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On powers of conjugacy classes in finite groups

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Abstract. Let K and D be conjugacy classes of a finite group G, and suppose that we have $K^n = D \cup D^{-1}$ for some integer $n \geq 2$. Under these assumptions, it was conjectured that $\langle K \rangle$ must be a (normal) solvable subgroup of G. Recently R. D. Camina has demonstrated that the conjecture is valid for any $n \geq 4$, and this is done by applying combinatorial results, the main of which concerns subsets with small doubling in a finite group. In this note, we solve the case $n = 3$ by appealing to other combinatorial results, such as an estimate of the cardinality of the product of two normal sets in a finite group as well as to some recent techniques and theorems.

Note 1: Red parts indicate major changes. Please check them carefully.

1 Introduction

There exist a number of revealing problems regarding the product of conjugacy classes of a finite group when these products are in some sense small. Perhaps, the most relevant is the celebrated and still unsolved Arad and Herzog conjecture, which asserts that the product of two non-trivial conjugacy classes cannot be a single conjugacy class in a nonabelian simple group. In addition, we might mention, for instance, that the square of the class of transpositions in the symmetric group is the union of three conjugacy classes, and a classification of groups satisfying a suitably restricted version of this condition was achieved by Fischer and led to the discovery of the three sporadic Fischer groups. Much more recently, Guralnick and Navarro [7] studied the case when the square K^2 of a conjugacy class K is a conjugacy class and obtained, among other results, the solvability of $\langle K \rangle$ by appealing to the Classification of Finite Simple Groups. This result is consistent with Arad and Herzog's conjecture. The same conclusion is attained in [4] when K^n, for some $n \geq 3$, is assumed to be a conjugacy class. Problems about products or powers of conjugacy classes that are the union of two classes have also been addressed (for instance, see also [1] or [2]). In particular, in [4], the authors proposed the following conjecture.

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Conjecture ([4, Conjecture 2]). *Let K and D be conjugacy classes of a finite group G and suppose that $K^n = D \cup D^{-1}$ for some integer $n \geq 2$. Then $\langle K \rangle$ is a (normal) solvable group of G.*

In this scenario, the authors prove that either $|D| = |K|/2$ or $|D| = |K|$, and moreover, in the first case, that $\langle K \rangle$ is solvable. So the case $|D| = |K|$ remained unsolved in [4]. Recently, R. D. Camina has shown in [5] that the assertion of the conjecture is true whenever $n \geq 4$. This is done by using, among others, a useful combinatorial result due to Freiman, which concerns finite subsets of nonabelian groups that have a small doubling. The purpose of this note is to study the case $n = 3$ and hence provide a solution to an open question in [5]. We introduce new ideas and techniques, some of which involve general results on products of sets in finite groups; for example, a quite elementary theorem of Arad, Fisman and Muzychuk appeared in [1] (see Theorem 2.1) that provides a lower bound for the cardinality of the product of two normal subsets. Other results that we employ, however, are based on the Classification of Finite Simple Groups. Thus, our contribution is the following.

Theorem A. *Suppose that G is a finite group and K is a conjugacy class of G such that $K^3 = D \cup D^{-1}$ for some conjugacy class D of G. Then $\langle K \rangle$ is solvable.*

When trying to prove the solvability of $\langle K \rangle$ in the case $K^2 = D \cup D^{-1}$, one easily reaches the arithmetical equation $|K^2| = 2|K|$. However, unfortunately, the combinatorial methods that we are using here (either Freiman's result or Theorem 2.1) cannot be satisfactorily applied in this situation. Therefore, this is the unique remaining open case for solving the conjecture entirely. We hope that the ideas of this note might inspire the reader to complete it. The notation and terminology are standard and follow, for instance, that appearing in [8].

2 Preliminaries

In this section, we collect the main results that we need in the proofs. We start with a generalization of a theorem of Kneser [9, Theorem 1.5], due to Arad, Fisman and Muzychuk, which plays a fundamental role in our development. We will apply it in the particular case when the subsets are conjugacy classes or powers of conjugacy classes of a finite group G, which of course are normal subsets of G.

Theorem 2.1 ([1, Theorem 1.11]). *Let G be a group with finite subsets A and B. If the product AB is a normal subset of G, then there exists a normal subgroup N of G such that*

(i) $ABN = AB$;

(ii) $|AB| \geq |AN| + |BN| - |N|$.

It is worth mentioning that the proof of the above result only uses elementary group-theoretical techniques. Indeed, the proof is a "non-commutative translation" of that given in [9] for the original Kneser theorem.

