

## Research Article

# Representation of Group Isomorphisms: The Compact Case

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Let  $G$  be a discrete group and let  $\mathcal{A}$  and  $\mathcal{B}$  be two subgroups of  $G$ -valued continuous functions defined on two 0-dimensional compact spaces  $X$  and  $Y$ . A group isomorphism  $H$  defined between  $\mathcal{A}$  and  $\mathcal{B}$  is called *separating* when, for each pair of maps  $f, g \in \mathcal{A}$  satisfying that  $f^{-1}(e_G) \cup g^{-1}(e_G) = X$ , it holds that  $Hf^{-1}(e_G) \cup Hg^{-1}(e_G) = Y$ . We prove that under some mild conditions every biseparating isomorphism  $H : \mathcal{A} \rightarrow \mathcal{B}$  can be represented by means of a continuous function  $h : Y \rightarrow X$  as a weighted composition operator. As a consequence we establish the equivalence of two subgroups of continuous functions if there is a biseparating isomorphism defined between them.

## 1. Introduction

Let  $G$  be a discrete group and let  $X$  and  $Y$  be topological spaces. If  $\mathcal{A}$  and  $\mathcal{B}$  are groups of  $G$ -valued continuous maps, we say that  $\mathcal{A}$  and  $\mathcal{B}$  are *equivalent* when there is a homeomorphism  $h : Y \rightarrow X$  and a continuous map  $\omega : Y \rightarrow \text{Aut}(G)$  such that the map  $H : \mathcal{A} \rightarrow \mathcal{B}$  defined as  $Hf(y) = \omega[y](f(h(y)))$ ,  $f \in \mathcal{A}$ ,  $y \in Y$ , is a group isomorphism of  $\mathcal{A}$  onto  $\mathcal{B}$ . Here  $\text{Aut}(G)$  is equipped with the pointwise convergence topology. We say in this case that  $H$  is represented as a *weighted composition operator*. There are many results that are concerned with the representation of linear operators as weighted composition maps and the equivalence of specific groups of continuous functions in the literature, which is vast in this regard. We will only mention here the classic Banach-Stone theorem that, when  $G$  is the field of real or complex numbers, establishes that if the Banach spaces of continuous functions  $C(X, G)$  and  $C(Y, G)$  are isometric, then they are equivalent and the isometry can be represented as a weighted composition map (cf. [1–10]). Another important example appears in coding theory, where the well-known MacWilliams Equivalence Theorem asserts that when  $G$  is a finite field and  $X$  and  $Y$  are finite sets, two codes (linear subspaces)  $\mathcal{A}$  and  $\mathcal{B}$  of  $G^X$  and  $G^Y$ , respectively,

are equivalent when they are isometric for the Hamming metric (see [11–13]). This result has been generalized to convolutional codes in [14] and it also makes sense in other areas, for example, functional analysis and linear dynamical systems (cf. [14–18]). The main motivation of this research has been to extend MacWilliams Equivalence Theorem to more general settings and explore the possible application of these methods to the study of convolutional codes or linear dynamical systems. However, throughout this paper, we will only deal with 0-dimensional compact spaces  $X$  and  $Y$  and a discrete group  $G$ . We will look at the possible application of this abstract approach elsewhere. There are many precedents in the study of the representation of group homomorphisms for group-valued continuous functions. Among them, the following ones are relevant here (cf. [19–26]). Most basic facts and notions related to topological properties may be found in [27].

Throughout this paper all spaces are assumed to be Hausdorff 0-dimensional and compact. That is to say, we only deal with Hausdorff compact spaces that contain an open basis consisting of closed and open (*clopen*) subsets. If  $X$  is a topological space and  $G$  is a topological (discrete) group, we denote by  $C(X, G)$  the group of continuous functions from  $X$  to  $G$ . Let  $e_G$  be the neutral element of  $G$ . For  $f \in C(X, G)$

the cozero of  $f$  is the set  $\text{coz}(f) = \{x \in X : f(x) \neq e_G\}$  and the zero of  $f$  is the set  $Z(f) = X \setminus \text{coz}(f)$ . Since  $G$  is discrete,  $\text{coz}(f)$  and  $Z(f)$  are both clopen subsets of  $X$ .

Let  $\mathcal{A}$  be a subgroup of  $C(X, G)$  and set  $Z(\mathcal{A}) \stackrel{\text{def}}{=} \{Z(f) : f \in \mathcal{A}\}$ . Then  $\sigma(Z(\mathcal{A}))$  denotes the minimum collection of subsets containing  $Z(\mathcal{A})$  that is closed under finite unions and intersections (resp.,  $\text{coz}(\mathcal{A}) \stackrel{\text{def}}{=} \{\text{coz}(f) : f \in \mathcal{A}\}$  and  $\sigma(\text{coz}(\mathcal{A}))$  denotes the minimum collection of subsets containing  $\text{coz}(\mathcal{A})$  that is closed under finite unions and intersections). It is said that  $\mathcal{A}$  separates points in  $X$  if for every pair of distinct points  $(x_1, x_2) \in X \times X$  there is a map  $f \in \mathcal{A}$  such that  $f(x_1) \neq e_G$  and  $f(x_2) = e_G$ . It is said that  $\mathcal{A}$  strongly separates points in  $X$  if, for every pair  $(x_1, x_2) \in X \times X$ , there are maps  $f_1, f_2 \in \mathcal{A}$  such that  $x_i \in \text{coz}(f_i)$ ,  $1 \leq i \leq 2$ , and  $\text{coz}(f_1) \cap \text{coz}(f_2) = \emptyset$ .

