

# WEIGHT-PRESERVING ISOMORPHISMS BETWEEN SPACES OF CONTINUOUS FUNCTIONS: THE SCALAR CASE

MARITA FERRER, MARGARITA GARY, AND SALVADOR HERNÁNDEZ

ABSTRACT. Let  $\mathbb{F}$  be a finite field (or discrete) and let  $\mathcal{A}$  and  $\mathcal{B}$  be vector spaces of  $\mathbb{F}$ -valued continuous functions defined on locally compact spaces  $X$  and  $Y$ , respectively. We look at the representation of linear bijections  $H : \mathcal{A} \rightarrow \mathcal{B}$  by continuous functions  $h : Y \rightarrow X$  as weighted composition operators. In order to do it, we extend the notion of Hamming metric to infinite spaces. Our main result establishes that under some mild conditions, every Hamming isometry can be represented as a weighted composition operator. Connections to coding theory are also highlighted.

## 1. INTRODUCTION

In this paper, we are concerned with the representation of linear isomorphisms defined on spaces of continuous functions taking values in a vector space  $\mathbb{F}^n$  over a finite field  $\mathbb{F}$ . The starting point, and our main motivation, stems from two very celebrated, and apparently disconnected, results, whose formulation is strikingly similar, namely: MacWilliams Equivalence Theorem and Banach-Stone Theorem. The former one completely describes the isometries between block codes (see [25, 26]). For the reader's sake, we recall its main features here.

Let  $\mathbb{F}$  be a finite field. Two linear codes  $C_1$  and  $C_2$  over  $\mathbb{F}$  of length  $n$  are *equivalent* if there is a monomial transformation  $H$  of  $\mathbb{F}^n$  such that  $H(C_1) = C_2$ . Here, a monomial

---

*Date:* 27 January 2015.

The first and third listed authors acknowledge partial support by the Generalitat Valenciana, grant code: PROMETEO/2014/062; and by Universitat Jaume I, grant P1·1B2012-05.

*2010 Mathematics Subject Classification.* Primary 46E10. Secondary 54C35, 47B38, 93B05, 94B10  
*Key Words and Phrases:* Banach-Stone Theorem, MacWilliams Equivalence Theorem, separating map, Hamming isometry, weighted composition operator, weight-preserving isomorphism, representation of linear isomorphisms, continuous functions taking values in a finite field.

transformation is a linear isomorphism  $H$  of the form

$$H(a_1, \dots, a_n) = (a_{\sigma(1)}w_1, \dots, a_{\sigma(n)}w_n), \quad (a_1, \dots, a_n) \in \mathbb{F}^n,$$

where  $\sigma$  is a permutation of  $\{1, 2, \dots, n\}$  and  $(w_1, \dots, w_n) \in (\mathbb{F} \setminus \{0\})^n$ .

The Hamming weight  $\text{wt}(x)$  of a vector  $x \in \mathbb{F}^n$  is defined as the number of coordinates that are different from zero. The following classical result establishes the relation between Hamming isometries and equivalent codes.

**Theorem 1.1** (MacWilliams). *Two linear codes  $C_1, C_2$  of dimension  $k$  in  $\mathbb{F}^n$  are equivalent if and only if there exists an abstract  $\mathbb{F}$ -linear isomorphism  $f : C_1 \rightarrow C_2$  which preserves weights,  $\text{wt}(f(x)) = \text{wt}(x)$ , for all  $x \in C_1$ .*

Hence, two block codes are *isometric* if and only if they are monomially equivalent. More precisely, if  $H$  is a weight-preserving isomorphism between two codes  $C_1$  and  $C_2$ , then  $H = W \cdot P$ , where  $W = \text{diag}(w_i)$  and  $P$  is a permutation matrix.

This fundamental result has been extended in different directions by many workers (cf. [6, 10, 31, 33]). In particular, Heide Gluesing-Luerssen has established a variant of MacWilliams theorem for 1-dimensional convolutional codes and the isometries defined between them that respect the module structure of the codes (see [21]). It remains open the representation of general  $\mathbb{F}$ -isometries defined between convolutional codes (cf. [21] and [28, Ch. 8]).

The second result we are concerned in this paper, the Banach-Stone Theorem, establishes that every linear isometry defined between the spaces of continuous functions of two compact spaces is a weighted composition operator (see [5, 30]). This celebrated theorem has now become a classical result that has been extended in many ways. Even though we approach this research using techniques of separating (disjoint preserving)

maps, we refer to the volumes by Fleming and Jamison [14, 15] and the survey article [19], which contain a current and comprehensive exposition on this topic.

**Theorem 1.2** (Banach-Stone Theorem). *Let  $X$  and  $Y$  be compact spaces and let  $H : C(X) \rightarrow C(Y)$  be a linear isometry. Then  $X$  and  $Y$  are homeomorphic and the isometry  $H$  has the following form: there is a homeomorphism  $h : Y \rightarrow X$ , and a scalar-valued continuous function  $w$  on  $C(Y)$  such that*

$$Hf(y) = w(y)f(h(y)), \quad \forall f \in C(X), \quad \forall y \in Y.$$

The analogy between MacWilliams and Banach-Stone theorems is blatant and our motivation has been to explore the application of functional analysis methods in order to extend MacWilliams Equivalence Theorem to a more general setting. All in all, there is a clear difference between these two important theorems. While, MacWilliams theorem applies to any two vector subspaces of  $\mathbb{F}^n$ , the Banach-Stone theorem, and most of its variants and generalizations, deal with *algebraic or analytical subspaces* that separate the points of the topological spaces where they are defined. In the presence of infinite topological spaces, the former approach takes us to more elaborated (and perhaps less elegant) results. However, this point of view raises the question of representing linear operator defined between general vector subspaces of continuous mappings without the constraint of separating points. Our overall goal is to clarify this question in this and subsequent papers. We are also concerned with the possible application of this approach to describe  $\mathbb{F}$ -isomorphisms defined between (possibly multi-dimensional) convolutional codes. In this sense, we include here an application of our results for discrete spaces. Finally, even though we have been concerned with finite fields along this paper, we remark that all results extend, *mutatis mutandis*,

for general discrete fields without any essential modification in the arguments (this is because we only work with compactly supported functions). We leave the verification of this fact to the interested reader (cf. [12]).

In the sequel, we look at continuous mappings defined on a locally compact space  $X$ . Since we are concerned here with finite fields, it is clear that we can assume without loss of generality that  $X$  is totally disconnected. Furthermore, being a locally compact space, it follows that  $X$  is also 0-dimensional. Thus, let  $X$  be a 0-dimensional locally compact space, equipped with a Borel regular, strictly positive, measure  $\mu$ , and let  $C_{00}(X, \mathbb{F}^n)$  designate the space of  $\mathbb{F}$ -valued, compactly supported, continuous functions defined on  $X$ . For any  $f \in C_{00}(X, \mathbb{F}^n)$  and  $x \in X$ , we define

$$\text{wt}(f(x)) \stackrel{\text{def}}{=} |\{j : \pi_j(f(x)) \neq 0\}|$$

and

$$\text{wt}(f) \stackrel{\text{def}}{=} \int_X \text{wt}(f(x)) d\mu(x).$$

Notice that this integral is finite because  $\text{wt}(f(x))$  is continuous and has compact support. Moreover, in the scalar case, i.e.  $n = 1$ , the *weight* of a function coincides with the measure of its support set, namely,

$$\text{wt}(f) = \mu(\text{supp}(f)).$$

The map

$$d(f, g) \stackrel{\text{def}}{=} \text{wt}(f - g)$$

defines a metric on the vector space  $C_{00}(X, \mathbb{F}^n)$  that is compatible with its additive group structure. Since this metric extends the well known distance introduced by Hamming in coding theory, we call it *Hamming metric*.

**Definition 1.3.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be vector subspaces of  $C_{00}(X, \mathbb{F}^n)$  and  $C_{00}(Y, \mathbb{F}^n)$ , where  $X$  and  $Y$  are 0-dimensional locally compact spaces equipped with Borel regular measures  $\mu_X$  and  $\mu_Y$ , respectively.

A linear map  $H : \mathcal{A} \rightarrow \mathcal{B}$  is called *Hamming isometry* if it is a linear isomorphism such that  $\text{wt}(f) = \text{wt}(Hf)$  for each  $f \in \mathcal{A}$ .

The map  $H$  is called *weighted composition operator* when there exist continuous functions  $h : Y \rightarrow X$  and  $w : Y \rightarrow \mathbb{F}$  such that  $Hf(y) = w(y)f(h(y))$  for all  $y \in Y$  and  $f \in \mathcal{A}$ .

