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PSEUDOCOMPACT GROUP TOPOLOGIES WITH PRESCRIBED TOPOLOGICAL SUBSPACES

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Abstract. We prove that every pseudocompact topological Abelian group G admits a pseudocompact topological group topology with a non-trivial convergent sequence.

Imposing some restrictions on the properties of G, stronger properties are also obtained. If, for instance, G is an Abelian group with $m(\beta) \leq r_0(G) \leq |G| \leq 2^\beta$ (see the Introduction below for unexplained terminology) for some uncountable cardinal β, and X is any topological space with $|X| \leq r_0(G)$ and $w(X) \leq \beta$, then G admits a pseudocompact topological group topology that contains X as a subspace.

If, on the other direction, G is a torsion Abelian group that admits a pseudocompact group topology, then, for every sequence $(a_n)_{n\in\mathbb{N}}$ of G there exists a pseudocompact group topology on G for which some subsequence of $(a_n)_{n\in\mathbb{N}}$ converges.

1 Introduction In this paper, all topological groups will be Hausdorff and all groups will be infinite and Abelian. The symbol $\mathbb{T}$ will stand for the unit circle, $\mathbb{N}$ the natural numbers and $\mathbb{P}$ the set of all prime numbers. The identity element of an arbitrary Abelian group will be simply denoted by 0. The order of an element $g \in G$ will be denoted by $o(g)$, this is the smallest integer m such that $mg = 0$. If G is a group and $n \in \mathbb{N}$, we let $nG = \{ng : g \in G\}$ and if $p \in \mathbb{P}$, we put $G_p = \{g \in G : \exists k \in \mathbb{N}\,(p^k g = 0)\}$, this is the so-called *p-component* of G. We recall that a finite subset $\{g_1, ..., g_k\}$ of an Abelian group G is called *independent* if $g_i \neq 0$, for all $1 \leq i \leq k$, and $n_1 g_1 + ... + n_k g_k = 0$ with $n_i \in \mathbb{Z}$, for each $1 \leq i \leq k$, implies that $n_i g_i = 0$, for every $1 \leq i \leq k$. An infinite subset A of G is called *independent* if every finite subset of A is independent. If G is an Abelian group, $t(G)$ will denote the *torsion* subgroup of G and $r_0(G)$ will stand for the *torsion-free rank* of G which is the cardinality of a maximal independent subset of G consisting entirely of non-torsion elements. For an infinite set X, we let $[X]^\omega = \{A \subseteq X : |A| = \omega\}$. When G_1 and G_2 are isomorphic groups (regardless the topology they may or may not carry) we will write $G_1 \approx G_2$.

It is well-known that every compact group G contains a copy of the Cantor set $\{0,1\}^{w(G)}$ (for a proof of this fact see [21]). Hence, every compact group has many non-trivial convergent sequences. S. M. Sirota [22] constructed (in ZFC) the first pseudocompact topological group without non-trivial convergent sequences answering a question posed by A. V. Arhangel'skiĭ in 1967. Several conditions for the existence of dense pseudocompact subgroups without non-trivial convergent sequences in a given compact group are given in [7] and [15]. It has been proved in [17] that it is independent from the axioms of ZFC that the compact group $\mathbf{Z}(2)^{\aleph_\omega}$ admits a dense pseudocompact subgroup without non-trivial convergent sequences. The question concerning the existence of a countably compact group

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without non-trivial convergent sequences is still unsolved in ZFC. Constructions of such a topological groups by using tools outside of ZFC are given in [9], [12], [18], [19] and [23]. In this paper, we mainly address our attention to the following question due to D. Dikranjan and D. Shakhmatov [9, Q. 13.17] and [10, Question 24]:

Question 1.1. *Does every pseudocompact group admit a pseudocompact group topology with a non-trivial convergent sequence?*

We obtain in this paper an affirmative answer for all Abelian groups. The same techniques also provide an answer to the countably compact version of this question when the torsion part of G is large. It may be worth mentioning that it has been proven in [16] that, under GCH, every pseudocompact group admits a pseudocompact group topology with no non-trivial convergent sequences.

