An Approach to the Plateau Problem in the Framework of Bézier and B-spline Surfaces

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INTRODUCTION

The term CAGD (Computer Aided Geometric Design) was coined by R Barnhill and R. Riesenfeld in 1974 when they organized a conference on that topic at the University of Utah. This concept deals with the mathematical description of shape for use in computer graphics, numerical analysis, approximation theory, data structures and computer algebra. Renaissance naval architects in Italy were the firsts to use drafting techniques that involved conic sections. These techniques were refined through the centuries, culminating in the use of B-splines.

At the end of the last century, CAGD focused its study of surfaces mainly on the theory of rectangular surface or tensor product patches, introduced by Coons and Bézier in the sixties.

This work is divided in two different parts. The first part is dedicated to a review in rectangular Bézier surfaces. Given the set of points $\mathcal{P} = \{P_{i,j}\} \in \mathbb{R}^3$, where $0 \leq i \leq m$ and $0 \leq j \leq n$, the rectangular Bézier surface of degree n, m associated to $\mathcal{P}$ is defined as the polynomial surface given by $\overrightarrow{x}(u,v)$: $[0,1] \times [0,1] \to \mathbb{R}^3$,

$$\overrightarrow{x}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(u) B_j^n(v) P_{i,j}.$$

The set of points $\mathcal{P}$ is called the control net of the Bézier surface and $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ is the ith Bernstein polynomial of degree n.

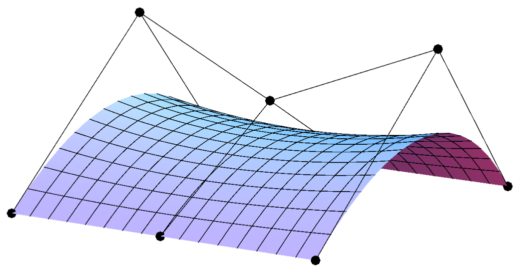


Figure 1: Example of Bézier surface

In [1], [3], [8], [9] and [10] the authors studied different methods to approach the rectangular Bézier surface of minimal area among all other Bézier surfaces with the same boundary curves.

The second part of the work is dedicated to the study of minimal surfaces for B-splines. The Bézier representation of surfaces has a main disadvantage, the number of control points is directly related with the degree. Therefore, to increase the complexity of the shape of the surface by adding control points requires increasing the degree of the surface. This disadvantage is remedied with the introduction of the B-spline (basis spline) representation.

A B-spline surface is surface defined by a set of control points $\{P_{i,j}\}_{i,j=0}^{n,m}$ and two knot vectors $U = (u_0, u_1, \dots, u_{n-1}, u_n, u_{n+1}, \dots, u_{n+k})$, $V = (v_0, v_1, \dots, v_{m-1}, v_m, v_{m+1}, \dots, v_{m+l})$ associated to each parameter u and v where $u_i \leq u_{i+1}$ and $v_j \leq v_{j+1}$. The corresponding B-spline surface is given by

$$\vec{x}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,k}(u) N_{j,l}(v) P_{i,j},$$

where

$$N_{i,1}(t) = \begin{cases} 1, & \text{for } t_i \leq t \leq t_{i+1}, \\ 0, & \text{otherwise} \end{cases}$$

for $k = 1$, and

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t),$$

for $k > 1$ and $i = 0, 1, \dots, n$.

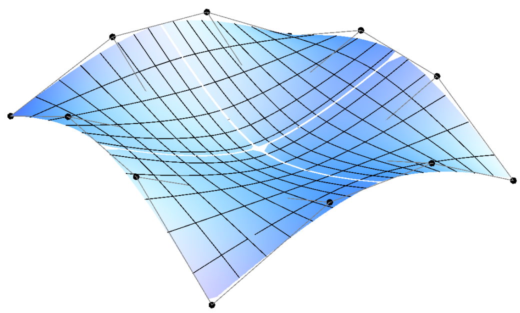


Figure 2: Bicuadratic B-spline surface

More than one hundred years ago, one of the most famous problems in Geometry was *the Plateau problem.* The problem of finding a surface that minimizes the area with prescribed border was called the Plateau problem, after the Belgian researcher Joseph-Antoine Ferdinand Plateau (1801-1883). When trying to solve the problem one has to minimize the area functional, regrettably this functional is highly nonlinear. This is one of the reasons that left the problem unsolved for more than a century. It was in 1931 when Douglas solved the problem by replacing the area functional by the Dirichlet functional which was easier to manage and has the same extremal under isothermal conditions.

In the case of Bézier surfaces, which are polynomial surfaces, it is possible to state the same problem (see [8]): given the border, or equivalently the boundary control points, the *Plateau-Bézier problem* consists on finding the inner control points in such a way that the resulting Bézier surface is of minimal area among all other Bézier surfaces with the same boundary control points.

The different methods we have studied throughout this notes to approach a solution to the Plateau-Bézier problem can be classified in three categories:

1. Functional minimization: Given the boundary curves, we determine the surface that minimizes some functionals among all the polynomial surfaces with that given boundary. We have considered two functionals: the Dirichlet functional and the Biharmonic functional. We conduct our study in terms of Bézier surfaces, and these functionals, restricted to the space of polynomials, turn into functions of the control points. Thus, the extremal of a functional $\mathcal{I}$ among all rectangular Bézier surfaces can be computed as the minimum of the real function

$$\mathcal{P} \to \mathcal{I}(\overrightarrow{x_{\mathcal{P}}}),$$

$\overrightarrow{x}_{\mathcal{P}}$ being the rectangular Bézier patch associated to the control net $\mathcal{P}$.

2. PDE surfaces: A rectangular Bézier surface satisfying a partial differential equation can be determined given some of its control points. The minimum set of prescribed control points depends on the PDE under study, mainly on the order of the PDE. In this notes we study the generation of rectangular Bézier surfaces satisfying the Laplace equation and the biharmonic equation. In the case of harmonic Bézier surfaces two boundary conditions were required to construct the surface while for biharmonic Bézier surfaces four boundary curves were needed as initial data.

3. Masks: Another way of building surfaces is by means of masks. A mask is a set of coefficients that define any control point of a Bézier surface in

terms of its neighbouring control points. Thus, the whole control net is obtained as a solution of a linear system. The use of masks has its origin in numerical methods to discretize and solve differential equations.

Let us introduce our work with a little more detail.

In the first chapter we introduce the concepts we shall use through the notes.

In Chapter 2, we study the way to generate harmonic surfaces given some of their control points as initial data. Harmonic surfaces are the PDE surfaces obtained as a solution of the equation $\Delta\overrightarrow{x}(u,v) = 0$ where Δ denotes the harmonic operator otherwise known as the Laplacian. Harmonic surfaces, which have found their way into various application areas of CAGD such as surface design, geometric mesh smoothing and fairing, are moreover related to surfaces minimizing the area: an isothermal parametric surface is minimal if and only if it is harmonic. The end of the chapter is dedicated to see some examples of harmonic surfaces using B-splines.

Chapter 3 is dedicated to the study of biharmonic surfaces. A biharmonic surface satisfies the PDE $\Delta^2\overrightarrow{x}(u,v) = 0$ where Δ^2 is the bilaplacian operator. The term *thin plate problem*, which is used to refer to the biharmonic boundary problem, comes from the physical analogy involving the bending of a thin sheet of metal. In this chapter we shall see that any biharmonic rectangular Bézier surface is fully determined by the boundary control points, that is, four boundary curves. The end of the chapter is dedicated to see some results in biharmonic surfaces using B-splines.

The last chapter is dedicated to the study of the Dirichlet functional and the use of masks. As we have said before, the Dirichlet functional was used to solve the Plateau problem. In this chapter we study the extremal of the Dirichlet functional for rectangular Bézier surfaces and compare the obtained results with harmonic and biharmonic surfaces and with surfaces associated to different masks. Finally, we study the extremal of the Dirichlet functional for B-splines.

Chapter 1

Curves and surfaces in CAGD

For Computer Aided Geometric Design (CAGD) it is convenient to use simple representations of curves and surfaces involving elementary operations, as addition and multiplication. Therefore, the most reasonable candidate at first sight are the polynomial parameterizations. As it is well known, we can represent polynomial curves of degree n as

$$c(t) = a_0 + a_1 t + \ldots + a_n t^n, \quad t \in [0,1],$$

where each coefficient a_i is a point in the plane or space, depending on the curve is flat or spatial. For example, the curve $c(t) : [0,1] \to \mathbb{R}^2$ of coefficients $a_0 = (0,1)$, $a_1 = (1,0)$ and $a_2 = (1,1)$ is parameterized by

$$c(t) = a_0 + a_1 t + a_2 t^2 = (0,1) + (1,0)t + (1,1)t^2 = (t + t^2, t^2 + 1).$$

This representation has the advantage of simplicity, however, it is not very practical. The interpretation of the coefficients is referred to the values of the curve in the neighborhood of the starting point $c(0) = a_0$. In fact, they are the derivatives of the parametrization for $t = 0$:

$$a_i = \frac{c^{(i)}(0)}{i!}.$$

Thereby it does not give us a clear idea of the overall behavior of the curve.

As it is common in CAGD, if we observe the curve from another point of view, for example after a rotation, transformation or a deformation, the behavior of the coefficients is varied. If we make a translation by a vector v, the coefficient a_0 is moved by the same vector, whereas the other coefficients are not affected. But if we make a rotation centered at a_0 occurs just the contrary, all coefficients except on a_0, experience a rotation. As we see, the behavior of the coefficients under affine applications is complex, because of all coefficients except on a_0 (which is a point) are vectors, since they are derivatives of the parametrization.

Therefore, it seems appropriate to use another polynomial basis where the coefficients of the curve can be computed easily after an affine map.

1.1 The Bézier curves

1.1.1 The Bernstein polynomials

A different basis for polynomials of degree n is provided by the Bernstein polynomials. These polynomials were used in approximation theory to demonstrate the Weierstrass theorem of uniform approximation of continuous functions by polynomials. Its construction is very simple from the Newton's binomial:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i},$$

where

$$\binom{n}{i} = \begin{cases} \frac{n!}{i!(n-i)!}, & 0 \le i \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Taking $a = t$ and $b = 1 - t$, we obtain:

$$1 = (t + 1 - t)^n = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} = \sum_{i=0}^{n} B_i^n(t),$$

where $B_i^n(t)$ is the ith Bernstein polynomial of degree n.

Example 1.1.1 *When $n = 0$ there is only one Bernstein polynomial:*

$$B_0^0(t) = 1:$$

When $n = 1$ there are two,

$$B_0^1(t) = 1 - t, \quad B_1^1(t) = t:$$

When $n = 2$, the Bernstein polynomials are:

$$B_0^2(t) = 1 - 2t + t^2, \quad B_1^2(t) = 2t - 2t^2, \quad B_2^2(t) = t^2.$$

Let us have a look to some properties of the Bernstein polynomials:

1. *Polynomial basis:* The set of Bernstein polynomials of degree n given by $\{B_0^n(t), B_1^n(t), \ldots, B_n^n(t)\}$ is a basis of the vector space of polynomials of degree $\leq n$.

2. *Recursion:* Bernstein's polynomials verify the following recursion formula:

$$B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t),$$

with $B_0^0(t) \equiv 1$ and $B_j^n(t) \equiv 0$ when $j \notin \{0, \ldots, n\}$.

3. *Partition of unity:* As we have seen before, for $n \geq 0$ and $t \in [0,1]$:

$$\sum_{i=0}^{n} B_i^n(t) = 1.$$

4. *Non-negativity:* Each Bernstein polynomial is non-negative within the interval $[0,1]$. For all $t \in [0,1]$,

$$B_i^n(t) \in [0,1].$$

5. *Symmetry:* The next relation follows directly from the definition:

$$B_i^n(t) = B_{n-i}^n(1-t).$$

6. *Interval end conditions:*

$$B_i^n(0) = \delta_i^0, \quad \text{and} \quad B_i^n(1) = \delta_i^n,$$

for all $i \in \mathbb{N}$, where δ_i^j is the Kronecker's Delta function.

7. *Area under the curve:* The area under any Bernstein polynomial in the interval $[0,1]$ is always the same for all polynomials of the same degree:

$$\int_0^1 B_i^n(t) = \frac{1}{n+1},$$

for all $i \in \{0, \ldots, n\}$.

8. *Derivatives:*

$$\frac{d}{dt}B_i^n(t) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)),$$

for any $n, i \in \mathbb{N}$ and having in mind that $B_{-1}^n(t) = B_n^{n-1}(t) = 0$.

9. The polynomial $B_i^n(t)$ has only one maximum in $[0,1]$, and this maximum occurs at $t = i/n$.

10. *Linear precision:* The Bernstein polynomials satisfy:
$$\sum_{i=0}^{n} \frac{i}{n} B_i^n(t) = t.$$

Finally, let us see how a Bernstein polynomial can be written as a linear combination of Bernstein polynomials of higher degree.

Lemma 1.1.2 *For any $n > 0$, $k \in \{0, \ldots, n\}$ and $i \in \{0, \ldots, n-k\}$ we have that*
$$B_i^{n-k}(t) = \sum_{l=0}^{k} \frac{\binom{n-i-l}{k-l}\binom{i+l}{l}}{\binom{n}{k}} B_{i+l}^n(t).$$

Notice that, as we have said at the beginning of the chapter, we can write a curve in the usual polynomial parameterization, so then it should be possible to change the basis from Bernstein's polynomial basis to the usual basis.

Lemma 1.1.3 *For each $n > 0$ and $i \in \{0, 1, \ldots, n\}$:*
$$B_i^n(t) = \sum_{k=0}^{n-i} (-1)^k \binom{n}{i}\binom{n-i}{k} t^{k+i}.$$

Conversely we have the following Lemma.

Lemma 1.1.4 *For each $n > 0$ and $i \in \{0, 1, \ldots, n\}$:*
$$t^i = \sum_{k=0}^{n-i} \frac{\binom{k+i}{i}}{\binom{n}{i}} B_{i+k}^n(t).$$

1.1.2 Definition and properties of the Bézier curves

In the following subsection we shall work in the plane, $\mathbb{R}^2$, but all definitions and properties can be immediately generalized to higher dimensions.

Definition 1.1.5 *(DeCasteljau's algorithm) Given $n+1$ points $P_0, \ldots, P_n \in \mathbb{R}^2$, we define the associated Bézier curve as $\alpha : [0,1] \to \mathbb{R}^2$ given by $\alpha(t) = P_0^n(t)$, where*

$$\begin{aligned} P_i^0(t) &= P_i \\ P_i^r(t) &= (1-t)P_i^{r-1}(t) + tP_{i+1}^{r-1}(t), \end{aligned}$$

for $r = 1, \ldots, n$ *and* $i = 0, \ldots, n-r$.

We shall call *control polygon* the polygon P given by the points $P_0, \ldots, P_n$ and these points are called *control points.*

The intermediate points in the DeCasteljau's algorithm, $P_i^r(t)$, can be directly obtained as:

$$P_i^r(t) = \sum_{j=0}^{r} B_j^r(t) P_{i+j}.$$

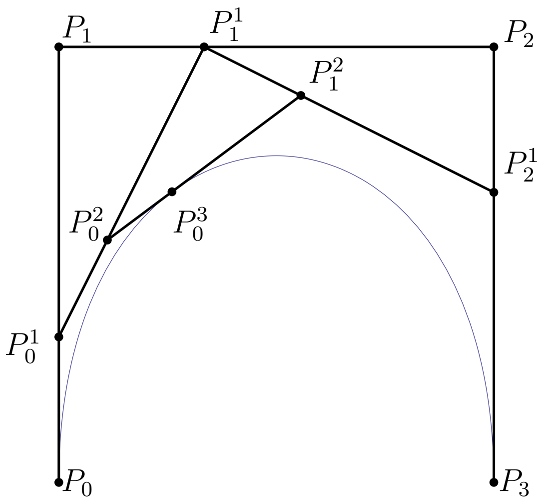


Figure 1.1: A cubic Bézier curve generated by DeCasteljau's algorithm

Definition 1.1.6 *Let* $\alpha(t)$ *be the Bézier curve defined by the control points* $P_0, \ldots, P_n$*, then, for all* $t \in [0, 1]$,

$$\alpha(t) = \sum_{i=0}^{n} B_i^n(t) P_i = B_0^n(t) P_0 + \ldots B_n^n(t) P_n.$$

Notice that this definition is equivalent to the Casteljau's algorithm when $r = n$.

As the Bézier curve is uniquely defined by its control polygon, sometimes it is denoted by $\alpha[P_0, P_1, \ldots, P_n](t)$.

Proposition 1.1.7 *Let* $\alpha[P_0, P_1, \ldots, P_n](t)$ *be a Bézier curve and let* $t_0 \in [0, 1]$. *The arcs of the curve* α, $\alpha_e = \alpha|_{[0,t_0]}$ *and* $\alpha_d = \alpha|_{[t_0,1]}$ *are again Bézier curves.*

The following properties characterize Bézier curves and most of them are consequences of corresponding properties of Bernstein polynomials.

1. *Endpoint interpolation:* The curve passes through the polygon endpoints:

$$P(0) = P_0 \quad \text{and} \quad P(1) = P_n.$$

2. *Symmetry:* The polygons $P_0, P_1, \ldots, P_n$ and $P_n, P_{n-1}, \ldots, P_0$ describe the same curve in different direction:

$$\alpha[P_0, P_1, \ldots, P_n](t) = \alpha[P_n, P_{n-1}, \ldots, P_0](1-t),$$

for all $t \in \mathbb{R}$.

3. *Affine invariance:* If an affine map, ϕ, is applied to the control polygon, then the curve is mapped by the same map. More precisely

$$\alpha[\phi(P_0), \phi(P_1), \ldots, \phi(P_n)](t) = \phi(\alpha[P_0, P_1, \ldots, P_n](t)).$$

This is a consequence of the fact that linear interpolation is preserved by affine maps.

4. *Convex hull:* The Bézier curve is always included in the convex hull of the control points.

5. *Variation diminishing:* If a straight line intersects a planar Bézier polygon m times, then the line can intersect the Bézier curve at most m times. For higher dimensions, the straight line should be substituted by hipersubespace.