Along this paper, we deal with vector subspaces of continuous functions that do not necessarily separate the points of the topological spaces where the functions are defined. Furthermore, this feature is essential in our approach as we have explained above. Since it is impossible to distinguish among the points that may not be separated by the functions we deal with, we need a more general definition of weighted composition operator in order to tackle this difficulty.

**Definition 1.4.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be vector subspaces of  $C_{00}(X, \mathbb{F}^n)$  and  $C_{00}(Y, \mathbb{F}^n)$ , respectively. We say that  $H : \mathcal{A} \rightarrow \mathcal{B}$  is a *general weighted composition operator* when there is a quotient map  $\pi : X \rightarrow \tilde{X}$ , continuous maps  $h : Y \rightarrow \tilde{X}$  and  $\omega : Gr[h] \rightarrow \mathbb{F}$  satisfying

$$Hf(y) = \omega(x, y)f(x)$$

for each  $y \in Y$ ,  $x \in h(y)$ , and every  $f \in \mathcal{A}$ .

Here

$$Gr[h] \stackrel{\text{def}}{=} \bigcup_{y \in Y} (h(y) \times \{y\})$$

denotes the graphic of  $h$  equipped with the topology inherited as a subspace of  $X \times Y$ .

The main question we address in this research is as follows:

**Question 1.5.** *Is every Hamming isometry  $H : \mathcal{A} \rightarrow \mathcal{B}$  representable as a general weighted composition operator?*

We now introduce some pertinent notions and terminology.

All spaces are assumed to be 0-dimensional and Hausdorff and throughout this paper the symbol  $\mathbb{F}$  denotes a finite (or discrete) field. If  $X$  is a locally compact space, then  $X^*$  denotes the *Alexandroff compactification* of  $X$ , that is,  $X^* = X \cup \{\infty\}$ , being  $\infty$  the point at infinity.

For  $f \in C(X, \mathbb{F}^n)$ , set

$$\text{coz}(f) \stackrel{\text{def}}{=} \{x \in X : f(x) \neq 0\}.$$

Since  $\mathbb{F}^n$  is discrete  $\text{coz}(f)$  and  $Z(f) = X \setminus \text{coz}(f)$  are open and closed (clopen) subsets of  $X$ .

Let  $\mathcal{A}$  be a linear subspace of  $C_{00}(X, \mathbb{F}^n)$ . For  $x \in X$ , let  $\delta_x : \mathcal{A} \rightarrow \mathbb{F}^n$  be the canonical *evaluation map*

$$\delta_x(f) \stackrel{\text{def}}{=} f(x) \quad \forall f \in \mathcal{A}.$$

and

$$I_x \stackrel{\text{def}}{=} \{f \in \mathcal{A} : f(x) = 0\}.$$

Set

$$S \stackrel{\text{def}}{=} \{x \in X : I_x \neq \mathcal{A}\} = \bigcup_{f \in \mathcal{A}} \text{coz}(f).$$

Therefore  $S$  is an open subset of  $X$  and, as a consequence, is also a locally compact space when it is equipped with the topology inherited from  $X$ . Hence we assume

WLOG that  $S = X$  throughout this paper. Thus, for each linear subspace of continuous functions considered from here on, it is assumed:

- (1) for every  $x \in X$  there exists  $f \in \mathcal{A}$  such that  $f(x) \neq 0$ .

Define  $Z(\mathcal{A}) \stackrel{\text{def}}{=} \{Z(f) : f \in \mathcal{A}\}$ ,  $\text{coz}(\mathcal{A}) \stackrel{\text{def}}{=} \{\text{coz}(f) : f \in \mathcal{A}\}$ , and let  $\mathcal{D}$  denote the smallest ring (with respect to finite unions and intersections) of subsets containing  $\text{coz}(\mathcal{A})$ .

In coding theory, it is said that a convolutional code is *controllable* when any code sequence can be reached from the zero sequence in a finite interval (see [13, 18, 29, 32]). The gist of controllability can be conveyed in a natural way to subspaces of continuous functions defined on a topological space. In an informal way, let us say that a vector subspace of continuous functions is controllable when any continuous functions can be reached from the zero function modulo a relatively compact open subset. It turns out that this notion is an essential ingredient in the approach we have taken in this paper.

**Definition 1.6.** We say that  $\mathcal{A}$  is *controllable* if for every  $f \in \mathcal{A}$  and  $D_1, D_2 \in \mathcal{D}$  with  $D_1 \cap D_2 = \emptyset$ , there exist  $f' \in \mathcal{A}$  and  $U \in \mathcal{D}$  such that

$$D_1 \subseteq U \subseteq X \setminus D_2, f|_{D_1} = f'|_{D_1}, \text{ and } f'|_{(Z(f) \cup (X \setminus U))} = 0.$$

We say that  $\mathcal{A}$  *separates* the points  $x_1, x_2 \in X$ , if there is  $f \in \mathcal{A}$  such that  $x_1 \in \text{coz}(f)$  and  $x_2 \in Z(f)$  or vice versa.

Along this paper, we deal with scalar-valued functions. The case of vector-valued functions will be considered in a subsequent paper. We now formulate the main result in this paper.

**Theorem 1.7.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  two vector spaces of  $\mathbb{F}$ -valued, compactly supported, continuous functions defined on two locally compact spaces  $X$  and  $Y$ , which are equipped with a Borel regular measures  $\mu_X$  and  $\mu_Y$ . If  $\mathcal{A}$  is controllable, then every Hamming isometry  $H : \mathcal{A} \rightarrow \mathcal{B}$  is a general weighted composition operator.*

As a consequence, it follows the following representation, as weighted composition operators, of Hamming isometries defined between vector subspaces of  $\mathbb{F}^X$  and  $\mathbb{F}^Y$  when  $X$  and  $Y$  are two discrete spaces and  $\mu_X$  and  $\mu_Y$  are the *counting measures* defined on them.

**Corollary 1.8.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  two vector spaces of  $\mathbb{F}$ -valued, finitely supported, functions defined on two discrete spaces  $X$  and  $Y$ . If  $\mathcal{A}$  is controllable, then every Hamming isometry  $H : \mathcal{A} \rightarrow \mathcal{B}$  is a weighted composition operator.*

We remark that convolutional codes are shift invariant subspaces of  $\mathbb{F}^X$  with  $X = \mathbb{Z}^k$ . The isometries considered by Gluesin-Luerssen in [21] are module homomorphisms with respect to the polynomial ring  $\mathbb{F}[z]$ . Here, we are considering the more general case of  $\mathbb{F}$ -linear isometries.

## 2. BASIC NOTIONS AND FACTS

In this section, we introduce some topological notions that will be needed in the rest of the paper. Some basic properties connecting them are also established.



**Definition 2.1.** Two points  $x_1$  and  $x_2$  in  $X$  are *related*, written  $x_1 \sim x_2$ , if for every  $f \in \mathcal{A}$  with  $f(x_1) \cdot f(x_2) = 0$ , it follows that  $f(x_1) = f(x_2) = 0$ . Let  $\tilde{X}$  be the set of equivalence classes  $X/\sim$  equipped with the quotient topology inherited from  $X$ . Every element  $\tilde{x} \in \tilde{X}$  is associated to the coset subset  $[x] \subseteq X$  consisting of all elements related to  $x$ . For simplicity's sake, we shall use the same symbol  $[x]$  to denote either the coset  $[x]$  or the element  $\tilde{x} \in \tilde{X}$ . Remark that  $I_{x_1} = I_{x_2}$  for every  $x_1$  and  $x_2$  belonging to the same coset.

