DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **22**, Number **3**, November **2008**

Website: http://aimSciences.org

pp. **587–604**

BOTT INTEGRABLE HAMILTONIAN SYSTEMS ON $S^2 \times S^1$

Alicia Cordero

Dpt. Matemàtica Aplicada. Universidad Politécnica de Valencia
Cno. de Vera s/n. 46022 Valencia, Spain

José Martínez Alfaro

Dpt. Matemàtica Aplicada. Facultat Matemàtiques
Universitat de València. Avda. Dr. Moliner, 50
46100 Burjassot (Valencia) Spain

Pura Vindel

Dpt. Matemàtiques. Universitat Jaume I. Campus Riu Sec.
12071 Castelló, Spain

(Communicated by Amadeu Delshams)

Abstract. In this paper, we study the topology of Bott integrable Hamiltonian flows on $S^2 \times S^1$ in terms of some types of periodic orbits, called NMS periodic orbits. The set of these periodic orbits can be identified by means of some operations applied on global and local links. These operations come from the round handle decomposition of these systems on $S^2 \times S^1$. We apply the results to obtain a non-integrability criterium.

1. **Introduction.** Let $v = sgrad\,(H)$ be a hamiltonian system defined by a smooth function H on a four-dimensional symplectic manifold. The function H is constant on trajectories of the vector field, i.e., it is a first integral. The system v is called Liouville integrable if another integral f is given in involution with H. On the constant-energy level set Q^3 Liouville theorem assures that the level surfaces corresponding to regular values of f that are connected and compact are tori.

A. T. Fomenko [10] defines this second integral f as a Bott integral if its critical points form non-degenerated critical manifolds. A Bott integral f of the system v implies that its critical submanifolds must be flow manifolds (see section 1.1), therefore, its Euler characteristic is zero. So, critical submanifolds in $S^2 \times S^1$ must be fixed points, circles, 2-dimensional tori or Klein bottles (see [10]).

The case $Q^3 \simeq S^2 \times S^1$ is interesting for different reasons. In particular, the manifold $S^2 \times S^1$ appears as phase space in problems from Celestial Mechanics, as the Two Fixed Centers problem (see [8]), and also in the study of the dynamics of a rigid body (see [17], for example). On the other hand, the topological characterization of the set of periodic orbits in terms of links allows the introduction of invariants of the Hamiltonian system (equivalent Hamiltonian systems have the

2000 *Mathematics Subject Classification.* Primary: 37J35, 37D15; Secondary: 34C25.

Key words and phrases. $S^2 \times S^1$, Bott integral, Hamiltonian systems, round handle decomposition.

Partially supported by MTM2004-03244 (DGES) and P1-1B2002-24 (Convenio Bancaixa-Universitat Jaume I).

same link of periodic orbits). Moreover, the non integrability of a Hamiltonian system can be inferred from the periodic orbits obtained in the phase space and the links they become.

In this paper we use the close relation between Non-singular Morse-Smale systems (NMS for short, see section 2.1) and integrable Hamiltonian systems and the results obtained in [9] about the round handle decomposition of NMS flows on $S^2 \times S^1$, to obtain the round handle decomposition of Bott integrable Hamiltonian systems on $S^2 \times S^1$ (section 2.3).

This close relation is proved in section **??** (lemma 2.1), by showing that it is possible to build a Non-singular Morse-Smale flow as closer as necessary to a Bott Hamiltonian flow in such a way that some periodic orbits belong to both, the NMS and the Hamiltonian systems. A small radial perturbation can be applied on a Bott integrable Hamiltonian field in such a way that the flow is transversal to the invariant 2-tori.

A Bott integral f on a surface Q^3 is defined as orientable if its critical manifolds are orientable. On the other hand, Bott integrable Hamiltonian fields on a non singular compact constant energy surface can be obtained as a small perturbation of a NMS field (see section 1.1). The periodic orbits that remain after the perturbation are called NMS type periodic orbits and belong to both, the NMS system and the Hamiltonian one.

When f is orientable only periodic orbits that are in the common axis of these invariant tori (those corresponding to the maximal or minimal critical circles of f) or in the intersection of two tori (those corresponding to the hyperbolic critical circles of f) remain; if f is non-orientable, some critical tori collapse into critical Klein bottles K^2. The embedding of a Klein bottle is admissible in $S^2 \times S^1$ but not in other 3-manifolds as S^3. Indeed, Klein bottles are found in $S^2 \times S^1$ as critical submanifolds of the Hamiltonian system (section 2.2).

The round handle decomposition of Bott integrable Hamiltonian systems on $S^2 \times S^1$ (section 2.3) allows to characterize the set of NMS periodic orbits of these systems in terms of knots and links that define the orbital structure of the system.

In section 3, we also use the round handle decomposition to build the phase portrait of these flows, with one saddle periodic orbit. We observe that a Klein bottle appears as the union of invariant manifolds of the saddle orbit and the corresponding orbits (proposition 5).

In section 4, the set of periodic orbits of NMS systems on $S^2 \times S^1$ is described in terms of some operations applied on links of periodic orbits (theorem 4.1).

Finally, in section 5 we apply the non-integrability criterium on a perturbed pendulum.

1.1. **Round handle decomposition and NMS flows.** We refer to [14] and [15] for definitions of the standard terms in 3-dimensional topology. In the following, we consider a compact, connected, prime, orientable manifold M.

Definition 1.1. Let M be a compact, orientable connected manifold. Let V be a non singular continuous vector field on M, transversal to the boundary of the manifold. The negative border of M, denoted by $\partial_- M$, is the union of components of ∂M such that the flow points inwards M. Let $\partial_+ M$ denote the complementary of the negative border in the boundary of the manifold, $\partial_+ M = \partial M - \partial_- M$ where V points outwards. Then, $(M, \partial_- M)$ is called a flow manifold.

As a consequence,

Proposition 1 ([18]). *Let $(M, \partial_- M)$ be a flow manifold. Then,*

$$\chi(M) = \chi(\partial_- M) = \chi(\partial_+ M) = 0$$

Regular neighborhoods of attractive and repulsive periodic orbits are flow manifolds, diffeomorphic to solid tori; nevertheless, a regular neighborhood of a saddle orbit is not a flow manifold. The study of this kind of neighborhoods yields to the definition of round handles.

We use the following additional definitions due to Asimov, where it is assumed that the flow held by a manifold M points outwards on the so called negative component of the boundary, $\partial_- M$, and points inwards on the complementary $\partial_+ M = \partial M \backslash \partial_- M$, called positive boundary.

Definition 1.2. A pair $(M, \partial_- M)$ of a manifold M and a compact submanifold $\partial_- M$ of ∂M, or by abuse of notation, a manifold M is called

a round 2-handle if $(M, \partial_- M) \cong \left(D^2 \times S^1, \emptyset\right)$.

a round 1-handle if $M \cong B_s \bigoplus_{S^1} B_u$ is a 2-disk bundle on S^1.

a round 0 -handle if $(M, \partial_- M) \cong \left(D^2 \times S^1, \partial D^2 \times S^1\right)$.