The next theorem is a consequence of the so-called "Freiman inverse problem for $\kappa < \frac{3}{2}$" (see [6]), which was referred to in the introduction. This theorem is particularly applied to the square and to the power of a conjugacy class, and actually is a key result in [5].

Theorem 2.2 ([5, Theorem A (a)]). *Suppose $K = x^G$ is a conjugacy class of an element x of a group G and K is finite. Let N denote the normal subgroup $[x, G]$. If $|K^2| = \mu|K|$ with $\mu < \frac{3}{2}$, then $K^r = x^r N$ for all $r \geq 2$.*

The following recent result is an extension of [7, Theorem B (a)], and is employed in [3, Theorem B] so as to confirm the conjecture in the particular case $K = D$. Hence, its proof depends on the Classification of Finite Simple Groups.

Theorem 2.3 ([3, Theorem A]). *Let G be a finite group, and let N be a normal subgroup of G. Let $K = x^G$ be the conjugacy class of an element $x \in G$. Suppose that $xN = K \cup K^{-1}$. Then N is solvable, and as a consequence so is $\langle K \rangle$.*

Finally, we state another useful consequence of [7, Theorem B (a)], which is thus based on the classification too.

Lemma 2.4 ([4, Lemma 2]). *Let G be a finite group and K, L and D non-trivial conjugacy classes of G such that $KL = D$ with $|D| = |K|$. Then $\langle L \rangle$ is solvable.*

3 Proof of Theorem A

We are ready to prove our result.

Proof of Theorem A. We argue by induction on the order of G. First, we prove that we can assume that G has no non-trivial normal solvable subgroups. Otherwise, if L is a non-trivial normal solvable subgroup of G and $\overline{G}$ denotes the factor group G/L, we have $\overline{K}^3 = \overline{D} \cup \overline{D}^{-1}$ (note this is still true if $\overline{D} = \overline{1}$), and by induction, $\langle \overline{K} \rangle$ is solvable, so the result follows as L is solvable. Furthermore, if $D = D^{-1}$, then the result follows from [4, Theorem A]. Also, by [4, Theorem C], either $|D| = |K|/2$ or $|D| = |K|$, and in the first case, the result follows too. Therefore, for the rest of the proof, we will also assume $D \neq D^{-1}$ and $|D| = |K|$.

We take $A = K$ and $B = K^2$ in Theorem 2.1 and obtain that there exists a normal subgroup N of G satisfying $K^3N = K^3$, that is, $D \cup D^{-1} = N(D \cup D^{-1})$ and

$$2|K| = |K^3| \geq |KN| + |K^2N| - |N|. \tag{3.1}$$

Let us consider the set KN, which is the union of (left) cosets of N. If KN is a single coset, say xN, with $x \in K$, then

$$D \cup D^{-1} = K^3 = K^3N = (KN)^3 = (xN)^3 = x^3N,$$

so by applying Theorem 2.3 (there is no loss in assuming $x^3 \in D$), we get the solvability of $\langle D \rangle$, contradicting our first assumption. Therefore, we can assume that KN consists of at least 2 cosets of N, and hence $|KN| \geq 2|N|$. Also, since $|K^2N| = |K(KN)| \geq |KN|$, from equation (3.1), we obtain

$$2|K| = |K^3| \geq 2|N| + 2|N| - |N| = 3|N|,$$

that is,

$$|N| \leq \frac{2}{3}|K|. \tag{3.2}$$

Next, we show that it can be assumed that $|K^2| \geq \frac{3}{2}|K|$. Indeed, if $|K^2| < \frac{3}{2}|K|$, by applying Theorem 2.2, we have $K^3 = x^3[x, G]$ with $x \in K$, and then Theorem 2.3 applies again to obtain the solvability of $\langle D \rangle$, which again contradicts our first assumption.