2 Pseudocompact Abelian groups It is shown in [6] that a topological group is G is pseudocompact if and only if it is a G_δ-dense subgroup of a compact group if and only if its completion, $\overline{G}$, equals $\beta(G)$, the Stone-Čech compactification of G. A useful feature of pseudocompact groups is that they are either torsion groups of bounded order or their torsion-free rank is not less than $\mathfrak{c}$ (see Remark 2.17 of [2]). For a compact Abelian group G, $m(G)$ is the smallest cardinality of a dense pseudocompact subgroup of G, this cardinal number coincides with the smallest cardinality of a dense pseudocompact subgroup of $\{0,1\}^{w(G)}$ (for a proof of this fact the reader is referred to [4]). If α is an uncountable cardinal, then we define $m(\alpha) = m(\{0,1\}^\alpha)$. We know from [4] that

$$log(\alpha) \leq m(\alpha) \leq log(\alpha)^\omega.$$

Following [8], we say that an infinite cardinal number α is called *admissible* if there is a pseudocompact topological group of size α. E. K. van Douwen [11] proved that the cardinality of every infinite pseudocompact group is at least $\mathfrak{c}$. Hence, every admissible cardinal is bigger than or equal to $\mathfrak{c}$.

Theorem 2.1. *Let G be an Abelian group such that*

$$m(\beta) \leq r_0(G) \leq |G| \leq 2^\beta,$$

for some uncountable cardinal β. If X is a Tychonoff space such that $|X| \leq r_0(G)$ and $w(X) \leq \beta$, then G admits a pseudocompact, connected topological group topology τ with $w(G,\tau) \leq \beta$ such that (G,τ) contains X as a subspace.

Proof. Let $\mathcal{B}$ be a base of X with $|\mathcal{B}| = w(X)$ and $\emptyset \notin \mathcal{B}$. Let us consider the set $\mathcal{D}$ consisting of all $(U_1, ..., U_k) \in \mathcal{B}^k$ such that $0 < k \in \mathbb{N}$, $U_i \in \mathcal{B}$ for each $0 < i \leq k$ and the elements of $\{U_1, ..., U_k\}$ are pairwise disjoint. By the proof of Theorem 3.6 from [20], for each $u = (U_1, ..., U_k) \in \mathcal{D}$ we can find a function $f_u : X \to \mathbb{T}$ such that $f_u[U_i] = \{t_i\}$, for every $1 \leq i \leq k$, where $\{t_1, ..., t_k\}$ is a fixed, arbitrarily chosen, independent subset of the group $\mathbb{T}$ whose elements are torsion-free. We also know that the family $\{f_u : u \in \mathcal{D}\}$ separates points from closed sets of X. Thus, this family of functions induces an embedding $f : X \to \mathbb{T}^{\mathcal{D}}$ given by $f(x) = (f_u(x))_{u\in\mathcal{D}}$, for all $x \in X$. We shall show that $f[X]$ is an independent subset of the group $\mathbb{T}^{\mathcal{D}}$ consisting of torsion-free elements. Let to that end $x_1, ..., x_k \in X$. Choose $v = (V_1, ..., V_k) \in \mathcal{D}$ so that $x_i \in V_i$, for each $1 \leq i \leq k$. Let $\pi_v : \mathbb{T}^{\mathcal{D}} \to \mathbb{T}$ be the projection map. Since $\pi_v(f(x_i)) = f_v(x_i) = t_i$, for every $1 \leq i \leq k$, we deduce that $\{f(x_1), ..., f(x_k)\}$ is an independent subset of $\mathbb{T}^{\mathcal{D}}$ made of torsion-free elements. Since $|\mathcal{D}| \leq w(X) \leq \beta$ we may assume that $f[X] \subseteq \mathbb{T}^\beta$. Now, choose a maximal independent torsion-free subset A of G and split A in two disjoint subsets A_0 and A_1 so that $|A_0| = |X|$ and $m(\beta) \leq |A_1|$. We can therefore define a one-to-one function $F : A \to \mathbb{T}^\beta$ so that $F[A_0] = f[X]$, $F[A_1]$ is an independent subset of $\mathbb{T}^\beta$ whose elements are torsion-free and $F[A_1]$ is G_δ-dense in $\mathbb{T}^\beta$. We extend F to a group monomorphism from $\langle A\rangle$ to $\mathbb{T}^\beta$. Corollary 4.2 from [8] guarantees, on the other hand, the existence of a monomorphism $F' : t(G) \to t(\mathbb{T}^\beta)$. The map $h : \langle A\rangle \oplus t(G) \to \mathbb{T}^\beta$ defined by $h(a + t) = F(a) + F'(t)$, for every $a \in \langle A\rangle$ and for every $t \in t(G)$, defines a one-to-one homomorphism of $\langle A\rangle \oplus t(G)$ into $\mathbb{T}^\beta$. We know, by [14, Lemma 16.2], that $\langle A\rangle \oplus t(G)$ is essential in G, and since $\mathbb{T}^\beta$ is a divisible group, we have that the homomorphism h can be extended to a one-to-one homomorphism $H: G \to \mathbb{T}^\beta$ [14, Lemma 24.4]. The homomorphism H will also have G_δ-dense range ($F[A_1] = h[A_1] \subseteq H[G]$) and therefore defines a connected pseudocompact group topology on G. Since $f[X] = F[A_0] = H[A_0]$, G with this topology has a subspace that is homeomorphic to X (namely, $H^{-1}[F[A_0]]$). □