So, a round handle, that is diffeomorphic to a torus, corresponds to a 0-handle when there is a repulsive periodic orbit in its core, to a 2-handle if there is an attractive periodic orbit in the core and to a 1-handle if the orbit is a saddle.

There are two possibilities for round 1-handles $B_s \bigoplus_{S^1} B_u$, where B_s and B_u are both trivial or are both non-trivial: in the first case, the round 1-handle is called orientable and in the second one it is called non-orientable.

Usually a round k-handle is denoted by R_k, where the index $k = 0, 1, 2$ is also known as the index of the periodic orbit.

Definition 1.3. A fattened round handle is obtained attaching a round 1-handle to a manifold $A \times [0, 1]$, where A is one torus or the union of two disjoint tori.

$$C = A \times [0, 1] \bigcup_{\partial_- R_1} \cup R_1 \tag{1}$$

Definition 1.4. [1] A round handle decomposition (RHD) is a sequence:

$$\partial_- M \times I = M_0 \subset M_1 \subset \cdots \subset M_i \subset M_{i+1} \subset \cdots \subset M_N = M \tag{2}$$

where each M_{i+1} is obtained from M_i by adding a handle.

On the other hand, Morse-Smale flows constitute a class of the simplest flows in the set of the structurally stable ones. This kind of flows are characterized by their non-wandering set consisting of a finite number of closed hyperbolic orbits and the transversal intersections of their stable and unstable manifolds.

The following proposition ensures the existence of a NMS flow when a manifold admits a round handle decomposition and vice versa:

Proposition 2 ([1], [16]). *If a manifold M admits a round handle decomposition*

$$\emptyset = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

there is a NMS flow on M such that: (1) the closed orbits of the flow coincide with the cores of round handles, and (2) the flow is pointing outward of ∂M_j. Conversely, if M has a NMS flow, M admits a round handle decomposition satisfying (1) and (2).

As an example, the 3-manifold $S^2 \times S^1$ can be obtained from the round handle decomposition:

$$\emptyset \subset V_1 \subset S^2 \times S^1 \tag{3}$$

attaching a 0-handle V_2 to the 2-handle V_1 by identifying their boundaries. This is the simplest round handle decomposition of $S^2 \times S^1$.

Given a surface S in W^3, it is said to be a separating surface if $W^3 - S$ is not a connected manifold. Let us note that $S^2 \times S^1$ is not irreducible and a non-separating sphere can be considered as a section of $S^2 \times S^1$ for a fixed point of S^1 ([3][Lemma 0.6]). Moreover, separating spheres allows us to distinguish between local and global submanifolds depending on they can be isolated by a 3-disk or not.

2. Round handle decomposition of Bott integrable Hamiltonian systems on $S^2 \times S^1$.

In this section we are going to study the close relation between NMS systems and Bott integrable Hamiltonian systems.

2.1. NMS flows on constant energy-level surfaces Q^3 .

Since submanifolds corresponding to regular values of f, are homeomorphic to T^2, several copies of $T^2 \times I$ are obtained as submanifolds of Q^3. The level sets that separate these copies of $T^2 \times I$ correspond to the critical values of f. The following lemma assures the round handle decomposition of a constant-energy level Q^3.

Lemma 2.1. *Let Q^3 be a compact, non singular constant-energy level surface of a Bott integrable Hamiltonian system. Then, Q^3 admits a Non singular Morse-Smale flow.*

Proof. Let $v = sgrad\ f$ be a smooth vector field on Q^3. As a NMS system holds a round handle decomposition, we are going to build a NMS vector field arbitrarially closer to v by continuous deformations.

Let $\left(S^1 \times I_1\right) \times I$ be a reference system on $T^2 \times I$. Polar coordinates (r, θ) can be considered for $\left(S^1 \times I_1\right) \subset D^2$. Let η be $v|_{T^2 \times I}$ and let $\overline{\eta}$ be the modified vector field obtained when a small radial perturbation is applied to η. Then, a modified vector field $\overline{v}$ is obtained corresponding to $\overline{\eta}$. The radial perturbation leads to $\overline{v}$ has not periodic orbits on $T^2 \times I$ and it is transversal to the invariant tori.

When f varies, the invariant tori can suffer one of the following changes ([10]):

1. A torus T^2 is contracted to the axial circle of a solid torus and then "vanishes" from the level surface of the integral f. The notation is: $T^2 \to S^1 \to \emptyset$.
2. Two tori T^2 move towards each other along a cylinder, flow together into one torus, and "vanish". The notation is: $2T^2 \to T^2 \to \emptyset$.
3. A torus T^2 splits into two tori as it passes through the centre of the trousers (oriented saddle) when they "stay" on the level surface of the integral f . The notation is: $T^2 \to 2T^2$.
4. A torus T^2 spirals twice round a torus T^2, following the topology of the non orientable saddle and then "stays" on the level surface of the integral f . The notation is: $T^2 \to T^2$.
5. A torus T^2 transforms into a Klein bottle (covering it twice) and "vanishes" from the level surface of the integral f. The notation is: $T^2 \to K^2 \to \emptyset$.

Let us analyze the modified flow $\overline{v}$ in each of these cases:

1. The radial component c goes smoothly to zero when it moves to the core of the solid torus. The critical manifold S^1 changes to be a repulsive or attractive periodic orbit.

2. Let us consider two $T^2 \times I$ separated by a critical torus. When a radial perturbation is applied these thick tori collapse in one $T^2 \times I$ with the modified flow $\overline{v}$.
3. The critical S^1 corresponds to one hyperbolic periodic orbit in the common boundary of two tori. We can introduce a small change in the flow in a neighbourhood of $T^2 \bigcup_{S^1} T^2$ (see figure 1) and we assign a radial perturbation c_i to each set of invariant tori in such a way that this radial perturbation yield to the modified flow $\overline{v}$.

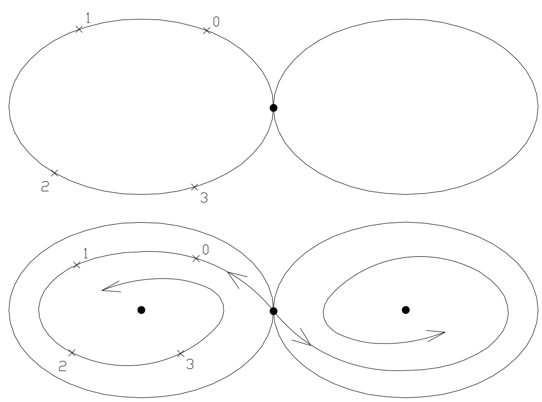

FIGURE 1. Section of the modified flow in a neighbourhood of $T^2 \bigcup_{S^1} T^2$

4. This case is similar to the previous one. The iterates $p_0, p_1, p_2, \ldots$ go from one invariant torus to the other one (see figure 2).