Now, we apply the fact that $|K^2| \geq \frac{3}{2}|K|$ to equation (3.1), and then

$$2|K| \geq |KN| + |K^2N| - |N| \geq |K| + |K^2| - |N| \geq |K| + \frac{3}{2}|K| - |N|, \tag{3.3}$$

so in particular,

$$|N| \geq \frac{1}{2}|K|.$$

Next, we prove that N cannot be trivial. Indeed, if $N = 1$, then $|K| \leq 2$, and so, for $x \in K$, it certainly follows that $\mathbf{Z}(\mathbf{C}_G(x))$ is a non-trivial normal solvable subgroup of G, a contradiction. In addition, if $|N| = \frac{1}{2}|K|$, then equation (3.3) forces that $|KN| = |K|$, and hence $KN = K$. Now, if we take C to be any non-trivial conjugacy class of G contained in N, we necessarily have $KC = K$, and then, by applying Lemma 2.4, we conclude that $\langle C \rangle$ is solvable, a contradiction too. Henceforth, we will assume that

$$|N| > \frac{1}{2}|K|. \tag{3.4}$$

We know that $D \cup D^{-1} = N(D \cup D^{-1})$. Since ND is a union of conjugacy classes of G, there exist exactly three possibilities: $ND = D$, $ND = D^{-1}$ or $ND = D \cup D^{-1}$. In the first two cases, using Lemma 2.4 and reasoning as above, we can deduce that N is (non-trivial) solvable, which is a contradiction. Therefore, we will assume that $ND = D \cup D^{-1}$. Now, since the set DN is the union of, say t, cosets of N, by using equation (3.2), we have

$$t|N| = |ND| = 2|D| = 2|K| \geq 3|N|, \quad \text{that is to say,} \quad t \geq 3. \tag{3.5}$$

Moreover, by taking into account that $2|K| = t|N|$ and applying equation (3.4), we obtain $2|K| > \frac{t}{2}|K|$, which means that $t \leq 3$. However, since $t \geq 3$ by equation (3.5), we conclude that $t = 3$, and hence

$$|N| = \frac{2}{3}|K|. \tag{3.6}$$

The rest of the proof consists of getting a contradiction. Let us consider again the set KN, which can be written as the union of $s \geq 2$ cosets of N. If $s \geq 3$, since $|K^2N| = |K(KN)| \geq |KN| \geq 3|N|$, then equations (3.1) and (3.6) give

$$2|K| \geq 3|N| + 3|N| - |N| = 5|N| = \frac{10}{3}|K|,$$

a contradiction. Therefore $s = 2$, and this means that $|KN| = 2|N|$. Write, for instance, $KN = xN \cup yN$, with $x, y \in K$. On the other hand, we have

$$2|N| = |KN| \leq |(KN)^2| \leq |(KN)^3| = |K^3N| = |K^3| = 2|K| = 3|N|.$$

Furthermore, $|K^2N| = |(KN)^2|$ is multiple of $|N|$ for K^2N being the union of cosets of N, so by the above inequality, there are exactly two possibilities for $|K^2N|$: either $|K^2N| = 2|N|$ or $3|N|$. We distinguish both possibilities. Suppose first that $|K^2N| = 2|N|$. Then, in particular, $|K^2N| = |KN|$, and this yields the following equalities:

$$K^2N = xKN = x(xN \cup yN) = x^2N \cup xyN,$$
$$K^2N = yKN = y(xN \cup yN) = yxN \cup y^2N.$$

These equalities imply that either $x^2N = yxN$ and $xyN = y^2N$, which clearly leads to $xN = yN$, a contradiction; or $x^2N = y^2N$ and $xyN = yxN$. However, in this case, we have

$$\begin{aligned} K^3 = K^3N &= (K^2N)(KN) = (x^2N \cup xyN)(xN \cup yN) \\ &= x^3N \cup x^2yN \cup xyxN \cup xy^2N = x^3N \cup y^3N. \end{aligned}$$

This implies that $|K^3| = 2|N|$, contradicting the fact that $|K^3| = 2|K|$ by equation (3.6). Finally, suppose that $|K^2N| = 3|N|$. Then equation (3.1) gives

$$2|K| \geq 2|N| + 3|N| - |N| = 4|N|,$$

which again contradicts equation (3.6). □

Note 2: Formula displayed to improve line-breaks.

Example. We give an easy example of a group satisfying the conditions of Theorem A with $|K| = |D|$ and $D \neq D^{-1}$. Observe that these are exactly the assumptions that we have at the beginning of the proof of Theorem A. Let

$$D = \langle a, b \mid a^8 = b^2 = 1, a^b = a^3 \rangle$$

be the semidihedral group of order 16 and $Z = \langle z \rangle$ a cyclic group of order 3. If we take $G = D \times Z$ and $K = (az)^G = \{az, a^3z\}$, then we have $K^3 = D \cup D^{-1}$ with $D = \{a, a^3\}$.

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Note 3: The bibliography has been cross-checked and updated with MathSciNet. Since entries may be mismatched and replaced by a wrong entry (especially: preprints, theses, and translations), please carefully double-check *each* entry.

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