6. *Linear precision:* If the control points $\{P_i\}_{i=0}^{n-1}$ are evenly spaced on the straight line between P_0 and P_n, then the degree n Bézier curve is the linear interpolant between P_0 and P_n.

7. *Pseudo-local control:* Suppose we move the i^{th} control point. The curve changes the most in the vicinity of $t = i/n$. In fact, all the points on the curve move in a direction parallel to the vector formed by the difference of the old and new control point.

Let us see some properties related with derivatives of Bézier curves.

Proposition 1.1.8 *The derivative of a Bézier curve* $\alpha[P_0, P_1, \ldots, P_n](t)$ *with control points* $P_0, \ldots, P_n$ *is again a Bézier curve of degree* $n-1$ *with control points* $n\Delta P_0, \ldots, n\Delta P_{n-1}$*, where* $\Delta P_i = P_{i+1} - P_i$*. This is,*

$$\alpha'(t) = n \sum_{i=0}^{n-1} B_i^{n-1}(t) \Delta P_i.$$

In particular, thanks to the endpoint interpolation property, we get

$$\alpha'(0) = n\Delta P_0 \qquad \text{and} \qquad \alpha'(1) = n\Delta P_{n-1}.$$

8. *Tangent lines at endpoints:* The tangent lines at the endpoints of a Bézier curve with control polygon $P_0, \ldots, P_n$, are determined by the vectors ΔP_0 and ΔP_{n-1}.

In general, to compute higher order derivatives, we have to apply the previous result iteratively:

$$\alpha^{(r)}(t) = \frac{n!}{(n-r)!} \sum_{i=0}^{n-r} B_i^{n-r}(t) \Delta^r P_i,$$

where $\Delta^r P_i = \Delta(\Delta^{r-1} P_i)$.

Finally, let us see a lemma which provides us a method of basis conversion.

Lemma 1.1.9 *Let* $\alpha[P_0, \ldots, P_n](t)$ *be a Bézier curve, then*

$$\alpha[P_0, \ldots, P_n](t) = \sum_{i=0}^{n} Q_i t^i,$$

where

$$Q_i = \binom{n}{i} \Delta^i P_0,$$

for all $i = 0, \ldots, n$.

1.2 B-spline curves

The Bézier representation of curves has two main disadvantages. First, the number of control points is directly related with the degree. Therefore, to increase the complexity of the shape of the curve by adding control points requires increasing the degree of the curve. Second, changing any control point affects the entire curve or surface, making design of specific sections very difficult. These disadvantages are remedied with the introduction of the B-spline (basis spline) representation.

B-spline curves consists of many polynomial pieces and the system of formulating a B-spline curve has the advantage that fewer control points are needed for definition compared to the usual Bézier curves. This implies that B-spline curves reduce calculation time when we are working with a computer.

An order k B-spline is formed by joining several pieces of polynomials of degree $k-1$ with at most $\mathcal{C}^{k-2}$ continuity at the end points. The *knot vector* is defined by

$$T = (t_0, t_1, \ldots, t_{n-1}, t_n, t_{n+1}, \ldots, t_{n+k}),$$

$t_0 \leq t_1 \leq \ldots \leq t_{n+k}$ and determines the parametrization of the basis function. The values t_i are called *knots* of the B-spline.

Given a knot vector T, the associated B-spline basis functions, $N_{i,k}(t)$, are defined as:

$$N_{i,1}(t) = \begin{cases} 1, & \text{for } t_i \leq t \leq t_{i+1}, \\ 0, & \text{otherwise} \end{cases}$$

for $k = 1$, and

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t), \tag{1.2.1}$$

for $k > 1$ and $i = 0, 1, \ldots, n$. This equations have the following properties:

1. *Positivity:* $N_{i,k}(t) > 0$, for $t \in [t_i, t_{i+k}]$ and $N_{i,k}(t) = 0$ otherwise.
2. *Partition of unity:* $\sum_{i=0}^{n} N_{i,k}(t) = 1$, for $t \in [t_{k-1}, t_{n+1}]$.
3. *Continuity:* $N_{i,k}(t)$ has $\mathcal{C}^{k-2}$ continuity at each simple knot t_i.
4. If $t_i = t_{i+k}$ then $N_{i,k} \equiv 0$.

The derivative of the B-spline basis function is given by:

$$\frac{dN_{i,k}(t)}{dt} = \frac{k-1}{t_{i+k-1} - t_i} N_{i,k-1}(t) - \frac{k-1}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t).$$

The following theorem shows B-splines are a generalization of the Bernstein's polynomials.

Theorem 1.2.1 *Given a knot vector of $2k$ knots $T = (0, \overset{(k)}{\ldots}, 0, 1, \overset{(k)}{\ldots}, 1)$, then the k order B-splines coincide with the Bernstein's polynomial $B_i^{k-1}(t)$ of degree $k-1$.*

Definition 1.2.2 *Given the knot vector $T = (t_0, t_1, \dots, t_{n-1}, t_n, t_{n+1}, \dots, t_{n+k})$, and $\{D_i\}_{i=0}^n$ be the control points, a B-spline curve of order k and knot vector T is defined as*

$$\alpha(t) = \sum_{i=0}^{n} N_{i,k}(t) D_i,$$

for $n \geq k-1$ and $\forall t \in [t_{k-1}, t_{n+1}]$. In this context the control points are called de Boor points.

The influence of the de Boor points is a consequence of the B-spline definition and it can be determined by the following lemma.

Lemma 1.2.3 *Let $\alpha(t) = \sum_{i=0}^{n} N_{i,k}(t) D_i$ be a B-spline curve with associated knot vector $T = (t_0, t_1, \dots, t_{n-1}, t_n, t_{n+1}, \dots, t_{n+k})$. Then the De Boor point D_j has only influence on $\alpha(t)$ for $t_j < t < t_{j+k}$. In fact, the associated curve to a given parameter $\bar{t}$ such that $t_r < \bar{t} < t_{r+1}$ is completely determined by the de Boor points $D_{r-(k-1)}, \dots, D_r$.*

The following image shows a cuadratic B-spline curve of control points $(1,2)$, $(1.5,3)$, $(3,0)$, $(4,3.5)$, $(5,3)$ and knot vector $T = (0,0,0,1,2,3,3,3)$.

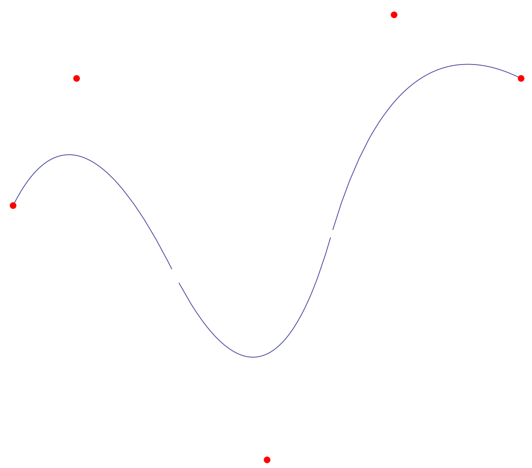


Figure 1.2: Cuadratic B-spline curve

A B-spline curve has the following properties:

1. *Geometry invariance property:* Partition of unity property of the B-spline assures the invariance of the shape of the B-spline curve under translation and rotation.

2. *End points geometric property:* B-spline curves with knot vector of the form

$$T = (\underbrace{t_0, t_1, \ldots, t_{k-1}}_{k \text{ equal knots}}, \underbrace{t_k, t_{k+1}, \ldots, t_{n-1}, t_n}_{n-k+1 \text{ internal knots}}, \underbrace{t_{n+1}, t_{n+2}, \ldots, t_{n+k}}_{k \text{ equal knots}}),$$

   are tangent to control polygon at their endpoints. In this case, the curve is called *clamped B-spline curve.*

3. *B-spline to Bézier property:* By the previous property, it can be seen that a Bézier curve of order k is a B-spline curve with no internal knots and the end knots repeated k times. The knot vector is thus

$$T = (\underbrace{t_0, t_1, \ldots, t_{k-1}}_{k \text{ equal knots}}, \underbrace{t_{n+1}, t_{n+2}, \ldots, t_{n+k}}_{k \text{ equal knots}}),$$

   where $n + k + 1 = 2k$ or $n = k - 1$.

In general, the derivative of a B-spline curve is again a new B-spline curve. The first derivative of a B-spline curve is given by:

$$\alpha'(t) = (k-1) \sum_{i=1}^{n} \frac{D_i - D_{i-1}}{t_{i+k-1} - t_i} N_{i,k-1}(t).$$

To compute higher order derivatives, we apply the following formula:

$$\alpha^{(j)}(t) = (k-1)(k-2) \ldots (k-j) \sum_i D_i^{[j]} N_{i,k-j}(t),$$

where

$$D_i^{[j]} = \begin{cases} D_i, & j = 0, \\ \frac{D^{[j-1]} - D_{i-1}^{[j-1]}}{t_{i+k-j} - t_i}, & j > 0. \end{cases}$$

1.3 The Bézier surfaces

The construction of a Bézier surface is very similar to the case of curves. In fact, many of the tools and algorithms that we developed for the curves will continue to be useful for surfaces.

As we have seen in the previous section a Bézier curve of control points $P_0, P_1, \ldots, P_n$ can be defined as:

$$\alpha(u) = \sum_{i=0}^{n} B_i^n(u) P_i.$$

Now, if we allow the control points move throughout parameterized curves $P_i(v)$,

$$\alpha(u,v) = \sum_{i=0}^{n} B_i^n(u) P_i(v),$$

then the Bézier curves $\alpha(u, v_0)$ of control points $P_0(v_0), P_1(v_0), \ldots, P_n(v_0)$ describe a surface in $\mathbb{R}^3$. As it seems natural, we will interest that the vertices $P_i(v)$ move along Bézier curves of control points $\{P_{i,0}, \ldots, P_{i,n}\}$.

As it occurs in the case of Bézier curves, there are two forms of Bézier surfaces. Let us see the DeCasteljau's algorithm for Bézier surfaces.

Definition 1.3.1 *Given the points* $\mathcal{P} = \{P_{i,j}\}_{0 \le i,j \le n}$ *and the parameters* $(u, v) \in \mathbb{R}^3$, *we define*

$$P_{i,j}^r(u,v) = [1-u, u] \left[\begin{array}{cc} P_{i,j}^{r-1}(u,v) & P_{i,j+1}^{r-1}(u,v) \\ P_{i+1,j}^{r-1}(u,v) & P_{i+1,j+1}^{r-1}(u,v) \end{array} \right] \left[\begin{array}{c} 1-v \\ v \end{array} \right],$$

where $P_{i,j}^0 = P_{i,j}$, $r = 1, \ldots, n$ *and* $i, j = 0, \ldots, n-r$. *Then, the Bézier surface associated to* $\{P_{i,j}\}_{i,j=0}^n$ *is given by* $\overrightarrow{x}(u,v) : [0,1] \times [0,1] \to \mathbb{R}^3$, $\overrightarrow{x}(u,v) = P_{0,0}^n(u,v)$. *The set of points* $\mathcal{P}$ *is called the control net of the Bézier surface.*

The disadvantage of this algorithm is that the polynomial surface has always the same degree n in both variables u and v. If we want to work with polynomial surfaces varying degrees in each variable, we must introduce the notion of tensor product of Bézier curves. This is the second possibility of definition of Bézier surfaces.

Definition 1.3.2 *Given the set of points* $\mathcal{P} = \{P_{i,j}\}$, *where* $0 \le i \le m$ *and* $0 \le j \le n$, *we define the Bézier surface associated to* $\mathcal{P}$ *as the parameterized surface given by* $\overrightarrow{x}(u,v) : [0,1] \times [0,1] \to \mathbb{R}^3$,

$$\overrightarrow{x}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(u) B_j^n(v) P_{i,j}.$$

The set of points $\mathcal{P}$ *is called the control net of the Bézier surface.*

Notice that this definition is equivalent to the Casteljau's algorithm when $m = n$.

In the same way as Bézier curves, it is possible to represent a Bézier surface by using as a basis the Bernstein polynomials or the usual basis.

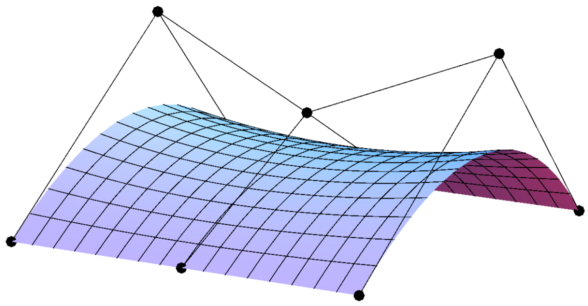


Figure 1.3: Example of a $(n, m) = (2, 2)$ Bézier surface

Lemma 1.3.3 *Let $\overrightarrow{x}(u, v)$ be a Bézier surface with control net $\{P^n_{k,\ell}\}^n_{k,\ell=0}$, then*

$$\overrightarrow{x}(u, v) = \sum_{k,\ell=0}^{n} B^n_k(u)B^n_\ell(v)P^n_{k,\ell} = \sum_{i,j=0}^{n} \frac{a_{i,j}}{i!j!}u^i v^j,$$

where $a_{i,j} = i!j!\binom{n}{i}\binom{n}{j}\Delta^{i,j}P_{0,0}$.

Let us remark that $\Delta^{i,j}$ denotes the difference operator

$$\begin{array}{ll} \Delta^{1,0}P_{i,j} = P_{i+1,j} - P_{i,j} & \Delta^{0,1}P_{i,j} = P_{i,j+1} - P_{i,j} \\ \Delta^{i,j}P_{i,j} = \Delta^{i-1,j}(\Delta^{1,0}P_{i,j}) & \Delta^{i,j}P_{i,j} = \Delta^{i,j-1}(\Delta^{0,1}P_{i,j}). \end{array}$$

The change from usual basis to Bézier control points is performed thanks to the following lemma.

Lemma 1.3.4 *Let $\overrightarrow{x}(u, v) = \sum_{i,j=0}^{n} \frac{a_{i,j}}{i!j!}u^i v^j$ be a polynomial patch of degree $\leq n$, then as a Bézier patch its control points are*

$$P_{k,\ell} = \sum_{s=0}^{k}\sum_{t=0}^{\ell} \frac{\binom{k}{s}\binom{\ell}{t}}{\binom{n}{s}\binom{n}{t}}\frac{a_{s,t}}{s!t!},$$

for all $k, \ell = 0, \ldots, n$.

Let us see some properties that Bézier surfaces satisfy:

1. Bézier surfaces are polynomial surfaces.

2. Bézier surfaces are invariants under affine maps.

3. As it occurs in Bézier curves, for $0 \leq u, v \leq 1$ the Bernstein polynomials $B_i^m(u)$ and $B_j^n(v)$ are non negative and verify the property of partition of unity:
$$\sum_{i=0}^{m}\sum_{j=0}^{n} B_i^m(u)B_j^n(v) \equiv 1.$$
As a consequence, the Bézier surfaces are included in the convex hull of its control net.

4. The coordinate curves are Bézier curves. The coordinate curves for u or v constant are Bézier curves of degree n and m respectively. For $v = v_0$ constant, we obtain:
$$\overrightarrow{x}(u, v_0) = \sum_{i=0}^{m}\sum_{j=0}^{n} B_i^m(u)B_j^n(v_0)P_{i,j} = \sum_{i=0}^{m} B_i^m(u)(\sum_{j=0}^{n} B_j^n(v_0)P_{i,j}).$$
In other words, $\overrightarrow{x}(u, v_0)$ is a Bézier curve of degree m and control points $\sum_{j=0}^{n} B_j^n(v_0)P_{i,j}$ for $i =, 0, 1, \ldots, m$. The same occurs for $u = u_0$ constant.

5. The partial derivatives of a Bézier surface is again a Bézier surface:
$$\begin{aligned}\frac{\partial}{\partial u}\overrightarrow{x}(u, v) =& m\sum_{j=0}^{n}\sum_{i=0}^{m-1} B_i^{m-1}(u)B_j^n(v)\Delta^{1,0}P_{i,j},\\ \frac{\partial}{\partial v}\overrightarrow{x}(u, v) =& n\sum_{i=0}^{m}\sum_{j=0}^{n-1} B_j^{n-1}(v)B_i^m(u)\Delta^{0,1}P_{i,j},\end{aligned}$$
where $\Delta^{1,0}P_{i,j} = P_{i+1,j} - P_{i,j}$ and $\Delta^{0,1}P_{i,j} = P_{i,j+1} - P_{i,j}$.

Partial derivatives of higher degree can be computed by the following formula:
$$\frac{\partial^{r+s}}{\partial u^r \partial v^s}\overrightarrow{x}(u, v) = \frac{m!n!}{(m-r)!(n-s)!}\sum_{i=0}^{m-r}\sum_{j=0}^{n-s} B_i^{m-r}(u)B_j^{n-s}(v)\Delta^{r,s}P_{i,j},$$
where $\Delta^{r,s}P_{i,j} = \Delta^{r,0}(\Delta^{0,s}P_{i,j})$ and:
$$\begin{aligned}\Delta^{r,0}P_{i,j} =& \Delta^{r-1,0}P_{i+1,j} - \Delta^{r-1,0}P_{i,j},\\ \Delta^{0,s}P_{i,j} =& \Delta^{0,s-1}P_{i+1,j} - \Delta^{0,s-1}P_{i,j}.\end{aligned}$$

1.4 B-spline surfaces

The surface analogue of the B-spline curve is the B-spline surface. This is a surface defined by a set of control points $\{P_{i,j}\}_{i,j=0}^{n,m}$ and two knot vectors $U =$

$(u_0, u_1, \ldots, u_{n-1}, u_n, u_{n+1}, \ldots, u_{n+k})$ and $V = (v_0, v_1, \ldots, v_{m-1}, v_m, v_{m+1}, \ldots, v_{m+l})$ associated to each parameter u and v where $u_i \leq u_{i+1}$ and $v_j \leq v_{j+1}$. The corresponding B-spline surface is given by

$$\overrightarrow{x}(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,k}(u) N_{j,l}(v) P_{i,j},$$

where $N_{i,k}(u)$ and $N_{j,l}(v)$ were defined in Equation 1.2.1.

