DISTRIBUTIONAL CHAOS FOR THE FORWARD AND BACKWARD CONTROL TRAFFIC MODEL

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Abstract. The interest in car-following models has increased in the last years due to its connection with vehicle-to-vehicle communications and the development of driverless cars. Some non-linear models such as the Gazes-Herman Rothery model were already known to be chaotic. We consider the linear Forward and Backward Control traffic model for an infinte number of cars on a track. We show the existence of solutions with a chaotic behaviour by using some results of linear dynamics of C_0-semigroups. In contrast, we also analyze which initial configurations lead to stable solutions.

1. Introduction

In this paper we study chaotic traffic patterns described by car-following models. Several notions of chaos can be considered, such as the one of Devaney [19] or the one of distributional chaos by Schweizer and Smítal [41]. Devaney chaos consists of 3 ingredients: transtivity, existence of a dense set of periodic points, and sensitive dependence on the initial conditions. Many ways have been used in order to explain this last notion. Here we refer to the one by Ethan Hunt due its connection with traffic. *For instance, when you hit the brakes for a second, just tap them on the freeway, you can literally track the ripple effect of that action across a 200-mile stretch of road, because traffic has a memory*, see [1]. In other words, when considering a number of cars on a track, the behavior of one of them can be transmitted and propagated to the ones in front (and behind) of it. The mathematical models used to described these interactions are known as *car-following* models.

The first ones were due to Greenshields [28, 29] in the 1930's. Car-following models were perfectioned in the 50's and 60's by taking into account considerations involved in driving a motor vehicle on a lane [17], such as the difference between the velocities of a car and the car in front of it, a distance of a car respect to the preceeding one, or the driver's reaction time, see for instance [26, 39]. An interested reader can find a historical evolution of these models in [14]. Recently, these models have attracted on the interest of researchers thanks to the development of vehicle-to-vehicle (V2V) and of vehicle-to-infrastructure (V2I) communications [31]. These models contribute not only to the study the possibility of allowing vehicles to talk or communicate with each other, but also to increase the efficiency of the communication of the vehicles with the communication networks.

One of the simplest models is the *Quick-Thinking Driver* (QTD) model, which states that the acceleration of a car depends on its distance respect to the car in fron of it. With just two cars, with one of them following the other, one can even find chaos relating its dynamics with certain the solutions of the the logistic equation [35]. Nevertheless, it has already been known for decades that chaotic behaviors exist in traffic flow systems. Gazis, Herman, and Rothery developed for

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General Motors a generalized car-following model, known as the (GHR) model. The discontinuous behavior of some of their solutions and the nonlinearity presented there suggested the existence of chaotic solutions for a certain range of input parameters [27, 40]. Later on, Disbro and Frame [20] showed the presence of chaos for (GHR) model without taking into account signals, bottlenecks, intersections, etc. or with a coordinated signal network. Chaos was also observed for a platoon of vehicles described by the traditional (GHR) model modified by adding a nonlinear inter-car separation dependent term [2, 3].

However, when taking an infinite number of cars on a lane, each of them following one another, a linear simplification of these models can even show chaotic phenomena. In [15] the authors show the existence of some Devaney and distributional chaotic solutions for the *Infinite Quick-Thinking-Driver (IQTD)* model. This situation can be represented by the following infinite system of ordinary differential equations,

$$u_i'(t) = \lambda_i(u_{i+1}(t) - u_i(t)) \text{ for } i \in \mathbb{N}, \tag{1}$$

in which u_1 stands for the velocity of the car 1, u_2 for the velocity of the car in front of car 1, namely car 2, t_1 denotes the reaction time of driver 1, and the positive number λ_1 is a sensitivity coefficient that measures how strong driver 1 responds to the acceleration of the car in front of her. Usually λ_i lies between $0.3 - 0.4s^{-1}$ [14]. We also assume that the velocities at $t = 0$, $(u_i(0))_i$, are given and belong to ℓ^1. Such a behavior is obtained by relating this model with the one of gene amplification–deamplification processes with cell proliferation. These models have been widely studied by Banasiak et al. [6, 7, 8, 10] (see also [5, 18, 30]). Such a behavior can also be found on certain size structured cell populations, c.f. [22, 23] and analyzing the growth of a cell population where cellular development is characterized by cellular size [33].