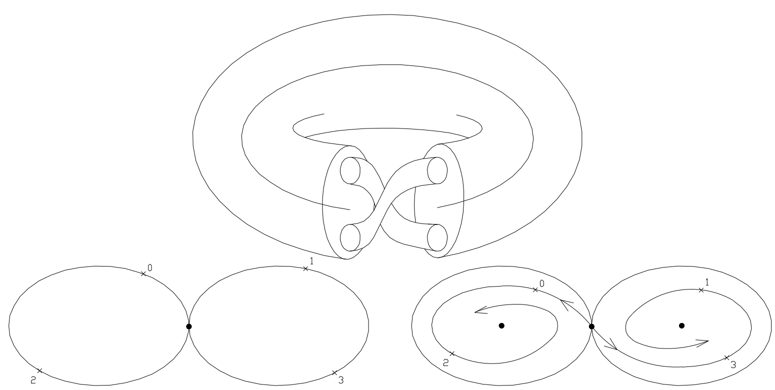

FIGURE 2. Modified flow in a neighbourhood of $T^2 \bigcup_{S^1} T^2$

5. In the last bifurcation of the Liouville tori, the flow is modified on a Klein bottle K^2 in such a way that only remain two periodic orbits of β type (section 2.2), one of the periodic orbits becames attractive and the other one repulsive.

In the complete system, the radial perturbation transforms these periodic orbits on the Klein bottle in one attractive (saddle) periodic orbit and one saddle (repulsive) periodic orbit (see figure 3).

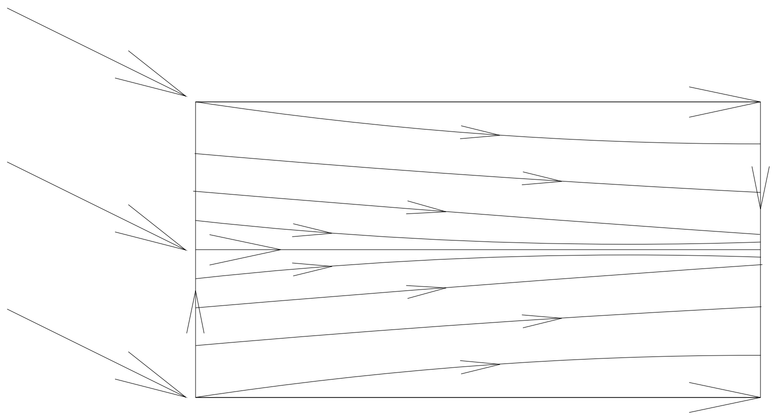

FIGURE 3. NMS periodic orbits on K^2.

As Q^3 is a compact 3-manifold, after a finite number of perturbations, the vector field v is modified to a close $\overline{v}$, but $\overline{v}$ is NMS field. □

Therefore, Q^3 admits a round handle decomposition ([1], [16]).

2.2. **NMS flow on the critical submanifold K^2.** In this section a NMS flow is built on a Klein bottle with two periodic orbits, one attractive and one repulsive, (proposition 4).

In figure 4, an embedding of the Klein bottle in $S^2 \times S^1$ is shown. Let us note that S^2 is obtained by identifying the upper lines of a rhomboid with the lower ones and $S^2 \times S^1$ is obtained when this rhomboid is multiplied by an interval, that becomes S^1 by identifying their extremal points.

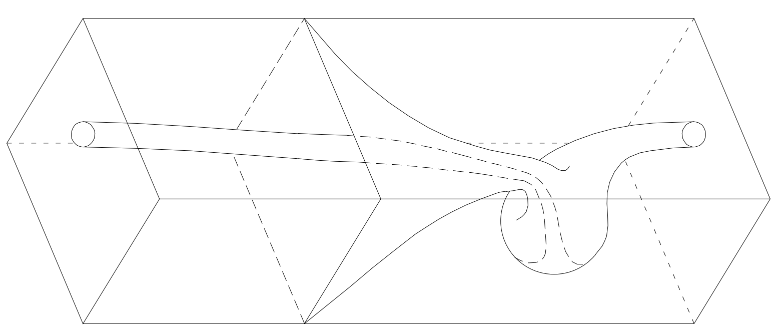

FIGURE 4. K^2 embedded in $S^2 \times S^1$

Let α and β denote the generators of $\pi_1\left(K^2\right)$ such that $\alpha\beta = \beta\alpha^{-1}$. A non-trivial closed curve on K^2 is represented by one of the following homotopy types (see Godbillón [12], for example)

$$\alpha, \alpha^{-1}, \beta^2, \beta^{-2}, \beta^{-1}\alpha^n, \quad n \in Z$$

and can be represented as in figure 5. Moreover, a Jordan curve can be either orientable or nonorientable depending on an arbitrary neighborhood of it is homeomorphic to the cylinder or to the Möbius band, respectively (see Soler [19]).

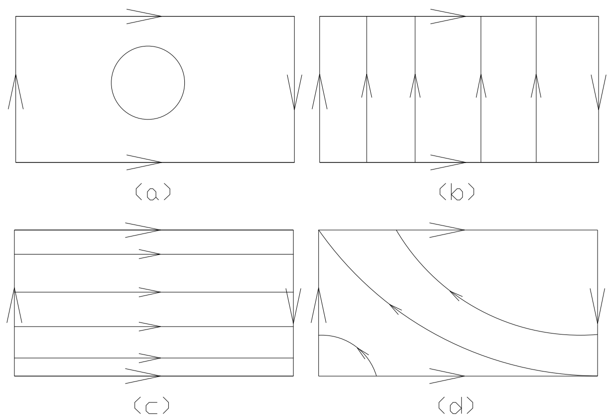

FIGURE 5. Different curves on K^2 :(a) Trivial curve; (b) α foliation; (c) β^2 foliation; (d) $\beta^{-1}\alpha^n$ foliation.

The following result tell us that all the trajectories on K^2 must be non trivial, closed, and also that α, α^{-1} curves are not admissible orbits.

Proposition 3. *Let K^2 be a critical Klein bottle in a non-singular compact contant-energy surface Q^3 of a Bott integrable Hamiltonian system on M^4. Then, all the orbits are periodic and have to be $\beta^{\pm 2}$ or $\beta^{-1}\alpha^n$ type, $n \in Z$.*

Proof. If the trajectory on K^2 is closed it is a periodic orbit. If it is open, its clausure is a 2-dimensional manifold. But, the Poincaré-Bendixon theorem can be applied on K^2, so limit sets of trajectories contain fixed points or they are closed orbits. Non singularity of the flow implies that all the limit sets must be closed orbits, so, it is not possible to have trajectories whose clausure are a 2-dimensional manifold. Therefore, all the trajectories defined on K^2 must be closed curves.

Let us show the second part of the proposition.

Firstly, trivial closed curves bound 2-disks D^2 in the Klein bottle, so an equilibrium point inside D^2 is needed and the flow is singular.

