *If $\operatorname{cof}(\kappa) = \aleph_0 < \kappa$, then $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) \geq \mathfrak{d} \cdot \kappa^+$.* □

Upper bounds require advanced methods.

8.1. **The easy way: Dismissing large cardinals.** Consider the following weakening of the Generalized Continuum Hypothesis.

Definition 8.8. *Shelah's Strong Hypothesis (SSH)* is the statement that for each uncountable κ with $\operatorname{cof}(\kappa) = \aleph_0$, $\operatorname{cof}([\kappa]^{\aleph_0}) = \kappa^+$.

Shelah's Strong Hypothesis is originally stated as "for each singular κ, the pseudopower of κ is κ^+". Its equivalence to the version presented here, which is much more convenient for our purposes, is due to Shelah.[11] The adjective "Strong" in SSH means that there is a yet weaker hypothesis, but SSH is in fact quite weak.

[11] The more involved direction follows from Theorem 6.3 of [36]. For the other direction: If κ is such that $\operatorname{pp}(\kappa) > \kappa^+$, then in particular $\operatorname{cof}[\kappa]^{\operatorname{cof}(\kappa)} > \kappa^+$, and we may (e.g., by Lemmata 3.4 and 3.8 in [33]) arrange that $\operatorname{cof}(\kappa) = \aleph_0$.

In particular, its failure implies the existence of large cardinals in the Dodd-Jensen core model.[12]

Following is the concluding Theorem 6.3 of [36]. The simplicity of the proof given here is due to the reformulation of SSH.

Theorem 8.9 (Shelah [36]). *Assume SSH. For each $\kappa > \aleph_0$, $\operatorname{cof}([\kappa]^{\aleph_0})$ is κ if $\operatorname{cof}(\kappa) > \aleph_0$, and κ^+ if $\operatorname{cof}(\kappa) = \aleph_0$.*

Proof. The case $\kappa = \aleph_1$ is Corollary 8.5. Continue by induction on κ: If $\operatorname{cof}(\kappa) = \aleph_0$, use Shelah's Strong Hypothesis (as reformulated in Definition 8.8). If $\operatorname{cof}(\kappa) > \aleph_0$, use Lemma 8.3 and the induction hypothesis to get

$$\operatorname{cof}([\kappa]^{\aleph_0}) = \kappa \cdot \sup\{\operatorname{cof}([\lambda]^{\aleph_0}) : \lambda < \kappa\} \leq \kappa \cdot \sup\{\lambda^+ : \lambda < \kappa\} = \kappa. \quad \square$$

Corollary 8.10. *Assume SSH. For each $\kappa > \aleph_0$:*

$$\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \begin{cases} \mathfrak{d} \cdot \kappa & \operatorname{cof}(\kappa) > \aleph_0 \\ \mathfrak{d} \cdot \kappa^+ & \operatorname{cof}(\kappa) = \aleph_0. \end{cases} \quad \square$$

Thus, under SSH, the value of $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}})$ is completely understood. We point out that in Theorem 8.9 and Corollary 8.10, it suffices to assume that Shelah's Strong Hypothesis holds for all $\lambda \leq \kappa$.

8.2. **The hard way: Bounds in ZFC.** Even without any hypotheses beyond the ordinary axioms of mathematics, nontrivial bounds on $\operatorname{Fin}(\kappa)^{\mathbb{N}}$ can be established in many cases, using Shelah's *pcf theory* [35]. There are several good introductions to pcf theory. A recent one is [1], whose references include some additional introductions. The following deep result appears as Theorem 7.2 in [1].

Theorem 8.11 (Shelah). *For each $\alpha < \aleph_\alpha$, $\operatorname{cof}([\aleph_\alpha]^{|\alpha|}) < \aleph_{|\alpha|^{+4}}$.*

In [1], Theorem 8.11 is stated for limit ordinals α, but taking $\delta = \alpha + \omega$, we have that $\delta < \aleph_\alpha < \aleph_\delta$, and applying Shelah's Theorem for the limit ordinal δ, $\operatorname{cof}([\aleph_\alpha]^{|\alpha|}) \leq \operatorname{cof}([\aleph_\delta]^{|\alpha|}) = \operatorname{cof}([\aleph_\delta]^{|\delta|}) < \aleph_{|\delta|^{+4}} = \aleph_{|\alpha|^{+4}}$.