**Proposition 2.2.** *Let  $[x]$  be an equivalence class in  $X$  and let  $x_1, x_2 \in [x]$ . Then there is a unique element  $\lambda(x_1, x_2) \in \mathbb{F} \setminus \{0\}$  such that  $f(x_1) = \lambda(x_1, x_2)f(x_2)$  for all  $f \in \mathcal{A}$ .*

*Proof.* We know that  $\mathcal{A} \setminus I_x \neq \emptyset$  by (1). On the other hand, if  $f \in \mathcal{A} \setminus I_x$ , it follows that  $[x] \subseteq \text{coz}(f)$ . Pick out  $x_1, x_2 \in [x]$ . Since  $f(x_1) = f(x_1)f(x_2)^{-1}f(x_2)$ , we define

$$\lambda_f(x_1, x_2) = f(x_1)f(x_2)^{-1},$$

which yields  $f(x_1) = \lambda_f(x_1, x_2)f(x_2)$ . It will suffice to verify that  $\lambda_f(x_1, x_2)$  does not depend on the selected  $f$  in  $\mathcal{A} \setminus I_x$ . Indeed, let  $g \in \mathcal{A} \setminus I_x$ . Then  $g(x_1) = \lambda_g(x_1, x_2)g(x_2)$ . The map  $h \stackrel{\text{def}}{=} f(x_2)^{-1}f - g(x_2)^{-1}g \in \mathcal{A}$  and  $h(x_2) = 0$ . Therefore  $[x] \subseteq Z(h)$  and

$$\begin{aligned} 0 &= h(x_1) \\ &= f(x_2)^{-1}f(x_1) - g(x_2)^{-1}g(x_1) \\ &= f(x_2)^{-1}\lambda_f(x_1, x_2)f(x_2) - g(x_2)^{-1}\lambda_g(x_1, x_2)g(x_2) \\ &= \lambda_f(x_1, x_2) - \lambda_g(x_1, x_2). \end{aligned}$$

As a consequence

$$\lambda_f(x_1, x_2) = \lambda_g(x_1, x_2) = \lambda(x_1, x_2) \in \mathbb{F} \setminus \{0\}.$$

□

It is readily seen that the map  $\lambda(, )$  has the following properties:

- $\lambda(x_2, x_1) = \lambda(x_1, x_2)^{-1}$ ,
- $\lambda(x_1, x_2) = \lambda(x_1, x)\lambda(x, x_2)$ .

**Lemma 2.3.** *If  $x_1, x_2 \in X$  and  $x_1 \not\sim x_2$ , then there is  $f_{x_1x_2}$  such that  $x_1 \in \text{coz}(f_{x_1x_2})$  and  $x_2 \in Z(f_{x_1x_2})$ .*

*Proof.* Since  $x_1 \not\sim x_2$  there is  $f \in \mathcal{A}$  such that  $f(x_1)f(x_2) = 0$  and  $f(x_1) \neq 0$  or  $f(x_2) \neq 0$ . If  $f(x_1) \neq 0$  and  $f(x_2) = 0$ , then  $f_{x_1x_2} = f$  and we are done. Otherwise, by (1), there is  $g \in \mathcal{A}$  such that  $g(x_1) \neq 0$ . Set  $h \stackrel{\text{def}}{=} g(x_2)f - f(x_2)g \in \mathcal{A}$ . Then  $h(x_2) = 0$  and  $h(x_1) = -f(x_2)g(x_1) \neq 0$ . In this case  $f_{x_1x_2} = h$ .  $\square$

**Definition 2.4.**  $A \subseteq X$  is called *saturated* if and only if  $x \in A$  implies  $[x] \subseteq A$ .

The proof of the next result is easy. We include it for the sake of completeness.

**Proposition 2.5.** *For every  $f \in \mathcal{A}$  and  $x \in X$ , we have:*

- (a)  $\text{coz}(f)$  and  $Z(f)$  are saturated subsets of  $X$ .
- (b)  $[x]$  is a saturated compact subset of  $X$ .

*Proof.* The proof of (a) is clear. (b) Let  $x \in X$ . We first proof that  $[x]$  is closed in  $X$ . Let  $x' \in X \setminus [x]$ . By Lemma 2.3 there is  $f \in \mathcal{A}$  such that  $x' \in \text{coz}(f)$  and  $x \in Z(f)$ . Applying (a), it follows that  $[x'] \subseteq \text{coz}(f)$  and  $[x] \subseteq Z(f)$ . Then  $x' \in \text{coz}(f) \subseteq X \setminus [x]$  and  $\text{coz}(f)$  is open in  $X$ .

On the other hand, by (1), there is  $g \in \mathcal{A}$  such that  $[x] \subseteq \text{coz}(g)$ . Since  $\text{coz}(g)$  is compact and  $[x]$  is closed in  $X$ , we have that  $[x]$  is compact.  $\square$

Let  $\pi: X \rightarrow \tilde{X}$  denote the canonical quotient map associated to the equivalence relation  $\sim$  and equip  $\tilde{X}$  with the canonical quotient topology. Using Proposition 2.5, it is easily seen that the subsets  $\pi(\text{coz}(f))$  and  $\pi(Z(f))$  are clopen in  $\tilde{X}$  for every  $f \in \mathcal{A}$  and, with a little more effort, it is proved that  $\tilde{X}$  is a Hausdorff, locally compact space. We leave the verification of this fact to the interested reader.

A standard compactness argument is used in the proof of the following lemma. We include it here for the sake of completeness.

**Lemma 2.6.** *Let  $K_1$  and  $K_2$  be compact subsets of  $X$  such that  $x_1 \not\sim x_2$  for every  $x_1 \in K_1$  and  $x_2 \in K_2$ . Then there are  $D_1, D_2 \in \mathcal{D}$  such that  $K_1 \subseteq D_1$ ,  $K_2 \subseteq D_2$  and  $D_1 \cap D_2 = \emptyset$ .*

*Proof.* Let  $x_1 \in K_1$  and  $x \in K_2$ , which implies  $x_1 \not\sim x$ . By Lemma 2.3, there is  $f_x \in \mathcal{A}$  such that  $[x_1] \subseteq \text{coz}(f_x)$  and  $[x] \subseteq Z(f_x)$ . We have  $K_2 \subseteq \bigcup_{[x] \in \pi(K_2)} Z(f_x)$  and  $[x_1] \subseteq$

$\bigcap_{[x] \in \pi(K_2)} \text{coz}(f_x)$ . Since  $K_2$  is compact and  $Z(f_x)$  is open, we have  $K_2 \subseteq \bigcup_{i=1}^n Z(f_{x^{(i)}})$  and  $[x_1] \subseteq \bigcap_{i=1}^n \text{coz}(f_{x^{(i)}}) = X \setminus \bigcup_{i=1}^n Z(f_{x^{(i)}}) \subseteq X \setminus K_2$ .

Define  $C_{x_1} = \bigcap_{i=1}^n \text{coz}(f_{x^{(i)}})$ , which is a clopen subset of  $X$ . Remark that  $[x_1] \subseteq C_{x_1}$  and  $C_{x_1} \cap K_2 = \emptyset$ . Consequently  $K_1 \subseteq \bigcup_{[x] \in \pi(K_1)} C_x$  and  $C_x \cap K_2 = \emptyset$  for every  $[x] \in \pi(K_1)$ .

Since  $K_1$  is compact, we have  $K_1 \subseteq \bigcup_{j=1}^m C_{x^{(j)}}$ .

Define  $D_1 = \bigcup_{j=1}^m C_{x^{(j)}} \in \mathcal{D}$  and observe that  $K_1 \subseteq D_1$  and  $D_1 \cap K_2 = \emptyset$ . Since  $D_1$  is a saturated compact subset of  $X$ , we repeat again the same procedure in order to obtain  $D_2 \in \mathcal{D}$  such that  $K_2 \subseteq D_2$  and  $D_1 \cap D_2 = \emptyset$ .  $\square$

We notice that the lemma above applies to any two disjoint saturated compact subsets of  $X$ . On the other hand, the following remark is easily seen.

*Remark 2.7.* Every  $D \in \mathcal{D}$  is a saturated compact subset of  $X$  and  $\pi(D)$  is clopen in  $\tilde{X}$ . Furthermore, the collection  $\{\pi(D) : D \in \mathcal{D}\}$  is an open base for  $\tilde{X}$ .

### 3. SEPARATING MAPS AND SUPPORT SUBSETS

**Definition 3.1.** A map  $H : \mathcal{A} \rightarrow \mathcal{B}$  is said to be *separating* (or *disjointness preserving*) when  $\text{coz}(f) \cap \text{coz}(g) = \emptyset$  implies  $\text{coz}(Hf) \cap \text{coz}(Hg) = \emptyset$ ,  $f, g \in \mathcal{A}$ .

A linear functional  $\varphi : \mathcal{A} \rightarrow \mathbb{F}$  is called *separating* when  $\text{coz}(f) \cap \text{coz}(g) = \emptyset$  implies  $\varphi(f) \cdot \varphi(g) = 0$ . The link between weight-preserving isomorphisms and separating maps is given by the next lemma. It follows easily taking into account that the weight of a function coincides with the measure of its support set. We sketch the proof for the sake of completeness.