Moreover, if all the closed trajectories of the flow follows curves of α and α^{-1} type, the topology of the Klein bottle implies that an odd number of inversions of the orientation on the closed orbits is made. So, the vector field must be canceled out at some point and this leads to a contradiction with the non-singularity of the flow. Therefore, all the periodic orbits have to be $\beta^{\pm 2}$ or $\beta^{-1}\alpha^n$ type, $n \in Z$. This conclusion is also obtained by applying some results of [19]. $\square$

As a consequence, for this kind of flow:

Corollary 1. *A NMS flow on K^2 is transversal to the α and α^{-1} type curves.*

Furthermore, proposition 3 allows us to build a NMS flow on the Klein bottle:

Proposition 4. *Let K^2 be a critical Klein bottle in a non-singular compact contant-energy surface Q^3 of a Bott integrable Hamiltonian system on M^4. A NMS flow can be built on K^2 with one attractive and one repulsive orbits.*

Proof. K^2 is obtained by identifying the two boundary circles S^1 of α type by means of a reversing homeomorphism and the two boundary circles of β type by means of an orientation preserving homeomorphism. Every reversing homeomorphism on S^1 has two fixed points. When these two fixed point are multiplied by S^1 two closed curves of β type are defined. The reversing homeomorphism implies that the other closed curves has to be $\beta^{\pm 2}$ type. So, a foliation of K^2 can be obtained with an infinite number of double-period orbits and two simple-period orbits, where the period denotes the exponent of β.

As we have seen before, the flow of the Hamiltonian system can be perturbed in order to obtain NMS type orbits. Then, it is observed that the double-period orbits on K^2 vanish and only the two simple-period orbits remain. Also, the reversing homeomorphism defined in order to obtain the Klein bottle involves that the trajectories of the flow go from one periodic orbit to the other. Therefore , these two simple-period orbits become one attractive and one repulsive orbits on K^2 (see figure 3) and a NMS flow is obtained. □

Let us remark that the two simple-period orbits that remain after the perturbation of the Hamiltonian system correspond to the only nonorientable Jordan curves admissible on K^2. Moreover, for any NMS flow on K^2 these two simple-period orbits must be attractive or repulsive periodic orbits. The other period orbits of the flow correspond to orientable Jordan curves.

In the following section, we observe that the NMS orbits on the critical K^2, due to the radial perturbation, become one saddle and one attractive (or repulsive) orbits, when it is the core of a fattened Klein bottle, $K^3 = K^2 \times [0,1]$, that appears in the round handle decomposition of $S^2 \times S^1$. Let us note that, in spite of K^2 is non-orientable, the boundary of K^3 is a global 2-torus.

2.3. **Fattened round handles in Bott integrable Hamiltonian systems on $S^2 \times S^1$.** Admissible fattened round handles for Bott integrable Hamiltonian systems come from essential attachment of the round 1-handle on tori, (see [7], Lemma 12). The following cases are obtained:

Theorem 2.2. *Let γ be a saddle periodic orbit in the core of a round 1-handle R_1. A fattened round 1-handle C of a Bott integrable Hamiltonian system on $S^2 \times S^1$, corresponds to one of the following cases:*

1. *$C \cong T^2 \times [0,1] - (N(k')) \cong F_{p+1} \times S^1$, $p \geq 1$, where k' and γ are (p,q)-cables of a component $k = \{*\} \times S^1$ for $\{*\} \in int(F_{p+1})$, and $\partial\_C = T_1 \times \{0\} \cup T_2 \times \{0\}$ or $\partial\_C = T^2 \times \{0\}$.*
2. *Let $C \cong T^2 \times [0,1] - \left(S^3 - N(k)\right)$ be local, where $\partial\_C = T^2 \times \{0\}$, $N(k)$ is a regular neighborhood of a nontrivial knot k and $\gamma = \{*\} \times S^1$ for a point $\{*\}$ of a transversal section of C.*
3. *$C \cong K^3 - int(W)$, where W is a global solid torus, $\partial W = \partial\_C = T^2 \times \{0\}$ and γ is a global knot.*
4. *$C \cong D^2 \times S^1 - int\left(\overline{W}\right)$, where $\overline{W}$ is a regular neighborhood of $(2,q)$-cable of $\{0\} \times S^1$ in $D^2 \times S^1$, $\partial\_C = \partial\overline{W}$ and γ is a local knot in the core of $D^2 \times S^1$.*

Proof. The fattened round handles $C \cong T^2 \times [0,1] - (N(k')) \cong F_{p+1} \times S^1$ and $C \cong T^2 \times [0,1] - \left(S^3 - N(k)\right)$ correspond to the attachment of a round 1-handle to one torus or two tori by using two essential attaching circles. When a non-separating round 1-handle is attached to a global torus by means of two longitudinal circles, $C \cong K^3 - int\,(W)$ is obtained. The last one, $C \cong D^2 \times S^1 - int\left(\overline{W}\right)$, is obtained when the round 1-handle is not orientable and it is attached to one torus by using one essential longitudinal circle.

Fattened handles corresponding to cases 1, 2 and 4 are equivalent to the obtained in S^3 ([20]). Let us study the third fattened handle. It is obtained when $A = T^2$ is a global torus and the round 1-handle R_1 is attached by means two essential circles. Different cases can be considered depending on the saddle orbit is local or global.

If A is one global torus, R_1 can be attached to the global torus by means of two transversal or two longitudinal circles. When the saddle orbit is local it is attached by means of two transversal circles, the global torus must go round $S^2 \times S^1$ twice (figure 6). The fattened handle has two boundary components: $\partial_- C = T^2 \times \{0\}$ global; the positive boundary of C is equivalent to a local torus built with two non-separating spheres glued by means of two handles, so it is linked to $T^2 \times \{0\}$. Then, the fattened handle is equivalent to $K^3 - int\,(W)$ and the flow is not a NMS flow, because the union of the invariant manifolds of the saddle, $W^s \cup W^u$, is a Klein bottle and the topology of K^2 requires an odd number of inversions of the orientation on the local closed orbits. So, the vector field must be canceled out at some point. This leads to a contradiction with the non-singularity of the flow.