Definition 8.12. Let π be the first fixed point of the $\aleph$ function, i.e., the first ordinal (necessarily, a cardinal) π such that $\pi = \aleph_\pi$.

π is quite big: Let $\pi_0 = \aleph_0$ and for each n, let $\pi_{n+1} = \aleph_{\pi_n}$. Then $\pi = \sup_n \pi_n$.
Shelah's Theorem has the following immediate corollaries.

Corollary 8.13. *For each $\alpha < \pi$, $\operatorname{cof}([\aleph_\alpha]^{\aleph_0}) < \aleph_{|\alpha|^{+4}}$.*

Proof. By induction on α. For $\alpha < \omega$ this follows from Corollary 8.5. Assume that the assertion is true for all $\beta < \alpha$, and prove it for α:

$$\operatorname{cof}([\aleph_\alpha]^{\aleph_0}) \leq \operatorname{cof}([\aleph_\alpha]^{|\alpha|}) \cdot \operatorname{cof}([|\alpha|]^{\aleph_0}).$$

As $\alpha < \pi$, Corollary 8.11 is applicable, and thus $\operatorname{cof}([\aleph_\alpha]^{|\alpha|}) < \aleph_{|\alpha|^{+4}}$. Let β be such that $|\alpha| = \aleph_\beta$. Then $\beta < \pi$, and thus $\beta < \aleph_\beta = |\alpha|$. By the induction hypothesis, $\operatorname{cof}([\aleph_\beta]^{\aleph_0}) < \aleph_{|\beta|^{+4}} \leq \aleph_{|\alpha|^{+3}}$. $\square$

Corollary 8.14. *For each successor cardinal $\kappa < \pi$ and each α with $\kappa \leq \alpha < \kappa + \omega$, $\operatorname{cof}([\aleph_\alpha]^{\aleph_0}) < \aleph_{\kappa^{+3}}$.*

[12] The failure of SSH at κ implies that in the Dodd-Jensen core model, there is a measurable $\lambda \leq \kappa$, moreover $o(\lambda) = \lambda^{++}$. The exact consistency strength of SSH was established by Gitik in [18, 19].

Proof. For each $\beta \in \{\kappa, \kappa+1, \kappa+2, \dots\}$, either $\beta = \kappa$ and $\mathrm{cof}(\aleph_\beta) = \mathrm{cof}(\kappa) > \aleph_0$, or β is a successor ordinal, and thus $\mathrm{cof}(\aleph_\beta) = \aleph_\beta > \aleph_0$. Thus, by Lemma 8.3,

$$\begin{aligned}\mathrm{cof}([\aleph_\alpha]^{\aleph_0}) &= \aleph_\alpha \cdot \sup\{\mathrm{cof}([\aleph_\beta]^{\aleph_0}) : \aleph_\beta \le \aleph_\alpha, \mathrm{cof}(\aleph_\beta) = \aleph_0\} = \\ &= \aleph_\alpha \cdot \sup\{\mathrm{cof}([\aleph_\beta]^{\aleph_0}) : \beta < \kappa, \mathrm{cof}(\beta) = \aleph_0\} \le \\ &\le \aleph_\alpha \cdot \sup\{\mathrm{cof}([\aleph_\beta]^{\aleph_0}) : \beta < \kappa\}.\end{aligned}$$

By Corollary 8.13, for each $\beta < \kappa$, $\mathrm{cof}([\aleph_\beta]^{\aleph_0}) < \aleph_{|\beta|^{+4}}$.

