

Gauss-Bonnet formulae and rotational integrals in constant curvature spaces

S. Barahona^a, X. Gual-Arnau^b

^a*Departament de Matemàtiques, Universitat Jaume I. 12071-Castelló, Spain.
barahona@uji.es*

^b*Departament de Matemàtiques, Institute of New Imaging Technologies, Universitat
Jaume I. 12071-Castelló, Spain. gual@uji.es*

Abstract

We obtain generalizations of the main result in [18], and then provide geometric interpretations of linear combinations of the mean curvature integrals that appear in the Gauss-Bonnet formula for hypersurfaces in space forms M_λ^n . Then, we combine these results with classical Morse theory to obtain new rotational integral formulae for the k -th mean curvature integrals of a hypersurface in M_λ^n .

Keywords: Gauss-Bonnet formula, integral of mean curvature, intrinsic volume, rotational integral formulas, space form.

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1. Introduction

Let M_λ^n denote a simply connected Riemannian manifold of constant sectional curvature λ . Further, let L_r^n denote a r -plane, ($r \leq n$), namely a totally geodesic submanifold of dimension r in M_λ^n , and let dL_r^n denote the corresponding density, invariant under the group of Euclidean and non Euclidean motions. A r -plane through a fixed point O in M_λ^n , and its invariant density, are denoted by $L_{r[0]}^n$ and $dL_{r[0]}^n$, respectively [16].

In [8] a new expression for the density of r -planes in M_λ^n has been obtained in terms of the density $dL_{r+1[0]}^n$, of the density dL_r^{r+1} of r -planes in $L_{r+1[0]}^n$ and the distance ρ from O to L_r^{r+1} . Thus, an invariant r -plane in M_λ^n may be generated by taking first an isotropic $(r+1)$ -plane through a fixed point O and then an invariant r -plane within this $(r+1)$ -plane, weighted

by a function of ρ .

This construction, called the invariator principle in M_λ^n ([19]), has opened the way to solve rotational integral equations for different quantities as the volume of a k -dimensional submanifold in M_λ^n [8], the k -th mean curvature integrals or k -th intrinsic volumes ([10] and [1], and different curvature measures ([19] for $\lambda = 0$)). The solutions of these equations allow to express these quantities as the integral of some functionals defined in sections produced by isotropic planes through a fixed point. Moreover, in [19], the authors, using classical Morse theory, rewrite the volume of compact submanifolds in \mathbb{R}^n of dimension $n - r$, in terms of critical values of the sectioned object with $(r + 1)$ -planes; and in [9] related generalizations valid for submanifolds in space forms of constant curvature are obtained.

On the other hand, in [18] it is proved that the Gauss-Bonnet defect of a hypersurface in M_λ^n is the measure of planes L_{n-2}^n meeting it, counted with multiplicity. From this result an integral-geometric proof of the Gauss-Bonnet theorem for hypersurfaces in M_λ^n is given.

The purpose of this paper is twofold: to obtain generalizations of the main result in [18], following a completely different route; and to combine these results with classical Morse theory to obtain new rotational integral formulae for the k -th mean curvature integrals of a hypersurface in M_λ^n .

2. The Gauss-Bonnet theorem in M_λ^n

Let $Q \subset M_\lambda^n$ be a compact domain with smooth boundary $S = \partial Q$. Let V denote the volume of Q , F the $(n - 1)$ -surface area of S , $\chi(Q)$ the Euler-Poincaré characteristic of Q , and M_i the i -th integral of mean curvature of S . The Gauss-Bonnet formula for S states that [16]

$$c_{n-1}M_{n-1} + \lambda c_{n-3}M_{n-3} + \cdots + \lambda^{\frac{n-2}{2}} c_1 M_1 + \lambda^{\frac{n}{2}} V = \frac{1}{2} O_n \chi(Q), \quad (1)$$

for n even, where $O_k = \text{vol}(\mathbb{S}^k)$ (surface area of the k -dimensional unit sphere), and

$$c_{n-1}M_{n-1} + \lambda c_{n-3}M_{n-3} + \cdots + \lambda^{\frac{n-3}{2}} c_2 M_2 + \lambda^{\frac{n-1}{2}} c_0 F = \frac{1}{2} O_n \chi(Q), \quad (2)$$

for n odd, where

$$c_h = \binom{n-1}{h} \frac{O_n}{O_h O_{n-1-h}}. \quad (3)$$

If n is odd, we can use the equality $2\chi(Q) = \chi(S)$, and for $\lambda = 0$, in any case, we obtain $M_{n-1} = O_{n-1}\chi(Q)$.

