

## SMOOTH FUNCTIONS WITH UNCOUNTABLY MANY ZEROS

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ABSTRACT. In this short note we show that there exist uncountably generated algebras every non-zero element of which is a smooth function having uncountably many zeros. This result complements some recent ones by Enflo et al. [7, 9].

As it nowadays is common terminology, a subset  $M$  of a topological vector space  $X$  is called *lineable* (respectively, *spaceable*) in  $X$  if there exists an infinite dimensional linear space (respectively, infinite dimensional *closed* linear space)  $Y \subset M \cup \{0\}$ . Recently there have been several results regarding the linear structure of certain subsets of real functions having a *large* set of zeros. For instance, in [9], Enflo et al. proved that, for every infinite dimensional closed subspace  $X$  of  $\mathcal{C}[0, 1]$ , the set of functions in  $X$  having infinitely many zeros in  $[0, 1]$  is *spaceable* in  $X$ . Also, in [7], Conejero et al. constructed an algebra of functions  $\mathcal{A}$  enjoying the following properties: (i)  $\mathcal{A}$  is uncountably infinitely generated (that is, the cardinality of a minimal system of generators of  $\mathcal{A}$  is uncountable), (ii) every nonzero element of  $\mathcal{A}$  is nowhere analytic, (iii)  $\mathcal{A} \subset \mathcal{C}^\infty(\mathbb{R})$ , (iv) every element of  $\mathcal{A}$  has infinitely many zeros in  $\mathbb{R}$ , and (v) for every  $f \in \mathcal{A} \setminus \{0\}$  and  $n \in \mathbb{N}$ ,  $f^{(n)}$  (the  $n$ -th derivative of  $f$ ) enjoys the same properties as the elements in  $\mathcal{A} \setminus \{0\}$ . Also, let us recall the notion of *algebrability* (see, e.g. [1–5, 10]). Given an algebra  $\mathcal{A}$ , a subset  $\mathcal{B} \subset \mathcal{A}$ , and a cardinal number  $\kappa$ , we say that  $\mathcal{B}$  is: (i) *algebrable* if there is a subalgebra  $\mathcal{C}$  of  $\mathcal{A}$  so that  $\mathcal{C} \subset \mathcal{B} \cup \{0\}$  and the cardinality of any system of generators of  $\mathcal{C}$  is infinite. (ii)  $\kappa$ -*algebrable* if there exists a  $\kappa$ -generated subalgebra  $\mathcal{C}$  of  $\mathcal{A}$  with  $\mathcal{C} \subset \mathcal{B} \cup \{0\}$ . (iii) *strongly  $\kappa$ -algebrable* if there exists a  $\kappa$ -generated free algebra  $\mathcal{C}$  contained in  $\mathcal{B} \cup \{0\}$ .

On a totally different framework, and somehow related to the study of the set of zeros of functions on a given interval, Aron and Gurariy in 2003, asked whether there exists an infinite dimensional subspace of  $\ell_\infty$  every non-zero element of which has a finite number of zeros. This question was recently answered, in the negative, in [6].

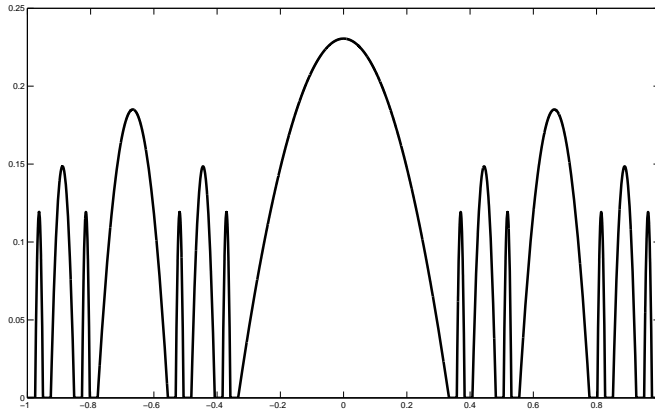
Let us also recall that both of the results from [7, 9] share a common ground: The cardinality of the considered set of zeros was always countable. Of course, by means of a Baire category argument (as seen in [8]) one can show that *almost every* continuous function having zeros has, actually, an uncountable amount of them.

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FIGURE 1. Sketch of  $d(x)$  on  $[-1, 1]$ .