Next, we give some immediate applications of Theorem 2.1. More refined consequences will be obtained in Section 3.

Lemma 2.2. *If G is a connected, pseudocompact Abelian group and $r_0(G) = r_0(G)^\omega$, then*

$$m(2^{r_0(G)}) \leq r_0(G) \leq |G| \leq 2^{2^{r_0(G)}}.$$

Proof. We know that the inequality $|G| \leq 2^{2^{r_0(G)}}$ holds for any connected, pseudocompact Abelian group G (see [8, Th. 1.8]). On the other hand, we have that

$$m(2^{r_0(G)}) \leq log(2^{r_0(G)})^\omega \leq r_0(G)^\omega = r_0(G).$$

Therefore, $m(2^{r_0(G)}) \leq r_0(G) \leq |G| \leq 2^{2^{r_0(G)}}$. □

Corollary 2.3. *Let G be a pseudocompact, connected, topological Abelian group with $r_0(G) = r_0(G)^\omega$. If X is a Tychonoff space such that $|X| \leq r_0(G)$, then G admits a pseudocompact, connected topological group topology that contains X as a subspace.*

Proof. According to Lemma 2.2, we only need to take $\beta = 2^{r_0(G)}$ in Theorem 2.1. □

Corollary 2.4. *Let G be a pseudocompact, connected, topological Abelian group. If X is a Tychonoff space such that $|X| \leq r_0(G)$ and $w(X) \leq 2^{w(G)}$, then G admits a pseudocompact, connected topological group topology that contains X as a subspace.*

Proof. Let $\beta = w(G)$, by Theorem 7.1 of [9], $m(\beta) \leq r_0(G) \leq |G| \leq 2^\beta$ and the Corollary follows at once from Theorem 2.1. □

Corollary 2.5. *Let G be a non-torsion, pseudocompact Abelian group with $|G| \leq 2^{2^\mathfrak{c}}$. If X is a Tychonoff space such that $|X| \leq \mathfrak{c}$, then G admits a pseudocompact, connected topological group topology that contains X as a subspace.*

Proof. Since $r_0(G) \geq \mathfrak{c}$ when G is non-torsion, we have that

$$m(2^\mathfrak{c}) \leq \mathfrak{c} \leq r_0(G) \leq |G| \leq 2^{2^\mathfrak{c}}.$$

Therefore, G admits a pseudocompact group topology that contains X as a subspace, by Theorem 2.1 (with $\beta = 2^\mathfrak{c}$). □

We end this section by remarking that Corollaries 2.4 and 2.5 show that connected pseudocompact groups and pseudocompact groups of small cardinality always admit pseudocompact group topologies with a non-trivial convergent sequence. We extend this to all pseudocompact groups in the next section.

3 Pseudocompact topologies with convergent sequences We give in this section an affirmative answer to Question 1.1 in the category of Abelian groups. Our arguments mainly rely on the structure of Abelian groups, we summarize here the basic facts that are needed.

It is well-known that every torsion group decomposes as the direct sum of its p-components (see [14]). Given an Abelian group G and $p \in \mathbb{P}$, we will denote by D_p a fixed maximal divisible subgroup of G_p (see [20, Th. A.6]). Thus $G_p = D_p \oplus C_p$ with C_p being a reduced subgroup of G_p, and

$$t(G) = \bigoplus_{p \in \mathbb{P}} (D_p \oplus C_p)\,.$$

For each $p \in \mathbb{P}$, we will further find necessary to bring into consideration, a basic subgroup B_p of G_p (for the existence of such a groups see [20, Th. A.24]). Recall that B is a *basic* subgroup of an Abelian group G provided that: (1) B is a direct sum of cyclic groups, (2) B is pure in G and (3) G/B is divisible.