Let \mathcal{L}_r be the space of r -dimensional totally geodesic submanifolds of M_λ^n . Our first result is the following theorem, which is a generalization of the main result in [18].

Theorem 2.1. *For n and r even, or n and r odd, we have*

$$\begin{aligned} & \frac{1}{2}O_n\chi(Q) - c_{n-1}M_{n-1} - \lambda c_{n-3}M_{n-3} - \dots - \lambda^{\frac{n-r-2}{2}}c_{r+1}M_{r+1} \\ & = \lambda^{\frac{n-r}{2}} \frac{O_r \dots O_1}{O_{n-1} \dots O_{n-r}} \int_{\mathcal{L}_r} \chi(Q \cap L_r^n) dL_r^n. \end{aligned} \quad (4)$$

Proof. We begin assuming that n and r are both even numbers. Given a r -plane L_r^n of M_λ^n , $Q_r = L_r^n \cap Q$ is, in general, a domain of dimension r in L_r^n . Applying Eq.(1) to Q_r we obtain

$$c'_{r-1}M'_{r-1} + \lambda c'_{r-3}M'_{r-3} + \dots + \lambda^{\frac{r-2}{2}}c'_1M'_1 + \lambda^{\frac{r}{2}}V(Q_r) = \frac{1}{2}O_r\chi(Q_r), \quad (5)$$

where M'_i is the i -th integral of mean curvature of ∂Q_r and

$$c'_h = \binom{r-1}{h} \frac{O_r}{O_h O_{r-1-h}}. \quad (6)$$

Eq.(14.69) for $q = n$ and Eq.(14.78) of [16], which are valid for M_λ^n , are

$$\int_{\mathcal{L}_r} V(Q_r) dL_r^n = \frac{O_{n-1} \dots O_{n-r}}{O_{r-1} \dots O_0} V(Q) \quad (7)$$

and

$$\int_{\mathcal{L}_r} M'_i dL_r^n = \frac{O_{n-2} \dots O_{n-r} O_{n-i}}{O_{r-2} \dots O_0 O_{r-i}} M_i. \quad (8)$$

Now, having the preceding equalities in mind, we integrate Eq.(5) and we obtain

$$\begin{aligned} & d_{r-1}M_{r-1} + \lambda d_{r-3}M_{r-3} + \dots + \lambda^{\frac{r-2}{2}}d_1M_1 + \lambda^{\frac{r}{2}}d_0V \\ & = \frac{1}{2}O_r \int_{\mathcal{L}_r} \chi(Q_r) dL_r^n, \end{aligned} \quad (9)$$

where

$$d_i = \binom{r-1}{i} \frac{O_r}{O_i O_{r-1-i}} \frac{O_{n-2} \cdots O_{n-r} O_{n-i}}{O_{r-2} \cdots O_0 O_{r-i}}; \quad i = 1, 3, \dots, r-1; \quad (10)$$

$$d_0 = \frac{O_{n-1} \cdots O_{n-r}}{O_{r-1} \cdots O_0}. \quad (11)$$

We multiply Eq.(9) by $\frac{\lambda^{(n-r)/2}}{d_0}$ to obtain

$$\begin{aligned} & \lambda^{\frac{n-r}{2}} k_{r-1} M_{r-1} + \lambda^{\frac{n-r+2}{2}} k_{r-3} M_{r-3} + \cdots + \lambda^{\frac{n-2}{2}} k_1 M_1 + \lambda^{\frac{n}{2}} V \\ & = \frac{1}{2} \lambda^{\frac{n-r}{2}} \frac{O_r}{d_0} \int_{\mathcal{L}_r} \chi(Q_r) dL_r^n, \end{aligned} \quad (12)$$

where

$$k_i = \binom{r-1}{i} \frac{O_r O_{r-1} O_{n-i}}{O_i O_{n-1} O_{r-i} O_{r-i-1}}. \quad (13)$$

If we compare the constants k_i and c_i in Eq.(1), using the equality $(k-1)O_k = O_1 O_{k-2}$, we have that

$$k_i = c_i; \quad (14)$$

then, Eq.(12) can be written as

$$\begin{aligned} & \lambda^{\frac{n-r}{2}} c_{r-1} M_{r-1} + \lambda^{\frac{n-r+2}{2}} c_{r-3} M_{r-3} + \cdots + \lambda^{\frac{n-2}{2}} c_1 M_1 + \lambda^{\frac{n}{2}} V \\ & = \frac{1}{2} \lambda^{\frac{n-r}{2}} \frac{O_r}{d_0} \int_{\mathcal{L}_r} \chi(Q_r) dL_r^n, \end{aligned} \quad (15)$$

and, from Eq.(1) we obtain the result for the case n and r even.