We want to emphasize that the models we shall study, which are based on simple linear equations, cannot describe all the highly complex situations that, as a matter of fact, occur on a roadway every morning on the way to work. These theoretical models work better on long stretches of road with dense traffic [15]. In this note we introduce the infinite version of the *Forward and Backward Control (FBC)* model. The (FBC) model was developed by [32] for General Motors. The infinite coupled system of ordinary differential equations that is required to model the behavior of these vehicles can be represented as a linear operator on a suitable infinite-dimensional separable Banach space. Then, using some results of linear dynamics of C_0-semigroups we can prove the existence of different chaotic behaviors for the solutions of these equations.

The paper is organized as follows: In Section 2 we describe the Infinite Forward and Backward Control (IFBC) model and introduce all the preliminaries required on linear dynamics and C_0-semigroups. Its representation as a C_0-semigroup and the main results are contained in Section 3. Finally, in Section 4 we deal with the stability of certain solutions.

2. Preliminaries

In the basic formulation of the (FBC) car-following model, there is a relation between the acceleration of a car and the speeds of the cars that go in front and behind of it. We consider this model for an infinite number of cars that circulate on a road, then the corresponding model for these cars is given by an infinite system of first-order differential equations. We will refer to it as the Infinite Forward and Backward Control (IFBC) model.

Definition 2.1 (The (IFBC) traffic model). *Let us consider $(u_i)_i$, the vector of speeds for an infinite number of cars, where u_i stands for the speed of the car i, u_{i+1} for the speed of the car* *behind the car i, and u_{i-1} for the car in front of it. The acceleration of each car u_i, $i \geq 2$, is given as a linear combination of the differences of speed of the i car respect to cars $i-1$ and $i+1$.*

$$
\begin{aligned}
u_1'(t) &= -\mu_1 u_1(t) + \mu_2(u_2(t) - u_1(t)), \\
u_i'(t) &= \mu_1(u_{i-1}(t) - u_i(t)) + \mu_2(u_{i+1}(t) - u_i(t)), \text{ for all } i \geq 2.
\end{aligned}
\tag{2}
$$

with control constants $\mu_1, \mu_2 > 0$, $\mu_1 < \mu_2$.

The vector of speeds $(u_i)_i$ will be considered in a weighted space of summable sequences. In particular, we will consider $\ell_1(s)$, with $0 < s \leq 1$, the weighted space of summable sequences defined as

$$
\ell_1(s) = \left\{ (v_i)_{i\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : ||(v_i)_{i\in\mathbb{N}}||_s = \sum_{i\in\mathbb{N}} |v_i| s^i < \infty \right\}. \tag{3}
$$

. If $s = 1$, we will simply denote it as ℓ_1. If $s < 1$, then any vector representing the velocities of all the cars in the (IFBC) model clearly belongs to $\ell_1(s)$. We point out that such a choice of weights gives more importance to the speeds of cars with low index i.

Let X be a separable infinite-dimensional Banach space. We assume that the reader is familiar with the terminology of C_0-semigroups on Banach spaces, see for instance [38, 25]. Next, let us recall some basic definitions on linear dynamics of C_0-semigroups. A C_0-semigroup $\{T_t\}_{t\geq 0}$ on X is said to be *hypercyclic* if there exists $x \in X$ such that the set $\{T_t x : t \geq 0\}$ is dense in X. An element $x \in X$ is called a *periodic point* for the semigroup $\{T_t\}_{t\geq 0}$ if there exists some $t > 0$ such that $T_t x = x$. A semigroup $\{T_t\}_{t\geq 0}$ is called *Devaney chaotic* if it is hypercyclic and the set of periodic points is dense in X. We point out that these two requirements also yield the sensitive dependence on the initial conditions, as it was seen by Banks et al [11, 30]. Further information on the linear dynamics of C_0-semigroups can be found in [30, Ch. 7].

Sometimes these properties do not hold in the whole space but they do on a closed subspace of X. Following the terminology of [9], we say that a C_0-semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ is called *sub-chaotic* (*sub-hypercyclic*) if there exists a closed subspace $\widetilde{X}$ invariant under $\mathcal{T}$, with $\{0\} \neq \widetilde{X} \subset X$, such that $\widetilde{\mathcal{T}} := \{T_t|_{\widetilde{X}}\}_{t\geq 0}$ is chaotic (hypercyclic) as a semigroup on $\widetilde{X}$.