**Lemma 3.2.** *Let  $f$  and  $g$  be two elements in  $\mathcal{A}$ . Then  $\text{coz}(f) \cap \text{coz}(g) = \emptyset$  if and only if  $\text{wt}(f + g) = \text{wt}(f) + \text{wt}(g)$ .*

*Proof.* It is obvious that  $\text{coz}(f) \cap \text{coz}(g) = \emptyset$  implies  $\text{wt}(f + g) = \text{wt}(f) + \text{wt}(g)$ . On the other hand, assume that  $\text{wt}(f + g) = \text{wt}(f) + \text{wt}(g)$ . From the inequality

$$\text{wt}(f + g) \leq \text{wt}(f) + \text{wt}(g) - \text{wt}(f \cdot g)$$

it follows that  $\text{wt}(f \cdot g) = 0$ , which implies  $\text{coz}(f) \cap \text{coz}(g) = \emptyset$ .  $\square$

**Corollary 3.3.** *Every Hamming isometry is a separating linear isomorphism.*

Separating isomorphisms have been studied by many workers and have found application to a variety of fields (cf. [1, 2, 3, 4, 7, 8, 9, 16, 17, 20, 22, 23, 24]). After Corollary 3.3, it is clear that, in order to prove Theorem 1.7, it suffices to deal with the broader case of separating isomorphisms and so we do in the rest of the paper.

The following definition makes sense for every subset of  $X$  but we have restricted it to saturated subsets, because it will only be applied to these subsets in this paper.

**Definition 3.4.** Let  $\varphi : \mathcal{A} \rightarrow \mathbb{F}$  be a map. A saturated closed subset  $K$  of  $X$  is said to be a *support* for  $\varphi$  if given  $f \in \mathcal{A}$  with  $K \subseteq Z(f)$ , it holds that  $\varphi(f) = 0$ .

Support subsets enjoy several nice properties.

**Proposition 3.5.** *Let  $\varphi : \mathcal{A} \rightarrow \mathbb{F}$  be a non null, separating, linear functional. Then the following assertions hold:*

- (a)  $X$  is a support for  $\varphi$ .
- (b) If  $K$  is a support for  $\varphi$  then  $K \neq \emptyset$ .
- (c) Let  $K$  be a support for  $\varphi$  and  $f, g \in \mathcal{A}$  such that  $f|_K = g|_K$ . Then  $\varphi(f) = \varphi(g)$ .
- (d) If  $\mathcal{A}$  is controllable and  $K_1$  and  $K_2$  are both supports for  $\varphi$ , then  $K_1 \cap K_2 \neq \emptyset$ .

*Proof.* (a) This is clear.

(b) Let  $K$  be a support for  $\varphi$  and suppose  $K = \emptyset$ . Then  $K = \emptyset \subseteq Z(f)$  for all  $f \in \mathcal{A}$ . Consequently  $\varphi(f) = 0$  for all  $f \in \mathcal{A}$ , which is a contradiction since  $\varphi$  is non null.

(c) Let  $K$  be a support for  $\varphi$ . If  $f, g \in \mathcal{A}$  and  $f|_K = g|_K$  then  $f - g \in \mathcal{A}$  and  $K \subseteq Z(f - g)$ . So  $0 = \varphi(f - g) = \varphi(f) - \varphi(g)$ .

(d) Let  $K_1$  and  $K_2$  be supports for  $\varphi$  and suppose that  $K_1 \cap K_2 = \emptyset$ . Since  $\varphi$  is non null, there is  $f \in \mathcal{A}$  such that  $\varphi(f) \neq 0$ . Remark that the set  $C_1 = \text{coz}(f) \cap K_1 \neq \emptyset$  because, otherwise,  $K_1 \subseteq Z(f)$  and then  $\varphi(f) = 0$ , which is not true. Since  $\text{coz}(f)$  is a saturated compact subset of  $X$  and  $K_1$  is also saturated and closed, it follows that  $C_1$  is a saturated compact subset of  $X$ . In like manner  $C_2 = \text{coz}(f) \cap K_2$  is non empty, saturated and compact. Furthermore  $C_1 \cap C_2 = \emptyset$  and by Lemma 2.6

there exist  $D_1, D_2 \in \mathcal{D}$  such that  $C_1 \subseteq D_1$ ,  $C_2 \subseteq D_2$  and  $D_1 \cap D_2 = \emptyset$ . Applying that  $\mathcal{A}$  is controllable to  $D_1$ ,  $D_2$  and  $f$ , we obtain  $U \in \mathcal{D}$  and  $f' \in \mathcal{A}$  such that  $C_1 \subseteq D_1 \subseteq U \subseteq X \setminus D_2 \subseteq X \setminus C_2$  and  $f|_{D_1} = f'|_{D_1}$  and  $f'|_{(Z(f) \cup (X \setminus U))} = 0$ .

Remark that  $\text{coz}(f) = C_1 \cup C_2 \cup (\text{coz}(f) \setminus (C_1 \cup C_2))$ . Evaluating  $f'$  yields:

If  $x \in C_1$  then  $f'(x) = f(x)$ .

If  $x \in K_1 \setminus C_1$  then  $f'(x) = 0 = f(x)$ .

If  $x \in K_2$  then  $f'(x) = 0$ .

As a consequence  $f'|_{K_1} = f|_{K_1}$  and  $f'|_{K_2} = 0$ . Applying (c), we deduce that  $\varphi(f') = \varphi(f) \neq 0$  and  $\varphi(f') = 0$ , which is a contradiction. This completes the proof.

□

Next it is proved that, when  $\mathcal{A}$  is controllable, every non null, separating, linear functional  $\varphi : \mathcal{A} \rightarrow \mathbb{F}$  has a minimum support set consisting of an equivalence class  $[x]$ . For that purpose, we define

$$\mathcal{S} = \{A \subseteq X : A \text{ is support for } \varphi\}.$$

It easily seen that  $\mathcal{S}$  has a  $\subseteq$ -minimal element  $K$ . Indeed, just take the intersection of all support sets. It follows from an easy compactness argument (each function  $f$  is compactly supported) that the intersection is again a (closed and saturated) support set.

**Proposition 3.6.** *Let  $\varphi : \mathcal{A} \rightarrow \mathbb{F}$  be a non null, separating, linear functional. If  $\mathcal{A}$  is controllable, then there exists  $x \in X$  such that  $K = [x]$  is a support for  $\varphi$ .*

*Proof.* By Proposition 3.5  $K \neq \emptyset$ . Suppose now that there are two different cosets  $[x_1], [x_2]$  that are contained in  $K$ . Since  $X$  is Hausdorff and  $K$  is saturated, using

Lemma 2.6, we can select two disjoint saturated open sets  $V_1, V_2 \subseteq X$  such that  $[x_1] \subseteq V_1$  and  $[x_2] \subseteq V_2$ . Since  $K$  is minimal, the subset  $K \setminus V_i$  is a saturated closed subset of  $X$  that is not a support for  $\varphi$ . Hence, there is  $f_i \in \mathcal{A}$  such that  $K \setminus V_i \subseteq Z(f_i)$  and  $\varphi(f_i) \neq 0$ ,  $1 \leq i \leq 2$ . As  $\varphi$  is a separating functional, the subset  $A = \text{coz}(f_1) \cap \text{coz}(f_2)$  is a nonempty saturated compact subset of  $X$ . We claim that  $K \cap A = \emptyset$ . Indeed, otherwise, pick out an element  $a \in K \cap A$ . Then  $[a] \subseteq K \cap A$ . If  $[a] \subseteq V_1$  then  $[a] \subseteq K \setminus V_2$  and  $[a] \subseteq Z(f_2)$ , which is a contradiction. On the other hand, if  $[a] \not\subseteq V_1$  then  $[a] \subseteq K \setminus V_1$  and  $[a] \subseteq Z(f_1)$ , which is a contradiction again. Therefore, we have proved that  $K \cap A = \emptyset$ .

Take now  $B = K \cap (\text{coz}(f_1) \cup \text{coz}(f_2))$ . If  $B = \emptyset$  then  $K \cap \text{coz}(f_i) = \emptyset$  and  $K \subseteq Z(f_i)$ , which implies  $\varphi(f_i) = 0$ ,  $1 \leq i \leq 2$ , and we obtain a contradiction. Therefore, we have  $B \neq \emptyset$ . Thus  $B$  is a saturated compact subset of  $X$  satisfying that  $A \cap B = \emptyset$ . Applying Lemma 2.6, we can select two disjoint subsets  $D_A, D_B \in \mathcal{D}$  such that  $A \subseteq D_A$  and  $B \subseteq D_B$ . Applying that  $\mathcal{A}$  is controllable to  $D_A, D_B$  and  $f_1$ , we can take  $U \in \mathcal{D}$  and  $f' \in \mathcal{A}$  such that  $B \subseteq D_B \subseteq U \subseteq X \setminus D_A \subseteq X \setminus A$ , which implies  $U \cap A = \emptyset$ ,  $f_{1|D_B} = f'_{|D_B}$  and  $f'_{|(Z(f_1) \cup (X \setminus U))} = 0$ .