As group  $G$  is finite, it could be thought that  $\text{coz}(\mathcal{A})$  and  $\sigma(\text{coz}(\mathcal{A}))$  should coincide. However, this is misleading as the following example shows. Obviously, by Morgan's laws, it suffices to prove that  $Z(\mathcal{A}) \neq \sigma(Z(\mathcal{A}))$ .

*Example 1.* Let  $G$  be the group with two elements  $\{0, 1\}$  and take  $X = G^{\mathbb{N}}$  equipped with the product topology. Clearly  $X$  is a compact space homeomorphic to the Cantor set, which is 0-dimensional. Let  $p_m \in C(X, G)$  be the  $m$ th projection on  $X$  and set  $\mathcal{A}$  as the subgroup of  $C(X, G)$  generated the collection  $\{p_m : m \in \mathbb{N}\}$ . Take  $n \in \mathbb{N}$  greater than 2 and  $H_n \stackrel{\text{def}}{=} \{x_j \in X : x_1 = \dots = x_n = 0\}$ . It is easily verified that  $H_n$  belongs to  $\sigma(Z(\mathcal{A}))$ . However, take an arbitrary but fixed element  $f \in \mathcal{A}$ . Denote by  $e_k$  the element in  $X$  such that  $p_m(e_k) = 0$  if  $m \neq k$  and  $p_k(e_k) = 1$  and set  $\text{sup}(f) \stackrel{\text{def}}{=} \{k \in \mathbb{N} : f(e_k) = 1\}$ . If  $\{1, \dots, n\} \not\subseteq \text{sup}(f)$ , then there is  $l < n$  such that  $l \notin \text{sup}(f)$ . Therefore  $f(e_l) = 0$ , which yields  $e_l \in Z(f) \setminus H_n$ . On the other hand, if  $\{1, \dots, n\} \subseteq \text{sup}(f)$ , take  $1 \leq n_1 < n_2 \leq n$ . We have that  $f(e_{n_1} + e_{n_2}) = 0$ , which again yields  $e_{n_1} + e_{n_2} \in Z(f) \setminus H_n$ . In either case, we obtain that  $Z(f) \neq H_n$ . Thus  $H_n \in \sigma(Z(\mathcal{A})) \setminus Z(\mathcal{A})$ , which completes the proof.

Denote by  $\delta_x : \mathcal{A} \rightarrow G$  the evaluation map; that is,  $\delta_x(f) = f(x)$  for every  $f \in \mathcal{A}$ . It is said that  $\mathcal{A}$  is pointwise dense when  $\delta_x(\mathcal{A})$  is dense in  $G$  for all  $x \in X$ . It is said that  $\mathcal{A} \subseteq C(X, G)$  is controllable if for every  $f \in \mathcal{A}$  and  $D_1, D_2 \in \sigma(Z(\mathcal{A}))$  such that  $D_1 \cap D_2 = \emptyset$  there exist a subset  $U \subseteq \sigma(\text{coz}(\mathcal{A}))$  and a function  $g \in \mathcal{A}$  such that  $D_1 \subseteq U \subseteq X \setminus D_2$ ,  $g|_{D_1} = f|_{D_1}$ , and  $g|_{Z(f) \cup (X \setminus U)} \equiv e_G$ .

We now formulate our main results.

**Theorem 2.** *Let  $X$  and  $Y$  be 0-dimensional compact Hausdorff spaces and let  $G$  be a discrete group. Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are controllable and pointwise dense subgroups of  $G$ -valued continuous functions separating the points of  $X$  and  $Y$ , respectively. If  $H : \mathcal{A} \rightarrow \mathcal{B}$  is a biseparating group isomorphism of  $\mathcal{A}$  onto  $\mathcal{B}$ , then there are continuous maps  $h : Y \rightarrow X$  and  $\omega : Y \rightarrow \text{Aut}(G)$  satisfying the following properties.*

- (1)  $h$  is a homeomorphism of  $Y$  onto  $X$ .
- (2) For each  $y \in Y$  and every  $f \in \mathcal{A}$  it holds

$$Hf(y) = \omega[y](f(h(y))). \quad (1)$$

- (3)  $H$  is a continuous isomorphism with respect to the pointwise convergence topology.
- (4)  $H$  is a continuous isomorphism with respect to the compact open topology.

**Corollary 3.** *Let  $X$  and  $Y$  be 0-dimensional compact Hausdorff spaces and let  $G$  be a discrete group. Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are controllable and pointwise dense subgroups of  $G$ -valued continuous functions separating the points of  $X$  and  $Y$ , respectively. If there is a biseparating group isomorphism  $H$  of  $\mathcal{A}$  onto  $\mathcal{B}$ , then  $\mathcal{A}$  and  $\mathcal{B}$  are equivalent.*

We notice that some of the requirements we have imposed on the previous results could be relaxed in general. However, this would take us to a wider setting in general. For instance, if we assume that  $\mathcal{A}$  does not separate points in  $X$ , then there must be some point  $x \in X$  such that  $f(x) = e_G$  for all  $f \in \mathcal{A}$ ; then we can replace  $X$  by the largest subspace  $X_{\mathcal{A}} \subseteq X$  where  $\mathcal{A}$  separates points. This subspace  $X_{\mathcal{A}}$  is open but not necessarily closed in general. Thus, the study of subgroups that does not separate points leads us to consider locally compact spaces. We will discuss these spaces in a subsequent paper.