$\aleph_\alpha < \aleph_{|\alpha|^+} = \aleph_{\kappa^+} < \aleph_{\kappa^{+3}}$. Now, for each $\beta < \kappa$, $\mathrm{cof}([\aleph_\beta]^{\aleph_0}) < \aleph_{|\beta|^{+4}} \le \aleph_{\kappa^{+3}}$. As $\mathrm{cof}(\aleph_{\kappa^{+3}}) = \kappa^{+3} > \kappa$, the supremum is also smaller than $\aleph_{\kappa^{+3}}$. □

Corollary 8.15. *For each cardinal κ with $\aleph_0 < \mathrm{cof}(\kappa) < \kappa < \pi$, and each α with $\kappa \le \alpha < \kappa + \omega$, $\mathrm{cof}([\aleph_\alpha]^{\aleph_0}) = \aleph_\alpha$.*

Proof. Replace, in the proof of Corollary 8.14, the last paragraph with the following one: For each $\beta < \kappa$, $|\beta|^{+4} < \kappa$, and thus $\aleph_{|\beta|^{+4}} < \aleph_\kappa \le \aleph_\alpha$. □

Example 8.16. For each $n \ge 1$:

(1) For each $\alpha < \omega_n + \omega$, $\mathrm{cof}([\aleph_\alpha]^{\aleph_0}) < \aleph_{\omega_{n+3}}$.
(2) $\mathrm{cof}([\aleph_{\aleph_{\omega_n}}]^{\aleph_0}) = \aleph_{\aleph_{\omega_n}}$.

Combining Theorem 4.15 and the estimations provided here for $\mathrm{cof}([\kappa]^{\aleph_0})$, we obtain estimations for $\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}})$.

9. Things that cannot be proved about $\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}})$

Bonanzinga and Matveev consider in [8] a property named *star Menger*, introduced by Kočinac in [28]. Among other results, they prove that for each almost disjoint family $\mathcal{A}$ of subsets of $\mathbb{N}$ with $\mathrm{cof}(\mathrm{Fin}(|\mathcal{A}|)^{\mathbb{N}}) = |\mathcal{A}|$, the Mrówka space $\Psi(\mathcal{A})$ is not star Menger. By Proposition 4.15, the condition can be reformulated as $\mathrm{cof}([|\mathcal{A}|]^{\aleph_0}) = |\mathcal{A}| \ge \mathfrak{d}$. Corollary 8.15 and Corollary 8.10 imply the following new result.

Corollary 9.1. *Let $\mathcal{A}$ be an almost disjoint family of subsets of $\mathbb{N}$.*

(1) *For each cardinal κ with $\aleph_0 < \mathrm{cof}(\kappa) < \kappa < \pi$, and each α with $\kappa \le \alpha < \kappa + \omega$: If $|\mathcal{A}| = \aleph_\alpha \ge \mathfrak{d}$, then $\Psi(\mathcal{A})$ is not star Menger.*
(2) *Assume SSH. If $|\mathcal{A}| \ge \mathfrak{d}$ and $|\mathcal{A}|$ has uncountable cofinality, then $\Psi(\mathcal{A})$ is not star Menger.* □

In this context, $|\mathcal{A}| \le \mathfrak{c}$, and the following problem is natural.

Problem 9.2 (Bonanzinga-Matveev [8]). *Is $\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d} \cdot \kappa$ for each infinite $\kappa \le \mathfrak{c}$? In particular, is $\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d}$ for each infinite $\kappa \le \mathfrak{d}$?*

It is pointed out in [8] that the answer is positive for $\kappa < \aleph_\omega$ and for $\kappa = \mathfrak{c}$ (see Corollaries 8.2 and 8.5). Thus, clearly the Continuum Hypothesis implies a positive answer, and the problem actually asks whether the statements are provable without special set theoretic hypotheses. We first observe that SSH implies a positive answer to the second part of this problem, and a conditional solution to its first part.

Theorem 9.3. *Assume SSH.*

(1) *For each infinite $\kappa \le \mathfrak{d}$, $\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d}$.*
(2) *$\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d} \cdot \kappa$ for all infinite $\kappa \le \mathfrak{c}$ if, and only if, there is $n \ge 0$ such that $\mathfrak{c} = \mathfrak{d}^{+n}$, the n-th successor of $\mathfrak{d}$.*

Proof. We use Corollary 8.10.