If we consider that n and r are both odd numbers the proof is similar to the preceding one but considering, instead of Eq.(7), the following equality (Eq.(14.69) of [16] with $q = n-1$):

$$\int_{\mathcal{L}_r} F(\partial Q_r) dL_r^n = \frac{O_n \cdots O_{n-r} O_{r-1}}{O_r \cdots O_0 O_{n-1}} F, \quad (16)$$

where $F(\partial Q_r)$ is the $(r-1)$ -surface area of $\partial Q \cap L_r^n = \partial(Q \cap L_r^n)$. \square

Remark. For $r = n-2$, Theorem 2.1 gives Theorem 1 of [18] and, as a result of Theorem 2.1, we obtain the following corollary which is equivalent to Proposition 7 of [18].

Corollary 2.2. *Let Q be a compact domain in M_λ^n and $L_r \in \mathcal{L}_r$, we have*

$$M_r = \frac{(n-r-1)O_r \dots O_0}{O_{n-2} \dots O_{n-r-2}} \int_{\mathcal{L}_{r+1}} \chi(Q \cap L_{r+1}^n) dL_{r+1}^n - \lambda \frac{rO_{r-2} \dots O_0}{O_{n-2} \dots O_{n-r}} \int_{\mathcal{L}_{r-1}} \chi(Q \cap L_{r-1}^n) dL_{r-1}^n. \quad (17)$$

Proof. When r is an even number, Eq.(15) divided by $\lambda^{\frac{n-r}{2}}$ is

$$c_{r-1}M_{r-1} + \lambda c_{r-3}M_{r-3} + \dots + \lambda^{\frac{r-2}{2}} c_1 M_1 + \lambda^{\frac{r}{2}} V = \frac{1}{2} \frac{O_r}{d_0} \int_{\mathcal{L}_r} \chi(Q_r) dL_r^n; \quad (18)$$

and the corresponding equation to Eq.(15) divided by $\lambda^{\frac{n-r}{2}}$ when r is an odd number is

$$c_{r-1}M_{r-1} + \lambda c_{r-3}M_{r-3} + \dots + \lambda^{\frac{r-3}{2}} c_2 M_2 + \lambda^{\frac{r-1}{2}} c_0 F = \frac{1}{2} \frac{O_r}{d_0} \int_{\mathcal{L}_r} \chi(Q_r) dL_r^n. \quad (19)$$

If r is odd, subtracting each part of Eq.(18), with $r \rightarrow r+1$, minus the corresponding part of λ multiplied by Eq.(18) with $r \rightarrow r-1$ we obtain the result. If r is even, we proceed in the same way but using Eq.(19) instead of the Eq.(18). \square

Remark. For $\lambda = 0$, Eq.(17) coincides with Eq.(14.79) of [16].

3. Rotational integrals and Morse representations for M_r

From rotational integral formulae we obtain quantitative properties (as M_r) of differential manifolds in M_λ^n , from the intersection of the manifold with planes (totally geodesic submanifolds) through a fixed point O . In this context, from Eq.(17), we will find measurement functions α_r defined on $L_{r+2[0]}^n \cap Q$ with rotational average equal to M_r , that is,

$$M_r = \int_{L_{r+2[0]}^n \cap Q \neq \emptyset} \alpha_r(L_{r+2[0]}^n \cap Q) dL_{r+2[0]}^n. \quad (20)$$

Theorem 3.1. *Let $Q \subset M_\lambda^n$ be a compact domain with smooth boundary $S = \partial Q$. The measurement functions α_r corresponding to the r -th integral*

of mean curvature of S , M_r , can be expressed as

$$\begin{aligned} \alpha_r(L_{r+2[0]}^n \cap Q) &= \frac{O_{r-2} \dots O_0}{O_{n-2} \dots O_{n-r-2}} \\ &\left[(n-r-1)O_r O_{r-1} \int \chi((Q \cap L_{r+2[0]}^n) \cap L_{r+1}^{r+2}) s_\lambda^{n-r-2}(\rho) dL_{r+1}^{r+2} \right. \\ &\left. - \lambda r O_1 O_0 \int \chi(((Q \cap L_{r+2[0]}^n) \cap L_{r[0]}^{r+2}) \cap L_{r-1}^r) s_\lambda^{n-r}(\rho) dL_{r-1}^r dL_{r[0]}^{r+2} \right], \end{aligned} \quad (21)$$

where, in both integrals, ρ is the distance from O to the planes L_{r+1}^{r+2} and L_{r-1}^r , respectively; and

$$s_\lambda(\rho) = \begin{cases} \lambda^{-1/2} \sin(\rho\sqrt{\lambda}), & \lambda > 0 \\ \rho, & \lambda = 0. \\ |\lambda|^{-1/2} \sinh(\rho\sqrt{|\lambda|}), & \lambda < 0 \end{cases} \quad (22)$$