For $p \in \mathbb{P}$, we will keep the meaning of G_p, D_p, C_p and B_p fixed for the rest of this section.

Lemma 3.1. *Let G be an Abelian group with $|G| = r_0(G)$. If G admits a pseudocompact group topology, then G also admits one with a non-trivial convergent sequence.*

Proof. Simply apply Theorem 2.1 with $\beta = |\overline{G}|$. Indeed, recall that $|K| = 2^{w(K)}$, for every compact group K, and that $w(X) \leq |X|$, for each compact space X. Since, necessarily, $m(\beta) \leq |G|$ and $|G| = r_0(G)$, we have that

$$m(\beta) \leq r_0(G) \leq |\overline{G}| = 2^{w(\overline{G})} = 2^{\beta},$$

and, by Theorem 2.1, G contains every compact space K with $|K| \leq r_0(G)$. □

Lemma 3.2. *Let G be an Abelian group such that $|B_p| \geq \mathfrak{c}$ for some $p \in \mathbb{P}$. If G admits a pseudocompact group topology, then G also admits one with a non-trivial convergent sequence.*

Proof. Since B_p is a direct sum of cyclic groups, there is a direct summand $L_p \subseteq B_p$ with

$$L_p \approx \bigoplus_{\alpha_n} \mathbb{Z}(p^n),$$

for some positive integer n and some cardinal $\alpha_n \geq \mathfrak{c}$. Now L_p is pure in B_p, B_p is pure in G_p, G_p is pure in $t(G)$ and $t(G)$ is pure in G (see [14, Section 26]); hence, L_p is pure in G. Being bounded and pure in G, L_p is a direct summand, thus

$$G = L_p \oplus H_p,$$

for some subgroup H_p of G. As $\alpha_n \geq \mathfrak{c}$, we obtain that

$$G \approx (\oplus_{\mathfrak{c}} \mathbb{Z}(p^n)) \oplus L_p \oplus H_p = (\oplus_{\mathfrak{c}} \mathbb{Z}(p^n)) \oplus G.$$

Now, the group $\oplus_c \mathbb{Z}(p^n)$ is isomorphic to $\mathbb{Z}(p^n)^{\omega}$ (see, for instance, Lemma 3.1 from [3]) and this latter group can be equipped with a compact metric topology τ_m. If G admits a pseudocompact group topology τ we see then that $\tau_m \times \tau$ is a pseudocompact group topology on $\mathbb{Z}(p^n)^{\omega} \times G \cong G$ with many non-trivial convergent sequences. □

The proof of Lemma 3.2 can be applied to produce a countably compact group topology on G that admits a non-trivial convergent sequence, provided G can be equipped with a countably compact group topology. To extend these results to more general Abelian groups we need some more information on the relation between G and $\widehat{\overline{G}}$. We recall that if G is a topological Abelian group, then $\widehat{G}$ will denote the group of continuous characters equipped with the compact-open topology.

Lemma 3.3. *If G is a pseudocompact Abelian group and $\overline{G}$ is its completion, then:*

1. $r_0(G) \geq m(r_0(\widehat{\overline{G}}))$.
2. *If $D \subset G$ is a divisible subgroup, then* $|D| \leq 2^{r_0\left(\widehat{\overline{G}}\right)}$.

Proof. (1) Let F be a maximal torsion-free independent subset of $\widehat{\overline{G}}$. According to Theorem 24.2 from [20], we know that $\widehat{\widehat{\overline{G}}} = \overline{G}$. Let us consider the projection map $\pi : \overline{G} \to \mathbb{T}^F$. Since G is G_δ-dense in $\overline{G}$, $\pi[G]$ is a G_δ-dense subgroup of $\mathbb{T}^F$. As a subgroup of $\mathbb{T}^F$, $\pi[G]$ is a connected pseudocompact group. By [8, Theorem 7.1], $r_0(\pi[G]) \geq m(|F|) = m\big(r_0(\widehat{\overline{G}})\big)$. As $r_0(G) \geq r_0(\pi[G])$, this finishes the proof.

(2) If $D \subset G$ is a divisible subgroup, then its closure in $\overline{G}$, is connected and is therefore contained in the connected component of $\overline{G}$, it only remains to notice that the cardinality of this component is exactly $2^{r_0(\widehat{\overline{G}})}$. □

Theorem 3.4. *Every pseudocompact Abelian group admits a pseudocompact group topology containing a non-trivial convergent sequence.*

Proof. We consider several cases.