(1) If $\operatorname{cof}(\kappa) > \aleph_0$, then $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d} \cdot \kappa = \mathfrak{d}$. Otherwise, as $\operatorname{cof}(\mathfrak{d}) \geq \mathfrak{b} > \aleph_0$, we have that $\kappa < \mathfrak{d}$, and $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d} \cdot \kappa^+ = \mathfrak{d}$.

(2) If there is such n, then each κ with $\mathfrak{d} \leq \kappa \leq \mathfrak{c}$ has uncountable cofinality, and thus $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d} \cdot \kappa$. Otherwise, take κ with $\operatorname{cof}(\kappa) = \aleph_0$ and $\mathfrak{d} < \kappa \leq \mathfrak{c}$. Then $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d} \cdot \kappa^+ = \kappa^+ > \kappa = \mathfrak{d} \cdot \kappa$. □

Theorem 9.3 indicates how to obtain a negative answer to the first part of Problem 9.2. We use some facts from the theory of forcing. A general introduction is available in Kunen's book [29], whose notation we follow. Some more details which are relevant for us here are available in Batoszyński and Judah's book [5], and in Blass's chapter [7].

Theorem 9.4. *It is consistent (relative to the consistency of ZFC) that Shelah's Strong Hypothesis holds (in particular,* $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d}$ *for each infinite* $\kappa \leq \mathfrak{d}$*), and there is* κ *with* $\mathfrak{d} < \kappa < \mathfrak{c}$*, such that* $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \operatorname{cof}([\kappa]^{\aleph_0}) = \kappa^+ > \mathfrak{d}$.

Proof. Let V be a model of (enough of) ZFC and of Shelah's Strong Hypothesis (e.g., a model of the Generalized Continuum Hypothesis). Let $\mathfrak{d} = \aleph_\alpha$ in V. Take $\beta > \alpha + \omega$, and let $\mathbb{B}(\aleph_\beta)$ be Solovay's forcing notion adding $\aleph_\beta$ random reals (see [5, Chapter 3]). $\mathbb{B}(\aleph_\beta)$ is ccc.

Lemma 9.5 (folklore)**.** *A generic extension by a ccc forcing notion does not change* $\operatorname{cof}([\kappa]^{\aleph_0})$*.*

Proof. Let P be a ccc forcing notion, and G be a P-generic filter over V.

Let λ be the cofinality of $[\kappa]^{\aleph_0}$ in $V[G]$. Take $f : \lambda \times \mathbb{N} \to \kappa$ such that $f \in V[G]$, and the sets $\{f(\alpha, n) : n \in \mathbb{N}\}$, $\alpha < \lambda$, are cofinal in $([\kappa]^{\aleph_0})^{V[G]}$.

As $\lambda \times \mathbb{N}$ and κ belong to V, there is $F : \lambda \times \mathbb{N} \to [\kappa]^{\aleph_0}$ such that $F \in V$, and $f(\alpha, n) \in F(\alpha, n)$ for all $(\alpha, n) \in \lambda \times \mathbb{N}$ [29, Lemma 5.5]. Let $\mathcal{F} = \{\bigcup_n F(\alpha, n) : \alpha < \lambda\}$. $\mathcal{F} \in V$, and $|\mathcal{F}| \leq \lambda$ in V.

For each $A \in ([\kappa]^{\aleph_0})^V$, $A \in ([\kappa]^{\aleph_0})^{V[G]}$, and thus there is α such that $A \subseteq \{f(\alpha, n) : n \in \mathbb{N}\} \subseteq \bigcup_n F(\alpha, n)$. Thus, $\mathcal{F}$ is cofinal in $([\kappa]^{\aleph_0})^V$. This shows that $\operatorname{cof}([\kappa]^{\aleph_0}) \leq \lambda$ in V.

This argument also shows that $([\kappa]^{\aleph_0})^V$ is cofinal in $([\kappa]^{\aleph_0})^{V[G]}$. Thus, $\operatorname{cof}([\kappa]^{\aleph_0})$ cannot be $< \lambda$ in V. □

Let G be $\mathbb{B}(\aleph_\beta)$-generic over V. By Lemma 9.5, $V[G]$ satisfies Shelah's Strong Hypothesis.