## 2. Basic Notions and Facts

The following lemma is easily verified using a standard compactness argument. Recall that we are assuming that all spaces are compact and 0-dimensional.

**Lemma 4.** *Let  $\mathcal{D}$  be a family of clopen subsets of  $X$  that is a subbase for the closed subsets of  $X$ . Then for every disjoint nonempty closed subsets  $A$  and  $B$  of  $X$  there are two disjoint subsets  $D_A$  and  $D_B$  in  $\sigma(\mathcal{D})$  such that  $A \subseteq D_A$  and  $B \subseteq D_B$ .*

*Example 5.* A specific example where Lemma 4 applies is given when  $\mathcal{A}$  separates points in  $X$ , where  $\mathcal{D} = Z(\mathcal{A})$ .

Next proposition shows that the notions of separating and strongly separating points are equivalent for controllable subgroups.

**Proposition 6.** *If  $\mathcal{A}$  is a controllable subgroup of  $C(X, G)$  that separates the points of  $X$ , then  $\mathcal{A}$  strongly separates the points of  $X$ .*

*Proof.* Set  $\mathcal{D} = Z(\mathcal{A})$  and take two distinct elements  $x_1 \neq x_2$  in  $X$ . Applying Lemma 4, since  $\mathcal{A}$  separates the points of  $X$ , there are  $D_1, D_2 \in \sigma(\mathcal{D})$  such that  $x_1 \in D_1$ ,  $x_2 \in D_2$ , and  $D_1 \cap D_2 = \emptyset$ . Take  $f_i \in \mathcal{A}$  such that  $f_i(x_i) \neq e_G$ ,  $i \in \{1, 2\}$ . Since  $\mathcal{A}$  is controllable, we have  $U_1 \in \sigma(\text{coz}(\mathcal{A}))$  and  $g_1 \in \mathcal{A}$  such that  $D_1 \subseteq U_1 \subseteq X \setminus D_2$ ,  $g_1|_{D_1} = f_1|_{D_1}$ , and  $Z(f_1) \cup (X \setminus U_1) \subseteq Z(g_1)$ . Therefore  $x_1 \in \text{coz}(g_1) \subseteq U_1$ .

Applying the fact that  $\mathcal{D}$  is a subbase of closed subsets and using a compactness argument, we deduce that there is  $D'_1 \in \sigma(\mathcal{D})$  such that  $U_1 \subseteq D'_1$  and  $D'_1 \cap D_2 = \emptyset$ . By the controllability of  $\mathcal{A}$  again, we have  $U_2 \in \sigma(\text{coz}(\mathcal{A}))$  and  $g_2 \in \mathcal{A}$  such that  $D_2 \subseteq U_2 \subseteq X \setminus D'_1$ ,  $g_2|_{D_2} = f_2|_{D_2}$ , and  $Z(f_2) \cup (X \setminus U_2) \subseteq Z(g_2)$ . Therefore  $x_2 \in \text{coz}(g_2) \subseteq U_2$ , which yields  $\text{coz}(g_1) \cap \text{coz}(g_2) = \emptyset$ . This completes the proof.  $\square$

*Definition 7.* Let  $\mathcal{A}$  be a subgroup of  $C(X, G)$  and let  $\varphi : \mathcal{A} \rightarrow G$  be a group homomorphism. A subset  $A$  of  $X$  is said to be a *support* for  $\varphi$  if, given  $f \in \mathcal{A}$  with  $A \subseteq Z(f)$ , it holds that  $\varphi(f) = e_G$ .

Some basic properties of support subsets are shown in the next proposition. Observe that since  $A \subseteq \overline{A^X} \subseteq Z(f)$ , we may assume without loss of generality that all support subsets are closed and therefore compact subsets of  $X$ .

**Proposition 8.** Let  $\varphi : \mathcal{A} \rightarrow G$  be a nonnull group homomorphism. The following assertions hold.

- (1)  $X$  is a support for  $\varphi$ .
- (2) If  $A$  is a support for  $\varphi$  then  $A \neq \emptyset$ .
- (3) If  $A$  is a support for  $\varphi$  and  $A \subseteq B$  then  $B$  is a support for  $\varphi$ .
- (4) Let  $A$  be a support for  $\varphi$  and  $f, g \in \mathcal{A}$  such that  $f|_A = g|_A$ . Then  $\varphi(f) = \varphi(g)$ .

If, in addition,  $\mathcal{A}$  is controllable and separates points in  $X$ , then we have the following.

- (5) Let  $A$  and  $B$  be supports for  $\varphi$ ; then  $A \cap B \neq \emptyset$ .

*Proof.* Assertions (1)–(4) are obvious.