Proof. The idea of the proof consists in generating the planes L_{r+1}^n and L_{r-1}^n , which appear in Eq.(17), by taking first an isotropic plane through O and then an invariant plane within this isotropic plane, weighted by a function of ρ ; that is, from Corollary 3.1 of [8] we have the identity

$$dL_{r+1}^n = s_\lambda^{n-r-2}(\rho) dL_{r+1}^{r+2} dL_{r+2[0]}^n, \quad (23)$$

and also

$$dL_{r-1}^n dL_{r+2[r]}^n = s_\lambda^{n-r}(\rho) dL_{r-1}^r dL_{r+2[r]}^n dL_{r[0]}^n, \quad (24)$$

where $dL_{r+2[r]}^n$ denotes the density for $(r+2)$ -planes about a about a r -plane L_r^n (see page 202 of [16]).

As justified in [16], p. 309, before Eq. (17.55), from the expressions of the densities of planes in M_λ^n it follows that some density decompositions (such as [16], Eq. (12.53)) have the same form whatever the sign of λ . Then, from Eq.(12.53) of [16], Eq.(24), can be expressed as

$$dL_{r-1}^n dL_{r+2[r]}^n = s_\lambda^{n-r}(\rho) dL_{r-1}^r dL_{r[0]}^{r+2} dL_{r+2[0]}^n. \quad (25)$$

Finally, substituting Eq.(23) and Eq.(25) in Eq.(17), having in mind that

$$\int dL_{r+2[r]}^n = \frac{O_{n-r-1} O_{n-r-2}}{O_1 O_0}, \quad (26)$$

we obtain the result. \square

Remark. For $\lambda = 0$, Eq.(21) coincides, up to a constant factor, with Eq.(18) of [10].

3.1. Morse representations for M_r

In this section a geometric interpretation is given of Eq.(21) in terms of the critical points of height functions. In particular, and in order to simplify, we will give a geometric interpretation of the function

$$\beta_r = \int \chi((Q \cap L_{r+1[0]}^n) \cap L_r^{r+1}) s_\lambda^{n-r-1}(\rho) dL_r^{r+1}. \quad (27)$$

The density dL_r^{r+1} may be decomposed as follows,

$$dL_r^{r+1} = c_\lambda^r(\rho) d\rho du_r, \quad (28)$$

where du_r denotes the surface area element of the r -dimensional unit sphere and $c_\lambda(\rho) = \frac{d}{d\rho} s_\lambda(\rho)$. Note that $\rho \geq 0$ for the cases $\lambda = 0$ (Euclidean) and $\lambda < 0$ (hyperbolic); however, for the case $\lambda > 0$ (spherical) ρ varies from 0 (which corresponds to the point O) to $\frac{\pi}{\sqrt{\lambda}}$ (which corresponds to the cut locus of O (i.e., the antipodal point of O)).

Therefore, for the cases $\lambda = 0$ (Euclidean) and $\lambda < 0$ (hyperbolic), we may write,

$$\beta_r = \int_{\mathbb{S}^r} du_r \int_0^\infty s_\lambda^{n-r-1}(\rho) c_\lambda^r(\rho) \chi((Q \cap L_{r+1[0]}^n) \cap L_r^{r+1}) d\rho, \quad (29)$$

whereas, for the case $\lambda > 0$ (spherical),

$$\beta_r = \int_{\mathbb{S}^r} du_r \int_0^{\frac{\pi}{\sqrt{\lambda}}} s_\lambda^{n-r-1}(\rho) c_\lambda^r(\rho) \chi((Q \cap L_{r+1[0]}^n) \cap L_r^{r+1}) d\rho, \quad (30)$$

where L_r^{r+1} is the r -plane expressed in terms of its distance ρ from the fixed point O , perpendicular to the geodesic defined from the direction u_r from O , and $\chi((Q \cap L_{r+1[0]}^n) \cap L_r^{r+1}) = 0$ whenever $(Q \cap L_{r+1[0]}^n) \cap L_r^{r+1} = \emptyset$.