For each $f \in \mathbb{N}^{\mathbb{N}} \cap V[G]$, there is $g \in V$ such that $f \leq^* g$ [5, Lemma 3.1.2]. Thus, in $V[G]$, $\mathfrak{d}$ is at most $\aleph_\alpha$ and $\mathfrak{c}$ is at least $\aleph_\beta$.[13] Theorem 9.3(2) applies. □

Thus, the answer to the second part of Problem 9.2 is "No.", and the answer to its first part is "Yes" if there are no (inner) models of set theory with large cardinals. To complete the picture, it remains to show that the answer is "No" (to both parts) when large cardinal hypotheses are available. For the following theorem, it suffices for example to assume the consistency of supercompact cardinals, or of so-called *strong cardinals*. More precise large cardinal hypotheses are available in [21].

[13] In fact, if we begin with a model of GCH, then in $V[G]$, $\mathfrak{d} = \aleph_1$ and $\mathfrak{c} = \aleph_\beta$, or $\aleph_{\beta+1}$ if $\operatorname{cof}(\beta) = \aleph_0$.

Theorem 9.6 (Gitik-Magidor [21])**.** *It is consistent (relative to the consistency of ZFC with an appropriate large cardinal hypothesis) that* $2^{\aleph_n} = \aleph_{n+1}$ *for all* n*, and* $2^{\aleph_\omega} = \aleph_{\omega+\gamma+1}$*, for any prescribed* $\gamma < \omega_1$*.*

This is related to our questions as follows. As $\aleph_\omega$ is a limit cardinal of cofinality $\aleph_0$, $2^{\aleph_\omega} = (2^{<\aleph_\omega})^{\aleph_0}$. If $2^{\aleph_n} = \aleph_{n+1}$ for all n, then $2^{<\aleph_\omega} = \aleph_\omega$, and thus $2^{\aleph_\omega} = (\aleph_\omega)^{\aleph_0} = 2^{\aleph_0} \cdot \mathrm{cof}([\aleph_\omega]^{\aleph_0}) = \mathrm{cof}([\aleph_\omega]^{\aleph_0})$.

Hechler's forcing $\mathbb{H}$ is a natural forcing notion adding a dominating real, i.e., $d \in \mathbb{N}^\mathbb{N}$ such that for each $f \in \mathbb{N}^\mathbb{N} \cap V$, where V is the ground model, $f \leq^* d$. $\mathbb{H} = \{(n, f) : n \in \mathbb{N}, f \in \mathbb{N}^\mathbb{N}\}$, and $(n, f) \leq (m, g)$ if $n \geq m$, $f \geq g$, and $f(k) = g(k)$ for all $k < m$. If G is $\mathbb{H}$-generic over V, then by a density argument, $d = \bigcup_{(n,f)\in G} f|_{\{1,\dots,n\}} \in \mathbb{N}^\mathbb{N}$ is as required. $\mathbb{H}$ is ccc, and thus so is the finite support iteration $P = (P_\alpha, \dot{Q}_\alpha : \alpha < \lambda)$, where for each α, P_α forces that $\dot{Q}_\alpha$ is Hechler's forcing.

Theorem 9.7. *It is consistent (relative to the consistency of ZFC with appropriate large cardinal hypotheses) that*

$$\aleph_\omega < \mathfrak{b} = \mathfrak{d} = \aleph_{\omega+1} < \mathrm{cof}(\mathrm{Fin}(\aleph_\omega)^\mathbb{N}) = \mathrm{cof}([\aleph_\omega]^{\aleph_0}) = \aleph_{\omega+\gamma+1} = \mathfrak{c},$$

for each prescribed γ *with* $1 \leq \gamma < \aleph_1$*.*

Proof. Use Theorem 9.6 to produce a model of set theory, V, satisfying $\mathfrak{c} = \aleph_1$ and $\mathrm{cof}([\aleph_\omega]^{\aleph_0}) = \aleph_{\omega+\gamma+1}$.