(5) Let  $A$  and  $B$  be closed supports for  $\varphi$ . Suppose  $A \cap B = \emptyset$ . Since  $\mathcal{A}$  separates points in  $X$ , by Lemma 4 and Proposition 6, there are two disjoint subsets  $D_A$  and  $D_B$  in  $\sigma(Z(\mathcal{A}))$  containing  $A$  and  $B$ , respectively. Take  $f \in \mathcal{A}$  such that  $\varphi(f) \neq e_G$ . Applying the controllability of  $\mathcal{A}$ , we obtain  $U \in \sigma(\text{coz}(\mathcal{A}))$  and  $g \in \mathcal{A}$  such that  $A \subseteq D_A \subseteq U \subseteq X \setminus D_B \subseteq X \setminus B$ ,  $g|_{D_A} = f|_{D_A}$ , and  $Z(f) \cup (X \setminus U) \subseteq Z(g)$ . This yields a contradiction as the evaluation of  $\varphi(g)$  shows. Indeed, since  $g(x) = f(x)$  for all  $x \in A$ , by item (4) it follows that  $\varphi(g) = \varphi(f) \neq e_G$ . On the other hand, we have that  $g(x) = e_G$  for all  $x \in B$ , which implies  $\varphi(g) = e_G$ . This contradiction completes the proof.  $\square$

*Definition 9.* A map  $H : \mathcal{A} \rightarrow \mathcal{B}$  is said to be *separating* or *disjointness preserving*, if, for each pair of maps  $f, g \in \mathcal{A}$  satisfying that  $f^{-1}(e_G) \cup g^{-1}(e_G) = X$ , it holds that  $Hf^{-1}(e_G) \cup Hg^{-1}(e_G) = Y$  (equivalently, if  $\text{coz}(f) \cap \text{coz}(g) = \emptyset$  implies  $\text{coz}(Hf) \cap \text{coz}(Hg) = \emptyset$  for all  $f, g \in \mathcal{A}$ ). In case  $H$  is bijective, the map  $H$  is said to be *biseparating* if both  $H$  and  $H^{-1}$  are separating. Remark that this definition makes sense and extends naturally to maps  $\varphi : \mathcal{A} \rightarrow G$ .

*Remark 10.* Originally, separating maps for scalar-valued continuous functions were defined as those maps  $H$  such that  $f g = 0$  implies  $HfHg = 0$ . If one interprets the null element 0 as the identity  $e$  of the group, then separating maps could be defined as those maps that satisfy  $f(x)g(x) = e$  for all  $x \in X$  which implies  $Hf(y)Hg(y) = e$  for all  $y \in Y$  or that  $H(f^{-1}) = H(f)^{-1}$  for all  $f \in \mathcal{A}$ . Obviously this definition would be vacuous here since we are assuming that  $H$  is a group homomorphism throughout this paper. Thus, this definition would take us to the more general question of representing group isomorphisms defined between groups of continuous

functions without any further requirement. Unfortunately, this is not possible in general. Indeed, a remarkable result due to Milutin [28] establishes that if  $K$  is an uncountable compact metric space, then  $C(K, \mathbb{C})$  is linearly isomorphic to  $C([0, 1], \mathbb{C})$ . Therefore, it is essential to impose some extra algebraic or geometrical condition on the isomorphisms  $H$  in order to be able to represent them by continuous maps defined on the compact spaces  $K$  and  $[0, 1]$ . In this sense, the connection with separating isomorphisms stems from [8], where it was proved that every linear isometry is a separating isomorphism.

Next we will see that every nonnull separating group homomorphism  $\varphi : \mathcal{A} \rightarrow G$ , where  $\mathcal{A}$  is controllable, has the smallest possible compact support set. For that purpose, set

$$\mathcal{S} = \{A \subseteq X : A \text{ is a compact support for } \varphi\}. \quad (2)$$

There is a canonical partial order that can be defined on  $\mathcal{S}$ :  $A \leq B$ ,  $A, B \in \mathcal{S}$ , if and only if  $B \subseteq A$ . A standard argument shows that  $(\mathcal{S}, \leq)$  is an inductive set and, by Zorn's lemma,  $\mathcal{S}$  contains a  $\subseteq$ -minimal element. Furthermore, this minimal element is in fact a minimum because of the next proposition.

**Proposition 11.** If  $\mathcal{A}$  separates points in  $X$ , then the minimum element  $\mathcal{S}$  consists of a singleton.

*Proof.* Let  $S$  be a minimal element of  $\mathcal{S}$ , which is nonempty by Proposition 8. Suppose now that there are two different elements  $x_1, x_2$  that are contained in  $S$ . As  $X$  is Hausdorff, we can select two disjoint open subsets  $V_1, V_2$  in  $X$  such that  $x_1 \in V_1$  and  $x_2 \in V_2$ . Since  $S$  is minimal, the compact subset  $S \setminus V_i$  is not a support for  $\varphi$ . Hence, there are  $f_i \in \mathcal{A}$  such that  $S \setminus V_i \subseteq Z(f_i)$  and  $\varphi(f_i) \neq e_G$ ,  $1 \leq i \leq 2$ . Since  $\varphi$  is separating, it follows that  $A = \text{coz}(f_1) \cap \text{coz}(f_2)$  is a nonempty compact subset of  $X$ .