Case 1: *G is a torsion group.* We know from [5] that G is a group of bounded order, and bounded Abelian groups are direct sums of cyclic groups (see [14, Theorem 17.2]). There is therefore a finite set of prime numbers P_1 such that $G_p = \{0\}$ for each $p \notin P_1$ while $G_p = B_p = C_p$ for all $p \in P_1$. Since $|G| \geq \mathfrak{c}$ we conclude from Lemma 3.2 that G admits a pseudocompact group topology with a non-trivial convergent sequence.

Case 2: *G is non-torsion and $|G| \leq 2^{\mathfrak{c}}$.* This case follows immediately from Corollary 2.5.

Case 3: $r_0(G) = |G|$. This is Lemma 3.1.

Case 4: $\max\{r_0(G), 2^{\mathfrak{c}}\} < |G|$ *and $|G_p| = |D_p|$, for all p with $|G_p| > 2^{\mathfrak{c}}$.* We then have, by Lemma 3.3, that $2^{r_0(\widehat{\overline{G}})} \geq |G_p|$ and so $|G| \leq 2^{r_0(\widehat{\overline{G}})}$. According to Lemma 3.3, we have that $r_0(G) \geq m(r_0(\widehat{\overline{G}}))$. Thus, we can apply Theorem 2.1 with $\beta = r_0(\widehat{\overline{G}})$ to find the desired pseudocompact group topology with a non-trivial convergent sequence.

Case 5. *$|D_p| < |G_p|$ for some $p \in \mathbb{P}$ with $|G_p| > 2^{\mathfrak{c}}$.* By Corollary 34.4 of [14] $|C_p| \leq |B_p|^\omega$, and we have in this case that $|B_p| \geq \mathfrak{c}$. Lemma 3.2 is thus applicable. □

We next give a direct proof of Theorem 3.4 for torsion groups. The countably compact version of Theorem 3.4 then follows easily. This proof will help us to understand the basic idea of the proof Theorem 3.7. First, we need to prove a preliminary lemma.

Lemma 3.5. *If the Abelian torsion group G admits a pseudocompact group topology, then there exists $n \in \mathbb{N}$ such that G is isomorphic to $\mathbb{Z}_n^\omega \times G$.*

Proof. Suppose that G is a pseudocompact torsion group. We know from [5] that G as a group is of bounded order and, by Theorem 6.2 of [8], we have that

$$G = \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{n_j} \bigoplus_{\gamma_{j,i}} \mathbb{Z}(p_j^{r_{j,i}}),$$

where $p_{j,i} \in \mathbb{P}$, $r_{j,i} \in \mathbb{N}$, $\gamma_{j,i}$ are cardinal numbers and $n_j \in \mathbb{N}$ such that $1 \leq r_{j,1} < ... < r_{j,n_j}$ for all $1 \leq j \leq m$ and for all $1 \leq i \leq n_j$, and $\max\{\gamma_{j,i+1}, ..., \gamma_{j,n_j}\}$ is either finite or admissible, for each $1 \leq i < n_j$ and for each $1 \leq j \leq m$. Hence, there are $1 \leq k \leq m$ and $1 \leq l \leq n_j$ such that $\gamma_{k,l}$ is admissible and then $\mathfrak{c} \leq \gamma_{k,l}$. So, we obtain that

$$\begin{aligned} G &\approx \bigoplus_{\gamma_{k,l}} \mathbb{Z}(p_k^{r_{k,l}}) \times \left(\bigoplus_{k\neq j=1}^{m} \bigoplus_{l\neq i=1}^{n_j} \bigoplus_{\gamma_{j,i}} \mathbb{Z}(p_j^{r_{j,i}}) \right) \\ &\approx \left(\bigoplus_{\mathfrak{c}} \mathbb{Z}(p_k^{r_{k,l}}) \times \bigoplus_{\gamma_{k,l}} \mathbb{Z}(p_k^{r_{k,l}}) \right) \times \left(\bigoplus_{k\neq j=1}^{m} \bigoplus_{l\neq i=1}^{n_j} \bigoplus_{\gamma_{j,i}} \mathbb{Z}(p_j^{r_{j,i}}) \right) \\ &\approx \bigoplus_{\mathfrak{c}} \mathbb{Z}(p_k^{r_{k,l}}) \times \left(\bigoplus_{\gamma_{k,l}} \mathbb{Z}(p_k^{r_{k,l}}) \times \bigoplus_{k\neq j=1}^{m} \bigoplus_{l\neq i=1}^{n_j} \bigoplus_{\gamma_{j,i}} \mathbb{Z}(p_j^{r_{j,i}}) \right) \\ &\approx \bigoplus_{\mathfrak{c}} \mathbb{Z}(p_k^{r_{k,l}}) \times G. \end{aligned}$$