Let $P = (P_\alpha, \dot{Q}_\alpha : \alpha < \aleph_{\omega+1})$ be the finite support iteration, where for each α, P_α forces that $\dot{Q}_\alpha$ is Hechler's forcing. Let G be P-generic over V, and for each $\alpha < \aleph_{\omega+1}$, let $G_\alpha = G \cap P_\alpha$ be the induced P_α-generic filter over V. For each α, let d_α be the dominating real added by Q_α in stage $\alpha + 1$, so that for each $f \in V[G_\alpha] \cap \mathbb{N}^\mathbb{N}$, $f \leq^* d_\alpha$.

As P is ccc, $\mathrm{cof}([\aleph_\omega]^{\aleph_0})$ remains $\aleph_{\omega+\gamma+1}$ in $V[G]$ (Lemma 9.5). As $\aleph_{\omega+1}$ has uncountable cofinality, we have that $\mathbb{N}^\mathbb{N} \cap V[G] = \bigcup_{\alpha<\aleph_{\omega+1}} \mathbb{N}^\mathbb{N} \cap V[G_\alpha]$ [5, Lemma 1.5.7]. It follows that $\{d_\alpha : \alpha < \aleph_{\omega+1}\}$ is dominating in $V[G]$. Moreover, it follows that for each $B \subseteq \mathbb{N}^\mathbb{N} \cap V[G]$ with $|B| < \aleph_{\omega+1}$, there is $\alpha < \aleph_{\omega+1}$ such that $B \subseteq \mathbb{N}^\mathbb{N} \cap V[G_\alpha]$, and thus B is $\leq^*$-bounded (by d_α). Thus, in $V[G]$, $\mathfrak{b} = \mathfrak{d} = \aleph_{\omega+1}$.

As the Continuum Hypothesis holds in V, $|P| = \aleph_{\omega+1}$, and as P is ccc, the value of $\mathfrak{c}$ in $V[G]$ is at most (by counting nice names [29, Lemma 5.13]) $|P|^{\aleph_0} = \aleph_{\omega+1}^{\aleph_0}$, evaluated in V. In V, $\aleph_{\omega+1}^{\aleph_0} \leq (2^{\aleph_\omega})^{\aleph_0} = 2^{\aleph_\omega} = \aleph_{\omega+\gamma+1}$. Thus, in $V[G]$, $\mathfrak{c} \leq \aleph_{\omega+\gamma+1}$. On the other hand, in V, as $\aleph_\omega < \mathfrak{d} \leq \mathfrak{c}$, $\aleph_{\omega+\gamma+1} = \mathrm{cof}([\aleph_\omega]^{\aleph_0}) \leq \aleph_\omega^{\aleph_0} \leq \mathfrak{c}^{\aleph_0} = \mathfrak{c}$. □

Remark 9.8. For finite γ, which are sufficient for our purposes, a simplified proof of the Gitik-Magidor Theorem 9.6 is available in Gitik's Chapter [20]. Following our proof, Assaf Rinot pointed out to us that starting with a supercompact cardinal (a stronger assumption than that in [20]), one may argue as follows: Start with a model of GCH with κ supercompact. Use Silver forcing to make $2^\kappa = \kappa^{++}$ [27, Theorem 21.4]. Since κ remains measurable, we can use Prikry forcing to make $\mathrm{cof}(\kappa) = \aleph_0$, without adding bounded subsets [27, Theorem 21.10]. Then GCH holds up to κ, and $\mathrm{cof}([\kappa]^{\aleph_0}) = \kappa^{\aleph_0} = 2^\kappa = \kappa^{++}$. Then, continue as in the proof of Theorem 9.7.

10. Concluding remarks

Most of the results provided here for complete groups, have natural extensions to incomplete groups. For these extensions, one needs to consider the dual group $\widehat{G}$ with $[P, \epsilon]$ a neighborhood of the identity for each *precompact* $P \subseteq G$. The extension is sometimes straightforward, using Theorem 3.20.