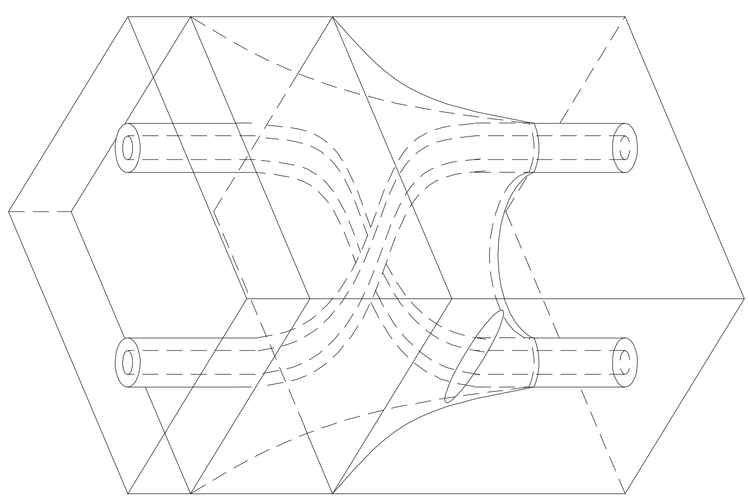

FIGURE 6. Non admissible $K^3 - int\,(W)$

When R_1 is attached by means of two longitudinal circles to a global torus, the boundary components are a $(2,1)$-torus and a separating torus. The $(2,1)$-torus is the boundary of a fattened Klein bottle K^3 embedded in $S^2 \times S^1$. So, the fattened round handle is again $C \cong K^3 - int\,(W)$, where W is a global solid torus (figure 7), such that $\partial W = \partial_- C$.

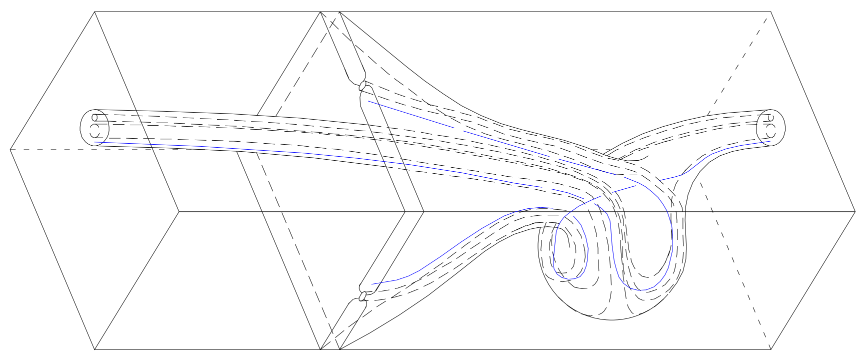

Figure 7. $K^3 - int\,(W)$

If A is a local torus, the attaching curves must be longitudinal; in this case, the boundary components are a $(2, 1)$-torus and a local separating torus. As before, the $(2, 1)$-torus defines a fattened Klein bottle K^3 embedded in $S^2 \times S^1$ but W is local, and the flow is not a NMS flow (figure 8).

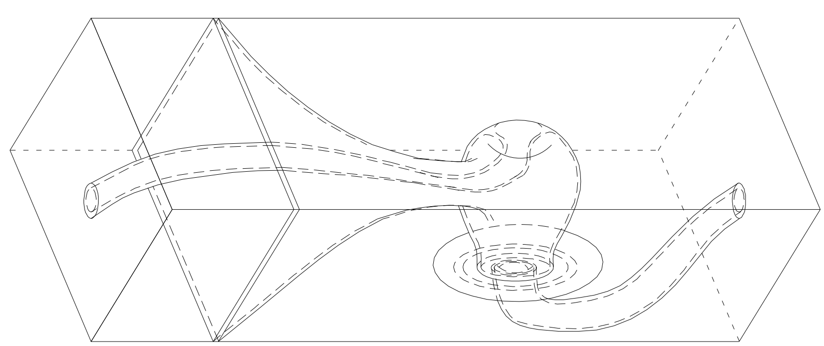

Figure 8. Non admissible $K^3 - int\,(W)$

Let us remark that when the saddle orbit is local, the flow is not NMS and the only admissible case is the obtained when the round 1-handle is attached to a global torus by means of two longitudinal circles. □

From this round handle decomposition, proposition 2 allows us to build a NMS flow on $S^2 \times S^1$, whose periodic orbits are located in the cores of the tori. The indexed links formed by these orbits are called admissible links.

3. **Building NMS flows on** $S^2 \times S^1$ **.** As it has been shown, fattened round handles corresponding to Bott integrable Hamiltonian systems come from the essential attachment of round 1-handles on one or two tori. In this section, we use them to build the corresponding NMS flows on $S^2 \times S^1$ with one saddle orbit. In the following figures, attractive, repulsive and saddle periodic orbits are denoted, respectively, by a, r and s.

Firstly, let us consider that the round 1-handle is orientable and separating. So, it is attached by using two attaching circles. Moreover, if its associated saddle orbit is global, the round 1-handle must be attached to global tori by means of longitudinal circles. When it is attached to one global torus, the fattened round handle is $F_{p+1} \times S^1$ and the saddle orbit is parallel to the (p, q)-toroidal hole left when the round 1-handle is attached. The corresponding flow can be seen in figure 9, for $p = 1$.

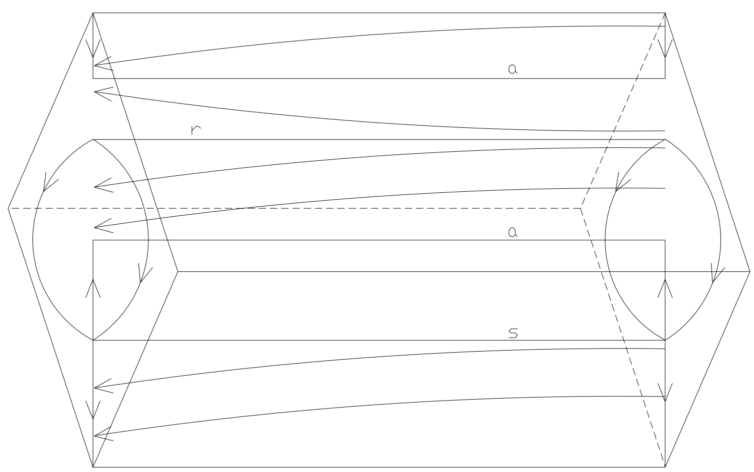

FIGURE 9. NMS flow for a global fattened handle $F_{p+1} \times S^1$

If the round 1-handle is attached to two global tori, the corresponding flow is the reverse of the previous one.

If the saddle orbit u is local, its associated round 1-handle must be attached to global tori by means of transversal circles. Then, the fattened handle is $F_2 \times S^1$. The flow shown in figure 10 corresponds to the attachment of the round 1-handle to one global torus by means of two transversal circles. The reverse flow is obtained when it is attached to one global and one local torus by means of one transversal and one longitudinal circles, respectively.