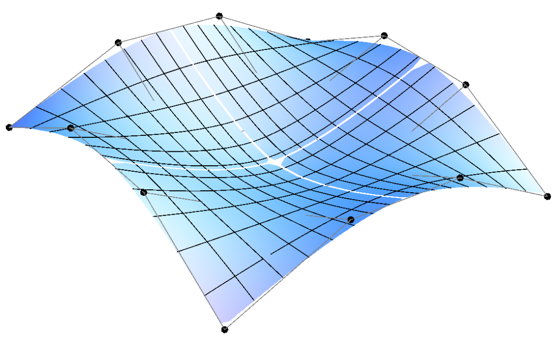
Notice that as it occurs in Bézier surfaces, for $u = u_0$ constant $\overrightarrow{x}(u_0, v)$ is a B-spline curve in v of knot vector V and control points $q_j = \sum_{i=0}^{m} N_{i,m}(u_0) P_{i,j}$, where $0 \leq j \leq n$.

Some of the properties of the B-spline curves can be extended to surfaces, such as:

1. Geometry invariance property.
2. End points geometric property.
3. B-spline to Bézier property.

Let us see an example, for $U = (0, 0, 0, 1, 2, 2, 2)$ and $V = (1, 1, 1, 2, 3, 3, 3)$ the following image shows a bicuadratic B-spline surface of control points

$$\begin{array}{cccc}
(-15, 10, 15) & (-5, 5, 15) & (5, 5, 15) & (15, 10, 15) \\
(-15, 5, 5) & (-5, 10, 5) & (5, 10, 5) & (15, 5, 5) \\
(-15, 5, -5) & (-5, 10, -5) & (5, 10, -5) & (15, 5, -5) \\
(-15, 10, -15) & (-5, 5, -15) & (5, 5, -15) & (15, 10, -15)
\end{array}$$



1.5 The Plateau-Bézier problem

The problem of finding a surface that minimizes the area with prescribed border is called the *Plateau problem,* after the Belgian researcher Joseph-Antoine Ferdinand Plateau (1801-1883). Such surfaces are characterized by the fact that the mean curvature vanishes and, in some real problems, the interest comes from the fact that minimal area means minimal cost of material used to build a surface.

In the case of Bézier surfaces, which are polynomial surfaces, it is possible to state the same problem (see [8]): given the border, or equivalently the boundary control points, the *Plateau-Bézier problem* consists on finding the inner control points in such a way that the resulting Bézier surface is of minimal area among all other Bézier surfaces with the same boundary control points.

Let $\overrightarrow{x} : U \to S$ be a chart on a surface $S \in \mathbb{R}^3$ and E, F, G be the coefficients of the first fundamental form given by:

$$E = < \overrightarrow{x}_u, \overrightarrow{x}_u >, \quad F = < \overrightarrow{x}_u, \overrightarrow{x}_v > \quad \text{and} \quad G = < \overrightarrow{x}_v, \overrightarrow{x}_v >,$$

where $\overrightarrow{x}_u, \overrightarrow{x}_v$ represent the first derivatives of $\overrightarrow{x}$ and $<,>$ defines the dot product of the vectors. The chart $\overrightarrow{x}$ is said to be *isothermal* when $F = 0$ and $E = G$.

As it is well known in the theory of minimal surfaces, if $\overrightarrow{x}$ is an isothermal map then $\overrightarrow{x}$ is minimal iff $\Delta\overrightarrow{x} = 0$, where Δ is the usual Laplacian operator given by:

$$\Delta\overrightarrow{x}(u, v) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\overrightarrow{x}(u, v).$$

In this case we say that S is a *harmonic* surface.

In the same way, the surfaces that satisfy the condition:

$$\Delta^2\overrightarrow{x}(u, v) = 0$$

are called *biharmonic* surfaces and the knowledge of the boundary and tangent planes of these surfaces enables to fully determine the entire surface. In Chapters 2 and 3 we shall analyze the relation between minimal surfaces with harmonic and biharmonic surfaces.

Finally, the area of the Bézier surface, S, is given by:

$$A(\mathcal{P}) = \int_R \|\overrightarrow{x}_u \times \overrightarrow{x}_v\| dudv = \int_R (EG - F^2)^{1/2} dudv,$$

where $R = [0,1] \times [0,1]$.

As it also happens in the theory of minimal surfaces, the area functional is highly nonlinear, so in order to find the minimal area associated to a boundary curve, we must consider other possibilities. Let us recall that under isothermal conditions the extremal of the functional area coincides with the extremal of the Dirichlet functional in the general case:

$$D(\mathcal{P}) = \frac{1}{2}\int_R (\|\overrightarrow{x}_u\|^2 + \|\overrightarrow{x}_v\|^2)dudv = \frac{1}{2}\int_R (E+G)dudv.$$

So then, instead of minimizing the area functional, in Chapter 4 we shall work with the Dirichlet functional.

There are other methods to find approximations to the solutions of the Plateau–Bézier problem, for example, the use of masks. A simple way of constructing Bézier surfaces with prescribed boundary consists in generating the inner control points by using a mask. Let us recall that a mask is a way of writing a linear relation between one inner control point and its eight neighboring control points. What one has to do is just to solve if possible a system of linear equations whose matrix of coefficients has just a few non-vanishing entries. For example, for an $n \times m$ Bézier surface, there are $(n-1) \times (m-1)$ linear equations and the same number of inner control points. At the end of Chapter 4 we shall compare the use of different masks.

Chapter 2

Harmonic surfaces

Throughout these notes we are thinking about different ways to address the Plateau-Bézier problem. As we have stated in the previous chapter, the Plateau-Bézier problem consists on finding the inner control points of a Bézier surface with prescribed boundary in such a way that the resulting Bézier surface is of minimal area among all other Bézier surfaces with the same boundary control points.

The first attempt to solve this problem is related to harmonic surfaces which satisfy the condition:

$$\Delta \overrightarrow{x}(u,v) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \overrightarrow{x}(u,v) = 0,$$

where Δ is the usual Laplacian operator. This operator has been widely used in many application areas such as physics. It is associated with a wide range of physical problems, for example gravity, electromagnetism and fluid flows.

Harmonic surfaces are related to minimal surfaces, i.e., surfaces that minimize the area among all surfaces with prescribed boundary conditions. The relation is as follows. Given a parametric surface patch $\overrightarrow{x}(u,v)$ satisfying the isothermal conditions, i.e., $< \overrightarrow{x}_u, \overrightarrow{x}_u >=< \overrightarrow{x}_v, \overrightarrow{x}_v >$, and $< \overrightarrow{x}_u, \overrightarrow{x}_v >= 0$, then the surface it represents is minimal if and only if it is harmonic.

2.1 Harmonic tensor product Bézier surfaces

In terms of control points, the harmonic condition of a polynomial surface is a linear system.

Theorem 2.1.1 *([8]) Given the control net in* $\mathbb{R}^3$, $\{P_{ij}\}_{i,j=0}^{n,m}$, *the associated Bézier surface* $\overrightarrow{x} : [0,1] \times [0,1] \to \mathbb{R}^3$, *is harmonic i.e.* $\Delta \overrightarrow{x} = 0$, *if and only if for any*

$i \in \{0, 1, \ldots, n\}$ and $j \in \{0, 1, \ldots, m\}$:

$$\begin{aligned} 0 =& b_{m,i,0}\, P_{i+2,j} + (b_{m,i-1,1} - 2b_{m,i,0})\, P_{i+1,j} + (b_{m,i-1,1} - 2b_{m,i-2,2})\, P_{i-1,j} \\ &+ b_{m,i-2,2}\, P_{i-2,j} + b_{n,j,0}\, P_{i,j+2} + (b_{n,j-1,1} - 2b_{n,j,0})\, P_{i,j+1} \\ &+ (b_{n,j-1,1} - 2b_{n,j-2,2})\, P_{i,j-1} + b_{n,j-2,2}\, P_{i,j-2} \\ &+ (b_{m,i,0} - 2b_{m,i-1,1} + b_{m,i-2,2} + b_{n,j,0} - 2b_{n,j-1,1} + b_{n,j-2,2})\, P_{i,j}, \end{aligned} \tag{2.1.1}$$

where, for $i \in \{0, 1, \ldots, n-2\}$

$$b_{n,i,0} = (n-i)(n-i-1) \quad b_{n,i,1} = 2(i+1)(n-i-1) \quad b_{n,i,2} = (i+1)(i+2),$$

and $b_{n,i,k} = 0$ otherwise, and with the convention $P_{ij} = 0$ if $i \notin \{1, 2, \ldots, n\}$ and $j \notin \{1, 2, \ldots, m\}$.

Corollary 2.1.2 *([3]) A bicuadratic Bézier surface is harmonic iff*

$$\begin{aligned} P_{01} =& \frac{1}{2}(2P_{00} + P_{02} - 2P_{10} + P_{20}), \\ P_{11} =& \frac{1}{4}(P_{00} + P_{02} + P_{20} + P_{22}), \\ P_{21} =& \frac{1}{2}(P_{00} - 2P_{10} + 2P_{20} + P_{22}), \\ P_{12} =& \frac{1}{2}(-P_{00} + P_{02} + 2P_{10} - P_{20} + P_{22}). \end{aligned}$$

In the same way, for $n = m = 3$ we have the following corollary.

Corollary 2.1.3 *([3]) A bicubic Bézier surface is harmonic iff*

$$\begin{aligned} P_{11} =& \frac{1}{9}(4P_{00} + 2P_{03} + 2P_{30} + P_{33}), \\ P_{21} =& \frac{1}{9}(2P_{00} + P_{03} + 4P_{30} + 2P_{33}), \\ P_{12} =& \frac{1}{9}(2P_{00} + 4P_{03} + P_{30} + 2P_{33}), \\ P_{22} =& \frac{1}{9}(P_{00} + 2P_{03} + 2P_{30} + 4P_{33}), \\ P_{10} =& \frac{1}{3}(4P_{00} - 4P_{01} + 2P_{02} + 2P_{30} - 2P_{31} + P_{32}), \\ P_{20} =& \frac{1}{3}(2P_{00} - 2P_{01} + P_{02} + 4P_{30} - 4P_{31} + 2P_{32}), \\ P_{13} =& \frac{1}{3}(2P_{01} - 4P_{02} + 4P_{03} + P_{31} - 2P_{32} + 2P_{33}), \\ P_{23} =& \frac{1}{3}(P_{01} - 2P_{02} + 2P_{03} + 2P_{31} - 4P_{32} + 4P_{33}). \end{aligned}$$

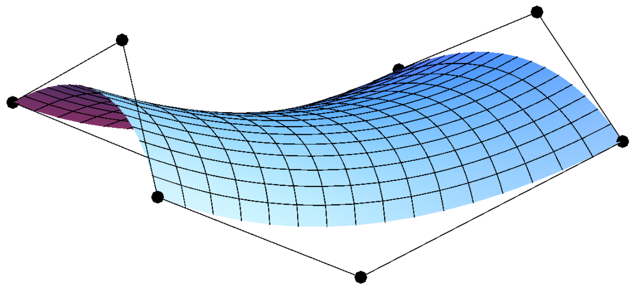


Figure 2.1: Bicuadratic harmonic surface

Let us recall that a Bézier surface can be written in two different ways depending on the basis:

$$\vec{x}(u,v) = \sum_{i,j=0}^{n} \frac{a_{i,j}}{i!j!} u^i v^j = \sum_{k,\ell=0}^{n} B_k^n(u) B_\ell^n(v) P_{k,\ell}^n.$$

The harmonic condition, in equation (2.1.1), with $\vec{x}$ written in the usual basis of polynomials can be translated into a system of linear equations in terms of the coefficients $\{a_{k,l}\}_{k,l=0}^{n}$

$$a_{k+2,l} + a_{k,l+2} = 0, \qquad k + l \leq n - 1 \tag{2.1.2}$$

with the convention $a_{k,l} = 0$ if $k + l > n + 1$. Its solution is given in the following lemma.

Lemma 2.1.4 *([10]) A polynomial function of degree $n \geq 2$, $\vec{x} = \sum_{i,j=0}^{n} \frac{a_{i,j}}{i!j!} u^i v^j$ is harmonic if and only if*

$$a_{k\ell} = (-1)^{\left[\frac{k}{2}\right]} a_{k\, mod\, 2, \ell + 2\left[\frac{k}{2}\right]}, \quad \forall k, \ell \tag{2.1.3}$$

that is,

$$a_{2\ell,j} = (-1)^{\ell}\, a_{0,2\ell+j}$$
$$a_{2\ell+1,j} = (-1)^{\ell}\, a_{1,2\ell+j},$$

with the convention $a_{k,\ell} = 0$ if $k + \ell > n + 1$. Therefore

1. *If n is odd, then all coefficients $\{a_{k,\ell}\}_{k=2,\ell=0}^{n}$ are totally determined by the coefficients $\{a_{0,\ell}, a_{1,\ell}\}_{\ell=0}^{n}$.*

2. *If n is even, then all coefficients $\{a_{k,\ell}\}_{k=2,\ell=0}^{n}$ and also $a_{1,n}$, which vanishes, are totally determined by the coefficients $\{a_{0,\ell}\}_{\ell=0}^{n}$ and $\{a_{1,\ell}\}_{\ell=0}^{n-1}$.*

The following proposition gives the Bézier version of the previous Lemma, but not the explicit solution of the system in equation (2.1.1).

Proposition 2.1.5 *([10]) Let $\overrightarrow{x}$ be a harmonic Bézier patch of degree n with control net $\{P^n_{k,\ell}\}^n_{k,\ell=0}$, then*

1. *If n is odd, the control points in the inner rows $\{P^n_{k,\ell}\}^{n-1,n}_{k=1,\ell=0}$ are determined by the control points in the first and the last rows, $\{P^n_{0,\ell}\}^n_{\ell=0}$ and $\{P^n_{n,\ell}\}^n_{\ell=0}$.*

2. *If n is even, the control points in the inner rows and also the corner control point $P^n_{n,n}$ are determined by the control points in the first and last rows, $\{P^n_{0,\ell}\}^n_{\ell=0}$ and $\{P^n_{n,\ell}\}^{n-1}_{\ell=0}$.*

If $\overrightarrow{x}$ is an $n\times n$ harmonic surface of even degree, from Lemma 2.1.4, the corner coefficients, $a_{0,n}$ and $a_{n,0}$, coincide. Then, from the basis conversion formula in Lemma 1.3.3 we have that $\Delta^{0,n}P_{0,0} = \Delta^{0,n}P_{n,0}$.

Therefore, if n is even, the boundary control points $\{P^n_{0,\ell}\}^n_{\ell=0}$ and $\{P^n_{n,\ell}\}^{n-1}_{\ell=0}$ determine the corner control point as follows

$$P_{n,n} = P_{0,n} + \sum_{w=0}^{n-1} \binom{n}{w} (-1)^{n-w} (P_{0,w} - P_{n,w}).$$

2.2 Generating function of harmonic Bézier surfaces

In the previous section, we have seen that we needed to change the polynomial basis because we were unable to obtain harmonic surfaces in Bézier form explicitly. Until now we had two options open to us in order to give a harmonic surface in Bézier form:

1. Solve the linear system in equation (2.1.1), i.e. compute the whole control net in terms of the given boundary control points.

2. Three steps. Compute the usual basis coefficients, $a_{k,l}$, prescribed by the given boundary control points $a_{0,i}$, $a_{1,i}$; Determine the $a_{k,l}$ that remain unknown with the explicit solution in usual basis, Equation (3.1.2); and finally come back to the Bézier basis.

In this section we shall see a new method to solve the harmonic condition in Bézier form by using a generating function. In general, a generating function is a formal power series in one indeterminate, whose coefficients encode information about a sequence a_n that is indexed by the natural numbers

$$g(a_n, t) = \sum_{n=0}^{\infty} a_n \, t^n.$$

It can be proved that the only rectangular harmonic tensor product patch is $(n+1) \times n$, with even n. Hence, thanks to this additional row of control points for the even case, we obtain the following result:

- If n is odd, given two rows of boundary control points $\{P^n_{0,\ell}\}_{\ell=0}^n$ and $\{P^n_{n,\ell}\}_{\ell=0}^n$, there is a unique harmonic tensor product Bézier surface of degree n,

- if n is even, given two rows of boundary control points $\{P^n_{0,\ell}\}_{\ell=0}^n$ and $\{P^n_{n+1,\ell}\}_{\ell=0}^n$, there is a unique harmonic tensor product Bézier surface of degree $n+1$, n.

Thus, from now on, to avoid writing the same things twice with the only difference being the parity of n, we will use the notation $\mathbf{n(n+1)}$ to express the fact that when n is odd, the choice is n, whereas when n is even then the choice is $n+1$.