Since we want to give a geometrical interpretation of β_r , based on critical points of height functions, from now on we will consider that ρ means signed distance and we will rewrite β_r as:

$$\beta_r = \frac{1}{2} \int_{\mathbb{S}^r} du_r \int_{-\infty}^\infty s_\lambda^{n-r-1}(|\rho|) c_\lambda^r(\rho) \chi((Q \cap L_{r+1[0]}^n) \cap L_r^{r+1}) d\rho, \quad \lambda \leq 0; \quad (31)$$

$$\beta_r = \frac{1}{2} \int_{\mathbb{S}^r} du_r \int_{-\frac{\pi}{\sqrt{\lambda}}}^{\frac{\pi}{\sqrt{\lambda}}} s_\lambda^{n-r-1}(|\rho|) c_\lambda^r(\rho) \chi((Q \cap L_{r+1[0]}^n) \cap L_r^{r+1}) d\rho \quad \lambda > 0. \quad (32)$$

Let u_r denote a unit vector in $\mathbb{S}^r \subset T_O L_{r+1[0]}^n$. The geodesic $\gamma_{u_r} : \mathbb{R} \rightarrow L_{r+1[0]}^n$, with $\gamma_{u_r}(0) = O$ and $\gamma'(0) = u_r$ is given by $\gamma_{u_r}(t) = c_\lambda(t)O + s_\lambda(t)u_r$, where $c_\lambda(t) = \frac{d}{dt}s_\lambda(t)$. Given u_r , let $h_{u_r} : L_{r+1[0]}^n \rightarrow \mathbb{R}$ be the height function whose level hypersurfaces are just the r -planes L_r^{r+1} perpendicular to the geodesic $\gamma_{u_r}(t)$. Note that in the Euclidean case ($\lambda = 0$) this height function coincides with the standard height function considered in [19]. We suppose that the level hypersurface L_r^{r+1} is oriented in such a way that the unit vector $\nu(p)$, perpendicular to the level set $L_r^{r+1} \subset L_{r+1[0]}^n$ at p is given by $\nu(p) = \text{grad}(h_{u_r})(p) / \|\text{grad}(h_{u_r})(p)\|$.

Let us denote $Q_{r+1} = Q \cap L_{r+1[0]}^n$ which is, in general, a domain with boundary in $L_{r+1[0]}^n$ (see Appendix A of [10]). In Section 5 (Appendix) we show that in Euclidean and hyperbolic cases; and in the spherical case, if the domain Q is contained in the hemisphere of M_λ^n with pole O , $h_{u_r}|_{Q_{r+1}}$ is a strong Morse function for almost all $u_r \in \mathbb{S}^r$, it means that all of the critical points in the direction u_r from O are non-degenerate, and no two of them lie on the same level hypersurface (i.e. they have different critical values). In particular, $h_{u_r}|_{Q_{r+1}}$ has not critical points in Q_{r+1} . Let $p_i \in \text{Crit}(h_{u_r}|_{\partial Q_{r+1}})$, $i = 1, \dots, m$, be the set of critical points, and

$$\rho_1 < \rho_2 < \dots < \rho_m, \quad (\text{with } \frac{-\pi}{2\sqrt{\lambda}} \leq \rho_1, \quad \rho_m \leq \frac{\pi}{2\sqrt{\lambda}} \quad \text{for } \lambda > 0)$$

the corresponding critical values ($h_{u_r}(p_i) = \rho_i$). To each critical point p_i we assign an index

$$\epsilon_i = \chi(Q_{r+1} \cap L_r^{r+1}(\rho_i - \varepsilon)) - \chi(Q_{r+1} \cap L_r^{r+1}(\rho_i + \varepsilon)), \quad (33)$$

where $L_r^{r+1}(\rho_i + \varepsilon)$ denotes the r -plane defined from the direction u_r at a signed distance $\rho_i + \varepsilon$ from O ; and ε is small enough to ensure that there are no critical points of $\text{Crit}(h_{u_r}|_{\partial Q_{r+1}})$ whose height function belongs to $(\rho_i - \varepsilon, \rho_i + \varepsilon)$.

For $r < n \in \{1, 2, \dots\}$, define:

$$\begin{aligned} I_{n-r-1,r}(\rho) &= \int s_\lambda^{n-r-1}(|\rho|) c_\lambda^r(\rho) d\rho \\ &= \begin{cases} \int s_\lambda^{n-r-1}(\rho) c_\lambda^r(\rho) d\rho, & \rho \geq 0, \\ (-1)^{n-r-1} \int s_\lambda^{n-r-1}(\rho) c_\lambda^r(\rho) d\rho, & \rho < 0. \end{cases} \end{aligned} \quad (34)$$

Then, for $\lambda = 0$,

$$I_{n-r-1,r}(\rho) = \int |\rho|^{n-r-1} d\rho = \begin{cases} \frac{\rho^{n-r}}{n-r}, & \rho \geq 0, \\ (-1)^{n-r-1} \frac{\rho^{n-r}}{n-r}, & \rho < 0. \end{cases} \quad (35)$$

For $\lambda \neq 0$, and for any given pair (n, r) , the integral $I_{n-r-1,r}(\rho)$ may be evaluated explicitly from [13], pages 114 and 159, or with the aid of a mathematical software package such as Mathematica[®].