Let us see that  $f'_{|K} = f_{1|K}$ . Indeed, if  $x \in K \setminus \text{coz}(f_1)$  then  $f'(x) = 0 = f_1(x)$  and if  $x \in K \cap \text{coz}(f_1) \subseteq D_B$  then  $f'(x) = f_1(x) \neq 0$ . By Proposition 3.5  $\varphi(f') = \varphi(f_1) \neq 0$ . Since  $\varphi$  is separating,  $\emptyset \neq \text{coz}(f') \cap \text{coz}(f_2) \subseteq \text{coz}(f_1) \cap \text{coz}(f_2) = A$ . But this is a contradiction because  $A \subseteq Z(f')$ . By Proposition 2.5, it follows that  $K$  may only contain an equivalence class  $[x] = K$ , for some point  $x$  in  $X$ . This completes the proof.  $\square$

## 4. PROOF OF MAIN RESULT

We have remarked after Corollary 3.3 that, in order to prove the main result formulated at the Introduction, it suffices to deal with separating linear isomorphisms. Therefore, assume that  $H : \mathcal{A} \rightarrow \mathcal{B}$  is a linear separating map defined between linear subspaces  $\mathcal{A}$  and  $\mathcal{B}$  of  $C_{00}(X, \mathbb{F})$  and  $C_{00}(Y, \mathbb{F})$ , respectively. Observe that for every  $y \in Y$ , the composition  $\delta_y \circ H$  is a separating linear functional of  $\mathcal{A}$  into  $\mathbb{F}$ . Conveying to  $Y$  and  $\mathcal{B}$  the equivalence relation we have defined above on  $X$  and  $\mathcal{A}$ , and applying to  $\delta_y \circ H$  the last two results in the previous section, we obtain:

**Proposition 4.1.** *Let  $H : \mathcal{A} \rightarrow \mathcal{B}$  be a linear separating map. If  $K$  is a support for  $\delta_y \circ H$  and  $y' \in [y]$  then  $K$  is a support to  $\delta_{y'} \circ H$ .*

*Proof.* It suffices to take into account that every  $Z \in Z(\mathcal{B})$  is saturated. □

Applying Proposition 3.6 to  $\delta_y \circ H$ , for each  $y \in Y$ , we are now in position of defining the *support map*  $h$  that is associated to  $H$ . This map is defined between the spaces  $Y$  and  $\tilde{X}$ . Again, in order to simplify the notation, we will use the same symbol  $h(y)$  to denote both, an element of  $\tilde{X}$ , and the equivalence class  $\pi^{-1}(h(y))$ , which is a subset of  $X$ .

**Proposition 4.2.** *Let  $H : \mathcal{A} \rightarrow \mathcal{B}$  a separating linear map satisfying that for every  $y \in Y$  there is  $f_y \in \mathcal{A}$  such that  $Hf_y(y) \neq 0$ . If  $\mathcal{A}$  is controllable, then there is a map  $h : Y \rightarrow \tilde{X}$  satisfying the following properties:*

- (a) *For every  $f \in \mathcal{A}$  with  $f|_{h(y)} = 0$ , it follows that  $Hf(y) = 0$ .*
- (b)  *$h(y') = h(y)$  for all  $y' \sim y$ .*
- (c) *If  $A \subsetneq \tilde{X}$  is open,  $f \in \mathcal{A}$  and  $\pi^{-1}(A) \subseteq Z(f)$  then  $h^{-1}(A) \subseteq Z(Hf)$ .*



(d)  $h(\text{coz}(Hf)) \subseteq \pi(\text{coz}(f))$  for every  $f \in \mathcal{A}$ .

*Proof.* We define  $h(y)$  as the smallest support associated to  $\delta_y \circ H$ .

(a) This is clear.

(b) It follows from  $\mathcal{S}_y = \mathcal{S}_{y'}$  when  $y \sim y'$ .

(c) Take  $y \in h^{-1}(A)$ . Then  $\pi^{-1}(\tilde{X} \setminus A)$  is a nonempty, saturated, and closed subset that it is not a support for  $\delta_y \circ H$ . Therefore, there is  $g \in \mathcal{A}$  such that  $\pi^{-1}(\tilde{X} \setminus A) \subseteq Z(g)$  and  $Hg(y) \neq 0$ . So we have  $\text{coz}(g) \subseteq \pi^{-1}(A)$  and  $\text{coz}(f) \subseteq X \setminus \pi^{-1}(A)$ . Since  $H$  is a separating map,  $\text{coz}(Hg) \cap \text{coz}(Hf) = \emptyset$ . As a consequence  $Hf(y) = 0$ .

(d) Let  $[x] \in h(\text{coz}(Hf))$ , then  $[x] = h(y)$  for some  $y \in \text{coz}(Hf)$ . Since  $h(y)$  is support for  $\delta_y \circ H$ , we have  $[x] \not\subseteq Z(f)$ . Since  $Z(f)$  is saturated, it follows that  $[x] \subseteq \text{coz}(f)$ .  $\square$

Let  $Gr[h] \stackrel{\text{def}}{=} \bigcup_{y \in Y} (h(y) \times \{y\})$  denote the graphic of  $h$  equipped with the topology inherited as a subspace of  $X \times Y$ . Using the map  $\lambda(.,.)$  defined in Proposition 2.2, we have the following representation of separating linear maps.

**Proposition 4.3.** *Let  $H : \mathcal{A} \rightarrow \mathcal{B}$  a separating linear map satisfying that for every  $y \in Y$  there is  $f_y \in \mathcal{A}$  such that  $Hf_y(y) \neq 0$ . If  $\mathcal{A}$  is controllable, then there is a map  $\omega : Gr[h] \rightarrow \mathbb{F} \setminus \{0\}$  satisfying the following properties:*

(a)  $Hf(y) = \omega(x, y)f(x)$  for all  $(x, y) \in Gr[h]$  and all  $f \in \mathcal{A}$ .

(b)  $\omega(x', y') = \lambda(y', y)\omega(x, y)\lambda(x, x')$  for all  $y' \sim y$  and  $(x, y), (x', y') \in Gr[h]$ .

(c)  $\omega$  is continuous.

*Proof.* (a) Let  $(x, y) \in Gr[h]$ . By hypothesis, there is  $f' \in \mathcal{A}$  such that  $Hf'(y) \neq 0$ . Then  $f'(x) \neq 0$  since  $h(y)$  is a support set for  $\delta_y \circ H$ . Set  $\alpha = f'(x) \in \mathbb{F} \setminus \{0\}$  and  $f_x = \alpha^{-1}f' \in \mathcal{A}$ , which implies  $f_x(x) = 1$ .

We define

$$\omega(x, y) = Hf_x(y) = \alpha^{-1}Hf'(y) \in \mathbb{F} \setminus \{0\}.$$

Observe that  $\omega(x, y)$  does not depend on the specific map  $f \in \mathcal{A}$  with  $f(x) = 1$  we select. Indeed, let  $g_x \in \mathcal{A}$  such that  $g_x(x) = 1$ . Take  $x' \in h(y)$ , then by Proposition 2.2  $f_x(x') = \lambda(x', x)f_x(x) = \lambda(x', x) = \lambda(x', x)g_x(x) = g_x(x')$ . Thus, we have shown that  $(f_x)_{|h(y)} = (g_x)_{|h(y)}$ . By Proposition 3.5, we have  $Hg_x(y) = Hf_x(y) = \omega(x, y)$ .