Once the degrees of the surfaces that we shall work with have been established, the explicit formula we are looking for to compute the inner control points as a linear combination of given boundary control points, for $k = 0, \ldots, \mathbf{n(n+1)}$ and $\ell = 0, \ldots, n$, is

$$P^n_{k,\ell} = \sum_{i=0}^{n} \lambda^n_{k,\ell,i} \; P^n_{0,i} + \sum_{i=0}^{n} \mu^n_{k,\ell,i} \; P^n_{\mathbf{n(n+1)},i},$$

where, in the limiting cases, we have

$$\begin{aligned} \lambda^n_{0,\ell,i} &= \delta^i_\ell, \quad \mu^n_{0,\ell,i} = 0, \\ \lambda^n_{\mathbf{n(n+1)},\ell,i} &= 0, \quad \mu^n_{\mathbf{n(n+1)},\ell,i} = \delta^i_\ell. \end{aligned}$$

Then we can write

$$\begin{aligned}
\overrightarrow{x}(u,v) &= \textstyle\sum_{k,\ell=0}^{\mathbf{n(n+1)},n} B_k^{\mathbf{n(n+1)}}(u)B_\ell^n(v)P_{k,\ell}^n \\
&= \textstyle\sum_{k,\ell=0}^{\mathbf{n(n+1)},n} B_k^{\mathbf{n(n+1)}}(u)B_\ell^n(v)\left(\sum_{i=0}^n \lambda_{k,\ell,i}^n\ P_{0,i}^n + \mu_{k,\ell,i}^n\ P_{\mathbf{n(n+1)},i}^n\right) \\
&= \textstyle\sum_{i=0}^n \left(\sum_{k,\ell=0}^{\mathbf{n(n+1)},n} B_k^{\mathbf{n(n+1)}}(u)B_\ell^n(v)\lambda_{k,\ell,i}^n\right)\ P_{0,i}^n \\
&+ \textstyle\sum_{i=0}^n \left(\sum_{k,\ell=0}^{\mathbf{n(n+1)},n} B_k^{\mathbf{n(n+1)}}(u)B_\ell^n(v)\mu_{k,\ell,i}^n\right)\ P_{\mathbf{n(n+1)},i}^n
\end{aligned}$$

and define

$$\begin{aligned}
f_i^n(u,v) &:= \textstyle\sum_{k,\ell=0}^{\mathbf{n(n+1)},n} B_k^{\mathbf{n(n+1)}}(u)B_\ell^n(v)\lambda_{k,\ell,i}^n \\
g_i^n(u,v) &:= \textstyle\sum_{k,\ell=0}^{\mathbf{n(n+1)},n} B_k^{\mathbf{n(n+1)}}(u)B_\ell^n(v)\mu_{k,\ell,i}^n.
\end{aligned}$$

Therefore,

$$\overrightarrow{x}(u,v) = \sum_{i=0}^n f_i^n(u,v)\ P_{0,i}^n + g_i^n(u,v)\ P_{\mathbf{n(n+1)},i}^n.$$

Since we assume that $\overrightarrow{x}(u,v)$ is harmonic, then f_i^n and g_i^n are harmonic polynomials. Notice that functions f_i^n and g_i^n are related because the change of variables $(u,v) \to (1-u,v)$ implies that

$$f_i^n(u,v) = g_i^n(1-u,v).$$

Now, let us determine what type of boundary conditions are satisfied by the harmonic functions f_i^n and g_i^n. Since

$$\overrightarrow{x}(0,v) = \sum_{\ell=0}^n B_\ell^n(v)P_{0,\ell}^n = \sum_{i=0}^n f_i^n(0,v)\ \ P_{0,i}^n + g_i^n(0,v)\ P_{\mathbf{n(n+1)},i}^n,$$

then

$$\begin{cases} f_i^n(0,v) &= B_i^n(v), \\ g_i^n(0,v) &= 0. \end{cases}$$

Analogously, but using

$$\overrightarrow{x}(1,v) = \sum_{\ell=0}^n B_\ell^n(v)P_{\mathbf{n(n+1)},\ell}^n = \sum_{i=0}^n f_i^n(1,v)\ \ P_{0,i}^n + g_i^n(1,v)\ P_{\mathbf{n(n+1)},i}^n,$$

we get

$$\begin{cases} f_i^n(1,v) &= 0, \\ g_i^n(1,v) &= B_i^n(v). \end{cases}$$

Now, we will determine the generating function for the sequence of polynomials $\{g_i^n\}_{n=0}^{\infty}$,

$$g_i(u,v,t) = \sum_{n=0}^{\infty} \frac{g_i^n(u,v)}{n!} t^n.$$

The generating function for the sequence of polynomials $\{g_i^n\}_{n=0}^{\infty}$ must satisfy a pair of constraints.

First, notice that since all the terms of the sequence $\{g_i^n\}_{n=0}^{\infty}$ are harmonic polynomials, the generating function is a harmonic function with polynomial n-th derivatives,

$$g_i^n(u,v) = \frac{d^n}{dt^n}\Big|_{t=0} g_i(u,v,t).$$

Second, the boundary conditions of the generating function:

$$\begin{aligned} g_i(0,v,t) &= \textstyle\sum_{n=0}^{\infty} \frac{g_i^n(0,v)}{n!}\, t^n = \sum_{n=0}^{\infty} \frac{0}{n!}\, t^n = 0 \\ g_i(1,v,t) &= \textstyle\sum_{n=0}^{\infty} \frac{g_i^n(1,v)}{n!}\, t^n = \sum_{n=0}^{\infty} \frac{B_i^n(v)}{n!}\, t^n = \frac{(vt)^i}{i!}\; e^{(1-v)t}. \end{aligned}$$

For $i = 0$, we look for a harmonic function

$$g_0(u,v,t) = \sum_{n=0}^{\infty} g_0^n(u,v) \frac{t^n}{n!}$$

the sequence terms $g_0^n(u,v)$ being polynomial harmonic functions and satisfying the boundary conditions $g_0(0,v,t) = 0$ and $g_0(1,v,t) = e^{(1-v)t}$.

A particular solution of this problem is

$$g_0(u,v,t) = \frac{\sin(u\,t)}{\sin(t)}\; e^{(1-v)t},$$

but, in fact, it can be proved that it is the unique solution.

Now, if we define the operator

$$D_i = \frac{1}{i+1}\left(-t\frac{\partial}{\partial t} + (i+t)\mathrm{Id}\right),$$

it is easy to check by induction that

$$g_{i+1}(1,v,t) = \frac{(vt)^{i+1}}{(i+1)!}\; e^{(1-v)t} = D_i\left(\frac{(vt)^i}{i!}\; e^{(1-v)t}\right) = D_i\left(g_i(1,v,t)\right).$$

Therefore, if the boundary conditions can be obtained by successive application of the operators D_i acting on an initial function, the same will happen with the generating function:

$$g_{i+1}(u,v,t) = D_i(g_i(u,v,t)), \qquad g_0(u,v,t) = \frac{\sin(u\,t)}{\sin(t)}\; e^{(1-v)t}.$$

In addition, D_i commutes with the Laplacian operator, $\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)$, then, since g_0 is harmonic, so is g_i, and moreover, thanks to the symmetry,

$$f_i(u, v, t) = g_i(1 - u, v, t).$$

Therefore we arrive to the following proposition.

Proposition 2.2.1 *([1]) The harmonic surfaces generating function*

$$g_i(u, v, t) = \sum_{n=0}^{\infty} \frac{g_i^n(u, v)}{n!} t^n,$$

can be recursively defined by

$$g_i(u, v, t) = D_{i-1}(g_{i-1}(u, v, t))$$

with

$$g_0(u, v, t) = \frac{\sin(u\,t)}{\sin(t)}\ e^{(1-v)t}, \qquad D_i = \frac{1}{i+1}\left(-t\frac{\partial}{\partial t} + (i+t)\mathrm{Id}\right).$$

The following theorem shows what our interest is in the generating function.

Theorem 2.2.2 *([1]) The control net of a harmonic Bézier surface*

$$\overrightarrow{x}(u, v) = \sum_{k,\ell=0}^{n} P_{k,\ell}^n\, B_k^{\mathbf{n(n+1)}}(u) B_\ell^n(v)$$

is determined by two rows of boundary control points $\{P_{0,\ell}^n\}_{\ell=0}^n$ and $\{P_{\mathbf{n(n+1)},\ell}^n\}_{\ell=0}^n$ as follows

$$P_{k,\ell}^n = \sum_{i=0}^{n} \lambda_{k,\ell,i}^n\ P_{0,i}^n + \sum_{i=0}^{n} \mu_{k,\ell,i}^n\ P_{\mathbf{n(n+1)},i}^n,$$

$\lambda_{k,\ell,i}^n = \mu_{\mathbf{n(n+1)}-k,\ell,i}^n$ and $\{\mu_{k,\ell,i}^n\}_{k,\ell=0}^{\mathbf{n(n+1)},n}$ being the control points of the harmonic polynomial

$$g_i^n(u, v) = \frac{d^n}{dt^n}\bigg|_{t=0} g_i(u, v, t).$$

The inner control points of a harmonic Bézier patch are determined from boundary control points in the following way:

Proposition 2.2.3 *([1]) If n is odd, given two rows of boundary control points, $\{P_{0,\ell}^n\}_{\ell=0}^n$ and $\{P_{n,\ell}^n\}_{\ell=0}^n$, the Bézier control net of an $n \times n$ harmonic patch is defined explicitly in terms of them as follows*

$$P_{k,\ell}^n = \sum_{w=0}^{n} \mu_{n-k,\ell,w}^n\ P_{0,w}^n + \sum_{w=0}^{n} \mu_{k,\ell,w}^n\ P_{n,w}^n$$

with $\mu^n_{k,\ell,w}$ given by:

$$\mu^n_{k,\ell,i} = \sum_{t=0}^{\ell} \sum_{s=0}^{\left[\frac{k-1}{2}\right]} \sum_{r=s}^{\left[\frac{n-t}{2}\right]} \frac{\binom{k}{2s+1}\binom{\ell}{t}\binom{n}{2r+t}\binom{2r+t}{i}(2r+t)!}{\binom{\mathbf{n(n+1)}}{2s+1}\binom{n}{t}(2s+1)!\,t!\,(2r-2s)!}(-1)^{s+t-i} B^{sin}_{2r-2s}$$

and $B^{sin}_{1,2n} = (-1)^{n-1}(2^{2n}-2)B_{2n}$, $B^{sin}_{1,2n+1} = 0$ be the generalized Bernoulli numbers introduced in [2], B_n being the Bernoulli numbers.

If n is even, given two rows of boundary control points $\{P^n_{0,\ell}\}^n_{\ell=0}$ and $\{P^n_{n+1,\ell}\}^n_{\ell=0}$, the Bézier control net of an $(n+1)\times n$ harmonic patch is defined explicitly in terms of them as follows,

$$P^n_{k,\ell} = \sum_{w=0}^{n} \mu^n_{n+1-k,\ell,w}\, P^n_{0,w} + \sum_{w=0}^{n} \mu^n_{k,\ell,w}\, P^n_{n+1,w}.$$

Proposition 2.2.4 *([1]) If n is even, given the boundary control points $\{P^n_{0,\ell}\}^n_{\ell=0}$ and $\{P^n_{n,\ell}\}^{n-1}_{\ell=0}$, the corner control point is*

$$P_{n,n} = P_{0,n} + \sum_{w=0}^{n-1} \binom{n}{w} (-1)^{n-w}(P_{0,w} - P_{n,w}),$$

and the whole Bézier control net of a harmonic $n\times n$ patch is defined explicitly as follows

$$P^n_{k,\ell} = \sum_{w=0}^{n} \alpha^n_{k,\ell,w}\, P^n_{0,w} + \sum_{w=0}^{n-1} \beta^n_{k,\ell,w}\, P^n_{n,w}, \tag{2.2.1}$$

where

$$\beta^n_{k,\ell,w} = \sum_{t=0}^{\ell} \sum_{s=0}^{\left[\frac{k-1}{2}\right]} \sum_{r=s}^{\left[\frac{n-t-1}{2}\right]} \frac{\binom{k}{2s+1}\binom{\ell}{t}\binom{n}{2r+t}\binom{2r+t}{w}(2r+t)!}{\binom{n}{2s+1}\binom{n}{t}(2s+1)!\,t!\,(2r-2s)!}(-1)^{s+t-w} B^{sin}_{2r-2s}.$$

$$\alpha^n_{k,\ell,w} = \sum_{t=0}^{\ell} \sum_{s=0}^{\left[\frac{k}{2}\right]} \frac{\binom{\ell}{t}\binom{k}{2s}\binom{n}{2s+t}\binom{2s+t}{w}\binom{2s+t}{t}}{\binom{n}{2s}\binom{n}{t}} (-1)^{3s+t-w}$$

$$+ \sum_{t=0}^{\ell} \sum_{s=0}^{\left[\frac{k-1}{2}\right]} \sum_{r=s}^{\left[\frac{n-t-1}{2}\right]} \sum_{m=2r+t}^{n+2r+t} \frac{\binom{\ell}{t}\binom{k}{2s+1}\binom{m}{2r+t}\binom{n}{m}\binom{m}{w}(2r+t)!}{\binom{n}{2s+1}\binom{n}{t}(2s+1)!\,t!\,(2r-2s)!}(-1)^{s-w+r+\frac{m+t}{2}+1} B^{sin}_{2r-2s}.$$

Throughout this chapter we have seen three equivalent methods to solve the harmonic condition:

1. First, to solve the harmonic condition $\Delta\overrightarrow{x}(u,v)=0$ by taking derivatives and solving the resulting system making the coefficients $u^k v^l$ equal to zero.

2. Second, to solve the harmonic equations in 2.1.1 in terms of Bézier control points.

3. Third, to compute the scalars $\mu^n_{k,l,w}$ that determine the inner control points.

The goal of this last method is to explicitly know the scalars that characterize a harmonic control net, and then avoid the change of basis. Notice that, this method is not the best one in terms of computation times, however, the harmonic surfaces generating function, $g_i(u, v, t)$, a function that generalizes harmonic surfaces of all degrees, is a new tool for the study of harmonic functions.

2.3 B-spline harmonic surfaces

As we have said in the previous chapter, to increase the complexity of a surface by adding control points requires increasing the degree. This disadvantage is remedied with the introduction of B-splines surfaces. These kind of surfaces are one of the most widespread methods in CAGD specially bicubic B-spline surfaces because they have $\mathcal{C}^2$ continuity. Therefore it has sense to ask about harmonic B-spline surfaces. However, since the Regularity theorem for harmonic functions states that harmonic functions are infinitely differentiable, this case has no many interesting because we always obtain a polynomial surface, a Bézier surface. Let us see some examples.

For knot the vectors $U = V = (0, 0, 0, 1, 2, 2, 2)$ and control points $P_{2,2} = (1, 2, 0)$, $P_{2,3} = (1, 3, 0)$, $P_{3,1} = (0, 1, 0)$, $P_{3,2} = (0, 2, 0)$ and $P_{3,3} = (0, 3, -0.5)$ we obtain the polynomial surface

$$\overrightarrow{x}(u, v) = (4 - 2u, \frac{1}{2}(-2 + 4u - u^2 + v^2), -4.5 + 0.5u^2 + u(1 - 2v) + 5v - 0.5v^2)$$

where $u, v \in [0, 2] \times [0, 2]$.

In the same way, by using bicubic B-splines, for the control points $P_{4,1} = (0, 0, -0.5)$, $P_{4,2} = (1, 0, 0)$, $P_{4,3} = (1.5, 0, 0)$, $P_{4,4} = (2, 0, -0.5)$, $P_{3,1} = (0, 1, -0.5)$, $P_{3,2} = (1, 1, 0)$, $P_{3,3} = (1.5, 1, 0)$, $P_{3,4} = (2, 1, -0.5)$ and knot vectors $U = (0, 0, 0, 0, 1, 2, 2, 2, 2)$ and $V = (0, 0, 0, 0, 1, 2, 2, 2, 2)$, we obtain the following poly-

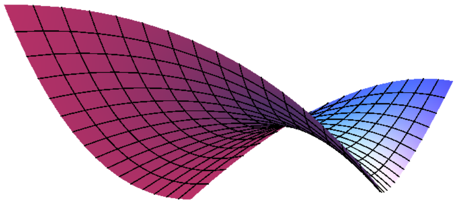


Figure 2.2: Harmonic surface using bicuadratic B-splines

nomial surface

$$\begin{aligned}\overrightarrow{x}(u,v) =& \frac{1}{4}(20 + 3u^2(2 - 1.5v) - 6v - 6v^2 + 1.5v^3 + 3u(-8 + 6v), 24 - 12u + \\ & 3u^2(12 + 6(-2 + v) - 6v), -14 - 3u^3(-1 - 0.5(-2 + v) + 0.5v) \\ & + 18v + 3v^2 - 1.5v^3 + 3u^2(-7 - 3(-2 + v) + 4.5v) + \\ & 3u(12 - 6v - 3v^2 + 0.5v^3 - 0.5(16 - 6v^2 + v^3))).\end{aligned}$$

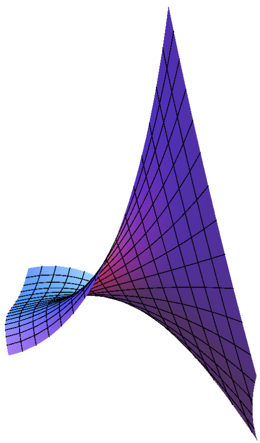


Figure 2.3: Harmonic surface using bicubic B-splines

Chapter 3

Biharmonic surfaces

Following a similar fashion to the harmonic case, let us now ask for the conditions that a Bézier surface must fulfil in order to be biharmonic. Biharmonic surfaces satisfies the condition:

$$\Delta^2 \overrightarrow{x}(u,v) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right)^2 \overrightarrow{x}(u,v) = 0,$$

where Δ is the usual Laplacian operator.

This equation is associated with a great variety of physical problems such as tension in elastic membranes and the study of stress and strain in physical structures. There are many mechanical problems concerning the bending of a thin elastic clamped rectangular plate, and they can all be formulated in terms of a two-dimensional biharmonic equation with prescribed values of the function and its normal derivative at the boundary. Hence, the biharmonic boundary problem is also known as the thin plate problem. From a geometric design point of view, which is our field of interest, this operator has found its way into various areas of application, such as surface design, geometric mesh, smoothing and fairing.

3.1 Existence of biharmonic Bézier surfaces

As we have seen in Proposition 2.1.5, a harmonic Bézier surface of odd degree is determined by two opposite rows of boundary control points. For the even case, the inner rows and in addition a corner control point are determined by control points in the first and last row.