We claim that  $S \cap A = \emptyset$ . Otherwise, pick up an element  $a \in S \cap A$ . If  $a \in V_1$  then  $a \in S \setminus V_2$  and  $a \in Z(f_2)$ , which is a contradiction; but if  $a \notin V_1$  then  $a \in S \setminus V_1$ , which implies that  $a \in Z(f_1)$  and we get a contradiction again. Therefore  $S \cap A = \emptyset$ . By Lemma 4, we can take two disjoint sets  $D_S, D_A \in \sigma(Z(\mathcal{A}))$  such that  $S \subseteq D_S$  and  $A \subseteq D_A$ . Applying the fact that  $\mathcal{A}$  is controllable to  $D_S, D_A$  and  $f_1$ , we obtain a set  $U \in \sigma(\text{coz}(\mathcal{A}))$  and a map  $g \in \mathcal{A}$  such that  $S \subseteq D_S \subseteq U \subseteq X \setminus D_A \subseteq X \setminus A$ ,  $g|_S = f_1|_S$ , and  $g|_{Z(f_1) \cup (X \setminus U)} \equiv e_G$ . Then  $U \cap A = \emptyset$ ,  $\varphi(g) = \varphi(f_1) \neq e_G$ , and  $A \subseteq Z(g)$ . Since  $\varphi$  is separating the set  $B = \text{coz}(g) \cap \text{coz}(f_2) \neq \emptyset$ , take  $b \in B$ . Then  $b \in \text{coz}(f_2)$  and  $b \in \text{coz}(g) \subseteq \text{coz}(f_1)$ ; that is,  $b \in \text{coz}(f_1) \cap \text{coz}(f_2) = A$ . As a consequence  $g(b) = e_G$ , which is a contradiction. Therefore we have proved that  $|S| = 1$ . This completes the proof.  $\square$

### 3. Proof of Main Results

Along this section  $\mathcal{A}$  (resp.,  $\mathcal{B}$ ) is a controllable subgroup of  $C(X, G)$  (resp.,  $C(Y, G)$ ) that separates points in  $X$  (resp.,  $Y$ ).

Let  $H : \mathcal{A} \rightarrow \mathcal{B}$  be a separating group homomorphism. The maps  $\delta_y \circ H$  are separating group homomorphisms of  $\mathcal{A}$  into  $G$  for all  $y \in Y$ . Furthermore, since  $\mathcal{A}$  is controllable and separates points in  $X$ , we can apply Proposition 11, in

order to obtain that each partial ordered set  $\mathcal{S}_y = \{A \subseteq X : A \text{ is a compact support for } \delta_y \circ H\}$  has a minimum element, which is a singleton denoted by  $h(y)$ . Therefore, by sending  $y \in Y$  to  $h(y) \in X$  for every  $y \in Y$ , we have defined the *support map* of  $Y$  into  $X$  that is associated with  $H$ .

**Proposition 12.** *Let  $H : \mathcal{A} \rightarrow \mathcal{B}$  be a separating group homomorphism. Then the support map  $h$  has the following properties.*

- (1)  $h$  is continuous.
- (2) If  $\emptyset \neq A \subsetneq X$  is open,  $f \in \mathcal{A}$ , and  $A \subseteq Z(f)$ , then  $h^{-1}(A) \subseteq Z(Hf)$ .
- (3)  $h(\text{coz}(Hf)) \subseteq \text{coz}(f)$ .
- (4) If  $H$  is one-to-one, then  $h$  is onto.

Moreover, when  $H$  is a bijection of  $\mathcal{A}$  onto  $\mathcal{B}$ , we have, in addition, the following.

- (5) If  $H$  is biseparating, then  $h$  is a homeomorphism of  $Y$  onto  $X$ .

*Proof.* (1) Let  $(y_d)_{d \in D}$  be a net in  $Y$  converging to  $y \in Y$ . By a standard compactness argument, we may assume without loss of generality that  $(h(y_d))_d$  converges to  $x \in X$ . Reasoning by contradiction, suppose  $h(y) \neq x$ . Since  $X$  is Hausdorff, we can take two disjoint open neighborhoods  $V_{h(y)}$  and  $V_x$  of  $h(y)$  and  $x$ , respectively. Using convergence, there is  $d_1 \in D$  such that  $h(y_d) \in V_x$  for all  $d \geq d_1$ .

As every support subset for  $\delta_{y'} \circ H$  contains  $h(y')$ , for all  $y' \in Y$ , the subset  $X \setminus V_{h(y)}$  may not be a support for  $\delta_y \circ H$ . Therefore there exists  $f \in \mathcal{A}$  such that  $X \setminus V_{h(y)} \subseteq Z(f)$  and  $Hf(y) \neq e_G$ . Moreover, since  $H(f)$  is a continuous function, the net  $(Hf(y_d))_d$  converges to  $Hf(y)$  and, because  $G$  is discrete, there is  $d_2 \geq d_1$  such that  $Hf(y_d) \neq e_G$  for all  $d \geq d_2$ . If we take and index  $d_3 \in D$  such that  $d_3 \geq d_2$ , then the subset  $X \setminus V_x$  may not be a support for  $\delta_{y_{d_3}} \circ H$ . Thus, there exists  $f_3 \in \mathcal{A}$  such that  $X \setminus V_x \subseteq Z(f_3)$  and  $Hf_3(y_{d_3}) \neq e_G$ . This means that  $y_{d_3} \in \text{coz}(Hf_3) \cap \text{coz}(Hf)$  and, since  $H$  is a separating map, it follows that  $\text{coz}(f_3) \cap \text{coz}(f) \neq \emptyset$ . But  $\text{coz}(f_3) \subseteq V_x$ , which is disjoint from  $\text{coz}(f) \subseteq V_{h(y)}$ . This is a contradiction that completes the proof.