Another variation of the definition of chaos is the notion of *distributional chaos* introduced by Schweizer and Smítal [41], see also [34, 37] for its presentation in the infinite-dimensional linear setting. A C_0-semigroup $\{T_t\}_{t\geq 0}$ on X is said to be *distributionally chaotic* if there exists an uncountable subset $S \subset X$ and $\delta > 0$ such that, for each pair of distinct points $x, y \in S$ and for every $\varepsilon > 0$, we have $\overline{\text{Dens}}(\{s \geq 0; ||T_s x - T_s y|| > \delta\}) = 1$ and $\overline{\text{Dens}}(\{s \geq 0; ||T_s x - T_s y|| < \varepsilon\}) = 1$, where $\overline{\text{Dens}}$ stands for the upper density of a set of real positive numbers. The semigroup is said to be *densely distributionally chaotic* if S is dense on X.

3. Chaos for the Forward and Backward model

In order to solve the infinite system of equations in (2), we pose the following abstract Cauchy problem on $\ell_1(s)$:

$$
\begin{cases}
u'(t) = Au(t), \\
u(0) = (u_i(0))_{i\in\mathbb{N}}.
\end{cases}
\tag{4}
$$

Here the operator A is defined as

$$(Au(t))_i = \begin{cases} au_i(t) + du_{i+1}(t), & i = 1, \\ bu_{i-1}(t) + au_i(t) + du_{i+1}(t), & i \geq 2, \end{cases} \tag{5}$$

with $b = \mu_1, a = -\mu_1 - \mu_2, d = \mu_2$; for $u = (u_i(t))_{i \in \mathbb{N}} \in \ell_1(s)$ for $t \geq 0$, being $(u_i(0))_{i \in \mathbb{N}}$ the vector of speeds of the cars at $t = 0$.

The solution to (2) can be represented by a C_0-semigroup $\{T_t\}_{t \geq 0}$ on $\ell^1(s)$ whose infinitesimal generator is A. If $A \in L(X)$, then the operators in the C_0-semigroup can be represented as $T_t = e^{tA} = \sum_{k=0}^{\infty} (tA)^n / n!$ for all $t \geq 0$, see for instance [25, Ch. I, Prop. 3.5].

The problem of determining Devaney chaos for the C_0-semigroup generated by A on $\ell_1(s)$ was analysed by Banasiak and Moszynski in [10], when studying the exponential decay of the drug resistant population of cells. There, $u_i(t)$, $i \geq 1$, stands for the number of copies of the drug resistant gene in the i-th subpopulation of cells.

Theorem 3.1. [10, Th. 4] *If $0 < |b| < |d|$ and $|a| < |b + d|$ hold, then $\{e^{tA}\}_{t \geq 0}$ is chaotic on ℓ_1.*

We will study the existence of distributional chaos for the solutions to this general model, and we will analyze, as a particular case, its consequences for the (IFBC) car-following model. In order to do this, we will use the following criterion which ensures distributional chaos and can be found in [4].

Criterion 3.2. ***Dense Distributionally Irregular Manifold Criterion*** *Let $\mathcal{T} = \{T_t\}_{t \geq 0}$ be a C_0-semigroup in $L(X)$ such that there exist a dense subset $X_0 \subset X$ such that $\lim_{t \to \infty} T_t x = 0$, for each $x \in X_0$, and a Lebesgue measurable set $A \subseteq [0, \infty)$ with $\overline{\mathrm{Dens}}(B) = 1$ satisfying*

(i) *either* $\displaystyle\int_B \frac{1}{\|T_t\|} \, dt < \infty$,

(ii) *or X is a complex Hilbert space and* $\displaystyle\int_B \frac{1}{\|T_t\|^2} \, dt < \infty$.

Then $\mathcal{T}$ has a dense distributionally irregular manifold. In particular, $\mathcal{T}$ is densely distributionally chaotic.