Theorem 3.2. *Let O be a point in M_λ^n and $Q \subset M_\lambda^n$ a compact domain which is contained in the hemisphere of M_λ^n with pole O when $\lambda > 0$. Let $Q_{r+1} = Q \cap L_{r+1[0]}^n$ be the domain with boundary in $L_{r+1[0]}^n$. Then, for $r \in \{0, 1, \dots, n-2\}$,*

$$\beta_r = \frac{1}{2} \int_{\mathbb{S}^r} \left(\sum_{k=1}^m \epsilon_k I_{n-r-1,r}(\rho_k) \right) du_r, \quad (36)$$

where m represents the number of points $\text{Crit}(h_{u_r}|_{\partial Q_{r+1}})$ corresponding to the direction u_r .

Proof. The fact that Q_{r+1} will be a domain with boundary in $L_{r+1[0]}^n$, for a generic $(r+1)$ -space $L_{r+1[0]}^n$, follows from Theorem A.1 of [10], and the fact that $h_{u_r}|_{Q_{r+1}}$ will in general be a strong Morse function for almost all $u_r \in \mathbb{S}^r$ follows from the appendix, having in mind that Q_{r+1} is contained in the hemisphere of $L_{r+1[0]}^n$ with pole O .

Then Eq.(31) and Eq.(32) may be written as follows,

$$\beta_r = \frac{1}{2} \int_{\mathbb{S}^r} du_r \sum_{k=1}^{m-1} \int_{\rho_k}^{\rho_{k+1}} s_\lambda^{n-r-1} (|\rho|) c_\lambda^r(\rho) \chi((Q \cap L_{r+1[0]}^n) \cap L_r^{r+1}) d\rho, \quad (37)$$

Thus,

$$\begin{aligned} \beta_r &= \frac{1}{2} \int_{\mathbb{S}^r} du_r \sum_{k=1}^{m-1} (I_{n-r-1,r}(\rho_{k+1}) - I_{n-r-1,r}(\rho_k)) \sum_{j=k+1}^m \epsilon_j \\ &= \frac{1}{2} \int_{\mathbb{S}^r} \left(\sum_{k=2}^m \epsilon_k I_{n-r-1,r}(\rho_k) - I_{n-r-1,r}(\rho_1) \sum_{k=2}^m \epsilon_k \right) du_r. \end{aligned} \quad (38)$$

Finally, since $\sum_{k=1}^m \epsilon_k = 0$, it means $\sum_{k=2}^m \epsilon_k = -\epsilon_1$, and the proposed result is obtained. \square

4. Applications

Let $Q \subset M_\lambda^3$ ($\lambda \neq 0$) be a compact domain with smooth boundary $S = \partial Q$; then, from Theorem 2.1 with $n = 3$ and $r = 1$, we have

$$2\pi\chi(S) - \int_S K(x)dx = \frac{2\lambda}{\pi} \int_{\mathcal{L}} \chi(Q \cap L_1^3) dL_1^3, \quad (39)$$

where $K(x)$ is the Gauss curvature of S at x , and χ denotes Euler characteristic.

Now, from Eq.(23) and the definition of β_1 (Eq.(27)), a rotational formula of the *defect* of the surface in $M^3(\lambda)$ is given by

$$2\pi\chi(S) - \int_S K(x)dx = \frac{2\lambda}{\pi} \int_{Q \cap L_{2[0]}^3 \neq \emptyset} \beta_1(Q \cap L_{2[0]}^3) dL_{2[0]}^3, \quad (40)$$

where, using Theorem 3.2,

$$\beta_1(Q \cap L_{2[0]}^3) = \frac{1}{2} \int_{\mathbb{S}^2 \cap L_{2[0]}^3} \sum_{k=1}^m \epsilon_k I_{1,1}(\rho_k) du. \quad (41)$$

Example. Let S be a geodesic sphere of radius ρ centered at O in $M^3(\lambda)$; then, $\chi(S) = 2$, and $\int_{M^2} K(x)dx = 4\pi c_\lambda^2(\rho)$.