(2) Let  $\emptyset \neq A \subsetneq X$  be an open subset,  $f \in \mathcal{A}$ , and  $A \subseteq Z(f)$ . If we take  $y \in h^{-1}(A)$ , then  $X \setminus A$  is a nonempty compact subset that is not a support for  $\delta_y \circ H$ . Then there is  $g \in \mathcal{A}$  such that  $X \setminus A \subseteq Z(g)$  and  $Hg(y) \neq e_G$ . Therefore  $\text{coz}(g) \subseteq A$  and  $\text{coz}(f) \subseteq X \setminus A$ . Since  $H$  is separating, we have that  $\text{coz}(Hg) \cap \text{coz}(Hf) = \emptyset$ . Therefore  $Hf(y) = e_G$ .

(3) Take  $x \in h(\text{coz}(Hf))$ . Then  $x = h(y)$  for some  $y \in \text{coz}(Hf)$ . Since  $h(y)$  is a support for  $\delta_y \circ H$ , it follows that  $x \notin Z(f)$  or, equivalently, we have  $x \in \text{coz}(f)$ .

(4) Suppose  $h(Y) \neq X$  and take  $x \in X$  such that  $x \notin h(Y)$ . Since  $h$  is continuous and  $Y$  is compact, we have that  $h(Y)$  is a proper compact subset of  $X$ . Applying Lemma 4, there are two disjoint subsets  $D_x, D_Y \in \sigma(Z(\mathcal{A}))$  such that  $x \in D_x$  and  $h(Y) \subseteq D_Y$ . Moreover, as  $\mathcal{A}$  separates points in  $X$ , there exists  $f \in \mathcal{A}$  such that  $f(x) \neq e_G$ . Again, by the controllability of  $\mathcal{A}$ , we may take a subset  $U \subseteq \sigma(\text{coz}(\mathcal{A}))$  and a map  $g \in \mathcal{A}$  such that  $x \in D_x \subseteq U \subseteq X \setminus D_Y \subseteq X \setminus h(Y)$ ,  $g|_{D_x} = f|_{D_x}$ , and

$Z(f) \cup (X \setminus U) \subseteq Z(g)$ . As a consequence  $g(x) = f(x) \neq e_G$ ,  $h(Y) \subseteq Z(g)$ , and  $H(f)(y) = e_G$  for all  $y \in Y$ . Then  $Hf \equiv e_G$ . Since  $H$  is an injective group homomorphism, this yields  $f \equiv e_G$ , which is a contradiction.

(5) Since  $X$  and  $Y$  are compact spaces, it will suffice to prove that  $h$  is one-to-one. Suppose there are two elements  $y_1 \neq y_2$  in  $Y$  such that  $h(y_1) = h(y_2) = x_0$ . Since  $\mathcal{B}$  separates the points of  $Y$ , there are  $D'_1, D'_2 \in \sigma(Z(\mathcal{B}))$  such that  $y_1 \in D'_1$ ,  $y_2 \in D'_2$ , and  $D'_1 \cap D'_2 = \emptyset$ . Since  $H(\mathcal{A}) = \mathcal{B}$ , there is  $f_i \in \mathcal{A}$  such that  $H(f_i)(y_i) \neq e_G$ , for  $i \in \{1, 2\}$ . Since  $\mathcal{B}$  is controllable, there are  $U_1 \in \sigma(\text{coz}(\mathcal{B}))$ ,  $g_1 \in \mathcal{A}$  such that  $y_1 \in D'_1 \subseteq U_1 \subseteq Y \setminus D'_2$ ,  $Hg_1|_{D'_1} = Hf_1|_{D'_1}$ , and  $Hg_1|_{Z(Hf_1) \cup (Y \setminus U_1)} \equiv e_G$ . Since  $D'_2 \cap U_1 = \emptyset$ , applying a compactness argument, there is  $D''_1 \in \sigma(Z(\mathcal{A}))$  such that  $U_1 \subseteq D''_1$  and  $D'_2 \cap D''_1 = \emptyset$ . Now, by the controllability of  $\mathcal{B}$ , there are  $U_2 \in \sigma(\text{coz}(\mathcal{B}))$  and  $g_2 \in \mathcal{A}$  such that  $y_2 \in D'_2 \subseteq U_2 \subseteq Y \setminus D''_1$ ,  $Hg_2|_{D'_2} = Hf_2|_{D'_2}$ , and  $Hg_2|_{Z(Hf_2) \cup (Y \setminus U_2)} \equiv e_G$ . Hence, since  $\text{coz}(Hg_i) \subseteq U_i$ ,  $U_1 \cap U_2 = \emptyset$ , and  $H$  is biseparating, it follows that  $\text{coz}(g_1) \cap \text{coz}(g_2) = \emptyset$ . On the other hand  $Hg_i(y_i) = Hf_i(y_i) \neq e_G$ ,  $i \in \{1, 2\}$ , and by item (3) above, we have that  $h(y_1) = h(y_2) = x_0 \in \text{coz}(g_1) \cap \text{coz}(g_2)$ , which is a contradiction.  $\square$