Similarly, the results of Section 6 extend to completely regular spaces which are not μ-spaces. Here, one should consider *functionally bounded* subsets of X instead of compact subsets of X, and the topology of $C(X, \mathbb{T})$ should be the functionally bounded-open topology. The main result of this section would then deal with spaces X having a cofinal family of functionally bounded sets, and whose topology is determined by its functionally bounded sets. We point out that in this case, the μ-completion of X is k_ω, and X is dense in this completion.

With some adaptation, the results presented here for k_ω groups also apply to locally convex vector spaces that have a countable cofinal family of bounded sets. For instance, any countable inductive limit of DF-spaces.

The present work is not the only one where pcf theory arises naturally in a study of a seemingly unrelated concept. Another recent example is in Feng and Gartside's paper [15], where pcf theory turned out essential in a study of a problem motivated by Hilbert's 13th problem.

Acknowledgments. We thank Maria Jesus Chasco, Jorge Galindo, and Mikhail Tkachenko for useful discussions on topological groups, Moti Gitik and Assaf Rinot for useful discussions on pcf theory, and Lydia Außenhofer and Adi Jarden for reading parts of this paper and making comments. Some of the work on this paper was carried out when the fourth named author was visiting the other authors at Universitat Jaume I, Castellón. This author thanks his hosts for their kind hospitality.

References

[1] U. Abraham and M. Magidor, *Cardinal arithmetic*, in: **Handbook of Set Theory** (M. Foreman, A. Kanamori, and M. Magidor, eds.), Kluwer Academic Publishers, Dordrecht, to appear. `http://www.cs.bgu.ac.il/~abraham/papers/math/Pcf.dvi`

[2] A. Arhangel'skiĭ and M. Tkachenko, **Topological groups and related structures**, Atlantis Studies in Mathematics 1, Hackensack, NJ: World Scientific; Paris: Atlantis Press, 2008.

[3] L. Außenhofer, **Contributions to the Duality Theory of Abelian Topological Groups and to the Theory of Nuclear Groups**, Dissertationes Mathematicae (Rozprawy Matematyczne) CCCLXXXIV, Polska Akademia Nauk, Instytut Matematyczny, Warszawa, 1999.

[4] W. Banaszczyk, *Additive subgroups of topological vector spaces*, Lecture Notes in Mathematics **1466** (1991), Springer-Verlag, Berlin, viii+178 pp.

[5] T. Bartoszyński and H. Judah, Set Theory: On the structure of the real line, A. K. Peters, Massachusetts: 1995.

[6] Gerald Beer, *Embeddings of bornological universes*, Set-Valued Analysis **16** (2008), 477–488.

[7] A. Blass, *Combinatorial cardinal characteristics of the continuum*, in: Handbook of Set Theory (M. Foreman, A. Kanamori, and M. Magidor, eds.), Kluwer Academic Publishers, Dordrecht, to appear. `http://www.math.lsa.umich.edu/~ablass/hbk.pdf`

[8] M. Bonanzinga and M. Matveev, *Some covering properties for Ψ-spaces*, Matematicki Vesnik **61** (2009), 3–11.

[9] M. Burke, S. Todorcevic, *Bounded sets in topological vector spaces*, Mathematische Annalen **105** (1996), 103–125.

[10] M. Chasco, *Pontryagin duality for metrizable groups*, Archive for Mathematics **70** (1998), 22–28.