In this short note we complement the previously mentioned results by proving, constructively, the following:

**Theorem.** *The subset of smooth functions in  $\mathbb{R}$  having a uncountable set of zeros is strongly  $\mathfrak{c}$ -algebrable.*

Let us start by fixing  $Z \subset \mathbb{R}$  with  $|Z| = \mathfrak{c}$  and a function  $0 \neq f \in \mathcal{C}^\infty(\mathbb{R})$  such that  $f(z) = 0$  for every  $z \in Z$  and  $f$  does not have horizontal asymptotes. Such a function can be defined as follows. Let  $\mathfrak{C}$  be a copy of the Cantor set in the interval  $[-1, 1]$ . Observe that  $[-1, 1] \setminus \mathfrak{C} = \bigcup_n I_n$ , where the  $I_n$ 's are pairwise disjoint open intervals. Now define the function  $d : [-1, 1] \rightarrow \mathbb{R}$  as

$$d(x) = \begin{cases} k_{a_n, b_n} \cdot (x - a_n)(b_n - x) & \text{if } x \notin \mathfrak{C}, \text{ and } x \in I_n = (a_n, b_n) \text{ for some } n, \\ 0 & \text{if } x \in \mathfrak{C}, \end{cases}$$

where  $k_{a_n, b_n}$  is a positive constant depending on  $a_n$  and  $b_n$ . Next, let  $g$  be the function, on  $[-1, 1]$ , given by:

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathfrak{C}, \\ e^{-1/d(x)} & \text{if } x \notin \mathfrak{C}. \end{cases}$$

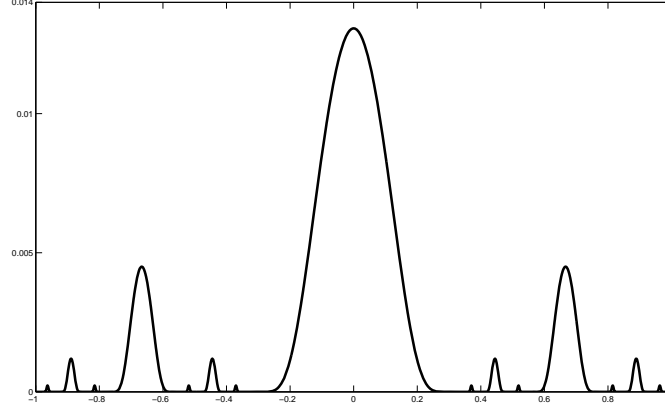
The value of the constant  $k_{a_n, b_n}$  does not affect at all the smoothness of  $g$ . For instance, in Figures 1 and 2 we used  $k_{a_n, b_n} = 1/(b_n - a_n)^{1.8}$ . This constant plays the role of a “scaling factor”.

We leave as an exercise to the reader to check that  $g$  is smooth. Next, we can define our function  $f : \mathbb{R} \rightarrow \mathbb{R}$  by extending  $g$  in a usual way by making it smooth on  $\mathbb{R}$  and by making it not have horizontal asymptotes.

Let us go back to our main construction now. Let  $\mathcal{H}$  be a Hamel basis of  $\mathbb{R}$  as a  $\mathbb{Q}$ -vector space such that all elements in  $\mathcal{H}$  are positive. Also, let (for  $r \in \mathcal{H}$  and  $x \in \mathbb{R}$ ),

$$f_r(x) = e^{rx} \sin(f(x)).$$

Our aim is to show that the algebra generated by the  $f_r$ 's,  $A = \mathcal{A}(f_r : r \in \mathcal{H})$ , is uncountably generated and that every element in  $A$  has an uncountable set of zeros.

FIGURE 2. Sketch of  $g(x)$  on  $[-1, 1]$ .

In order to do so, let  $k \in \mathbb{N}$ ,  $P \in \mathbb{R}[z_1, z_2, \dots, z_k]$  be any non-constant polynomial in  $k$  real variables, and  $r_1, r_2, \dots, r_k \in \mathcal{H}$ . Now we need to see that:

- (i.-)  $\phi(z) := P(f_{r_1}, f_{r_2}, \dots, f_{r_k})(z) = 0$  for every  $z \in Z$ .
- (ii.-) The algebra  $A$  is  $\mathfrak{c}$ -generated.