Pick out now an arbitrary map  $f \in \mathcal{A}$ . If  $f(x) = 0$  then, since  $Z(f)$  is saturated,  $h(y) = [x] \subseteq Z(f)$  and  $Hf(y) = 0$ . Obviously  $Hf(y) = \omega(x, y)f(x) = 0$ . Therefore, suppose WLOG that  $f(x) = \beta \neq 0$  and set  $g'_x = \beta^{-1}f \in \mathcal{A}$ . Then we have  $g'_x(x) = 1$  and, since  $\omega(x, y)$  does not depend on  $g'_x$ , it follows that  $Hg'_x(y) = Hf_x(y) = \omega(x, y)$ . Taking into account that  $H$  is a linear map, we get  $Hg'_x = \beta^{-1}Hf$ . Thus  $\beta^{-1}Hf(y) = \omega(x, y)$ , which yields  $Hf(y) = \beta\omega(x, y) = \omega(x, y)f(x)$ . This completes the proof.

(b) This is clear after making some straightforward evaluations.

(c) Let  $((x_d, y_d))_d$  be a net converging to  $(x, y)$  in  $Gr[h]$  and take  $f_x \in \mathcal{A}$  such that  $f_x(x) = 1$ . Since  $\mathbb{F}$  is discrete and  $f_x$  and  $Hf_x$  are continuous, there exists  $d_0$  such that  $f_x(x_d) = 1$  and  $Hf_x(y_d) = Hf_x(y)$  for all  $d \geq d_0$ . Thus  $\omega(x_d, y_d) = \omega(x_d, y_d)f_x(x_d) = Hf_x(y_d) = Hf_x(y) = \omega(x, y)f_x(x) = \omega(x, y)$  for all  $d \geq d_0$ . This implies that the net  $(\omega(x_d, y_d))_d$  converges to  $\omega(x, y)$ .  $\square$

As a consequence of the previous result, we obtain a converse to Proposition 4.2.

**Corollary 4.4.**  *$Hf(y) = 0$  implies  $f(x) = 0$  for all  $(x, y) \in Gr[h]$ .*

Our next goal is to verify that the support map  $h$  is continuous and surjective assuming the same conditions as in Proposition 4.2 if  $H$  is also one-to-one. We split the proof in several lemmata for the reader's sake.

**Lemma 4.5.** *Assuming the same conditions as in Proposition 4.2, the support map  $h: Y \rightarrow \tilde{X}$  is continuous.*

*Proof.* Let  $(y_d)_{d \in D}$  be a net in  $Y$  converging to  $y \in Y$ . Since  $\tilde{X}$  is locally compact and Hausdorff, its Alexandroff compactification  $\tilde{X}^*$  is also Hausdorff. By a standard compactness argument, we may assume WLOG that  $(h(y_d))_d$  converges to  $t \in \tilde{X}^*$ . Reasoning by contradiction, suppose  $h(y) \neq t$  and take two disjoint open neighborhoods  $V_{h(y)}$  and  $V_t$  of  $h(y)$  and  $t$  respectively. Take  $d_1$  such that  $h(y_d) \in V_t \cap \tilde{X}$  for all  $d \geq d_1$ .

Since the support sets for  $\delta_z \circ H$  contains  $h(z)$  for all  $z \in Y$ , it follows that the subset  $\pi^{-1}(\tilde{X} \setminus (V_{h(y)} \cap \tilde{X}))$  may not be a support set for  $\delta_y \circ H$ . Therefore, there exists  $f \in \mathcal{A}$  such that  $\pi^{-1}(\tilde{X} \setminus (V_{h(y)} \cap \tilde{X})) \subseteq Z(f)$  and  $Hf(y) \neq 0$ . Moreover, since  $H(f)$  is continuous, the net  $(Hf(y_d))_{d \in D}$  converges to  $Hf(y)$  and, since  $\mathbb{F}$  is discrete, there is  $d_2 \geq d_1$  such that  $Hf(y_d) \neq 0$  for all  $d \geq d_2$ . Therefore, the subset  $\pi^{-1}(\tilde{X} \setminus (V_t \cap \tilde{X}))$  may not be a support set for  $\delta_{y_{d_3}} \circ H$  for some index  $d_3 \geq d_2$ . As a consequence, there exists  $f_3 \in \mathcal{A}$  such that  $\pi^{-1}(\tilde{X} \setminus (V_t \cap \tilde{X})) \subseteq Z(f_3)$  and  $Hf_3(y_{d_3}) \neq 0$ . Thus, we have  $y_{d_3} \in \text{coz}(Hf_3) \cap \text{coz}(Hf)$  and, since  $H$  is a separating map,  $\text{coz}(f_3) \cap \text{coz}(f) \neq \emptyset$ . But  $\text{coz}(f_3) \subseteq \pi^{-1}(V_t \cap \tilde{X})$  is disjoint from  $\text{coz}(f) \subseteq \pi^{-1}(V_{h(y)} \cap \tilde{X})$ . This contradiction completes the proof.  $\square$

**Lemma 4.6.** *Assuming the same conditions as in Proposition 4.2, if  $H$  is also one-to-one, then  $h(Y)$  is dense in  $\tilde{X}$ .*

*Proof.* Reasoning by contradiction again, suppose there is  $x \in X$  such that  $[x] \notin \overline{h(Y)}^{\tilde{X}}$ . Set  $A = \overline{h(Y)}^{\tilde{X}}$ , which implies  $[x] \cap \pi^{-1}(A) = \emptyset$ . On the other hand, by (1), there is  $f \in \mathcal{A}$  such that  $[x] \subseteq \text{coz}(f)$ . Define  $B = \pi^{-1}(A) \cap \text{coz}(f)$ , which is a saturated compact subset because  $\pi^{-1}(A)$  is closed and  $\text{coz}(f)$  is compact and

saturated. Moreover, we have that  $B \neq \emptyset$ . Otherwise,  $\pi^{-1}(h(Y)) \subseteq \pi^{-1}(A) \subseteq Z(f)$ . This implies that  $Hf \equiv 0$  and  $f \equiv 0$ , which is a contradiction. Since  $[x] \cap B = \emptyset$ , by Lemma 2.6, there are two disjoint subsets  $D_x, D_B \in \mathcal{D}$  such that  $[x] \subseteq D_x$  and  $B \subseteq D_B$ . Then the subset  $D = D_x \cap \text{coz}(f) \in \mathcal{D}$  contains  $[x]$  and  $D \cap \pi^{-1}(A) = \emptyset$ . We now apply that  $\mathcal{A}$  is controllable to  $D, D_B$  and  $f$  in order to obtain  $U \in \mathcal{D}$  and  $f' \in \mathcal{A}$  such that  $[x] \subseteq D \subseteq U \subseteq X \setminus D_B \subseteq X \setminus B$ ,  $f|_D = f'|_D$  and  $f'_{|(Z(f) \cup (X \setminus U))} = 0$ . Hence  $\text{coz}(f') \subseteq U \cap \text{coz}(f)$ ,  $U \cap B = \emptyset$  and  $\text{coz}(f') \cap \pi^{-1}(A) = \emptyset$ . As a consequence  $\pi^{-1}(h(Y)) \subseteq \pi^{-1}(A) \subseteq Z(f')$  and  $Hf(y) = 0$  for all  $y \in Y$ . Since  $H$  is a linear monomorphism we have  $f \equiv 0$ , which is a contradiction. Therefore  $\overline{h(Y)}^{\tilde{X}} = \tilde{X}$ , which completes the proof.  $\square$

Let  $Y^*$  and  $\tilde{X}^*$  be the Alexandroff compactification of  $Y$  and  $\tilde{X}$  respectively. Then there is a canonical way of extending  $h$  to a map  $h^*: Y^* \rightarrow \tilde{X}^*$  by  $h^*|_Y = h$  and  $h^*(\infty) = \infty$ . It turns out that this canonical extension is a continuous onto map.

**Lemma 4.7.** *Assuming the same conditions as in Proposition 4.2, if  $H$  is also one-to-one, then  $h^*$  is continuous and onto.*

*Proof.* Since  $h^*|_Y = h$  is continuous, in order to prove the continuity of  $h^*$ , it suffices to verify the continuity of  $h^*$  at  $\infty$ . Reasoning by contradiction, suppose that  $h^*$  is not continuous at  $\infty$ . Then, there must be a compact subset  $K_0 \subseteq \tilde{X}$  such that  $\infty \in \overline{h^{-1}(K_0)}^{Y^*}$ . Otherwise, we would have  $\infty \notin \overline{h^{-1}(K)}^{Y^*}$  for every compact subset  $K$  of  $\tilde{X}$ . Since  $h^{-1}(K)$  is closed in  $Y$ , it follows that  $h^{-1}(K) = \overline{h^{-1}(K)}^Y = \overline{h^{-1}(K)}^{Y^*}$ . However, every closed subset of  $Y^*$  is either the union of  $\{\infty\}$  and a closed subset of  $Y$ , or a compact subset of  $Y$ . Hence  $h^{-1}(K)$  is compact in  $Y$  for every compact subset  $K$  in  $\tilde{X}$  and, as a consequence, we have  $\infty \in Y^* \setminus h^{-1}(K)$ , which is open in

$Y^*$ . Thus, we have proved that  $\tilde{X}^* \setminus K$  is an open neighborhood of  $\infty = h^*(\infty)$  and  $h^*(\infty) \in h^*(Y^* \setminus h^{-1}(K)) \subseteq \tilde{X}^* \setminus K$  for every compact subset  $K$  of  $\tilde{X}$ , which would yield the continuity of  $h^*$  at  $\infty$ .