If some component of the complementary of the fattened round handle in $S^2 \times S^1$ is not a solid torus (see [4]) and the fattened round handle is local, it is equivalent to $T^2 \times [0, 1] - \left(S^3 - N(k)\right)$ where $N(k)$ is a regular neighborhood of a non trivial

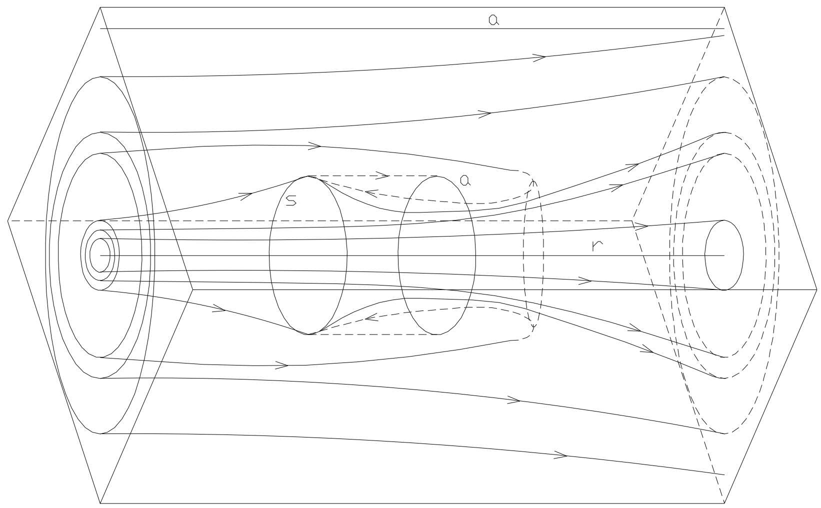

FIGURE 10. NMS flow associated to a local round 1-handle attached to a global tori by means of two transversal circles

knot k. In this case the complementary of the fattened round handle must also be a fattened round handle and more than one saddle orbit is needed.

Secondly, let us consider a non separating round 1-handle attached to a global torus by means of two longitudinal circles. In this case the boundary components are two global tori and one of them must be a $(2,1)$-torus. This $(2,1)$-torus is the boundary of a fattened Klein bottle K^3 embedded in $S^2 \times S^1$. So, the fattened round handle is $K^3 - int\,(W)$, where W is a global solid torus.

Finally, let the round 1-handle be not orientable. It must be attached to one torus by using one attaching circle and the fattened round handle is equivalent to $D^2 \times S^1 - int\left(\bar{W}\right)$; the corresponding flow is the reverse of the previous one and it can be seen in figure 11.

Moreover, in the last two phase portraits, invariant manifolds of the saddle orbit are not orientable. In fact, it can be observed that a Klein bottle appears as critical submanifold.

Proposition 5. *The Klein bottle K^2 is a critical submanifold of the fattened round handles $D^2 \times S^1 - int\left(\overline{W}\right)$ and $K^3 - int\,(W)$.*

Proof. When the fattened round handle is $K^3 - int\,(W)$, a non separating round 1-handle is attached to a global torus by using two longitudinal circles. The saddle and the repulsive orbits must be of the same homotopy type. Then the stable manifold of the saddle orbit and the repulsive orbit, $W^s \cup r$, form a Klein bottle, that is found in the core of K^3. The attractive orbit is the core of a $(2,1)$ −solid torus, that is the complementary of K^3 in $S^2 \times S^1$.

The fattened round handle $D^2 \times S^1 - int\left(\overline{W}\right)$ is obtained by attaching a non orientable round 1-handle to one global torus by means of one attaching circle. In this case, the stable and unstable manifolds of the saddle follow a Möbius band, being the saddle orbit γ a trivial knot in the core of the fattened Möbius band. The attractive orbit a is also a trivial knot parallel to the saddle orbit; so, to build the

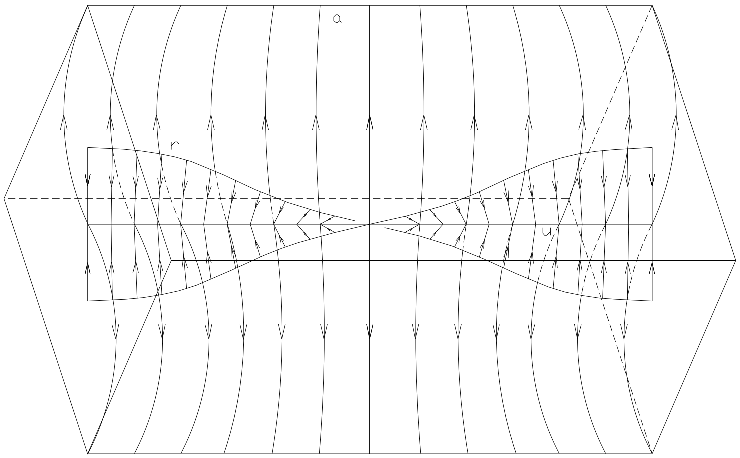

FIGURE 11. NMS flow associated to $D^2 \times S^1 - int\left(\bar{W}\right)$

flow two Möbius bands are glued along their boundaries, with the saddle and the attractive orbits in their cores. Then, a Klein bottle is obtained as $K^2 = W^u \cup a$. □

4. **Characterization of the set of periodic orbits in Bott integrable Hamiltonian systems on $S^2 \times S^1$.** The set of periodic orbits in a NMS system are in the core of the handles in a round handle decomposition of the manifold. Periodic orbits on a 3-dimensional manifold can be characterized as knots and the set of periodic orbits is characterized as a link. In the following, we see that fattened handles defined in section 2.3 can be associated to different operations that describe the indexed links corresponding to the orbits is the core of the round handles for a RHD of $S^2 \times S^1$ involving these fattened handles.

A closed orbit of a flow on a 3-manifold is repulsive, saddle or attractive if the dimension of its stable manifold is equal to one, two or three, respectively. The dimension of this manifold minus one, is called the index of the periodic orbit.

The topological characterization of NMS links and Bott integrable Hamiltonian systems on the sphere S^3 is already made ([20], [7]), although it is not the case of $S^2 \times S^1$. In this section we are going to study the characterization of the periodic orbits in $S^2 \times S^1$ for Bott integrable Hamiltonian systems in terms of operations on local and global links. Let O_4, O_5 and O_6 denote these operations.

Theorem 4.1. *Let $Q^3 \cong S^2 \times S^1$ be a non-singular level surface of a Bott integrable Hamiltonian. The link of indexed periodic orbits formed by the set of NMS periodic orbits of the Hamiltonian on Q^3 can be obtained from global and local links by means of the following operations:*

1. $O_4(L, l) = (L \# l) \cup m$. *The connected sum $L \# l$ is obtained by composing a component K of L and a component k of l each of which has index 0 or 2. The index of the composed component is the index of K, m is a meridian of $K \# k$ with index 1.*
2. $O_5(L)$. *Choose a component K_1 of L of index 0 or 2, and replace a neighborhood of K_1 in $S^2 \times S^1$, $N\left(K_1, S^2 \times S^1\right)$, by $D^2 \times S^1$ with three indexed*

circles in it; $\{0\} \times S^1$ *(where* $\{0\} \in D^2$*),* K_2 *and* K_3*. Here,* K_2 *and* K_3 *are* (p,q)*-cables on the boundary of the neighborhood* $\partial N\left(\{0\} \times S^1, D^2 \times S^1\right)$*. The indices of* $\{0\} \times S^1$ *and* K_2 *are either* 0 *or* 2*, and one of them is equal to the index of* K_1*. The index of* K_3 *is* 1*.*

3. $O_6\,(L)$*. Choose a component* K_1 *of* L *of index* 0 *or* 2*, and replace* $N\left(K_1, S^2 \times S^1\right)$ *by* $D^2 \times S^1$ *with two indexed circles in it;* $\{0\} \times S^1$ *and the* $(2,q)-$*cable* K_2 *of* $\{0\}\times S^1$*, the index of* $\{0\}\times S^1$ *is* 1 *and* $ind\,(K_1) = ind\,(K_2)$*.*

where L *is an admissible link and* l *denotes a local admissible link.*

Proof. Let

$$S^2 \times S^1 = \bigcup_{j=1}^{n} C_j$$

be a round handle decomposition of $S^2 \times S^1$ and let L denote the indexed link formed by the periodic orbits in the core of this decomposition.