The biharmonic case is similar to harmonic surfaces. In this case, the inner control points are determined by the boundary control points for both even and odd degree. For a rectangular Bézier surface of degree n, m the boundary control points are given by:

$$\begin{array}{cccccc} P_{00} & P_{01} & P_{02} & \dots & P_{0,m-1} & P_{0m} \\ P_{10} & * & * & \dots & * & P_{1m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{n-1,0} & * & * & \dots & * & P_{n-1,m} \\ P_{n0} & P_{n1} & P_{n2} & \dots & P_{n,m-1} & P_{nm} \end{array}$$

Proposition 3.1.1 *([9]) Let $\overrightarrow{x}(u,v) = \sum_{k,l}^{n,m} B_k^n(u) B_l^m(v) P_{kl}$ be a biharmonic Bézier surface of degree n, m with control net $\{P_{kl}\}_{k,l=0}^{n,m}$. Then all the inner control points $\{P_{kl}\}_{k=1,l=1}^{n-1,m-1}$ are determined by the boundary control points, $\{P_{0l}\}_{l=0}^{m}$, $\{P_{nl}\}_{l=0}^{m}$, $\{P_{k0}\}_{k=0}^{n}$ and $\{P_{kn}\}_{k=0}^{n}$.*

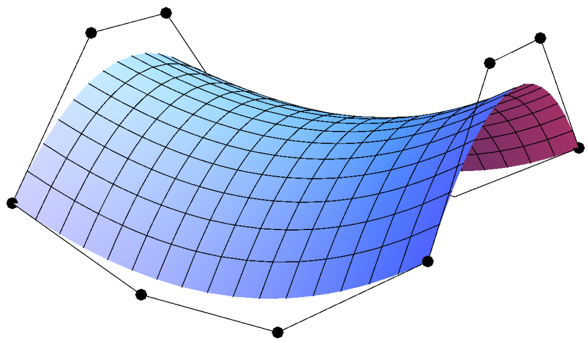


Figure 3.1: Biharmonic surface generated by the boundary control points

In terms of control points, the biharmonic condition of a polynomial surface is a linear system.

Theorem 3.1.2 *([9]) Given a control net in $\mathbb{R}^3$, $\{P_{ij}\}_{i,j=0}^{n,m}$, the associated Bézier surface, $\overrightarrow{x} : [0,1] \times [0,1] \to \mathbb{R}^3$, is biharmonic, i.e, $\Delta^2 \overrightarrow{x} = 0$ if and only if for any $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$*

$$\sum_{k=0}^{4} b_{n,i-k,k} \Delta^{4,0} P_{i-k,j} + 2 \sum_{k,l=0}^{2} a_{n,i-k,k} a_{m,j-l,l} \Delta^{2,2} P_{i-k,j-l} + \sum_{l=0}^{4} b_{m,j-l,l} \Delta^{0,4} P_{i,j-l} = 0, \tag{3.1.1}$$

where, for $i \in \{0, ..., n-2\}$

$$a_{ni0} = (n-i)(n-i-1),$$

$$a_{ni1} = 2(i+1)(n-i-1),$$

$$a_{ni2} = (i+1)(i+2),$$

and $a_{nik} = 0$ *otherwise, and for* $i \in \{0, \ldots, n-4\}$

$$\begin{aligned}
b_{ni0} &= (n-i)(n-i-1)(n-i-2)(n-i-3),\\
b_{ni1} &= 4(i+1)(n-i-1)(n-i-2)(n-i-3),\\
b_{ni2} &= 6(i+1)(i+2)(n-i-2)(n-i-3),\\
b_{ni3} &= 4(i+1)(i+2)(i+3)(n-i-3),\\
b_{ni4} &= (i+1)(i+2)(i+3)(i+4),
\end{aligned}$$

and $b_{nik} = 0$ *otherwise.*

Note that the first case where the biharmonic equation makes sense is for $n = m = 3$. In this case, the solution of equation 3.1.1 is

$$\begin{aligned}
P_{11} &= \frac{1}{9}(-4P_{00} + 6P_{01} - 2P_{03} + 6P_{10} + 3P_{13} - 2P_{30} + 3P_{31} - P_{33}),\\
P_{12} &= \frac{1}{9}(-2P_{00} + 6P_{02} - 4P_{03} + 3P_{10} + 6P_{13} - P_{30} + 3P_{32} - 2P_{33}),\\
P_{21} &= \frac{1}{9}(-2P_{00} + 3P_{01} - P_{03} + 6P_{20} + 3P_{23} - 4P_{30} + 6P_{31} - 2P_{33}),\\
P_{22} &= \frac{1}{9}(-P_{00} + 3P_{02} - 2P_{03} + 3P_{20} + 6P_{23} - 2P_{30} + 6P_{32} - 4P_{33}).
\end{aligned}$$

We can translate the biharmonic condition into a system of linear equations of the form

$$a_{k+4,l} + 2a_{k+2,l+2} + a_{k,l+4} = 0, \qquad k = 0, \ldots, m, \quad l = 0, \ldots, n, \tag{3.1.2}$$

with $a_{k,l} = 0$ if $k > m$ or $l > n$. For $k, \ell > 1$, we obtain that:

$$a_{k,\ell} := \frac{1}{\left[\frac{k}{2}\right] + \left[\frac{\ell}{2}\right]} \left((-1)^{\left[\frac{\ell}{2}\right]} \left[\frac{k}{2}\right] a_{k+2\left[\frac{\ell}{2}\right], \ell \bmod 2} + (-1)^{\left[\frac{k}{2}\right]} \left[\frac{\ell}{2}\right] a_{k \bmod 2, 2\left[\frac{k}{2}\right]+\ell} \right).$$

Let us see now the necessary conditions for the existence of biharmonic surfaces. Let $\overrightarrow{x}(u,v) : [0,1] \times [0,1] \to \mathbb{R}^3$ be the Bézier surface in the usual polynomial basis:

$$\overrightarrow{x}(u,v) = \sum_{i,j=0}^{n} \frac{a_{i,j}}{i!j!} u^i v^j$$

with boundary curves:

$$\overrightarrow{x}(0,v) = \sum_{j=0}^{n} p_j v^j, \qquad \overrightarrow{x}(1,v) = \sum_{j=0}^{n} q_j v^j,$$

$$\vec{x}(u,0) = \sum_{i=0}^{m} r_i u^i, \qquad \text{and} \qquad \vec{x}(u,1) = \sum_{i=0}^{m} s_i u^i,$$

with given coefficients p_j, q_j, r_i, s_i. Without loss of generality we shall assume that the degrees satisfy $n \leq m$.

The given coefficients of the boundary curves have to satisfy

$$\vec{x}(0,0) = p_0 = r_0, \qquad \vec{x}(1,0) = q_0 = \sum_{i=0}^{m} r_i,$$

$$\vec{x}(0,1) = \sum_{j=0}^{n} p_j = s_0, \qquad \text{and} \qquad \vec{x}(1,1) = \sum_{j=0}^{n} q_j = \sum_{i=0}^{m} s_i.$$

The boundary curves $\vec{x}(0,v)$ and $\vec{x}(u,0)$ determine the coefficients

$$p_j = a_{0,j}, \qquad j = 0,\ldots,n, \quad \text{and} \quad r_i = a_{i,0}, \qquad i = 0,\ldots,m. \tag{3.1.3}$$

The remaining two boundary curves $\vec{x}(1,v)$ and $\vec{x}(u,1)$ determine sums of the coefficients $a_{i,j}$,

$$q_j = \sum_{i=0}^{m} \frac{a_{i,j}}{i!j!}, \quad j = 0,\ldots,n, \quad \text{and} \quad s_i = \sum_{j=0}^{n} \frac{a_{i,j}}{i!j!}, \quad i = 0,\ldots,m. \tag{3.1.4}$$

In order to analyze the resulting conditions for the boundary curves we need to distinguish between several cases, (see [7]).

Case 1: n is even

a) If $m = n$ all coefficients $a_{i,j}$ with $i + j \geq n + 2$ vanish. This does not imply any conditions for the given boundary curves.

b) If $m = n + 1$ all coefficients with $2\left[\frac{i}{2}\right] + j \geq n + 2$ vanish. This does not imply any conditions for the given boundary curves.

c) If $m = n + 2$ all coefficients $a_{i,j}$ with $i + j \geq n + 3$ vanish. In this case, by 3.1.3 and 3.1.4 a biharmonic patch exists only if the given boundaries satisfy $r_{n+2} = s_{n+2}$ or equivalently,

$$\frac{\partial^{n+2}}{\partial u^{n+2}} \vec{x}(u,v)|_{(0,0)} = \frac{\partial^{n+2}}{\partial u^{n+2}} \vec{x}(u,v)|_{(0,1)}.$$

d) If $m \geq n+3$, all coefficients $a_{i,j}$ with $2\left[\frac{i}{2}\right]+j \geq n+3$ vanish. Again by 3.1.3 and 3.1.4 a biharmonic patch exists only if the given boundaries satisfy $r_{n+3} = s_{n+3}$ or equivalently,

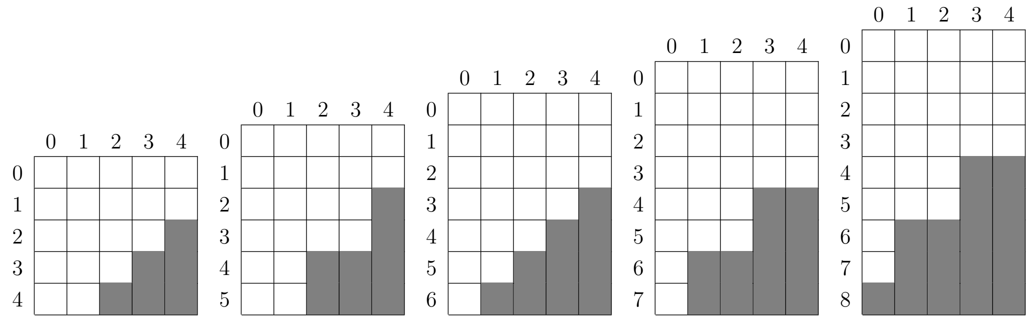
$$\frac{\partial^{n+3}}{\partial u^{n+3}} \overrightarrow{x}(u,v)|_{(0,0)} = \frac{\partial^{n+3}}{\partial u^{n+3}} \overrightarrow{x}(u,v)|_{(0,1)}.$$

Moreover, due to $a_{i,j} = 0$ for $i = n+4, \ldots, m$ and $j = 0, \ldots, n$, a biharmonic patch exists only if the given boundaries satisfy $r_i = s_i = 0$ or, equivalently,

$$\frac{\partial^{i}}{\partial u^{i}} \overrightarrow{x}(u,v)|_{(0,0)} = \frac{\partial^{i}}{\partial u^{i}} \overrightarrow{x}(u,v)|_{(0,1)} = 0, \tag{3.1.5}$$

for $i = n+4, \ldots, m$ and $m > n+3$.

The following image shows the matrices of coefficients $a_{i,j}$ for $n = 4$. The gray boxes correspond to coefficients which vanishes.



$m = 4$ Subcase 1.1 | $m = 5$ Subcase 1.2 | $m = 6$ Subcase 1.3 | $m = 7$ Subcase 1.4 | $m = 8$ Subcase 1.4

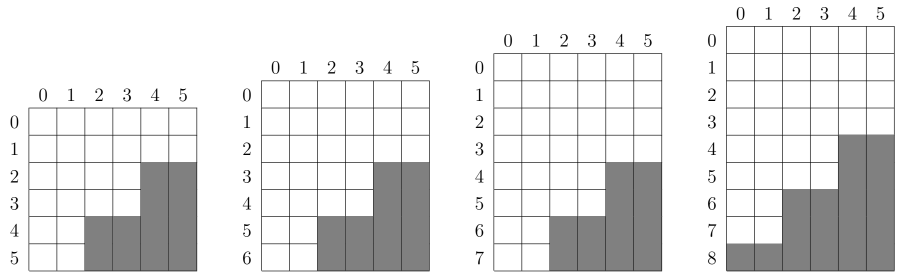
Case 2: n is odd

a) If $m = n$ all coefficients $a_{i,j}$ with $2\left[\frac{i}{2}\right]+j \geq n+1$ vanish. This does not imply any conditions for the given boundary curves.

b) If $m = n+1$ all coefficients $a_{i,j}$ with $2\left[\frac{i}{2}\right]+j \geq n+3$ vanish. This does not imply any conditions for the given boundary curves.

c) If $m \geq n+2$ all coefficients $a_{i,j}$ with $2\left[\frac{i}{2}\right]+j \geq n+3$ vanish. Due to $a_{i,j} = 0$ for $i = n+3, \ldots, m$ and $j = 0, \ldots, n$, a biharmonic patch exists only if the given boundaries satisfy 3.1.5 for $i = n+3, \ldots, m$ and $m > n+2$.

The following image shows the matrices of coefficients $a_{i,j}$ for $n = 5$. The gray boxes correspond to coefficients which vanishes.



$m = 5$ Subcase 2.1 $m = 6$ Subcase 2.2 $m = 7$ Subcase 2.3 $m = 8$ Subcase 2.3

3.2 Biharmonic generating function problem

Until now, we have seen that if we wanted to solve the biharmonic condition, we had two options:

1. Solve the linear system in equation (3.1.2), i.e. compute the whole control net in terms of the given boundary control points.

2. Compute the unknown basis coefficients, $a_{k,l}$ by 3.1.2 and come back to the Bézier basis.

As we have seen before, an inner control point of a biharmonic Bézier surface can be expressed as a linear combination of the boundary control points, then it has sense to ask about the biharmonic generating function as in the harmonic case. Let us see the equations must be fulfilled in order to find the generating function. Notice that, until now this problem is unsolved.

By Proposition 3.1.1, we can write

$$P^n_{k,\ell} = \sum_{i=0}^{n} \lambda^n_{k,\ell,i}\ P^n_{0,i} + \sum_{i=0}^{n} \mu^n_{k,\ell,i}\ P^n_{n,i} + \sum_{i=0}^{n} \alpha^n_{k,\ell,i}\ P^n_{i,0} + \sum_{i=0}^{n} \beta^n_{k,\ell,i}\ P^n_{i,n}$$

Then we can write

$$\begin{aligned}
\overrightarrow{x}(u,v) &= \textstyle\sum_{k,\ell=0}^{n} B^n_k(u) B^n_\ell(v) P^n_{k,\ell} \\
&= \textstyle\sum_{k,\ell=0}^{n} B^n_k(u) B^n_\ell(v) \big(\sum_{i=0}^{n} \lambda^n_{k,\ell,i}\ P^n_{0,i} + \sum_{i=0}^{n} \mu^n_{k,\ell,i}\ P^n_{n,i} \\
&\quad + \textstyle\sum_{i=0}^{n} \alpha^n_{k,\ell,i}\ P^n_{i,0} + \sum_{i=0}^{n} \beta^n_{k,\ell,i}\ P^n_{i,n} \big) \\
&= \textstyle\sum_{i=0}^{n} \left(\sum_{k,\ell=0}^{n} B^n_k(u) B^n_\ell(v) \lambda^n_{k,\ell,i} \right)\ P^n_{0,i} \\
&\quad + \textstyle\sum_{i=0}^{n} \left(\sum_{k,\ell=0}^{n} B^n_k(u) B^n_\ell(v) \mu^n_{k,\ell,i} \right)\ P^n_{n,i} \\
&\quad + \textstyle\sum_{i=0}^{n} \left(\sum_{k,\ell=0}^{n} B^n_k(u) B^n_\ell(v) \alpha^n_{k,\ell,i} \right)\ P^n_{i,0} \\
&\quad + \textstyle\sum_{i=0}^{n} \left(\sum_{k,\ell=0}^{n} B^n_k(u) B^n_\ell(v) \beta^n_{k,\ell,i} \right)\ P^n_{i,n}
\end{aligned}$$

and define

$$\begin{aligned} f_i^n(u,v) &:= \textstyle\sum_{k,\ell=0}^n B_k^n(u)B_\ell^n(v)\lambda_{k,\ell,i}^n \\ g_i^n(u,v) &:= \textstyle\sum_{k,\ell=0}^n B_k^n(u)B_\ell^n(v)\mu_{k,\ell,i}^n \\ h_i^n(u,v) &:= \textstyle\sum_{k,\ell=0}^n B_k^n(u)B_\ell^n(v)\alpha_{k,\ell,i}^n \\ j_i^n(u,v) &:= \textstyle\sum_{k,\ell=0}^n B_k^n(u)B_\ell^n(v)\beta_{k,\ell,i}^n. \end{aligned}$$

Therefore,

$$\overrightarrow{x}(u,v) = \sum_{i=0}^n f_i^n(u,v)\ P_{0,i}^n + g_i^n(u,v)\ P_{n,i}^n + h_i^n(u,v)\ P_{i,0}^n + j_i^n(u,v)\ P_{i,n}^n.$$

Since we assume that $\overrightarrow{x}(u,v)$ is biharmonic, then f_i^n, g_i^n, h_i^n and j_i^n are biharmonic polynomials. As in the harmonic case $f_i^n(u,v) = g_i^n(1-u,v)$ and the same occurs with h_i^n and j_i^n. Then, we can write:

$$\begin{aligned} \overrightarrow{x}(u,v) &= \textstyle\sum_{i=0}^{n-1} f_i^n(u,v)P_{0,i}^n + f_i^n(1-u,v)P_{n,i}^n + f_i^n(v,u)P_{i,0}^n \\ &\quad + f_i^n(1-v,u)P_{i,n}^n + (f_0^n(u,v) + f_0^n(v,u))P_{0,0}^n \\ &\quad + (f_n^n(u,v) + f_0^n(1-v,u))P_{0,n}^n + (f_0^n(1-u,v) + f_n^n(v,u))P_{n,0}^n \\ &\quad + (f_n^n(1-u,v) + f_0^n(1-v,u))P_{n,n}^n. \end{aligned}$$

The previous symmetries imply that, in fact, to solve the problem it would be enough to compute f_i control points, $\lambda_{k,l,i}$

$$\begin{aligned} P_{k,\ell}^n = &\sum_{i=0}^n \lambda_{k,\ell,i}\ P_{0,i}^n + \sum_{i=0}^n \lambda_{n-k,n-\ell,n-i}\ P_{n,i}^n + \sum_{i=0}^n \lambda_{\ell,k,i}\ P_{i,0}^n \\ &+ \sum_{i=0}^n \lambda_{\ell,k,i}\ P_{i,n}^n. \end{aligned}$$

Now, let us determine what type of boundary conditions should satisfied the biharmonic function f_i^n. Since

$$\overrightarrow{x}(0,v) = \sum_{\ell=0}^n B_\ell^n(v)P_{0,\ell}^n = \sum_{i=0}^n f_i^n(0,v)P_{0,i}^n + f_0^n(v,0)P_{0,0}^n + f_0^n(1-v,0)P_{0,n}^n,$$

then

$$\begin{cases} f_i^n(0,v) & = B_i^n(v), \\ f_0^n(0,v) + f_0^n(v,0) & = B_0^n(v), \\ f_n^n(0,v) + f_0^n(1-v,0) & = B_n^n(v). \end{cases}$$

Then $f_i^n(0,v) = B_i^n(v)$. In the same way, but using

$$\overrightarrow{x}(1,v) = \sum_{\ell=0}^{n} B_\ell^n(v) P_{n,i}^n = \sum_{i=0}^{n} f_i^n(0,v) P_{n,i}^n + f_n^n(v,1) P_{n,0}^n + f_n^n(1-v,1) P_{n,n}^n,$$

we obtain

$$\begin{cases} f_0^n(0,v) + f_n^n(v,1) & = B_0^n(v), \\ f_n^n(0,v) + f_n^n(1-v,1) & = B_n^n(v). \end{cases}$$

Finally, by computing $\overrightarrow{x}(u,0)$ and $\overrightarrow{x}(u,1)$ we obtain the conditions:

$$\begin{cases} f_i^n(0,u) & = B_i^n(u), \\ f_0^n(0,u) + f_0^n(u,0) & = B_0^n(u), \\ f_n^n(0,u) + f_0^n(1-u,0) & = B_n^n(u) \end{cases}$$

and

$$\begin{cases} f_0^n(0,u) + f_n^n(u,1) & = B_0^n(u), \\ f_n^n(0,u) + f_n^n(1-u,1) & = B_n^n(u). \end{cases}$$

From these conditions we obtain that $f_i^n(1,v) = f_i^n(u,0) = f_i^n(u,1) = 0$.