According to Lemma 3.1 from [3], we have that $\bigoplus_{\mathfrak{c}} \mathbb{Z}(p_k^{r_{k,l}}) \approx \mathbb{Z}(p_k^{r_{k,l}})^\omega$ and so $G \approx \bigoplus_{\mathfrak{c}} \mathbb{Z}(p_k^{r_{k,l}}) \times G \approx \mathbb{Z}(p_k^{r_{k,l}})^\omega \times G$. □

Corollary 3.6. *Let G be an infinite torsion Abelian group that admits a pseudocompact group topology (countably compact group topology). Then, G admits a pseudocompact group topology topology (countably compact group topology) with a non-trivial convergent sequence.*

Proof. By Lemma 3.5, we may find $n \in \mathbb{N}$ so that $G \approx \mathbb{Z}_n^\omega \times G$. By Theorem 3.10.26 of [13], we have that $\mathbb{Z}_n^\omega \times G$ is pseudocompact (resp. countably compact by Theorem 3.10.14 loc. cit.) and contains a copy of the metric compact group $\mathbb{Z}_n^\omega$. □

Corollary 3.6 can finally be improved as follows.

Theorem 3.7. *Let G be a torsion Abelian group that admits a pseudocompact group topology. Then, for every sequence $(a_n)_{n\in\mathbb{N}}$ of G there exists a pseudocompact group topology on G for which some subsequence of $(a_n)_{n\in\mathbb{N}}$ converges in this topology.*

Proof. As indicated above, pseudocompact torsion groups are direct sums of cyclic groups, that is there is a finite set of prime numbers $P = \{p_j \colon 1 \leq j \leq m\}$ such that $G = \bigoplus_{p\in P} G_p$ and G_p is a direct sum of cyclic groups of order a power of p.

We first fix $p \in P$ and shall show that the theorem holds for G_p. Recall, by Theorem 6.2 of [8], that G_p admits a pseudocompact topological group topology. Now, we know that

$$G_p = \bigoplus_{k=1}^{l} \bigoplus_{\gamma_k} \mathbb{Z}(p^{r_k})$$

where $1 \leq r_1 < \cdots < r_l$ are natural numbers and $\{\gamma_1, \gamma_2, ..., \gamma_l\}$ are cardinal numbers such that $\max\{\gamma_{i+1}, ..., \gamma_l\}$ is either finite or admissible, for all $1 \leq i < l$. Let $(a_n)_{n\in\mathbb{N}}$

be a sequence of G_p. The support of an element $g \in G_p$ will be denoted by $s(g)$. We may assume that the support of an element of G_p is a finite subset of $\bigcup_{k=1}^{l}\{k\} \times \gamma_k$. Put $S = \bigcup\{s(a_n) : n \in \mathbb{N}\}$. Let $I = \{k : \mathfrak{c} \leq \gamma_k\}$, $J = \{k : \omega \leq \gamma_k < \mathfrak{c}\}$ and $K = \{k : \gamma_k \text{ is finite}\}$. Then, $G_p = \left(\prod_{k\in K} \mathbb{Z}(p^{r_k})^{\gamma_k}\right) \oplus \left(\bigoplus_{k\in I\cup J} \bigoplus_{\gamma_k} \mathbb{Z}(p^{r_k})\right)$. For each $k \in J$, we choose $j_k \geq k$ for which γ_{j_k} is admissible (this is possible by Theorem 3.17 of [3]) and choose $J_k \subseteq \gamma_{j_k}$ of cardinality $\mathfrak{c}$ so that $(\{j_k\} \times J_k) \cap S = \emptyset$. We can also assume that $|\gamma_j \setminus (\bigcup_{j_k=j} J_k)| = \gamma_j$ and $\{J_k : j_k = j\}$ are pairwise disjoint, for each $1 \leq j \leq l$. For every $j \in I$ find $I_j \subseteq \gamma_j \setminus (\bigcup_{j_k=j} J_k)$ of cardinality $\mathfrak{c}$ such that $(\{j\} \times \gamma_j) \cap S \subseteq \{j\} \times I_j$. For each $n \in \mathbb{N}$ and for each $1 \leq k \leq l$, let $a_n^k \in \bigoplus_{\gamma_k} \mathbb{Z}(p^{r_k})$ for which $\sum_{k=1}^{l} a_n^k = a_n$. For every $1 \leq k \leq l$, we then have the following:

1. If $k \in I$, then $a_n^k \in \bigoplus_{I_k} \mathbb{Z}(p^{r_k})$.
2. If $k \in J$, then $a_n^k \in \left(\bigoplus_{\gamma_k} \mathbb{Z}(p^{r_k})\right) \oplus \left(\bigoplus_{J_k} \mathbb{Z}(p^{r_{j_k}})\right)$.
3. If $k \in K$, then $a_n^k \in \mathbb{Z}(p^{r_k})^{\gamma_k}$.

Claim. For each $k \in J$, the group $\left(\bigoplus_{\gamma_k} \mathbb{Z}(p^{r_k})\right) \oplus \left(\bigoplus_{J_k} \mathbb{Z}(p^{r_{j_k}})\right)$ can be equipped with a pseudocompact group topology in which every subsequence of $(a_n^k)_{n\in\mathbb{N}}$ has a convergent subsequence.

Proof: Fix $k \in J$. Let $\{J_k^0, J_k^1\}$ be a partition of J_k into two subsets of cardinality $\mathfrak{c}$. We first observe that, since $r_k < r_{j_k}$, we can embed $\mathbb{Z}(p^{r_k})$ in $\mathbb{Z}(p^{r_{j_k}})$ in such a way that the former contains all elements of the latter whose order is p^j with $j \leq r_k$. Using that embedding we have that

$$\left(\bigoplus_{\gamma_k} \mathbb{Z}(p^{r_k})\right) \oplus \left(\bigoplus_{J_k^0} \mathbb{Z}(p^{r_{j_k}})\right) \subset$$

$$\left(\bigoplus_{\gamma_k} \mathbb{Z}(p^{r_{j_k}})\right) \oplus \left(\bigoplus_{J_k^0} \mathbb{Z}(p^{r_{j_k}})\right) \approx \bigoplus_{\mathfrak{c}} \mathbb{Z}(p^{r_{j_k}}).$$

Being isomorphic to $\mathbb{Z}(p^{r_{j_k}})^\omega$ (see Theorem 3.1 of [3]), $\left(\bigoplus_{\gamma_k} \mathbb{Z}(p^{r_{j_k}})\right) \oplus \left(\bigoplus_{J_k^0} \mathbb{Z}(p^{r_{j_k}})\right)$ admits a metric compact group topology. Let us consider this topology and equip $\bigoplus_{\gamma_k} \mathbb{Z}(p^{r_k}) \oplus \bigoplus_{J_k^0} \mathbb{Z}(p^{r_{j_k}})$ with the subspace topology. In this topology, every sequence will admit a convergent subsequence. Since the order a_n^k is not greater than p^{r_k} for any n, the limit of any subsequence of $(a_n^k)_{k\in\mathbb{N}}$ will lie in $\bigoplus_{\gamma_k} \mathbb{Z}(p^{r_k}) \oplus \left(\bigoplus_{J_k^0} \mathbb{Z}(p^{r_{j_k}})\right)$.

We now equip $\left(\bigoplus_{\gamma_k} \mathbb{Z}(p^{r_k})\right) \oplus \left(\bigoplus_{J_k} \mathbb{Z}(p^{r_{j_k}})\right)$ with a pseudocompact group topology. We know from Theorem 3.2 from [3] that $\bigoplus_{J_k^1} \mathbb{Z}(p^{r_{j_k}})$ can be embedded in $(\mathbb{Z}_{p^{r_{j_k}}})^{\mathfrak{c}}$ as a G_δ-dense subset. Hence, the group $\bigoplus_{\gamma_k} \mathbb{Z}(p^{r_k}) \oplus \bigoplus_{J_k^0} \mathbb{Z}(p^{r_{j_k}}) \oplus \bigoplus_{J_k^1} \mathbb{Z}(p^{r_{j_k}})$ can be embedded in $\mathbb{Z}(p^{r_{j_k}})^{\mathfrak{c}}$ as a G_δ-dense subset so that $\bigoplus_{\gamma_k} \mathbb{Z}(p^{r_k}) \oplus \bigoplus_{J_k^0} \mathbb{Z}(p^{r_{j_k}})$ maintains the topology previously defined.