First, let us denote by A_s the following matrix, which will permit to relocate our problem into ℓ_1.

$$A_s = \begin{pmatrix} a & d/s & & & \\ sb & a & d/s & & \\ & sb & a & d/s & \\ & & sb & a & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{6}$$

Let us define the linear and continuous operator $A_s := aI + C_s$, $s > 0$, on ℓ_1, where C_s is the linear and continuous operator also defined on ℓ_1 as

$$C_s = \begin{pmatrix} 0 & d/s & & & \\ sb & 0 & d/s & & \\ & sb & 0 & d/s & \\ & & sb & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{7}$$

The following lemma will be helpful in the proof of our main theorem in order to compute the powers of the operator C_s.

Lemma 3.3 (See Lemma 1 in [10])**.** *We have*

$$(C_s^k u)_n = \sum_{i=0}^{k} \left[\binom{k}{i} - \binom{k}{k-(n+i)} \right] (sb)^{k-i} \left(\frac{d}{s}\right)^i u_{n-k+2i}, \tag{8}$$

where $u = (u_i)_i$, $f_i = 0$ for $i \leq 0$, and the Newton symbol is also 0 for negative entries.

We will first prove that $\mathcal{T}_s = \{e^{tA_s}\}_{t\geq 0}$ is distributionally chaotic on ℓ_1 and then via conjugation we will obtain the analogous result for $\mathcal{T} = \{e^{tA}\}_{t\geq 0}$ on $\ell_1(s)$. We recall that the operator A can be represented by the infinite matrix

$$A = \begin{pmatrix} a & d & & & \\ b & a & d & & \\ & b & a & d & \\ & & b & a & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{9}$$

This matrix A is tridiagonal with constant coefficients and thus it represents a bounded operator on $\ell_1(s)$ for any $s > 0$. We are interested in studying distributional chaos for the (IFBC) where $a = -\mu_1 - \mu_2 < 0$, so we will focus our attentionon the case when $a < 0$. The main result of this paper is the following :

Theorem 3.4. *The C_0-semigroup $\mathcal{T}_s = \{e^{tA_s}\}_{t\geq 0}$ is distributionally chaotic on ℓ_1 for all $s > 0$ provided that a, b, d, s satisfy:*

$$0 < b < d, \quad a < 0, \tag{10}$$

$$0 < a + bs + \frac{d}{s}. \tag{11}$$

Its proof is based on an application of the Dense Distributionally Irregular Manifold Criterion (Criterion 3.1). The condition on the existence of a dense set of elements whose orbits by the C_0-semigroup tend to 0 is implicit in the proof of [10, Th. 4]. This proof is a consequence of an application of Criterion 1.2 and 1.3 from [9]. Before presenting these results, let us introduce some notation.

Let (Ω, μ) be a measure space and $f : \Omega \to X$. Given a non-empty set $U \subset \Omega$, we denote $\mathcal{L}(f, U) := \overline{\operatorname{span} f(U)}$ and $\mathcal{L}(f) := \mathcal{L}(f, \Omega)$. If U is a measurable ser in Ω we define

$$\mathcal{L}_{ess}(f, U) := \bigcap_{\substack{\Omega' \subset U, \\ \mu(\Omega')=0}} \mathcal{L}(f, U \setminus \Omega'), \tag{12}$$

and, as before, $\mathcal{L}_{ess}(f) = \mathcal{L}_{ess}(f, \Omega)$.

The next criterion permits to find subspaces of chaoticity and hypercyclicity for a C_0-semigroup. We recall that f is a selection of eigenvectors A in Ω, that means $f(\lambda) \in \operatorname{Dom}(A)$ and $Af(\lambda) = \lambda f(\lambda)$ for any $\lambda \in \Omega$.

Criterion 3.5. [9, Crit. 1.2]. *Suppose that there exists a measurable subset $I \subseteq \mathbb{R}$ and a strongly measurable selection f of eigenvectors of A on iI which is not almost everywhere equal to zero. Then*

$$\mathcal{L}_{ess}(f) \neq \{0\}, \tag{13}$$

$\mathcal{T}$ is sub-hypercyclic, and $\mathcal{L}_{ess}(f)$ is a space of hypercyclicity for $\mathcal{T}$.

Criterion 3.6. [9, Crit. 1.3]. *Suppose that I is an interval of $\mathbb{R}$ of non-zero length and f is a weakly continuous selection of eigenvectors of A on iI which is not constantly equal to zero. Then* (13) *holds, $\mathcal{T}$ is sub-chaotic, and $\mathcal{L}_{ess}(f)$ is a space of chaoticity for T. Moreover*

$$\mathcal{L}_{ess}(f) = \mathcal{L}(f).$$

Finally, let us proceed with the proof of Theorem 3.4.