On the other hand, $S \cap L_{2[0]}^3$ is a geodesic circle (boundary of a geodesic ball) in $L_{2[0]}^3$; that is, all the points in $S \cap L_{2[0]}^3$ are a distance ρ apart from O . Then, for all directions $u \in \mathbb{S}^1$, $m = 2$, $\epsilon_1 = 1$, $\epsilon_2 = -1$, $I_{1,1}(\rho_1) = I_{1,1}(\rho) = \frac{1}{2}s_\lambda^2(\rho)$ and $I_{1,1}(\rho_2) = I_{1,1}(-\rho) = -\frac{1}{2}s_\lambda^2(\rho)$, $\beta_1(S \cap L_{2[0]}^3) = \pi s_\lambda^2(\rho)$; and Eq.(40) is satisfied.

If we consider a domain Q in \mathbb{R}^3 ($\lambda = 0$), Corollary 2.2, with $r = 1$ and $n = 3$, coincides with Eq.(12) of [6], Theorem 2.1 coincides with Eq.(12) of [6], and, since

$$2\chi(Q_2 \cap L_1^2) = N(\partial Q_2 \cap L_1^2), \quad (42)$$

where N denotes number, Theorem 3.2 coincides with the integrand of Eq.(50) in [6]; but now, for each axial direction $u \in [0, 2\pi)$ in the pivotal plane $L_{2[0]}^3$, the pivotal section is scanned entirely from top to bottom by a sweeping straight line parallel to the axis Ou , in search of critical points.

5. Appendix

Let X be a smooth manifold with boundary. We say that a smooth function $f : X \rightarrow \mathbb{R}$ is a *strong Morse function* if

1. all critical points of $f : X \rightarrow \mathbb{R}$ are non-degenerate and are contained in the interior of X ,
2. all critical points of the restriction $f : \partial X \rightarrow \mathbb{R}$ are also non-degenerate,
3. if $x, y \in X$ are distinct critical points of either $f : X \rightarrow \mathbb{R}$ or $f : \partial X \rightarrow \mathbb{R}$, then $f(x) \neq f(y)$.

5.1. Preliminary results for the Euclidean case ($\lambda = 0$)

Assume now that $X \subset \mathbb{R}^n$ is a submanifold with boundary and for each unit vector $v \in \mathbb{S}^{n-1}$, let us denote by $h_v : X \rightarrow \mathbb{R}$ the height function defined as $h_v(x) = \langle x, v \rangle$.

Theorem 5.1. *Let $X \subset \mathbb{R}^n$ be a compact submanifold with boundary. For almost any $v \in \mathbb{S}^{n-1}$, $h_v : X \rightarrow \mathbb{R}$ is a strong Morse function.*

Proof. We consider $S = X$ or $S = \partial X$ which are compact spaces in \mathbb{R}^n . From Theorem 3 of [14], since $(1, p)$ is in the nice range for all $p = \dim(S)$, the linear map $h_a : S \rightarrow \mathbb{R}$ given by $h_a(x) = \sum_i a_i x_i$ is stable for almost any $a \in \mathbb{R}^n \setminus \{0\}$.

Let $W \subset \mathbb{R}^n \setminus \{0\}$ be the set of points a such that $h_a : S \rightarrow \mathbb{R}$ is not stable. Since W is a null set in $\mathbb{R}^n \setminus \{0\}$, $p(W)$ is a null set in \mathbb{S}^{n-1} , where $p : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{S}^{n-1}$ is the normalization map. Then, for any $v \in \mathbb{S}^{n-1} \setminus p(W)$, $h_v : S \rightarrow \mathbb{R}$ is stable.

In the case of functions, it is well known that stability is equivalent to that all critical points are non-degenerate with distinct critical values (see [4]). Therefore $h_v : X \rightarrow \mathbb{R}$ and $h_v : \partial X \rightarrow \mathbb{R}$ are Morse functions with distinct critical values for almost any $v \in \mathbb{S}^{n-1}$. Since $h_v : X \rightarrow \mathbb{R}$ has not critical points, critical values of $h_v : \partial X \rightarrow \mathbb{R}$ cannot coincide with critical values of $h_v : X \rightarrow \mathbb{R}$. Then, $h_v : X \rightarrow \mathbb{R}$ is a strong Morse function for almost any $v \in \mathbb{S}^{n-1}$. \square

Corollary 5.2. *Let $Q \subset \mathbb{R}^n$ be a compact domain with boundary. For almost any $v \in \mathbb{S}^{n-1}$, $h_v : Q \rightarrow \mathbb{R}$ is a strong Morse function.*

5.2. General case M_λ^n ($\lambda \neq 0$)

Lemma 5.3. *Let $X \subset M_\lambda^n$ be a submanifold and let $\psi : I \rightarrow \mathbb{R}$ be a diffeomorphism, where I is an open interval in \mathbb{R} . If $f : X \rightarrow I$ is a strong Morse function, then $g := \psi \circ f$ is a strong Morse function.*