Therefore, $\bigoplus_{\gamma_k} \mathbb{Z}(p^{r_k}) \oplus \bigoplus_{J_k} \mathbb{Z}(p^{r_{j_k}})$ can be equipped with a pseudocompact group topology in which every subsequence of $(a_n^k)_{k\in\mathbb{N}}$ has a convergent subsequence. This proves the claim.

Next, we shall define the required pseudocompact group topology on G_p. Observe that

$$G_p = \prod_{k\in K} \mathbb{Z}(p^{r_k})^{\gamma_k} \oplus \bigoplus_{k\in J} \left(\bigoplus_{\gamma_k} \mathbb{Z}(p^{r_k}) \oplus \bigoplus_{J_k} \mathbb{Z}(p^{r_{j_k}}) \right) \oplus$$
$$\bigoplus_{k\in I} \left(\left(\bigoplus_{I_k} \mathbb{Z}(p^{r_k}) \right) \oplus \left(\bigoplus_{j\in I} \bigoplus_{\gamma_j \setminus (I_j \cup \{J_k : j_k = j\})} \mathbb{Z}(p^{r_j}) \right) \right).$$

According to Theorem 6.2 from [8], for each $k \in I$,the group

$$\left(\bigoplus_{I_k} \mathbb{Z}(p^{r_k}) \right) \oplus \left(\bigoplus_{j\in I} \bigoplus_{\gamma_j \setminus (I_j \cup \{J_k : j_k = j\})} \mathbb{Z}(p^{r_j}) \right)$$

admits a pseudocompact group topology so that $\bigoplus_{I_k} \mathbb{Z}(p^{r_k})$ is metrizable (it is isomorphic to $\mathbb{Z}(p^{r_k})^{\omega}$). Since the product of pseudocompact groups is pseudocompact ([6]), the product topology on G_p will be pseudocompact and will satisfy all the requirements.

Let us consider the general case. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in G. We know that $G = \bigoplus_{j=1}^m G_{p_j}$. For each $n \in \mathbb{N}$ and for each $1 \le j \le m$, choose $a_n^j \in G_{p_j}$ so that $\sum_{j=1}^m a_n^j = a_n$. Inductively, for each $1 \le j \le m$, we can find a topology τ_j on G_{p_j} and $A_j \in [\mathbb{N}]^{\omega}$ so that (G_{p_1}, τ_j) is a pseudocompact group, $A_{j+1} \subseteq A_j$, for all $1 \le j < m$ and the sequence $(a_n^j)_{n\in A_j}$ is τ_j-convergent, for all $1 \le j \le m$. By Theorem 6.2 from [8], the topological group $G = \bigoplus_{j=1}^m (G_{p_j}, \tau_j)$ is pseudocompact and it is clear that the subsequence $(a_n)_{n\in A_m}$ converges in this space.

We remark that if $\gamma_k \ge \mathfrak{c}$, for all $1 \le k \le l$ then the sequence will be inside a compact metric subgroup and the last subgroup in the sum above would be algebraically the same as the original group.

□

We have seen that every Abelian bounded torsion group G can be expressed as

$$G = \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{n_j} \bigoplus_{\gamma_{j,i}} \mathbb{Z}(p_j^{r_{j,i}}),$$

where $p_{j,i} \in \mathbb{P}$, $r_{j,i} \in \mathbb{N}$, $\gamma_{j,i}$ are cardinal numbers and $n_j \in \mathbb{N}$ such that $1 \le r_{j,1} < ... < r_{j,n_j}$ for all $1 \le j \le m$ and for all $1 \le i \le n_j$, and $\max\{\gamma_{j,i+1}, ..., \gamma_{j,n_j}\}$ is either finite or admissible, for each $1 \le i < n_j$ and for each $1 \le j \le m$. The cardinal numbers $\gamma_{j,i}$ are called the Ulm-Kaplanski invariants of G.

With a few minor modifications the proof of the previous theorem can be used to establish the following countably compact version:

Theorem 3.8. *Let G a torsion Abelian group that admits a countably compact group topology. If all Ulm-Kaplansky invariants of G are at least $\mathfrak{c}$ then for every sequence $(a_n)_{n\in\mathbb{N}}$ in G there exists a countably compact group topology on G for which some subsequence of $(a_n)_{n\in\mathbb{N}}$ converges.*

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