Proof of Theorem 3.4. We denote by $\{e_m\}_m$ the canonical basis of ℓ_1. As in [13], we compute the ℓ^1-norm of C_s^k acting over a sequence e_m with $m > k$ then,

$$\|C_s^k e_m\|_{\ell^1} = \sum_{n=0}^{\infty} \left| \sum_{i=0}^{k} \left[\binom{k}{i} - \binom{k}{k-(n+i)} \right] (sb)^{k-i} \left(\frac{d}{s} \right)^i \delta_{n-k+2i,m} \right|. \tag{14}$$

Since $\delta_{n-k+2i,m} = 0$ for $n < m-k$ or $n > m+k$, then,

$$\|C_s^k e_m\|_{\ell^1} = \sum_{n=m-k}^{m+k} \left| \sum_{i=0}^{k} \left[\binom{k}{i} - \binom{k}{k-(n+i)} \right] (sb)^{k-i} \left(\frac{d}{s} \right)^i \delta_{n-k+2i,m} \right|. \tag{15}$$

Making $j = k - i$ we obtain

$$\sum_{n=m-k}^{m+k} \left| \sum_{j=0}^{k} \left[\binom{k}{j} - \binom{k}{j-n} \right] (sb)^j \left(\frac{d}{s} \right)^{k-j} \delta_{n+k-2j,m} \right|. \tag{16}$$

Changing, also, $n' = n + k - m$, we have

$$\sum_{n'=0}^{2k} \left| \sum_{j=0}^{k} \left[\binom{k}{j} - \binom{k}{j+k-n'-m} \right] (sb)^j \left(\frac{d}{s} \right)^{k-j} \delta_{n'+m-2j,m} \right|. \tag{17}$$

If n' is odd, $\delta_{n'+m-2j,m} = 0$, then we are left with the even terms, getting

$$\sum_{j=0}^{k} \left| \left[\binom{k}{j} - \binom{k}{k-j-m} \right] (sb)^j \left(\frac{d}{s}\right)^{k-j} \right|. \tag{18}$$

Since $m > k$, we only have

$$\sum_{j=0}^{k} \binom{k}{j} \left| (sb)^j \left(\frac{d}{s}\right)^{k-j} \right|, \tag{19}$$

which is $\left(sb + \frac{d}{s}\right)^k$. Therefore, $\|C_s^k\| \geq \left(sb + \frac{d}{s}\right)^k$.

With the estimates above, we can also approximate the norm of e^{tC_s} on $L(\ell_1)$.

$$\|e^{tC_s}\| = \left\| \sum_{k=0}^{\infty} \frac{(tC_s)^k}{k!} \right\|. \tag{20}$$

Since C_s is a positive operator, for every $m > 0$ we have

$$\left\| \sum_{k=0}^{\infty} \frac{(tC_s)^k}{k!} \right\| \geq \left\| \sum_{k=0}^{m-1} \frac{(tC_s)^k}{k!} \right\| \geq \left\| \sum_{k=0}^{m-1} \frac{(tC_s)^k}{k!} e_m \right\| = \sum_{k=0}^{m-1} \frac{t^k \left(sb + \frac{d}{s}\right)^k}{k!}. \tag{21}$$

Therefore, taking the supremum over m we get $\|e^{tC_s}\| \geq e^{t(sb+\frac{d}{s})}$, and hence

$$\frac{1}{\|e^{tA_s}\|} \leq \frac{1}{e^{t(a+sb+\frac{d}{s})}}. \tag{22}$$

By (11) we have

$$\int_{\mathbb{R}^+} \frac{1}{\|e^{tA_s}\|} < \infty. \tag{23}$$

In order to apply Criterion 3.2 it only remains to show that there exists a dense subset $X_0 \subset X$ such that $\lim_{t\to\infty} T_t x = 0$ for each $x \in X_0$. Let us also denote $W_0(\mathcal{T}) := \{x \in X : \lim_{t\to\infty} T_t x = 0\}$.