Proof. Since ψ is a diffeomorphism and f is a strong Morse function, it is deduced that g is also a strong Morse function. Note that the critical points of f coincide with the critical points of g . \square

Let $Q \subset M_\lambda^n$ be a compact domain with boundary, $O \in M_\lambda^n$ and v denote a unit vector in $\mathbb{S}^{n-1} \subset T_O Q$. The geodesic $\gamma_v : I \subset \mathbb{R} \rightarrow Q$ is given by $\gamma_v = c_\lambda(t)O + s_\lambda(t)v$, where $I =] -\frac{\pi}{\sqrt{\lambda}}, \frac{\pi}{\sqrt{\lambda}}[$ for $\lambda > 0$ and $I = \mathbb{R}$ for $\lambda < 0$.

Then, given v , let $h_v : Q \subset M_\lambda^n \rightarrow \mathbb{R}$ be the height function in M_λ^n , whose level hypersurfaces are perpendicular to the geodesic γ_v .

Theorem 5.4. *Let $Q \subset M_\lambda^n$ be a compact domain with boundary which, for $\lambda > 0$, it is contained in the hemisphere of M_λ^n with pole O . Then, for almost any $v \in \mathbb{S}^{n-1}$, $h_v : Q \rightarrow \mathbb{R}$ is a strong Morse function.*

Proof. It is useful to consider the embedding of the space form M_λ^n into $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle_\lambda)$ as follows:

$$\begin{cases} x_0 = 1, & \lambda = 0, \\ x_0^2 + x_1^2 + \dots + x_n^2 = \frac{1}{\lambda}, & \lambda > 0, \\ -x_0^2 + x_1^2 + \dots + x_n^2 = \frac{1}{\lambda}, \quad x_0 > 0, & \lambda < 0, \end{cases} \quad (43)$$

where (x_0, x_1, \dots, x_n) denote the coordinates of a point in \mathbb{R}^{n+1} , and $\langle \cdot, \cdot \rangle_\lambda$ is the appropriate metric to the embedding, which depends on the sign of λ .

Using this embedding, $Q \subset M_\lambda^n \subset \mathbb{R}^{n+1}$ can be considered as a compact submanifold with boundary in \mathbb{R}^{n+1} . Then, the height function of \mathbb{R}^{n+1} with respect to the direction v , restricted to Q is:

$$\begin{aligned} h_{v,\lambda}^{\mathbb{R}^{n+1}} : \quad Q &\longrightarrow \mathbb{R} \\ x &\longrightarrow \langle x, v \rangle_\lambda \end{aligned} \quad (44)$$

From Theorem 5.1, $h_{v,\lambda}^{\mathbb{R}^{n+1}}$ is a strong Morse function for almost any $v \in S_+^{n-1}$. Moreover, we note $h_{v,\lambda}^{\mathbb{R}^{n+1}}(Q) \subset \bar{I}$.

Since $\langle v, O \rangle_\lambda = 0$, we have that,

$$h_{v,\lambda}^{\mathbb{R}^{n+1}}(\gamma_v(\rho)) = \langle \gamma_v(\rho), v \rangle_\lambda = s_\lambda(\rho) = \begin{cases} \lambda^{-1/2} \sin(\rho\sqrt{\lambda}), & \lambda > 0, \\ |\lambda|^{-1/2} \sinh(\rho\sqrt{|\lambda|}), & \lambda < 0. \end{cases} \quad (45)$$

Eq.(45) gives a relation between the height function $h_v(\gamma_v(\rho)) = \rho$ of Q in M_λ^n and the height function $h_{v,\lambda}^{\mathbb{R}^{n+1}}$ of Q in \mathbb{R}^{n+1} . That is,

$$h_v(x) = \psi(h_{v,\lambda}^{\mathbb{R}^{n+1}}(x)) = \begin{cases} \frac{1}{\sqrt{\lambda}} \arcsin(\sqrt{\lambda} h_{v,\lambda}^{\mathbb{R}^{n+1}}(x)), & \lambda > 0, \\ \frac{1}{\sqrt{-\lambda}} \operatorname{arcsinh}(\sqrt{-\lambda} h_{v,\lambda}^{\mathbb{R}^{n+1}}(x)), & \lambda < 0. \end{cases} \quad (46)$$

Finally, since Q is contained in the hemisphere of M_λ^n with pole O for $\lambda > 0$, we have that ψ is a diffeomorphism from I to \mathbb{R} when $I =] -\frac{\pi}{\sqrt{\lambda}}, \frac{\pi}{\sqrt{\lambda}}[$ for $\lambda > 0$ and when $I = \mathbb{R}$ for $\lambda < 0$; therefore from Lemma 5.3 we obtain the result. \square

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