As in the harmonic case, to determine the generating function for the sequence of polynomials $\{f_i^n\}_{n=0}^\infty$,

$$f_i(u,v,t) = \sum_{n=0}^{\infty} \frac{f_i^n(u,v)}{n!} t^n$$

it would be sufficient to find the initial condition $f_0(u,v,t)$ because Δ^2 commutes with

$$D_i = \frac{1}{i+1}\left(-t\frac{\partial}{\partial t} + (i+t)\mathrm{Id}\right),$$

where $f_i(u,v,t) = D_{i-1}(f_{i-1}(u,v,t))$. In this case, we have that

$$f_i(0,v,t) = \sum_{n=0}^{\infty} \frac{f_i^n(0,v)}{n!} t^n = \sum_{n=0}^{\infty} \frac{B_i^n(v)}{n!} t^n = \frac{(vt)^i}{i!} e^{(1-v)t}.$$

For $i = 0$, we look for a biharmonic function

$$f_0(u,v,t) = \sum_{n=0}^{\infty} f_0^n(u,v) \frac{t^n}{n!},$$

where $f_0(0,v,t)$ satisfies the previous condition. However, by the previous conditions it has not jet possible to determine $f_0(u,v,t)$ and therefore until now the problem of finding the generating function for biharmonic surfaces is unsolved.

3.3 B-spline biharmonic surfaces

As we have seen in the previous chapter, B-spline harmonic surfaces had no many interesting because we always obtain a Bézier surface. Unlike the harmonic case, biharmonic B-spline surfaces are not Bézier surfaces. In [5] the authors study biquadratic B-splines for biharmonic surfaces. Let us see how they obtain the inner controls point as a linear combination of the boundary control points.

Given the knots $U = (u_0, u_1, \ldots, u_{n+3})$ and $V = (v_0, v_1, \ldots, v_{m+3})$ in which $u_i \leq u_{i+1}$, $v_j \leq v_{j+1}$, for U let

$$h_i = u_{i+1} - u_i, \qquad \alpha_i = \frac{h_i}{h_{i-1} + h_i}, \qquad \beta_i = \frac{h_i}{h_i + h_{i+1}}.$$

Note that $\alpha_{i+1} + \beta_i = 1$. Thus they introduce B-spline basis function as

$$B_i(u) = \begin{cases} b_{i,2}(t_i(u)), & u \in [u_i, u_{i+1}), \\ b_{i+1,1}(t_{i+1}(u)), & u \in [u_{i+1}, u_{i+2}), \\ b_{i+2,0}(t_{i+2}(u)), & u \in [u_{i+2}, u_{i+3}), \\ 0 & u \notin [u_i, u_{i+3}), \end{cases}$$

where

$$t_i(u) = \frac{u - u_i}{h_i}$$

and

$$\begin{aligned} b_{i,0}(t) =& \alpha_i (1-t)^2, \\ b_{i,1}(t) =& \beta_{i-1}(1-t)^2 + 2(1-t)t + \alpha_{i+1} t^2, \\ b_{i,2}(t) =& \beta_i t^2. \end{aligned}$$

For knot V, denote similar notations with a bar on the top.

Given the control points $\mathcal{P} = \{P_{i,j}\}_{i,j=0}^{n,m}$, then the biquadratic B-spline surface is

$$\overrightarrow{x}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_i(v) \overline{B}_j(v) P_{i,j},$$

where $u \in [u_2, u_{n+1}]$ and $v \in [v_2, v_{m+1}]$.

As $\overrightarrow{x}(u,v)$ is biquadratic, then biharmonic condition is equivalent to the following,

$$\Delta^2 \overrightarrow{x}(u,v) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right)^2 \overrightarrow{x}(u,v) = \frac{\partial^4 \overrightarrow{x}(u,v)}{\partial u^2 \partial v^2} = 0.$$

The biharmonic condition only relates to second order derivatives along u and v direction respectively. Note that $b''_{i,k}(t)$, $k = 0, 1, 2$ is constant,

$$\begin{aligned} b''_{i,0}(t) =& 2\alpha_i, \\ b''_{i,1}(t) =& 2\beta_{i-1} + 2\alpha_{i+1} - 4, \\ b''_{i,2}(t) =& 2\beta_i. \end{aligned}$$

If we denote

$$c_{i,k} = \frac{1}{2} b''_{i,k}(t), \quad \overline{c}_{j,k} = \frac{1}{2} \overline{b}''_{j,k}(t),$$

then

$$\begin{aligned} c_{i,0} =& \alpha_i, & \overline{c}_{j,0} =& \overline{\alpha}_j, \\ c_{i,1} =& -(\alpha_i + \beta_i), & \overline{c}_{j,1} =& -(\overline{\alpha}_j + \overline{\beta}_j), \\ c_{i,2} =& \beta_i, & \overline{c}_{j,2} =& \overline{\beta}_j. \end{aligned}$$

Thereupon

$$B''_i(u) = \begin{cases} \frac{2}{h_i^2} c_{i,2}, & u \in [u_i, u_{i+1}), \\ \frac{2}{h_{i+1}^2} c_{i+1,1}, & u \in [u_{i+1}, u_{i+2}), \\ \frac{2}{h_{i+2}^2} c_{i+2,0}, & u \in [u_{i+2}, u_{i+3}), \\ 0 & u \notin [u_i, u_{i+3}). \end{cases}$$

Knots U and V divide the surface $\overrightarrow{x}(u, v)$ into nm patches and $\overrightarrow{x}(u, v)$ has $\mathcal{C}^1$ continuity between the patches because of the continuity of biquadratic B-spline basis function. For a patch on $[u_i, u_{i+1}] \times [v_j, v_{j+1}]$, applying biharmonic condition we have

$$\frac{\partial^2 \overrightarrow{x}(u, v)}{\partial u^2 \partial v^2} = \sum_{k=i-2}^{i} \sum_{l=j-2}^{j} B''_k \overline{B}''_l P_{k,l} = 0,$$

hence we can obtain the inner control points as following

$$\begin{aligned} P_{i-1,j-1} =& -\frac{1}{c_{i,1}\overline{c}_{j,1}} \left(c_{i,0}\overline{c}_{j,0} P_{i,j} + c_{i,0}\overline{c}_{j,1} P_{i,j-1} \right. \\ & + c_{i,0}\overline{c}_{j,2} P_{i,j-2} + c_{i,1}\overline{c}_{j,0} P_{i-1,j} \\ & + c_{i,1}\overline{c}_{j,2} P_{i-1,j-2} + c_{i,2}\overline{c}_{j,0} P_{i-2,j} \\ & \left. + c_{i,2}\overline{c}_{j,1} P_{i-2,j-1} + c_{i,2}\overline{c}_{j,2} P_{i-2,j-2} \right). \end{aligned}$$

Let us see some examples for the cuadratic and the bicubic case. For the knot vectors $U = (0,0,0,1,2,2,2)$, $V = (0,0,0,1,2,2,2)$ and boundary control points

$$\begin{array}{cccc} (-15,-2,15) & (-5,5,15) & (5,5,15) & (15,-2,15) \\ (-15,5,5) & * & * & (15,5,5) \\ (-15,5,-5) & * & * & (15,5,-5) \\ (-15,-2,-15) & (-5,5,-15) & (5,5,-15) & (15,-2,-15) \end{array}$$

we obtain that the inner control points are given by $P_{1,1} = (-5,12,5)$, $P_{1,2} = (5,12,5)$, $P_{2,1} = (-5,12,-5)$ and $P_{2,2} = (5,12,-5)$.

The biharmonic surface using bicuadratic B-splines is given by:

$$\vec{x}(u,v) = \begin{cases} A(u,v), & 0 \le u < 1 \text{ and } 0 \le v < 1, \\ B(u,v), & 1 \le u < 2 \text{ and } 0 \le v < 1, \\ C(u,v), & 0 \le u < 1 \text{ and } 1 \le v < 2, \\ D(u,v), & 1 \le u < 2 \text{ and } 1 \le v < 2, \end{cases}$$

where

$$\begin{aligned} A(u,v) =& (-5(3-4v+v^2), -2+14u-7u^2+14v-7v^2, 5(3-4u+u^2)), \\ B(u,v) =& (-5(3-4v+v^2), -2+14u-7u^2+14v-7v^2, 5-5u^2), \\ C(u,v) =& (5(-1+v^2), -2+14u-7u^2+14v-7v^2, 5(3-4u+u^2)), \\ D(u,v) =& (5(-1+v^2), -2+14u-7u^2+14v-7v^2, 5-5u^2). \end{aligned}$$

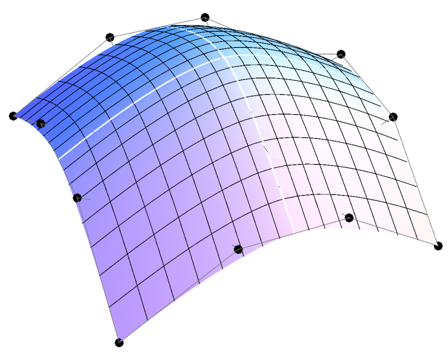


Figure 3.2: Biharmonic surface using bicuadratic B-splines

Notice that, as we have said before, the resulting surface is not a Bézier surface. In this example, the lines $\vec{x}(u,0)$ and $\vec{x}(u,1)$ join patches with $\mathcal{C}^1$ continuity. The same occurs with $\vec{x}(0,v)$ and $\vec{x}(1,v)$.

Let us see an example for the bicubic case. For the knot vectors $U = (0,0,0,0,1,2,2,2,2)$, $V = (0,0,0,0,1,2,2,2,2)$ and boundary control points

$$\begin{array}{ccccc}
(0,0,0) & (1,0,0.5) & (2,0,0) & (3,0,0.5) & (4,0,0) \\
(0,1,0.5) & * & * & * & (4,1,0.5) \\
(0,2,0) & * & * & * & (4,2,0) \\
(0,3,0.5) & * & * & * & (4,3,0.5) \\
(0,4,0) & (1,4,0.5) & (2,4,0) & (3,4,0.5) & (4,4,0)
\end{array}$$

we obtain

$$\begin{array}{ccc}
P_{1,1} = (1,1,1), & P_{1,2} = (2,1,0.5), & P_{1,3} = (3,1,1), \\
P_{2,1} = (1,2,0.5), & P_{2,2} = (2,2,0), & P_{2,3} = (3,2,0.5), \\
P_{3,1} = (1,3,1), & P_{3,2} = (2,3,0.5), & P_{3,3} = (3,3,1).
\end{array}$$

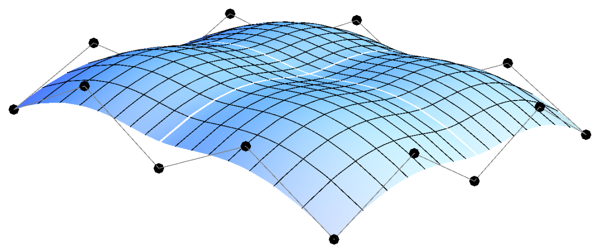


Figure 3.3: Biharmonic surface using bicubic B-splines

As in the previous example, the resulting surface is not a Bézier surface. In this case, the lines $\overrightarrow{x}(u,0)$ and $\overrightarrow{x}(u,1)$ join patches with $\mathcal{C}^2$ continuity. The same occurs with $\overrightarrow{x}(0,v)$ and $\overrightarrow{x}(1,v)$.

Chapter 4

The Dirichlet approach to the Plateau-Bézier problem

When trying to solve the Plateau problem, one has to minimize the area functional but this functional is highly nonlinear. This is one of the reasons that left the Plateau problem unsolved for more than a century. It was in 1931 when Douglas obtained the solution minimizing the Dirichlet functional instead of the area functional. This was easier to manage and has it one important property: both functionals have the same extremals for isothermal surfaces.

In this chapter, we shall compute the extremal of the Dirichlet functional for the Plateau-Bézier problem which gives an approximation to the extremal of the area functional.

There are other methods to find approximations to the solutions of the Plateau–Bézier problem. For example, the use of masks that we shall study at the end of the chapter.

4.1 The Dirichlet functional

Let $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{m,n}$ be the control net of the Bézier surface:

$$\overrightarrow{x}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} B_i^m(u)B_j^n(v)P_{ij}.$$

The area of the Bézier surface is given by:

$$A(\mathcal{P}) = \int_R ||\overrightarrow{x}_u \wedge \overrightarrow{x}_v||dudv = \int_R (EG - F^2)^{\frac{1}{2}} dudv,$$

where $R = [0,1] \times [0,1]$ and E, F, G are the coefficients of the first fundamental form of $\overrightarrow{x}$ given by $E =< \overrightarrow{x}_u, \overrightarrow{x}_u >$, $F =< \overrightarrow{x}_u, \overrightarrow{x}_v >$ and $G =< \overrightarrow{x}_v, \overrightarrow{x}_v >$.

As we have pointed out before, the area functional is highly nonlinear, so if we want to find the minimal surface associated to a given boundary, we shall start by studying instead the Dirichlet functional which is given by:

$$D(\mathcal{P}) = \frac{1}{2}\int_R (||\overrightarrow{x}_u||^2 + ||\overrightarrow{x}_v||^2)dudv = \int_R \frac{E+G}{2} dudv.$$

Let us recall that:

$$(EG - F^2)^{\frac{1}{2}} \leq (EG)^{\frac{1}{2}} \leq \frac{E+G}{2}.$$

Then, for any control net, $\mathcal{P}$, $A(\mathcal{P}) \leq D(\mathcal{P})$. Notice that the equality in the previous expression is given under isothermal conditions, i.e. for $E = G$ and $F = 0$. Also, both functionals have a minimum in the Bézier case (see [8]). In fact, both functionals have the same extremal in the isothermal case. But this main property is no longer true in general, what we shall obtain instead is that the Dirichlet extremals are an approximation to the extremals of the area functional, i.e., the resulting Bézier surface does not minimize area, but its area is close to the minimum.

4.1.1 Relation with harmonic patches

The Dirichlet functional can be defined for just Bézier (or polynomial) patches, $\overrightarrow{x}^{\mathcal{P}} : [0,1] \times [0,1] \to \mathbb{R}^3$, being $\mathcal{P}$ the associated control net. Or it could also be considered in the unrestricted case, i.e. for arbitrary patches, $\overrightarrow{x}$.

In the unrestricted case, the extremals of the Dirichlet functional are given by differentiable patches verifying its Euler-Lagrange equation, $\Delta\overrightarrow{x} = 0$, i.e. by harmonic patches. In general, we have that:

$$D(\overrightarrow{x}^{ext}) \leq D(\overrightarrow{x}^{\mathcal{P}^{ext}}) = D(\mathcal{P}^{ext}),$$

where $\overrightarrow{x}^{ext}$ is the extremal of the Dirichlet functional in the unrestricted case, $\mathcal{P}^{ext}$ is the control net extremal of the Dirichlet functional in the restricted polynomial case and $\overrightarrow{x}^{\mathcal{P}^{ext}}$ is its associated Bézier patch. Therefore if a polynomial patch is harmonic, then it is an extremal of the Dirichlet functional both in the unrestricted and the restricted case.

In addition, let us remark that when the boundary conditions are polynomial curves, the Dirichlet extremal for the unrestricted case is not necessarily a polynomial in general.

Theorem 4.1.1 *([8]) Let $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{n,m}$ be the control net of a Bézier surface. If the associated Bézier patch $\overrightarrow{x}$ is harmonic, then it is an extremal of the Dirichlet functional from among all the Bézier patches with the same boundary.*

Obviously, the converse, in general is not true, not all extremal patches of the Dirichlet functional in the restricted case are harmonic patches i.e. a polynomial extremal of the Dirichlet functional is not harmonic in general.

In Chapter 2, we saw the conditions that a control net must satisfy for the associated Bézier surface in order to be harmonic. We saw that for n odd, the inner rows are determined by the control points in the first and las rows. For example, if $n = m = 3$, we obtained

$$\begin{array}{cccc} P_{00} & P_{01} & P_{02} & P_{03} \\ * & * & * & * \\ * & * & * & * \\ P_{30} & P_{31} & P_{32} & P_{33} \end{array}$$

where,

$$\begin{aligned} P_{10} &= \frac{1}{3}(4P_{00} - 4P_{01} + 2P_{02} + 2P_{30} - 2P_{31} + P_{32}), \\ P_{20} &= \frac{1}{3}(2P_{00} - 2P_{01} + P_{02} + 4P_{30} - 4P_{31} + 2P_{32}), \\ P_{13} &= \frac{1}{3}(2P_{01} - 4P_{02} + 4P_{03} + P_{31} - 2P_{32} + 2P_{33}), \\ P_{23} &= \frac{1}{3}(P_{01} - 2P_{02} + 2P_{03} + 2P_{31} - 4P_{32} + 4P_{33}). \end{aligned}$$

Then, only those configurations of the boundary control points that verify such relations can produce extremals of the Dirichlet functional of the restricted case which are harmonic. The same occurs when n is even.