Take a net  $(y_d)_{d \in D} \subseteq h^{-1}(K_0)$  converging to  $\infty$ . By the compactness of  $K_0$ , we may assume WLOG that  $(h(y_d))_{d \in D}$  converges to  $[x_0] \in K_0$ . But  $\text{coz}(Hf)$  is compact and  $\infty \in Y^* \setminus \text{coz}(Hf)$  for all  $f \in \mathcal{A}$ . Therefore, for every  $f \in \mathcal{A}$ , there is an index  $d(f)$  such that  $y_d \in Y \setminus \text{coz}(Hf)$  for all  $d \geq d(f)$ . That is  $Hf(y_d) = 0$  and, by Corollary 4.4, we have  $f|_{h(y_d)} = 0$  for all  $d \geq d(f)$ . Thus  $(h(y_d))_{d \geq d(f)}$  is contained in  $\pi(Z(f))$  and, as a consequence, we have  $[x_0] \in \overline{\pi(Z(f))}^{\tilde{X}} = \pi(Z(f))$  for all  $f \in \mathcal{A}$ . This implies that  $f(x_0) = 0$  for all  $f \in \mathcal{A}$ , which is a contradiction.

Now, it is easy to show that  $h^*$  is an onto map. Indeed, since  $Y^*$  is compact,  $h^*$  is continuous and  $\tilde{X}^*$  is Hausdorff, we have that  $h^*(Y^*)$  is a compact subset of  $\tilde{X}^*$ . Therefore  $\overline{h^*(Y^*)}^{\tilde{X}^*} = h^*(Y \cup \{\infty\}) = h(Y) \cup \{\infty\} \subseteq \overline{h(Y)}^{\tilde{X}^*} \cup \{\infty\} = \overline{h^*(Y^*)}^{\tilde{X}^*}$  and, by Lemma 4.6, it follows that  $h^*(Y^*) = \overline{h^*(Y^*)}^{\tilde{X}^*} = \overline{h(Y)}^{\tilde{X}} \cup \{\infty\} = \tilde{X} \cup \{\infty\} = \tilde{X}^*$ .  $\square$

From Proposition 4.7, it follows a main partial result.

**Corollary 4.8.** *Assuming the same conditions as in Proposition 4.2, if  $H$  is also one-to-one, then  $h: Y \rightarrow \tilde{X}$  is continuous and onto.*

Set  $\tilde{h}: \tilde{Y} \rightarrow \tilde{X}$  by  $\tilde{h}([y]) = h(y)$  for all  $[y] \in \tilde{Y}$ , which is clearly well defined. A straightforward consequence of Corollary 4.8 is:

**Proposition 4.9.** *Assuming the same conditions as in Proposition 4.2, if  $H$  is also a bijection, then  $\tilde{h}$  is a homeomorphism of  $\tilde{Y}$  onto  $\tilde{X}$ .*

*Proof.* The continuity of  $\tilde{h}$  follows from the continuity of  $h$  and  $\pi$ .

Take  $[y_1] \neq [y_2]$  in  $Y$ . By Lemma 2.3, there is  $f \in \mathcal{A}$  such that  $[y_1] \subseteq Z(Hf)$  and  $[y_2] \subseteq \text{coz}(Hf)$ . Applying Corollary 4.4 and Proposition 4.2, we obtain  $h(y_1) \subseteq Z(f)$  and  $h(y_2) \subseteq \text{coz}(f)$ , which implies  $\tilde{h}([y_1]) \neq \tilde{h}([y_2])$ . Thus  $\tilde{h}$  is 1-to-1. On the other hand, the map  $\tilde{h}$  is onto because so is  $h$ .

Now, we can proceed as in Lemma 4.7, in order to extend  $\tilde{h}$  to a continuous map  $\tilde{h}^* : \tilde{Y}^* \rightarrow \tilde{X}^*$ . Clearly the map  $\tilde{h}^*$  is a continuous bijection and, therefore a homeomorphism between compact spaces. This automatically implies that  $\tilde{h}$  is a homeomorphism.  $\square$

We can now establish the representation of separating isomorphisms as weighted composition operator, which implies Theorem 1.7.

**Theorem 4.10.** *Let  $H : \mathcal{A} \rightarrow \mathcal{B}$  a separating, linear, onto, map. If  $\mathcal{A}$  is controllable, then there are continuous maps  $h : Y \rightarrow \tilde{X}$  and  $\omega : Gr[h] \rightarrow \mathbb{F}$  satisfying the following properties:*

(a) *For each  $y \in Y$ ,  $x \in h(y)$ , and every  $f \in \mathcal{A}$  it holds*

$$Hf(y) = \omega(x, y)f(x).$$

(b)  *$H$  is continuous with respect to the pointwise convergence topology.*

(c)  *$H$  is continuous with respect to the compact open topology.*

*Proof.* Since both  $\mathcal{A}$  and  $\mathcal{B}$  satisfy the initial assumption (1), it follows that item (a) is a direct consequence from Proposition 4.3. On the other hand, it is readily seen that (a) implies (b). Thus only (c) needs verification.

(c) Let  $(f_d)_d \subseteq \mathcal{A}$  be a net uniformly converging to 0 in the compact open topology. If  $K$  is a compact subset of  $Y$ , then  $h(K)$  is a compact subset of  $\tilde{X}$  by the

continuity of  $h$ . Furthermore, by Remark 2.7, the subset  $\pi^{-1}(h(K))$  is compact in  $X$ . Indeed, for every  $[x] \in h(K)$ , there is  $f_x \in \mathcal{A}$  such that  $[x] \in \pi(\text{coz}(f_x))$ . Hence  $h(K) \subseteq \bigcup_{[x] \in h(K)} \pi(\text{coz}(f_x))$ . By compactness, there is a finite subcover, say  $h(K) \subseteq \bigcup_{1 \leq i \leq n} \pi(\text{coz}(f_i))$ . Thus  $\pi^{-1}(h(K)) \subseteq \bigcup_{1 \leq i \leq n} \text{coz}(f_i)$ , which yields the compactness of  $\pi^{-1}(h(K))$ .

Since  $(f_d)_d$  converges to 0 uniformly on  $\pi^{-1}(h(K))$ , it follows that  $(f_d)_d$  is eventually equal to 0 on  $\pi^{-1}(h(K))$ . Applying (a), it follows that  $(Hf_d)_d$  is eventually 0 on  $K$ . This completes the proof.  $\square$

We are now in position of establishing the main result formulated at the Introduction.

*Proof of Theorem 1.7.* Since  $H$  is a Hamming isometry of  $\mathcal{A}$  onto  $\mathcal{B}$ , it is separating by Corollary 3.3. Thus  $H$  must be a general weighted composition operator by Theorem 4.10.  $\square$

*Proof of Corollary 1.8.* Applying Proposition 4.9 and Theorem 1.7, it follows that there is a homeomorphism (in fact, bijection)  $\tilde{h} : \tilde{Y} \rightarrow \tilde{X}$  such that  $Hf(y) = \omega(x, y)f(x)$  for each  $x \in \tilde{h}([y])$ ,  $[y] \in \tilde{Y}$ , and  $f \in \mathcal{A}$ .

We claim that  $\mu([y]) = \mu(\tilde{h}([y]))$  for all  $[y] \in \tilde{Y}$ . Indeed, take  $[x] \in \tilde{X}$  and consider  $f \in \mathcal{A}$  such that  $[x] \subseteq \text{coz}(f)$ . For every  $z \in \text{coz}(f)$  such that  $z \notin [x]$ , there is  $f_z \in \mathcal{A}$  such that  $f_z(z) = 0$  and  $f_z(x) \neq 0$ . Hence  $[x] = \text{coz}(f) \cap \{\text{coz}(f_z) : z \in \text{coz}(f), z \notin [x]\}$ . Since  $\text{coz}(f)$  is finite, this implies that  $[x] \in \mathcal{D}$ .