The proof of Theorem 3.1 is based on Criterion 3.6. As it is indicated in [10, pg. 74] we can find a selection of eigenvectors of A defined as $f : iS(b,d,a) \to \ell^1$, where $S(b,d,a)$ is the set of the values $y \in]c, c[$, with $c = \frac{|bs-\frac{d}{s}|}{|bs+\frac{d}{s}|}\sqrt{(bs+\frac{d}{s})^2 - a^2}$, such that $(iy-a)^2 - 4bd \neq 0$, see also [9, p. 579-580]. Taking S' as an arbitrary non-empty connected component of $S(b,d,a)$, it can be seen that the set $f(iS')$ is linearly dense in ℓ^1. Therefore :

$$\ell^1 = \mathcal{L}(f, iS') \subseteq \mathcal{L}(f) \subseteq \ell^1. \tag{24}$$

Now, proceeding with the same technique as in the proof of Criteria 3.5 and 3.6 in [9], let us re-scale the selection f by defining $\widetilde{f} := \rho \cdot f$, where $\rho : iS' \to \mathbb{R}$ is given by $\rho(\lambda) := [(1+\|f(\lambda)\|)(1+|\lambda|^2)]^{-1}$, for $\lambda \in iS'$. Let us also define $F : \mathbb{R} \to \ell^1$ as

$$F(t) := \int_{S'} e^{its} \tilde{f}(is) ds, \quad t \in \mathbb{R}. \tag{25}$$

Denote $Y_F := \mathrm{lin}(F(\mathbb{R}))$. We have, by Criterion 3.6, $\overline{Y_F} = \mathcal{L}(F)$, $Y_F \subset W_0(\mathcal{T}_s)$, and $\mathcal{L}(F) = \mathcal{L}_{ess}(\mathbf{f})$. Moreover we obtain that $\mathcal{L}_{ess}(\mathbf{f}) = \mathcal{L}(\mathbf{f})$ and, hence, we have found Y_F, a set of points which orbits by the semigroup tend to 0, dense in ℓ^1, that is

$$\overline{Y_F} = \mathcal{L}(F) = \mathcal{L}_{ess}(\mathbf{f}) = \mathcal{L}(\mathbf{f}) = \ell^1. \tag{26}$$

Therefore, it only remains to apply the Dense Distributionally Irregular Manifold Criterion (Criterion 3.2) and we get the conclusion. □

Remark 3.7. *In the proof of the previous result the existence of W_0 was deduced using techniques that requires the use of complex numbers. Nevertheless, for the case when a, b, d where real numbers, the same result holds just taking the restriction of W_0 to the real numbers. For further details we refer the reader to the proof of Criterion 3.2, see* [4, 12, 36]. *This approach can be compared with* [24, Th. 3.7].

This result can be transferred to the C_0-semigroup via the conjugation lemma.

Corollary 3.8. *The C_0-semigroup $\{e^{tA_s}\}_{t\geq 0}$ is distributionally chaotic on ℓ_1 if and only if $\{e^{tA}\}_{t\geq 0}$ is distributionally chaotic on $\ell^1(s)$.*

Proof. Let us define the operator $U_s : \ell^1 \to \ell^1(s)$, for all $s > 0$, as

$$U_s u := \left(\frac{u_n}{s_n}\right)_{n\geq 1} \quad \text{for every } u = (u_n)_n \in \ell^1. \tag{27}$$

This is an isometry from ℓ_1 onto $\ell_1(s)$ and it holds that

$$A_s = U_s^{-1} \tilde{A} U_s, \tag{28}$$

By an application of the conjugation lemma for distributional chaos, see for instance [34, Th. 2], we get the distributional chaos for the C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ from the C_0-semigroup $\{e^{tA_s}\}_{t\geq 0}$. □

As a consequence, we have the expected results for the weighted $\ell^1(s)$-spaces.

Corollary 3.9. *The solution C_0-semigroup of* (4), *$\mathcal{T} = \{e^{tA}\}_{t\geq 0}$ is distributionally chaotic on $\ell_1(s)$ for each $s > 0$ provided that a, b, d, s satisfy* (10) *and* (11).