4.1.2 Extremals of the Dirichlet functional

Let us see a proposition that gives the condition a control net, $\mathcal{P}$, must satisfy in order to be an extremal of the Dirichlet functional. Notice that we will simply compute the points where the gradient of a real function defined on $\mathbb{R}^{3(n-1)(m-1)}$ vanishes. In other words, what we are studying are the critical points of the function, $\mathcal{P} \to \mathcal{D}(\overrightarrow{x}^{\mathcal{P}})$, where $\overrightarrow{x}^{\mathcal{P}}$ denotes the Bézier patch associated to the control net $\mathcal{P}$.

Proposition 4.1.2 *([8]) A control net, $\mathcal{P} = \{P_{i,j}\}_{i,j=0}^{n,m}$, is an extremal of the Dirichlet functional with prescribed border if and only if*

$$\begin{aligned}0 =&\frac{n^2}{(2m+1)(2n-1)}\binom{n-1}{i}\binom{m}{j}\sum_{k,l=0}^{n-1,m} A_{ni}^k \frac{\binom{m}{l}}{\binom{2m}{j+l}}\Delta^{1,0}P_{kl}\\ &+\frac{m^2}{(2m-1)(2n+1)}\binom{n}{i}\binom{m-1}{j}\sum_{k,l=0}^{n,m-1}\frac{\binom{n}{k}}{\binom{2n}{i+k}}A_{mj}^l\Delta^{0,1}P_{kl},\end{aligned}$$

for any $i \in \{1,\dots,n-1\}$ and $j \in \{1,\dots,m-1\}$ where A_{ni}^k is defined by

$$A_{ni}^k = \frac{ni-nk-i}{(n-i)(2n-1-i-k)}\frac{\binom{n-1}{k}}{\binom{2n-2}{i+k-1}}.$$

Corollary 4.1.3 *([8]) A squared control net, $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{n,n}$, is an extremal of the Dirichlet functional with prescribed border if and only if*

$$0 = \sum_{k,l=0}^{n-1,n}\frac{\binom{n}{l}}{\binom{2n}{j+l}}C_{ni}^k\Delta^{10}P_{kl} + \sum_{k,l=0}^{n,n-1}\frac{\binom{n}{k}}{\binom{2n}{i+k}}C_{mj}^l\Delta^{01}P_{kl},$$

for any $i,j \in \{1,\dots,n-1\}$, where $C_{ni}^k = \frac{(n-1)i-nk}{i+k}\frac{\binom{n-1}{k}}{\binom{2n-2}{i+k}}$.

Let us see some particular cases. For example, for $n = m = 2$ there is just one equation corresponding to the inner control point P_{11}.

Proposition 4.1.4 *([8]) A biquadratic Bézier surface is an extremal of the Dirichlet functional with prescribed border if and only if*

$$P_{11} = \frac{1}{8}(3P_{00} - P_{01} + 3P_{02} - P_{10} - P_{12} + 3P_{20} - P_{21} + 3P_{22}).$$

In the following image we have taken:

$$\begin{array}{ccc} P_{00} & P_{01} & P_{02}\\ P_{10} & P_{11} & P_{12}\\ P_{20} & P_{21} & P_{22}\end{array} \qquad \begin{array}{ccc} (0,0,0) & (1,0,1) & (2,0,0)\\ (0,1,1) & P_{11} & (2,1,1)\\ (0,2,0) & (1,2,1) & (2,2,0)\end{array}$$

In this case $P_{11} = (1,1,-1/2)$.

If $n = m = 3$, there are four equations corresponding to the inner control points $P_{11}, P_{12}, P_{21}, P_{22}$.

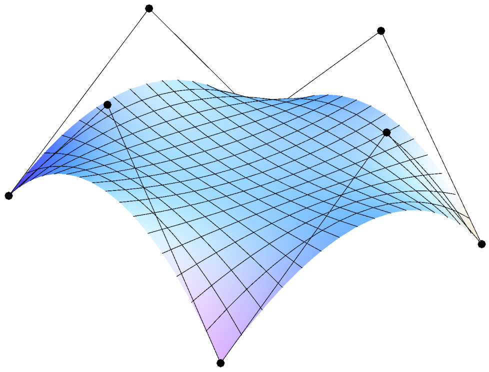


Figure 4.1: Extremal biquadratic Bézier surface of the Dirichlet functional

Proposition 4.1.5 *([8]) A bicubic Bézier surface is an extremal of the Dirichlet functional with prescribed border if and only if*

$$\begin{aligned}
P_{11} =& \frac{1}{78}(48P_{00} - 22P_{01} + 24P_{02} - 22P_{10} + 15P_{13} + 24P_{20} - 4P_{23} \\
&+ 15P_{31} - 4P_{32} + 4P_{33}), \\
P_{12} =& \frac{1}{78}(24P_{01} - 22P_{02} + 48P_{03} + 15P_{10} - 22P_{13} - 4P_{20} + 24P_{23} \\
&+ 4P_{30} - 4P_{31} + 15P_{32}), \\
P_{21} =& \frac{1}{78}(15P_{01} - 4P_{02} + 4P_{03} + 24P_{10} - 4P_{13} - 22P_{20} + 15P_{23} \\
&+ 48P_{30} - 22P_{31} + 24P_{32}), \\
P_{22} =& \frac{1}{78}(4P_{00} - 4P_{01} + 15P_{02} - 4P_{10} + 24P_{13} + 15P_{20} - 22P_{23} \\
&+ 24P_{31} - 22P_{32} + 48P_{33}).
\end{aligned}$$

In the following image we have taken:

$$\begin{array}{cccc} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{array} \qquad \begin{array}{cccc} (0,0,0) & (1,0,-0.5) & (2,0,-0.6) & (3,0,0) \\ (0,1,1) & P_{11} & P_{12} & (3,1,1) \\ (0,2,1) & P_{21} & P_{22} & (3,2,1) \\ (0,3,0) & (1,3,-0.5) & (2,3,-0.6) & (3,3,0) \end{array}$$

In this case $P_{11} = (1,1,0.057), P_{12} = (2,1,0.09), P_{21} = (1,2,0.05)$ and $P_{22} = (2,2,0.09)$.

Let us recall that, as we have said before, a minimum of the Dirichlet functional with prescribed border always exists. In fact, it is possible to prove the uniqueness.

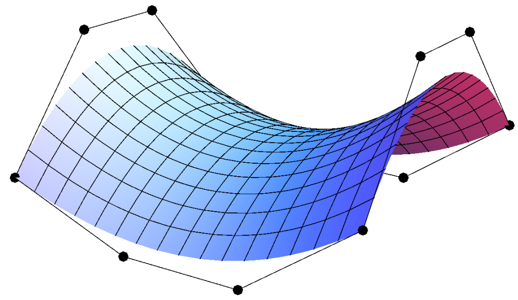


Figure 4.2: Extremal bicubic Bézier surface of the Dirichlet functional

Theorem 4.1.6 *([10]) The Dirichlet extremal is unique.*

Finally, the following theorem gives a method to reach the minimal area with prescribed boundary by a sequence of Bézier surfaces which are Dirichlet extremals.

Theorem 4.1.7 *([10]) Let* $\overrightarrow{x} : [0,1] \times [0,1] \to \mathbb{R}^3$ *be an isothermal chart of a surface of minimal area among all surfaces with the same boundary. Let* $\overrightarrow{y}_n$ *be the Dirichlet extremal of degree* n *with boundary defined by the exterior control points of the control net* $\mathcal{P}_n = \{\overrightarrow{x}(\frac{i}{n}, \frac{j}{n})\}_{i,j=0}^{n}$. *Then,*

$$\lim_{n\to\infty} \mathcal{A}(\overrightarrow{y}_n) = \mathcal{A}(\overrightarrow{x}).$$

4.1.3 The biharmonic functional

In a similar fashion to the harmonic functional or Dirichlet functional, the biharmonic functional is defined as

$$\mathcal{B}(\overrightarrow{x}) = \frac{1}{2}\int_R (\|\overrightarrow{x}_{uu}\|^2 + 2\|\overrightarrow{x}_{uv}\|^2 + \|\overrightarrow{x}_{vv}\|^2)dudv,$$

where $= [0,1] \times [0,1]$.

As the extremal of the Dirichlet functional in 4.1.2, the extremals of the biharmonic functional can also easily be computed. Again, as with the harmonic case what we are studying are the critical points of the function $\mathcal{P} \to \mathcal{B}(\overrightarrow{x}^{\mathcal{P}})$, where $\overrightarrow{x}^{\mathcal{P}}$ denotes the Bézier patch associated with the control net $\mathcal{P}$.

To present our comparative study here we discuss several examples of biharmonic Bézier surfaces. For each example the results are presented in a tabulated form.

As a first example we take the catenoid which is parameterized by $\overrightarrow{x}(u, v) = (\cosh(v)\cos(u), \cosh(u)\sin(u), v), u \in [0, \pi], v \in [0, argcosh(2)]$, and it is a minimal surface. In this example, the boundary curves are degree 5 Bézier curves approximations to the boundary curves of the catenoid.

In this case, we obtain:

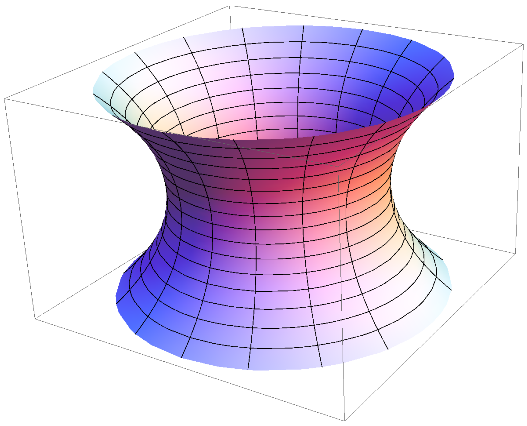


Figure 4.3: The catenoid surface

The following table compares different functionals for a Dirichlet extremal surface, biharmonic surface and biharmonic extremal surface.

Functional	Catenoid	Dirichet extremal	Bihar. surface	Bihar. extremal
Area	7.51007	8.31778	<u>7.77847</u>	7.79836
Dirichlet	7.51007	<u>9.18563</u>	9.82239	9.88235
Biharmonic	10.8828	113.941	89.2635	<u>86.1143</u>

Notice that the equality between the area and the value of the Dirichlet functional for the catenoid is a consequence of the fact that for isothermal parameterizations, the area and the harmonic functional agree. In this case the better approximation to the true area is the biharmonic surface.

As a second example we take the same boundary control points as in Figure 4.2 and we obtain the following values:

Functional	Dirichlet extremal	Bihar. surface	Bihar. extremal
Area	<u>10.6264</u>	10.7184	10.6466

As can be noted, in this example we found that the smaller area was obtained for the surface corresponding to the Dirichlet extremal.

We can see that there is no better choice. Depending on the boundary control points, i.e., the boundary curves, an approximation method is better than another.

4.2 The use of masks

Another way of building surfaces is by means of masks. A mask is a set of coefficients that define a control point of a Bézier surface in terms of its neighboring control points. Thus, the whole control net is obtained as a solution of a linear system. The use of masks has its origin in numerical methods to discretize and solve differential equations. One way of obtaining an approximated solution to a differential equation is by performing its finite difference discretization, and then the discrete solutions can be represented by masks. In [4], G. Farin and D. Hansford present a new class of control net generation schemes based on a special kind of masks that they call *permanence patches*. These masks have the following form

$$\begin{array}{ccc} \alpha & \beta & \alpha \\ \beta & \bullet & \beta \\ \alpha & \beta & \alpha \end{array}$$

where $\beta = 1/4 - \alpha$. We denote this mask by M_α. They are called permanence patches because the case $\alpha = -1/4$ gives the control net generation scheme used to generate Coons patches which satisfy the permanence principle (see [4]).

This mask implies that, in general, we can write:

$$P_{i,j} = \beta(P_{i-1,j}+P_{i,j-1}+P_{i,j+1}+P_{i+1,j})+\alpha(P_{i-1,j-1}+P_{i-1,j+1}+P_{i+1,j-1}+P_{i+1,j+1}).$$

For example, for $n = m = 2$:

$$\begin{array}{ccc} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{array}$$

the inner control point P_{11} is given by:

$$\begin{aligned} P_{11} =& \beta(P_{01} + P_{10} + P_{12} + P_{21}) + \alpha(P_{00} + P_{02} + P_{20} + P_{22}) \\ =& \alpha(P_{00} + P_{02} + P_{20} + P_{22} - (P_{01} + P_{10} + P_{12} + P_{21})) \\ &+ \frac{1}{4}(P_{01} + P_{10} + P_{12} + P_{21}). \end{aligned}$$

In this section we study different masks by applying different guiding principles also related with surfaces of minimal area.

1. The discrete Laplacian mask

It can be found in [4] that the mask M_0 is the discrete form of the Laplacian operator. Such a mask is used in the cited reference to obtain control nets resembling minimal surfaces that fit given boundary polygons.

For $\alpha = 0$ (and therefore $\beta = 1/4$), we obtain:

$$P_{ij} = \frac{1}{4}(P_{i+1,j} + P_{i-1,j} + P_{i,j+1} + P_{i,j-1}).$$

Note that $M_0(P_{ij})$ is the center of gravity of the four neighboring points of P_{ij}, which are not at the corners.

Should also be noted that what we would really obtain is an approximation of a *harmonic* control net, but not, in principle, an approximation of a harmonic Bézier patch.

2. The harmonic mask

Instead of discretizing the Laplacian operator, let us demand that, at least at one point, the Laplacian of the patch vanishes. So, we are not doing an approximation to a harmonic control net. What we are trying to do is to transfer the harmonic condition of the patch into a condition on the control net.

Proposition 4.2.1 *([8]) The Bézier patch $\overrightarrow{x}$, associated to a biquadratic control net, $\mathcal{P} = \{P_{i,j}\}_{i,j=0}^{2,2}$, verifies $\Delta\overrightarrow{x}(\frac{1}{2}, \frac{1}{2}) = 0$ if and only if*

$$P_{11} = M_{1/4}(P_{11}).$$

3. The Dirichlet mask

The third mask is given by the Dirichlet equations for $n = m = 2$.

Proposition 4.2.2 *([8]) A biquadratic control net, $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{2,2}$, is an extremal of the Dirichlet functional with prescribed border if and only if*

$$P_{11} = M_{3/8}(P_{11}).$$

The Dirichlet mask corresponds to the value $\alpha = 3/8$ and notice that coincides with the results obtained in Proposition 4.1.4 for $n = m = 2$.

The obvious question then is to determine which mask is the best, or even more generally, whether there is or not a better mask. The answer is negative. The highly nonlinearity of the area functional makes the dependence of the minimal surface from the boundary conditions highly nonlinear too. So, one cannot expect a mask, i.e., a linear expression, to be able to give a good approximation in all cases. This means that, depending on the boundary control points a mask is better than another.

Let us see some different examples depending on the grade of a Bézier surface. We start the comparison by studying some examples in the biquadratic case.

Example 4.2.3 *Taking the same boundary control points as in Figure 1.1. for a Bézier surface of degree n, the following figure shows an example of boundary conditions and the three Bézier surfaces obtained by the different masks. The resulting areas are*

Mask	*Area*
$\alpha = 0$	*4.5505*
$\alpha = 0.25$	*4.23649*
Dirichlet extr.	*4.19972*

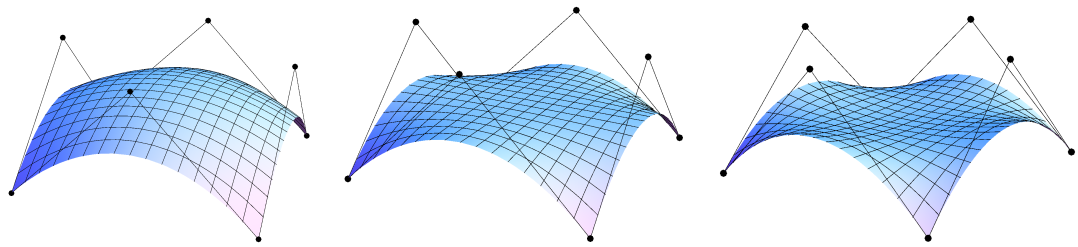


Figure 4.4: Left, the discretization of the Laplacian operator ($\alpha = 0$). Center, the harmonic mask ($\alpha = 0.25$). Right, the Dirichlet mask ($\alpha = 0.375$)

In this example the approximation given by the Dirichlet mask is better that the other two masks.

There is one interesting case. As we have seen before, we can write:

$$\begin{aligned} P_{11} = M_\alpha(P_{11}) =& \alpha(P_{00} + P_{02} + P_{20} + P_{22} - (P_{01} + P_{10} + P_{12} + P_{21})) \\ &+ \frac{1}{4}(P_{01} + P_{10} + P_{12} + P_{21}). \\ =& \alpha(M_{1/4}(P_{11}) - M_0(P_{11})) + \frac{1}{4}M_0(P_{11}). \end{aligned}$$

So, if the configuration of the boundary control points of a biquadratic control net is such that both centers of gravity are located at the same point, i.e. $M_{1/4}(P_{11}) = M_0(P_{11})$, then the central point P_{11} does not depend on α. Therefore, for such a configuration of the boundary, any mask will define the same Bézier surface.

Let us have a look at the behavior of the masks comparing with the Dirichlet extremal for rectangular Bézier surfaces. The next example corresponds to a Bézier surface of degree $n = 2$, $m = 3$.