We have just seen how a separating group homomorphism  $H$  has associated a continuous map  $h$  that assigns to each point  $y \in Y$  the support subset of  $\delta_y \circ H$ . Our next goal now is to obtain a complete representation of  $H$  by means of the support map  $h$ . Having this in mind, set

$$G_{h(y)} \stackrel{\text{def}}{=} \text{Im}(\delta_{h(y)}) = \{f(h(y)) : f \in \mathcal{A}\} \quad (3)$$

which is a subgroup of  $G$  for all  $y \in Y$ , and denote by  $\text{Hom}(G_{h(y)}, G_y)$  the set of all group homomorphisms on  $G_{h(y)}$  into  $G_y$ . Consider now the set

$$\mathcal{G} \stackrel{\text{def}}{=} \bigcup_{y \in Y} \text{Hom}(G_{h(y)}, G_y). \quad (4)$$

We can think of the elements of  $\mathcal{G}$  as partial functions on  $G$ . That is, functions  $\alpha : \text{Dom}(\alpha) \subseteq G \rightarrow G$  whose domain is a (not necessarily proper) subset of  $G$ . Since the group  $G$  is discrete, we can equip  $\mathcal{G}$  with the product (or pointwise convergence) topology as follows.

Let  $[\alpha; g_1, \dots, g_n] \stackrel{\text{def}}{=} \{\beta \in G^G : \alpha(g_i) = \beta(g_i), g_i \in G, 1 \leq i \leq n\}$  be a basic neighborhood of a map  $\alpha \in G^G$ . If now  $\alpha$  is a partial map, we can restrict this basic neighborhood to  $\mathcal{G}$  by letting  $[\alpha; g_1, \dots, g_n]$  be the set of all partial maps  $\beta : \text{Dom}(\beta) \subseteq G \rightarrow G$  such that  $g_1, \dots, g_n \in \text{Dom}(\beta)$  and  $\alpha(g_i) = \beta(g_i)$ ,  $1 \leq i \leq n$ . It is easily verified that this procedure extends the pointwise convergence topology on  $\mathcal{G}$  (cf. [29]).

With this notation, we define  $\omega : Y \rightarrow \mathcal{G}$  by

$$\omega[y](f(h(y))) \stackrel{\text{def}}{=} Hf(y) \quad (5)$$

for each  $y \in Y$ . We will see next that  $\omega$  is well defined and continuous.

**Proposition 13.** *With the terminology established above, the following assertions are true.*

- (1)  $\omega[y]$  is a well-defined group homomorphism of  $G_{h(y)}$  into  $G_y$  for all  $y \in Y$ .
- (2)  $\omega$  is continuous when  $\mathcal{G}$  is equipped with the pointwise convergence topology.

*Proof.* (1) In order to prove that  $\omega[y]$  is well defined, take  $f_1, f_2 \in \mathcal{A}$  such that  $f_1(h(y)) = f_2(h(y))$ . By Proposition 8, we have  $\omega[y](f_1(h(y))) = Hf_1(y) = Hf_2(y) = \omega[y](f_2(h(y)))$ . The verification that  $\omega[y]$  is a group homomorphism is easy and it is left to the reader.

(2) Let  $(y_d)_{d \in D}$  be a net converging to  $y$  in  $Y$ . If  $g$  is an arbitrary element in  $\text{Dom}(\omega[y]) = G_{h(y)}$ , then  $g = f(h(y)) \in G_{h(y)}$  for some  $f \in \mathcal{A}$ . Since  $h$  continuous by Proposition 12,  $f \circ h \in C(Y, G)$  and  $(f \circ h)^{-1}(g)$  is a clopen neighborhood of  $y \in Y$ . Since  $G$  is discrete, there is  $d_1(g) \in D$  such that  $f(h(y_d)) = g$  for all  $d \geq d_1(g)$ . Thus  $g \in \text{Dom}(\omega[y_d])$  for all  $d \geq d_1(g)$ . In like manner, as  $\omega[y](g) = Hf(y) = g_y \in G$  and  $Hf \in C(Y, G)$ , we have that  $(Hf)^{-1}(g_y)$  is a clopen neighborhood of  $y$ . As a consequence there is  $d_2 \geq d_1(g)$  such that  $Hf(y_d) = g_y$  for all  $d \geq d_2$ . Thus  $\omega[y_d](g) = Hf(y_d) = g_y = Hf(y) = \omega[y](g)$  for all  $d \geq d_2$ . This means that the net  $(\omega[y_d])_{d \in D}$  converges to  $\omega[y]$  in the pointwise convergence topology over  $\mathcal{G}$ .  $\square$

Observe that since  $G$  is discrete, the compact subsets in  $G$  are all finite. Therefore, we have also proved that  $\omega$  is also continuous if we consider the compact open topology on  $\mathcal{G}$ . We are in position now of establishing a main result in this paper.