In particular, for the $(IFBC)$ car-following model we have

Corollary 3.10. *Let $0 < \mu_1 < \mu_2$ the coefficients of the (IFBC) model in* (2). *The solution C_0-semigroup to the (IFBC) model is distributionally chaotic on $\ell_1(s)$ with $s > 0$ provided that*

$$(\mu_1 + \mu_2) < (s\mu_1 + \frac{\mu_2}{s}). \tag{29}$$

4. Study of stability

In this section we analyze the stability of the (IFBC) car-following model in our setting of weighted spaces of summable sequences. We recall that a C_0-semigroup of the form $\left\{e^{tA}\right\}_{t\geq 0}$ defined on a Banach space X is *exponentially stable*, [25, p. 296], if there exists $\varepsilon > 0$ such that

$$\lim_{t\to\infty} e^{\varepsilon t} ||e^{tA}|| = 0, \tag{30}$$

and *uniformly stable* if

$$\lim_{t\to\infty} ||e^{tA}|| = 0. \tag{31}$$

In fact, Eisner show that both notions are equivalent [21]. We recall that a vector $x \in X$ is said to be *distributionally irregular* for the C_0-semigroup $\mathcal{T}$ if the following holds: for every $\delta > 0$

$$\limsup_{t\to\infty} \frac{\mu(\{s \in [0,t] : ||T_s x|| < \delta\})}{t} = 1, \tag{32}$$

$$\limsup_{t\to\infty} \frac{\mu(\{s \in [0,t] : ||T_s x|| \geq \delta\})}{t} = 1. \tag{33}$$

The existence of a distributionally irregular vector is equivalent to the existence of distributional chaos [4]. So, it is clear that under the conditions expressed in (10) and (11), the C_0-semigroup will not be exponentially stable.

Nevertheless, a weaker version of stability can also be considered, in the same way as it has been done for considering sub-chaos and sub-hypercyclicity. We say that $\{e^{tA}\}_{t\geq 0}$ is *exponentially stable on a subspace* $Y \subset X$ if there exists $\varepsilon > 0$ such that for any $y \in Y$ we have

$$\lim_{t\to\infty} e^{\varepsilon t} ||e^{tA} y|| = 0. \tag{34}$$

Such analysis has been already performed when studying chaos of C_0-semigroups in [10, 16].

Theorem 4.1. *The solution C_0-semigroup of the (IFBC) model where μ_1, μ_2 and s_0 satisfy conditions in corollary 3.10, is exponentially stable on the subspace $Y_\delta := span\{y : T_t y = \mu y, \quad \mu \in \mathbb{K}, \quad \Re(\mu) < \delta\}$, for every $\delta < 0$.*

Proof. Fix $0 < \varepsilon < -\delta$ and $y \in Y_\delta$ of the form $y = \sum_{i=1}^k \alpha_i y_{\mu_i}$. Define $\delta_y = \max\{\Re(\mu_i) : 1 \leq i \leq k\}$. Clearly, $\varepsilon + \delta_y < 0$, then

$$\begin{aligned} e^{\varepsilon t} ||e^{tA} y||_s &= e^{\varepsilon t} \left|\left| \sum_{i=1}^k \alpha_i e^{t\mu_i} y_{\mu_i} \right|\right|_s \leq e^{\varepsilon t} \left(\sum_{i=1}^k e^{t\Re(\mu_i)} ||\alpha_i y_{\mu_i}||_s \right) \\ &< e^{t(\varepsilon+\delta_y)} \left(\sum_{i=1}^k ||\alpha_i y_{\mu_i}||_s \right), \end{aligned}$$

which tends to 0 when t tends to ∞. □

Remark 4.2. *This analysis of stability can be compared with the one carried out in* [32]*, where the condition for asymptotic stability is given by*

$$\frac{(\mu_1 - \mu_2)^2}{\mu_1 + \mu_2} < \frac{1}{2}. \tag{35}$$

5. Final comments

In this note we point out that the forward and backward traffic models can be related withbirth-and-death process of cell proliferation. It is fair to note that the chaotic properties reported in this paper refer to the whole-space of solutions whereas only positive solutions make sense. There- fore, being important, e.g., in the analysis of the stability of numerical schemes, does not necessarily say anything important about the actual behaviour of the physical trajectories of the cars. To sum up, in dense traffic drivers follow one another very closely and small disturbances such as acceleration or deceleration of one vehicle might be passed over or amplied along the line of vehicles on the road. Chaotic flow appears and this can lead to an impredictible dynamical behaviour and, in some cases, to accidents.

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