Example 4.2.4 *In this example, for the first surface we have taken the same boundary control points as in Figure 1.2, for the second surface we have taken:*

$$\begin{array}{cccc} (0,0,2) & (1,0,1) & (2,0,1) & (3,0,-1) \\ (0,1,1) & P_{11} & P_{12} & (3,1.25,1) \\ (0,2,1) & P_{21} & P_{22} & (3,1.75,1) \\ (0,3,-1) & (1,3,1) & (2,3,1) & (3,3,2) \end{array}$$

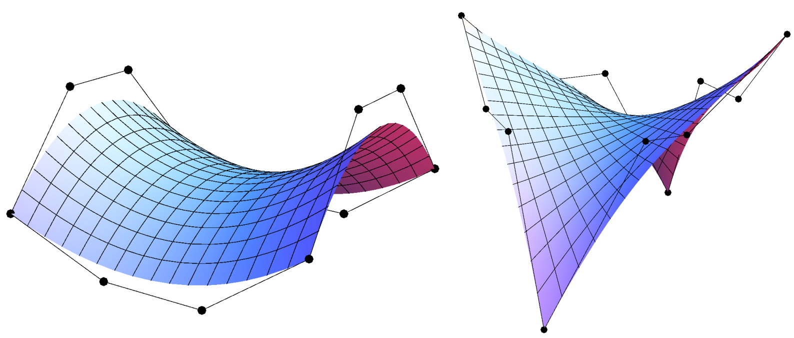


Figure 4.5: Different boundary conditions for $n = m = 3$ and the associated Bézier surface. The drawn surfaces are Dirichlet extremals

We obtain the following values for the areas:

Mask	*Left*	*Right*
$\alpha = 0$	*10.6425*	*11.2214*
$\alpha = 1/4$	*10.6310*	*11.1774*
$\alpha = 3/8$	*10.6289*	*11.2303*
Extremal Dirichlet	*10.6264*	*11.1891*

In this case we can see that in the left example the minimum of the area is provided by the extremal of Dirichlet functional. However, in the second example the mask $\alpha = 1/4$ is the best approximation.

As we have said before, there is no best choice, but the examples and theoretical arguments point out that when the first fundamental form, IFF, of the Bézier surface at the corners (at these points the IFF depends on just the boundary conditions) is close to being isothermal, i.e. $E = F$ and $G = 0$, then the Dirichlet extremal is a better approximation than the ones obtained by the use of masks.

In the opposite case, suppose that the first fundamental form of the Bézier surface at the corners is far from being isothermal. Then, since we are making a mistake starting, the results obtained by the use of a mask can be better than the result obtained by the Dirichlet extremal.

In [8] we can find different examples where, depending on the boundary conditions and for $n = m = 4$ the minimum of the area is provided by other masks.

Example 4.2.5 *In this example, for each of the images we have taken the boundary control points:*

$$\begin{array}{cccc} (0,0,0) & (1,0,1) & (2,0,1) & (3,0,0) \\ (0,1,0) & P_{11} & P_{12} & (3,1,0) \\ (0,2,0) & (1,2,1) & (2,2,1) & (3,2,0) \end{array} \qquad \begin{array}{cccc} (0,0,0) & (1,0,1) & (2,0,1) & (3,0,0) \\ (0,1,1) & P_{11} & P_{12} & (3,1,1) \\ (0,2,0) & (1,2,1) & (2,2,1) & (3,2,0) \end{array}$$

We have obtained the following values:

Mask	*Left*	*Right*
$\alpha = 0$	*6.68046*	*6.74866*
$\alpha = 1/4$	*6.42351*	*6.70932*
$\alpha = 3/8$	*6.38108*	*6.70057*
Dirichlet extr.	*6.37128*	*6.76700*

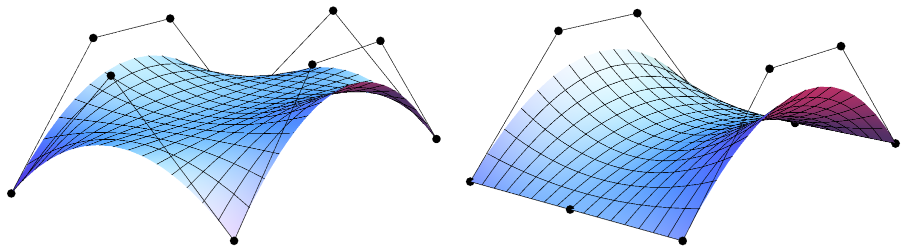


Figure 4.6: Different boundary conditions for $n = 2$, $m = 3$ and the associated Bézier surface. The drawn surfaces are Dirichlet extremals

In this example we can see that in the first example the approximation given by the extremal Dirichlet is better than the masks. However, in the second example, the best approximation is given by the mask $\alpha = 3/8$.

4.3 The extremal of the Dirichlet functional for B-splines

As we have seen in the previous chapters, in some cases is useful to work with B-splines. As for biharmonic surfaces, we can ask about the extremal of the Dirichlet functional for B-splines surfaces. Remember that for the knot vectors $U = (u_0, u_1, \ldots, u_{n+p})$ and $V = (v_0, v_1, \ldots, v_{m+q})$ associated to each parameter u and v the corresponding B-spline surface of control points $\{P_{i,j}\}_{i,j=0}^{n,m}$ is given by

$$\overrightarrow{x}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} N_i(u)N_j(v)P_{i,j}.$$

For an unknown basis coefficient $P_{i,j} = (x^1_{i,j}, x^2_{i,j}, x^3_{i,j})$ we have

$$\frac{\partial \overrightarrow{x}}{\partial u} = \sum_{i,j} N_i'(u)N_j(v)P_{i,j}, \quad \frac{\partial \overrightarrow{x}}{\partial v} = \sum_{i,j} N_i(u)N_j'(v)P_{i,j},$$
$$\frac{\partial}{\partial x^t_{i,j}}\left(\frac{\partial \overrightarrow{x}}{\partial u}\right) = N_i'(u)N_j(v)e^t, \quad \frac{\partial}{\partial x^t_{i,j}}\left(\frac{\partial \overrightarrow{x}}{\partial v}\right) = N_i(u)N_j'(v)e^t,$$

where e^t denotes the t-th vector of the canonical basis, i.e., $e^1 = (1,0,0)$, $e^2 = (0,1,0)$ and $e^3 = (0,0,1)$.

Remember that the Dirichlet functional is given by:

$$D(\mathcal{P}) = \frac{1}{2}\int_R (||\overrightarrow{x}_u||^2 + ||\overrightarrow{x}_v||^2)dudv = \frac{1}{2}\int_R (< \overrightarrow{x}_u, \overrightarrow{x}_u > + < \overrightarrow{x}_v, \overrightarrow{x}_v >)dudv.$$

Therefore, for an interval $\Omega = [a,b] \times [a,b]$

$$\begin{aligned}
\frac{\partial D(\mathcal{P})}{\partial x^t_{i,j}} &= \int_\Omega (< \frac{\partial \overrightarrow{x}_u}{\partial x^t_{i,j}}, \overrightarrow{x}_u > + < \frac{\partial \overrightarrow{x}_v}{\partial x^t_{i,j}}, \overrightarrow{x}_v >) dudv \\
&= \int_\Omega (N'_i(u)N_j(v) < e^t, \overrightarrow{x}_u > + N_i(u)N'_j(v) < e^t, \overrightarrow{x}_v >) dudv \\
&= \int_\Omega N'_i(u)N_j(v) \sum_{k,l} N'_k(u)N_l(v) < e^t, P_{k,l} > dudv \\
&\quad + \int_\Omega N_i(u)N'_j(v) \sum_{k,l} N_k(u)N'_l(v) < e^t, P_{k,l} > dudv \\
&= \sum_{k,l} < e^t, P_{k,l} > (E_{i,k}G_{j,l} + G_{i,k}E_{j,l}),
\end{aligned}$$

where

$$E_{i,k} = \int_a^b N'_i(u)N'_k(u)du, \quad G_{i,k} = \int_a^b N_i(u)N_k(u)du.$$

Then we have the following proposition.

Proposition 4.3.1 *A control net $\mathcal{P} = \{P_{i,j}\}_{i,j=0}^{n,m}$ of a B-spline is an extremal of the Dirichlet functional with prescribed border if and only if*

$$0 = \sum_{k,l} (E_{i,k}G_{j,l} + G_{i,k}E_{j,l}) P_{i,j},$$

where

$$E_{i,k} = \int_a^b N'_i(u)N'_k(u)du, \quad G_{i,k} = \int_a^b N_i(u)N_k(u)du,$$

for $i, j = 1, 2, \dots, n-1$.

Let us see some examples. For a biquadratic B-spline of knot vectors $U = (0,0,0,1,2,2,2)$, $V = (0,0,0,1,2,2,2)$, $n = 3$ and boundary control points:

$$\begin{array}{cccc}
(-15,-2,15) & (-5,5,15) & (5,5,15) & (15,-2,15) \\
(-15,5,5) & * & * & (15,5,5) \\
(-15,5,-5) & * & * & (15,5,-5) \\
(-15,-2,-15) & (-5,5,-15) & (5,5,-15) & (15,-2,-15)
\end{array}$$

we obtain $P_{1,1} = (-\frac{15}{2}, \frac{13}{4}, \frac{15}{2})$, $P_{1,2} = (\frac{15}{2}, \frac{13}{4}, \frac{15}{2})$, $P_{2,1} = (-\frac{15}{2}, \frac{13}{4}, -\frac{15}{2})$ and $P_{2,2} = (\frac{15}{2}, \frac{13}{4}, -\frac{15}{2})$. Figure 4.7 shows the corresponding bicuadratic B-spline surface.

For bicubic B-splines of knot vectors $U = (0,0,0,0,1,2,2,2,2)$ and $V = (0,0,0,0,1,2,2,2,2)$, and $n = 4$, the Figure 4.8 shows the bicubic B-spline surface

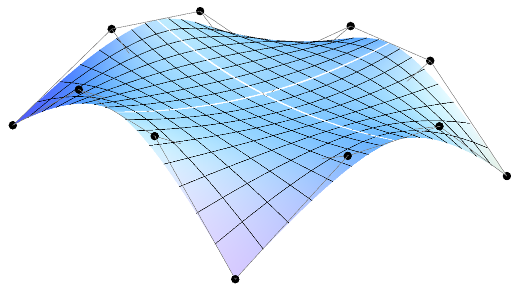


Figure 4.7: Surface obtained by the extremal of Dirichlet functional using bicuadratic B-splines

for the boundary control points:

$$\begin{array}{ccccc}
(0,0,1) & (1,0,0) & (2,0,-1) & (3,0,0) & (4,0,1)\\
(0,1,0) & * & * & * & (4,1,0)\\
(0,2,-1) & * & * & * & (4,2,-1)\\
(0,3,0) & * & * & * & (4,3,0)\\
(0,4,1) & (1,4,0) & (2,4,-1) & (3,4,0) & (4,4,1)
\end{array}$$

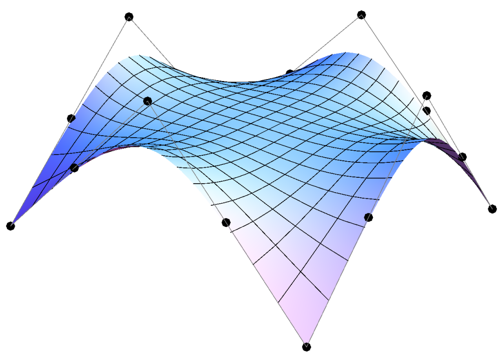


Figure 4.8: Surface obtained by the extremal of Dirichlet functional using bicubic B-splines

In this case, the inner control points are given by

$$\begin{array}{rrr}
P_{1,1}=\left(\frac{3719}{5469},\frac{3719}{5469},\frac{29}{141}\right), & P_{1,2}=\left(\frac{3929}{5469},-\frac{82}{141}\right), & P_{1,3}=\left(\frac{18157}{5469},\frac{3719}{5469},\frac{29}{141}\right),\\
P_{2,1}=\left(\frac{3929}{5469},2,-\frac{82}{141}\right), & P_{2,2}=\left(2,2,-\frac{19}{1833}\right), & P_{2,3}=\left(\frac{17947}{5469},2,-\frac{82}{141}\right),\\
P_{3,1}=\left(\frac{3719}{5469},\frac{18157}{5469},\frac{29}{141}\right), & P_{3,2}=\left(2,\frac{17947}{5469},-\frac{82}{141}\right), & P_{3,3}=\left(\frac{18157}{5469},\frac{18157}{5469},\frac{29}{141}\right).
\end{array}$$

In [6] the authors give an algorithm to compute the extremal of the Dirichlet functional from a Multiresolution Analysis point of view (MRA), where they prove the existence and the unique solution for linear and bicuadratic cardinal B-spline. Let us see how they study the extremal in the MRA case.

The theory of MRA provides the possibility to represent a function f with different degrees of accuracy by means of projection onto a nested sequence of approximation spaces $\{V_n\}_{n\in\mathbb{Z}}$, $V \subseteq V_{n+1}$. For a given sequence of subspaces $\{V_n\}_{n\in\mathbb{Z}}$, we say $\{V_n\}$ forms a MRA for $L^2(\mathbb{R})$ of square integrable functions, if the following conditions are satisfied:

$$V_n \subseteq V_{n+1},\ n\in\mathbb{Z};\quad \overline{\bigcup_n V_n} = L_2(\mathbb{R});\quad \bigcap_n V_n = \{0\}.$$

Any compactly supported refinable function $\phi \in L^2(\mathbb{R})$ with $\widehat{\phi} \neq 0$ will generate an MRA $\{V_n\}$, where $V_n = \overline{span(\phi(2^n x - k), k\in\mathbb{Z})}$, $\widehat{\phi}$ is the Fourier transform of ϕ.

The simplest refinable function is B-spline. The B-spline B_m of order m is compactly supported function $\mathcal{C}^{m-2}(\mathbb{R})$ with length of support being m. If the separation $x_{r+1} - x_r$, where r is any integer, between the successive knots in the set of knot vectors is a constant, the spline is called a *cardinal spline*. The set of integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is a standard choice for the set of knots of a cardinal spline. For this kind of knots, they use $B_2(x)$ and $B_3(x)$ respectively as the refinable function $\phi(x)$:

$$B_2(x) = \begin{cases} 1 - |x|, & x \in [-1, 1], \\ 0, & x \notin [-1, 1], \end{cases}$$

and

$$B_3(x) = \begin{cases} \frac{1}{2}x^2, & x \in [0, 1], \\ \frac{1}{2}(-3 + 6x - 2x^2), & x \in [1, 2], \\ \frac{1}{2}(3 - x)^2, & x \in [2, 3], \\ 0, & x \in [0, 3]. \end{cases}$$

The different refinement levels are given respectively by:

$$\phi^{(2)}_{n,k}(x) = 2^{\frac{n}{2}} B_2(2^n(x+1) - k),\ x \in \left[-1 + \frac{k-1}{2^n}, -1 + \frac{k+1}{2^n}\right], k \in \mathbb{Z}.$$

$$\phi^{(3)}_{n,k}(x) = 2^{\frac{n}{2}} B_3(2^n x - k),\ x \in \left[\frac{k}{2^n}, \frac{k+3}{2^n}\right], k \in \mathbb{Z}.$$

Define

$$\begin{aligned}\widetilde{V}^{(2)}_n =& span\{h^{(2)}_i(u)h^{(2)}_i(v), i, j = 1, 2, \dots, 2^{n+1} - 1\}, \\ \widetilde{V}^{(3)}_n =& span\{h^{(3)}_i(u)h^{(3)}_i(v), i, j = 1, 2, \dots, 3(2^{n+1} - 1)\},\end{aligned}$$

where $h_i^{(2)}(u) = \phi_{n,i}^{(2)}(u)$, $h_j^{(2)}(v) = \phi_{n,j}^{(2)}(v)$ and $h_i^{(3)}(u) = \phi_{n,i}^{(3)}(u)$, $h_j^{(3)}(v) = \phi_{n,j}^{(3)}(v)$. In the paper, the authors prove that for a given n, finding $g_n \in \widetilde{V}_n$ such that

$$g_n = arg\ min_{\widehat{g}\in\widehat{V_n}} D(\varphi + \widetilde{g}), \qquad (4.3.1)$$

where $\varphi(u,v)$ is a parametric smooth surface with the same boundary curves as f, implies that

$$\lim_{n\to\infty} D(\varphi + g_n) = D(f).$$

Then, the general solution of 4.3.1 can be expressed in the form

$$g = \sum_{i,j} P_{i,j} h_i(u) h_j(v),$$

where $h_i(u)h_j(v)$ are the corresponding functions in $\widetilde{V}_n^{(2)}$ or $\widetilde{V}_n^{(3)}$ for B-spline of order 2 or 3 respectively. By making the gradient of the functional $D(\varphi + g)$ vanish, they derive the solution of 4.3.1 as follows.

Theorem 4.3.2 *([6]) For the given boundary control points* $\{P_{i,0}\}_{i=0}^n$, $\{P_{i,n}\}_{i=0}^n$ *and* $\{P_{0,j}\}_{j=0}^n$ $\{P_{n,j}\}_{j=0}^n$*, the function* $g = \sum_{i,j} P_{i,j} h_i(u) h_j(v)$ *is the solution of 4.3.1 if and only if for any i,j,*

$$\sum_{k,l} P_{k,l}(E_{i,k}G_{j,l} + G_{i,k}E_{j,l}) = -(A_{i,j} + B_{i,j}),$$

where

$$E_{i,k} = \int_a^b h_i'(u)h_k'(u)du \quad G_{i,k} = \int_a^b h_i(u)h_k(u)du,$$

and

$$A_{i,j} = \int_\Omega h_i'(u)h_j(v)\varphi_u dudv \quad B_{i,j} = \int_\Omega h_i(u)h_k'(v)\varphi_v dudv,$$

where $\Omega = [a,b] \times [a,b]$.

It should pointed out that this result is equivalent to our method for the linear and cuadratic case. However, since we are using B-spline basis of any degree, we obtain a more general result. We have been able to compute higher degree B-splines.

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