Applying that  $\mathcal{A}$  is controllable, there exist  $f' \in \mathcal{A}$  and  $U \in \mathcal{D}$  such that

$$[x] \subseteq U \subseteq X \setminus [z], \quad f_{|[x]} = f'_{|[x]}, \quad \text{and} \quad f'_{|(Z(f) \cup (X \setminus U))} = 0.$$

Thus  $[x] \subseteq \text{coz}(f') \subsetneq \text{coz}(f)$ . Again, since  $\text{coz}(f)$  is finite, we can repeat this argument finitely many times in order to obtain a map  $g \in \mathcal{A}$  such that  $[x] = \text{coz}(g)$ . The claim is now verified by applying that  $H$  is a Hamming isometry and Theorem 1.7. Therefore, we have proved that  $|[y]| = |\tilde{h}([y])|$  for all  $[y] \in \tilde{Y}$ . Let  $h_y$  be any bijection from  $[y]$  onto  $\tilde{h}([y])$  for every  $[y] \in \tilde{Y}$ . The map  $h : Y \rightarrow X$  defined as  $h(y') \stackrel{\text{def}}{=} h_y(y')$  for  $y' \in [y]$ ,  $[y] \in \tilde{Y}$ , is clearly a bijection of  $X$  onto itself. Now, set

$$w(y') \stackrel{\text{def}}{=} w(h(y'), y'), \quad y' \in [y], \quad [y] \in \tilde{Y}.$$

By Theorem 1.7, we have that  $Hf(y') = w(y')f(h(y'))$  for all  $y' \in X$  and  $f \in \mathcal{A}$ , which completes the proof.  $\square$

## 5. ACKNOWLEDGEMENT

We thank Juan J. Font for helpful information and the referee for many valuable suggestions.

## REFERENCES

- [1] Abramovich, Y. A. and Kitover, A. K., *Inverses of disjointness preserving operators*, Mem. Amer. Math. Soc., **143**, (2000), no. 679.
- [2] Alaminos, J. and Brešar, M. and Extremera, J. and Villena, A. R., *Maps preserving zero products*, Studia Math., **193**, no. 2, (2009), 131–159.
- [3] Araujo, J., Beckenstein, E., Narici, L., *On biseparating maps between realcompact spaces*, Journal of Mathematical Analysis and Applications, **193**, no. 2, (1995), 258–265.
- [4] Araujo, J. and Jarosz, K., *Biseparating maps between operator algebras*, Journal of Mathematical Analysis and Applications, **282**, (1), (2003), 48–55.
- [5] Banach, S.: *Théorie des opérations linéaires*, Chelsea, 1955.
- [6] Bogart, K., Goldberg, D., and Gordon, J., *An elementary proof of the MacWilliams theorem on equivalence of codes*, Information and Computation, **37**, no. 1, (1978), 19–22.
- [7] Burgos, María and Sánchez-Ortega, Juana, *On mappings preserving zero products*, Linear Multilinear Algebra, **61**, no. 3, (2013), 323–335.
- [8] Chebotar, M. A., Ke, W.-F., Lee, P.-H., and Wong, N.-C., *Mappings preserving zero products*, Studia Math., **155**, no. 1, (2003), 77–94.
- [9] Chebotar, M. A. and Ke, Wen-Fong and Lee, Pjek-Hwee, *Maps characterized by action on zero products*, Pacific J. Math., **216**, no. 2, (2004), 217–228.
- [10] Dinh, H. Q. and López-Permouth, S. R.: *On the equivalence of codes over rings and modules*, Finite Fields and their Applications, **10**, (4), (2004), 615–625.



- [11] Engelking, R.: *General Topology*, Polish Scientific Publishers, Warszawa (1977).
- [12] Ferrer, M., Gary, M., and Hernández, S.: *Representation of group isomorphisms: The compact case* (2014), *Journal of Function Spaces*, Vol. 2015, Article ID 879414, 6 pages, 2015. doi:10.1155/2015/879414
- [13] Ferrer, M., Hernández, S., and Shakhmatov, D.: *Subgroups of direct products closely approximated by direct sums*, (2014), Pending. arXiv:1306.3954 [math.GN].
- [14] Fleming, R. J., Jamison, J.E., *Isometries on Banach spaces: function spaces*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, **129**, Chapman & Hall/CRC, Boca Raton, FL, (2003).
- [15] Fleming, R. J., Jamison, J.E., *Isometries on Banach spaces. Vol. 2*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, **138**, Chapman & Hall/CRC, Boca Raton, FL, (2008).
- [16] Font, J.J. and Hernández, S.: *On separating maps between locally compact spaces*, *Arch. Math.* **63**(1994), 158-165.
- [17] Font, J.J. and Hernández, S.: *Automatic continuity and representation of certain linear isomorphisms between group algebras*, *Indag. Mathem.* **6**(4)(1995), 397-409.
- [18] Forney, G. D. Jr. and Trott M. D.: *The Dynamics of Group Codes: Dual Abelian Group Codes and Systems*, Proc. IEEE Workshop on Coding, System Theory and Symbolic Dynamics (Mansfield, MA), pp. 35-65 (2004).
- [19] Garrido, M. I., Jaramillo, J.A., *Variations on the Banach-Stone theorem*, IV Course on Banach Spaces and Operators (Spanish) (Laredo, 2001), *Extracta Math.*, **17** (3), 2002, 351–383.
- [20] Gau, H-L, Jeang, J-S, and Wong, N. C.: *An algebraic approach to the Banach-Stone theorem for separating linear bijections* *Taiwanese J. Math.* **6**(3)(2002), 399-403.
- [21] Gluesing-Luerssen, H.: *On isometries for convolutional codes*, *Advances in Mathematics of Communications*, **3**, (2) (2009), 179–203.
- [22] Hernández, S., Beckenstein, E. and Narici. L.: *Banach-Stone theorems and separating maps*, *Manuscripta Math.* **86**(1995), 409-416.
- [23] Jarosz, K.: *Automatic continuity of separating linear maps*, *Canad. Math. Bull.* **33**(2)(1990), 139-144.
- [24] Lau, Anthony To-Ming and Wong, Ngai-Ching, *Orthogonality and disjointness preserving linear maps between Fourier and Fourier-Stieltjes algebras of locally compact groups*, *J. Funct. Anal.*, **265**, no. 4, (2013), 562–593.
- [25] MacWilliams, F. J.: *Combinatorial problems of elementary abelian groups*, PhD thesis, Harvard University, 1962.
- [26] MacWilliams, F. J.: *A theorem on the distribution of weights in a systematic code*, *Bell Syst. Tech. J.*, **42**, (1963), 79-94.
- [27] Ohta, H.: *Chains of strongly non-reflexive dual groups of integer-valued continuous functions*, *Proc. Amer. Math. Soc.* **124**(3)(1996), 961-967.
- [28] Piret, P.: *Convolutional Codes; An Algebraic Approach*, MIT Press, Cambridge, MA, 1988.
- [29] Rosenthal, J., Schumacher, S. M., and York, E. V.: *On Behaviors and Convolutional Codes*, *IEEE Trans. on Information Theory*, 42 (1996), no. 6, 1881–1891.
- [30] Stone, M.H.: *Applications of the theory of boolean rings to General Topology*, *Trans. Amer. Math. Soc.*, **41** (1937), 375-481.
- [31] Ward, H. N. and Wood, J. A.: *Characters and the equivalence of codes*, *Journal of Combinatorial Theory. Series A*, **73**, no. 2, (1996), 348–352.
- [32] Willems, J. C.: *From time series to linear systems*, Parts I-III, vol. 22, pp. 561-580 and 675-694, 1986.

- [33] Wood, J. A.: *The structure of linear codes of constant weight*, Trans. Amer. Math. Soc., **354**, no. 3, (2001), 1007-1026.

UNIVERSITAT JAUME I, INSTITUTO DE MATEMÁTICAS DE CASTELLÓN, CAMPUS DE RIU SEC,  
12071 CASTELLÓN, SPAIN.

*E-mail address:* mferrer@mat.uji.es

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA METROPOLITANA, IZTAPALAPA,  
MÉXICO DF, MÉXICO

*E-mail address:* gary@uji.es

UNIVERSITAT JAUME I, INIT AND DEPARTAMENTO DE MATEMÁTICAS, CAMPUS DE RIU SEC,  
12071 CASTELLÓN, SPAIN.

*E-mail address:* hernande@mat.uji.es