[11] W. Comfort, *Topological groups*, in: **Handbook of Set Theoretic Topology** (eds. K. Kunen and J. Vaughan), North-Holland, Amsterdam: 1984, 1143–1264.
[12] J. Dieudonné, *Bounded sets in (F)-spaces*, Proceedings of the American Mathematical Society **6** (1955), 729–731.
[13] E. van Douwen, *The integers and topology*, in: **Handbook of Set Theoretic Topology** (eds. K. Kunen and J. Vaughan), North-Holland, Amsterdam: 1984, 111–167.
[14] R. Engelking, **General Topology**, PWN – Polish Scientific Publishers, 1977.
[15] Z. Feng and P. Gartside, *Minimal size of basic families*, submitted.
[16] J. Ferrando, J. Ka̧kol, and M. López Pellicer, *Necessary and sufficient conditions for precompact sets to be metrizable*, Bulletin of the Australian Mathematical Society **74** (2006), 7–13.
[17] J. Galindo and S. Hernández, *Pontryagin-van Kampen reflexivity for free topological Abelian groups*, Forum Mathematicum **11** (1999), 399–415.
[18] M. Gitik, *The negation of the singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$*, Annals of Pure and Applied Logic **43** (1989), 209–234.
[19] M. Gitik, *The Strength of the Failure of the Singular Cardinal Hypothesis*, Annals of Pure and Applied Logic **51** (1991), 215–240.
[20] M. Gitik, *Prikry-type Forcings*, in: Handbook of Set Theory (M. Foreman, A. Kanamori, and M. Magidor, eds.), Kluwer Academic Publishers, Dordrecht, to appear. `http://www.math.tau.ac.il/ gitik/hbg1.07.pdf`
[21] M. Gitik and M. Magidor, *The singular cardinal hypothesis revisited*, in: Set Theory of the Continuum (Berkeley, CA, 1989), volume 26 of Mathematical Sciences Research Institute Publications, 243–279, Springer, New York, 1992.
[22] H. Glöckner, T. Hartnick, and R. Gramlich, *Final group topologies, KacMoody groups and Pontryagin duality*, Israel Journal of Mathematics **177** (2010), 49-101.
[23] A. Grothendieck, *Sur la espaces (F) et (DF)*, Summa Brasiliensis Mathematicae **3** (1954), 57–123.
[24] J. Hejcman, *Boundedness in uniform spaces and topological groups*, Czechoslovak Mathematics Journal **9** (1959), 544–563.
[25] S. Hernández, F. Trigos-Arrieta, *Group duality with the topology of precompact convergence*, Journal of Mathematical Analysis and its Applications **303** (2005), 274–287.
[26] K. Hofmann and S. Morris, *The Structure of Connected Pro-Lie groups*, EMS Tracts in Mathematics **2**, European Mathematical Society Publication House, Zürich, 2007.
[27] T. Jech, **Set theory**, The Third Millennium Edition, Springer-Verlag 2002.
[28] L. Kočinac, *Star-Menger and related spaces*, Publicationes Mathematicae Debrecen **55** (1999), 421–431.
[29] K. Kunen, **Set Theory: An introduction to independence proofs**, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Company, 1980.
[30] P. Nickolas and M. Tkachenko, *The character of free topological groups, I*, Applied General Topology **6** (2005), 15–41.
[31] P. Nickolas and M. Tkachenko, *The character of free topological groups, II*, Applied General Topology **6** (2005), 43–56.
[32] V. Pestov, *Free Abelian topological groups and the Pontryagin-van Kampen duality*, Bulletin of the Australian Mathematical Society **52** (1995), 297–311.
[33] A. Rinot, *On the consistency strength of the Milner-Sauer conjecture*, Annals of Pure and Applied Logic **140** (2006), 110–119.
[34] S. Saxon and L. Sánchez-Ruiz, *Optimal cardinals for metrizable barrelled spaces*, Journal of the London Mathematical Society **51** (1995), 137–147.
[35] S. Shelah, **Cardinal arithmetic**, Oxford Logic Guides 29, Clarendon Press, Oxford University Press, New York, 1994.
[36] S. Shelah, *Advances in Cardinal Arithmetic*, Finite and Infinite Combinatorics in Sets and Logic (Banff, AB, 1991), 355–383, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 411, Kluwer Academic Publishers, Dordrecht, 1993.
[37] N. Vilenkin, *The theory of characters of topological Abelian groups with boundedness given*, Izvestiya Akad. Nauk SSSR. Ser. Mat. **15** (1951), 439–462.

Index

(Chis, Ferrer, Hernández) Universitat Jaume I, Departamento de Matemáticas, Campus de Riu Sec, 12071 Castellón, Spain.
E-mail address: `[chis, mferrer, hernande]@mat.uji.es`

(Tsaban) Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel
E-mail address: `tsaban@math.biu.ac.il`