**Theorem 14.** *Let  $H : \mathcal{A} \rightarrow \mathcal{B}$  be a separating group homomorphism. Then there are continuous maps*

$$\begin{aligned} h : Y &\longrightarrow X, \\ \omega : Y &\longrightarrow \bigcup_{y \in Y} \text{Hom}(G_{h(y)}, G_y) \end{aligned} \quad (6)$$

satisfying the following properties.

- (1) For each  $y \in Y$  and every  $f \in \mathcal{A}$  it holds
 
$$Hf(y) = \omega[y](f(h(y))). \quad (7)$$
- (2)  $H$  is continuous with respect to the pointwise convergence topology.
- (3)  $H$  is continuous with respect to the compact open topology.
- (4) If  $H$  is biseparating bijection of  $\mathcal{A}$  onto  $\mathcal{B}$ , then  $h$  is a homeomorphism.

*Proof.* Item (1) is consequence of the definition of  $\omega$  and (2) follows from assertion (2) in Proposition 13. Thus, only (3) needs to be verified.

Let  $(f_d)_{d \in D} \in \mathcal{A}$  be a net converging to  $e_G$  in the compact open topology. If  $K$  is a compact subset of  $Y$ , then  $h(K)$  is

a compact subset in  $X$  by the continuity of  $h$ . Therefore  $(f_d)_d$  is eventually the constant function  $e_G$  on  $h(K)$ . Applying (1), it follows that  $(Hf_d)_{d \in D}$  is eventually  $e_G$  on  $K$ , which completes the proof.  $\square$

**Corollary 15.** *Let  $H : \mathcal{A} \rightarrow \mathcal{B}$  be a separating group homomorphism, where  $\mathcal{A}$  is pointwise dense. Then there are continuous maps  $h : Y \rightarrow X$  and  $\omega : Y \rightarrow \text{End}(G)$  satisfying the following properties.*

- (1) For each  $y \in Y$  and every  $f \in \mathcal{A}$  it holds

$$Hf(y) = \omega[y](f(h(y))). \quad (8)$$

- (2)  $H$  is continuous with respect to the pointwise convergence topology.
- (3)  $H$  is continuous with respect to the compact open topology.
- (4) If  $H$  is a biseparating bijection of  $\mathcal{A}$  onto  $\mathcal{B}$ , then  $h$  is a homeomorphism.

We are now in position of establishing the results formulated at Introduction.

*Proof of Theorem 2.* After Theorem 14 and Corollary 15, we only need to verify that  $\omega[y] \in \text{Aut}(G)$  for all  $y \in Y$ . Applying Corollary 15 to  $H^{-1}$ , we obtain maps

$$\begin{aligned} \rho : X &\longrightarrow \text{End}(G), \\ k : X &\longrightarrow Y \end{aligned} \quad (9)$$

such that for every  $x \in X$  and  $g \in \mathcal{B}$ , we have

$$H^{-1}g(x) = \rho[x](g(k(x))). \quad (10)$$

Thus, for every  $f \in \mathcal{A}$  and  $x \in X$ , we have

$$\begin{aligned} f(x) &= H^{-1} \circ (Hf)(x) = \rho[x](Hf(k(x))) \\ &= \rho[x](\omega[k(x)](f(h(k(x)))))) \end{aligned} \quad (11)$$

which means that the support subset of  $\delta_x \circ (H^{-1} \circ H)$  is both  $x$  and  $h(k(x))$ . Since the minimum support is unique, this means that  $h \circ k = id_X$ , which implies that  $k$  is a right inverse of  $h$ . Analogously, for every  $g \in \mathcal{B}$  and  $y \in Y$ , we have

$$\begin{aligned} g(y) &= H \circ (H^{-1}g)(y) = \omega[y](H^{-1}(g(h(y)))) \\ &= \omega[y](\rho[h(y)](g(k(h(y)))))) \end{aligned} \quad (12)$$

which means that the support subset of  $\delta_y \circ (H \circ H^{-1})$  is both  $y$  and  $k(h(y))$ . Again, this implies that  $k$  is a left inverse of  $h$ . Since  $k$  is both a left and right inverse of  $h$ , it follows that  $k = h^{-1}$ . Therefore

$$\begin{aligned} f(x) &= H^{-1} \circ (Hf)(x) = \rho[x](Hf(k(x))) \\ &= \rho[x](\omega[k(x)](f(x))), \\ g(y) &= H \circ (H^{-1}g)(y) = \omega[y](H^{-1}(g(h(y)))) \\ &= \omega[y](\rho[h(y)](g(y))). \end{aligned} \quad (13)$$

Applying the former equality to  $x = h(y)$ , it follows that  $\rho[h(y)] \circ \omega[y] = id_G$  for all  $y \in Y$ , and from the latter, we also have that  $\omega[y] \circ \rho[h(y)] = id_G$ . This means that  $\omega[y]$  has left and right inverse and, therefore, it is an automorphism on  $G$ , which completes the proof.  $\square$

*Proof of Corollary 3.* It follows directly from Theorem 2.  $\square$

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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