First, notice that, since  $P$  can be written as

$$P(z_1, \dots, z_k) = \sum_{j=1}^q a_j \cdot z_1^{n_{1,j}} \cdot \dots \cdot z_k^{n_{k,j}},$$

with  $q \in \mathbb{N}$ ,  $\{n_{i,j} : 1 \leq i \leq k, 1 \leq j \leq q\} \subset \mathbb{N}$ , and  $a_j \in \mathbb{R} \setminus \{0\}$  for every  $j \in \{1, \dots, q\}$ , then  $\phi$  can be expressed as

$$\phi(z) = \sum_{j=1}^q a_j \cdot (\sin f(z))^{\sum_{i=1}^k n_{i,j}} \cdot e^{\sum_{i=1}^k (r_i n_{i,j} z)} = \sum_{j=1}^q a_j \cdot (\sin f(z))^{m_j} \cdot e^{z s_j},$$

where  $m_j = \sum_{i=1}^k n_{i,j}$  and  $s_j = \sum_{i=1}^k r_i n_{i,j}$  for  $j \in \{1, \dots, q\}$ .

Once we have that, it is straightforward to check that  $\phi(z) = 0$  for every  $z \in Z$ . Next, let us check some properties of the  $s_j$ 's that appear in the expression of  $\phi$ . First of all, notice that  $s_j \neq 0$  for every  $j \in \{1, \dots, q\}$ . Indeed, suppose that for some  $j \in \{1, \dots, q\}$  we have  $s_j = 0$ . Then, it would be

$$r_1 n_{1,j} + r_2 n_{2,j} + r_3 n_{3,j} + \dots + r_k n_{k,j} = 0,$$

which contradicts the fact that  $\mathcal{H}$  is a Hamel basis. Similarly it can be also shown that  $s_i \neq s_j$  if  $i \neq j$ . Thus, we can assume without loss of generality, that  $s_1 < s_2 < \dots < s_q$ .

Now, let us show that the set  $\{f_r : r \in \mathcal{H}\}$  is algebraically independent. To achieve this, suppose that  $\phi \equiv 0$ , we shall show that  $a_j = 0$  for every  $j \in \{1, \dots, q\}$ . This will amount to  $P \equiv 0$ , and we will be done.

If  $\phi \equiv 0$ , then we would have that

$$\frac{\phi(z)}{e^{s_1 z}} = a_1 (\sin z)^{m_1} + \sum_{j=2}^q a_j \cdot (\sin z)^{m_j} \cdot e^{z(s_j - s_1)}$$

is also 0 for every  $z \in \mathbb{R}$ .

Let, now, take the limit when  $z \rightarrow -\infty$ . Then, we have that

$$\begin{aligned} 0 &= \lim_{z \rightarrow -\infty} a_1 (\sin f(z))^{m_1} + \sum_{j=2}^q a_j \cdot \lim_{z \rightarrow -\infty} (\sin f(z))^{m_j} \cdot e^{z(s_j - s_1)} \\ &= a_1 \cdot \lim_{z \rightarrow -\infty} (\sin f(z))^{m_1} + \sum_{j=2}^q a_j \cdot \lim_{z \rightarrow -\infty} \sin(f(z))^{m_j} e^{z(s_j - s_1)} = \\ &= a_1 \cdot \sin \left( \lim_{z \rightarrow -\infty} f(z) \right)^{m_j} + 0, \end{aligned}$$

and we obtain that  $a_1 = 0$  (since  $f$  has no horizontal asymptotes). We can now proceed similarly (dividing now the expression  $\sum_{j=2}^q a_j \cdot (\sin z)^{m_j} \cdot e^{s_j z}$  by  $e^{s_2 z}$  and taking again limits when  $z \rightarrow -\infty$ ) and we would obtain that all the  $a_j$ 's are 0. Thus,  $P \equiv 0$ , the set  $\{f_r : r \in \mathcal{H}\}$  is algebraically independent, and we are done.

**Remark.** Notice that this result is the best possible in terms of dimension, since the set of continuous functions has dimension  $\mathfrak{c}$ . Let us also recall that this construction can also be done using any other types of fractal sets with arbitrary fractal dimension. We chose the Cantor set for convenience.

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