Let $C \equiv C_j$ be a component associated to a round 1-handle. The fattened handle C must be equivalent to one of the manifolds described in theorem 2.2. Our aim is to associate each one of the operations in the statement of the theorem to the different fattened handles in order to characterize the indexed link associated to the set of periodic orbits in the cores of the round handles.

Let us consider global fattened handles, that allow us to complete $S^2 \times S^1$ by gluing solid tori.

1. Let C be equivalent to $F_{p+1} \times S^1$. Two cases must be distinguished depending on all the components of the boundary of C are global tori or one of them is a local torus.
   (a) When all the components of ∂C are global tori, let C_1, C_2 and C_3 denote the connected components of the complementary of C in $S^2 \times S^1$. This fattened handle has been obtained by attaching a round 1-handle by means of two longitudinal circles either to one global torus or to two global tori.
   When it is attached to one global torus, $\partial C_1 = T^2 \times \{0\}$ and $C \cup C_1 \cup C_2 \cong D^2 \times S^1$ are global, so $D^2 \times S^1$ and C_3 give rise to a RHD of $S^2 \times S^1$. Moreover, the periodic orbit in the core of C_2 and the saddle orbit are (p,q)-cables of the core of C_1. Then, the corresponding indexed link is described by operation O_5.
   When it is attached to two global tori, $A = T_1^2 \cup T_2^2$, $\partial C_1 = T_1^2 \times \{0\}$ and $\partial C_2 = T_2^2 \times \{0\}$; $C \cup C_1 \cup C_2 \cong D^2 \times S^1$ is global and a RHD of $S^2 \times S^1$ is obtained by gluing C_3. The periodic orbit in the core of C_2 and the saddle orbit are $(1,0)$-cables of the core of C_1, and the corresponding indexed link is also described by operation O_5 with $p = 1$ (see figure 9). Let us notice that the saddle orbit is parallel to the periodic orbit in the core of the solid torus that fills C_2.
   (b) Now, let us consider that one of the boundary components of C is a local torus: as we have seen before, the fattened handle has been obtained by the attachment of the round 1-handle to a global torus by means of transversal circles or when it is attached to one global and one local tori by means of one transversal and one longitudinal circles, respectively.
   Let C_1, C_2 and C_3 be the connected components of the complementary of C in $S^2 \times S^1$; in both cases, $C \cup C_1 \cup C_2 \cong D^2 \times S^1$ can replace

$\left(D^2 \times S^1, K_1\right)$ in a RHD of $S^2 \times S^1$ formed by this solid torus joint with (C_3, K_2). In this RHD, the saddle orbit and the periodic orbit in the core of C_2 are meridians of the orbit in the core of C_1 (see figure 10). The index of K_1 depends on $\partial C_2 \subset \partial_- C$. So, the corresponding link can be characterized by O_4.

Let us notice that the saddle orbit that appear in O_4 is local and a meridian of the connected component.

2. Finally, when $C \cong K^3 - int\,(W)$ or $C \cong D^2 \times S^1 - int\left(\overline{W}\right)$, let P denote $C - int\,(N\,(\gamma))$ where γ is the global saddle orbit. Then, $P \cong F_2 \times S^1$. Let C_2 be the component in the complementary of C in $S^2 \times S^1$ such that ∂C_2 is a $(2,1)$-torus. So, $P \cup C_2 \cong T^2 \times [0,1]$.

   Therefore, $C \cup C_2$ is equivalent to $D^2 \times S^1$ with two indexed knots: $\gamma \equiv \{*\} \times S^1$, with index 1, in the core of a Möbius band and K_2, that is a $(2,q)$-cable of γ with index i, $i = 0$ or 2 depending on $\partial_0 \subset \partial_- C$ or not (see figure 11). In a RHD of $S^2 \times S^1$ a round i-handle $\left(D^2 \times S^1, K_1\right)$ can be substituted by $C \cup C_2$, so a new round handle decomposition is obtained. Let L denote the indexed link that consist of the cores of the first RHD. Then, a new link is obtained from L by the operation O_6.

Let us notice that when a local fattened handle C is considered, other fattened handles are needed in order to complete the manifold; the set of periodic orbits will have more than one saddle orbit and the resulting link can be written in terms of the previous operations. In this case, at least one of the periodic orbits have to be global in order to obtain a round handle decomposition of $S^2 \times S^1$. Therefore, the resulting link must be obtained from global and local links. ☐

Let us notice that these three operations are similar to operations IV, V and VI of Wada on S^3, but in these case Klein bottles appear as invariant manifolds, that can not been embedded in S^3. Moreover, global and local links are admissible in $S^2 \times S^1$, meanwhile only local links appear in S^3.

The following corollaries can be obviously deduced from the previous result:

Corollary 2. *Let H be a Hamiltonian associated to an integrable system on $Q^3 \cong S^2 \times S^1$ by means of an orientable Bott integral. If L is a link of periodic orbits of H then L is obtained from global and local links through connected sums and cabling.*

A connected sum of iterated toroidal knots it is called a generalized toroidal knot.

Corollary 3. *Let H be a Hamiltonian associated to an integrable system on $Q^3 \cong S^2 \times S^1$ by means of an orientable Bott integral. Then, the set of NMS periodic orbits of H on Q^3 is a non separable link and all the periodic orbits are generalized toroidal knots.*

5. **Nonintegrability of a perturbed pendulum.**

As a consequence of corollary 3 the presence of a periodic orbit in a Hamiltonian system on $S^2 \times S^1$ whose associated knot is not a generalized toroidal knot implies the non-integrability of the system.

This is also true in a Hamiltonian system defined on a phase space that is a piece of a decomposition of $S^2 \times S^1$ by means of solid or thick tori.

As an application, consider the time-perturbed pendulum defined by

$$H = \frac{y^2}{2} - \cos x + \varepsilon \sin t \sin x$$

and $-1 \leq H \leq a$, $a > 1$.

If $\varepsilon = 0$ we have an autonomous integrable system defined on $S^2 \times [0, 1]$. The enlarged phase space is a thick torus.

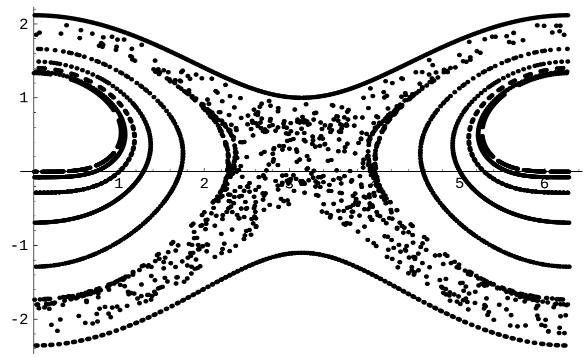

FIGURE 12. Poincare section

Let $0 < \varepsilon$ and small enough and consider the system defined on a thick torus bounded by invariant KAM tori; in figure 12 a Poincare section ($t = 0$) is shown for $\varepsilon = \frac{1}{10}$. Numerically we found (see in figure 13 the initial conditions as a function of ε) that the system has a periodic orbit whose associated knot is an eight knot (see figure 14). Therefore, the system is not integrable.

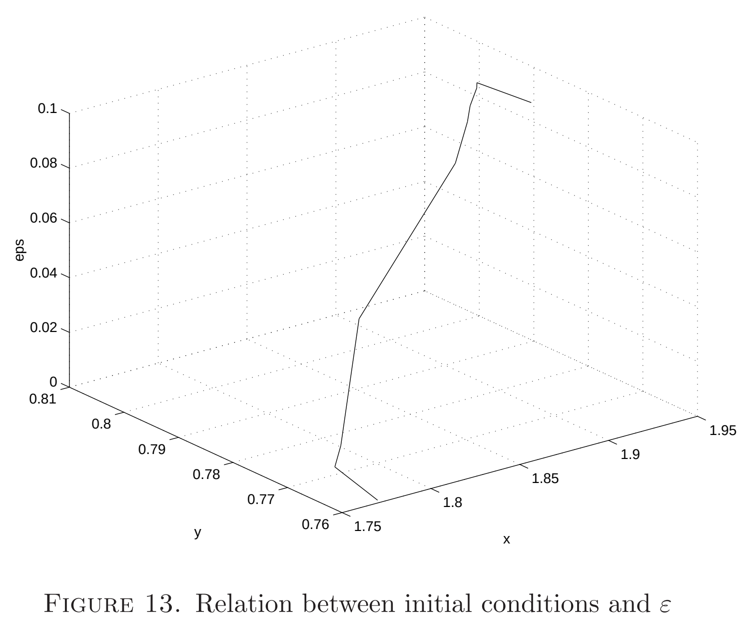

FIGURE 13. Relation between initial conditions and ε

Acknowledgements. The authors thank to Vicente Gil Ferrandis for his help in the figures and also to the referees for their useful comments.

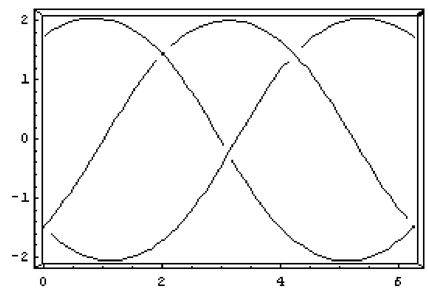

FIGURE 14. Periodic orbit associated to an eight knot

REFERENCES

[1] D. Asimov, *Round handles and non-singular Morse-Smale flows*, Ann. Math., **102** (1975), 41–54.

[2] A. V. Bolsinov, S. V. Matveev and A. T. Fomenko, *Topological classification of integrable Haimltonian systems with two degrees of freedom. List of systems of small complexity*, Russian Math. Surveys, **45** (1990), 59–94.

[3] C. Bonatti and V. Z. Grines, *Knots as topological invariants for gradient-like diffeomorphisms on the sphere* S^3, J. Dynam. Control Syst., **6** (2000), 579–602.

[4] C. Bonatti, V. Z. Grines and E. Pecou, *Two dimensional links and diffeomorfisms on* 3-*manifolds*, Ergod. Th. & Dynam. Syst., **22** (2002), 687–710.

[5] R. Bott, *Nondegenerate critical manifolds*, Annals of Mathematics, **60** (1954), 248–261.

[6] B. Campos, J. Martínez Alfaro and P. Vindel, *Topology of bifurcations on symmetric integrable Hamiltonian systems on* S^3, in "Equadiff99. Proc. of the International Conference on Differential Equations" (eds. B. Fiedler et al), World Scientific, (2000), 1467–1470.

[7] J. Casasayas, J. Martínez Alfaro and A. Nunes, *Knots and links in integrable Hamiltonian systems*, J. of Knot Theory and its Ramifications, **7** (1998), 123–153.

[8] A. Cordero, J. Martínez Alfaro and P. Vindel, *Topology of the two fixed centres problem*, Celestial Mechan. Dynam. Astr., **82** (2002), 203–223.

[9] A. Cordero, J. Martínez Alfaro and P. Vindel, *Round handle decomposition of* $S^2 \times S^1$, Dynamical Systems: An International Journal, **22** (2007), 179–202.

[10] A. T. Fomenko, "Integrability and Non Integrability in Geometry and Mechanics," Kluwer Academic Publishers, 1988.

[11] A. T. Fomenko and T. Z. Nguyen, *Topological classification on integrable non degenerate Hamiltonians on the isoenergy three-dimensional sphere*, Advances in Soviet Math., **6** (1991), 267–297.

[12] C. Godbillon, "Dynamical Systems on Surfaces," Springer-Verlag, 1983.

[13] A. Hatcher, *Notes on basic 3-manifold topology*, Available in http://math.cornell.edu/\char126\relaxhatcher.

[14] J. Hempel, *3-manifolds*, Ann. of Math. Studies, **86**, Princeton University Press, 1976.

[15] W. Jaco, *Lectures on three manifold topology,* CBMS Regional Conf. Series in Math., **43**, 1980.

[16] J. W. Morgan, *Non-singular Morse-Smale flows on 3-dimensional manifolds*, Topology, **18** (1978), 41–53.

[17] T. Z. Nguyen, *The complexity of integrable Hamiltonian systems on a prescibed three-dimensional constant-energy submanifold*, Russian Acad. Sci. Sb. Math., **75** (1993), 507–532.

[18] B. L. Reinhart, *Cobordism and the Euler number*, Topology, **2** (1963), 173–177.

[19] G. Soler López, *Accumulation points of flows on the Klein bottle*, Discrete and Continuous Dynamical Systems, **9** (2003), 497–503.

[20] M. Wada, *Closed orbits of non-singular Morse-Smale flows on* S^3, J. Math. Soc. Japan, **41** (1989), 405–413.

[21] K. Yano, *The homotopy class of non-singular Morse-Smale vector fields on 3-manifolds*, Invent. Math., **80** (1985), 435–451.

Received June 2007; revised February 2008.

E-mail address: acordero@mat.upv.es
E-mail address: Jose.Mtnez.Alfaro@uv.es
E-mail address: vindel@mat.uji.es