APPROXIMABLE $\mathcal{WAP}$- AND $\mathcal{LUC}$-INTERPOLATION SETS

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ABSTRACT. Extending and unifying concepts extensively used in the literature, we introduce the notion of approximable interpolation sets for algebras of functions on locally compact groups, especially for weakly almost periodic functions and for uniformly continuous functions. We characterize approximable interpolation sets both in combinatorial terms and in terms of the $\mathcal{LUC}$- and $\mathcal{WAP}$-compactifications and analyze some of their properties.

CONTENTS

1. INTRODUCTION

Interpolation sets have been a key technique for the construction of functions of various types on infinite discrete or, more generally, locally compact groups. They have the crucial property that any bounded function defined on them extends to the whole group as a function of the required type.

If we require the extended functions to be almost periodic, then interpolation sets are usually known as I_0-sets and were introduced by Hartman and Ryll-Nardzewski [27]. For further details and recent results on I_0-sets, see for example the papers by Galindo and Hernández [20, 21] Graham and Hare [22, 24, 25], Graham, Hare and Körner [23] or Hernández [28].

Interpolation sets for the functions in the Fourier-Stieltjes algebra $B(G)$ are usually known as Sidon sets when the group G is discrete and Abelian and weak Sidon sets in general, see for instance the works by Lopez and Ross [32], and Picardello [33]. Sidon sets are in fact uniformly approximable as proved by Drury in [9]. This means that in addition to being interpolation sets for the Fourier-Stieltjes

algebra, the characteristic function of the set can be uniformly approximated by members of the algebra; in other words, the characteristic function of the set belongs to the Eberlein algebra $\mathcal{B}(G) = \overline{B(G)}^{\|\cdot\|_\infty}$. This fact has important consequences as for instance Drury's union theorem: the union of two Sidon subsets of a discrete Abelian group remains Sidon.

Ruppert [39] and Chou [7] considered interpolation sets for the algebra of weakly almost periodic functions on discrete groups and semigroups, again with the extra condition that the characteristic function of the set is weakly almost periodic. This is equivalent to the property that all bounded functions vanishing off the set are weakly almost periodic. These interpolation sets were called *translation-finite* sets (after their combinatorial characterization) by Ruppert and R_W-sets (after their interpolation properties) by Chou.

Let $\ell_\infty(G)$ be the C*-algebra of bounded, scalar-valued functions on G with the supremum norm and let $\mathcal{A}(G) \subseteq \ell_\infty(G)$. In the present paper we introduce the notion of approximable $\mathcal{A}(G)$-interpolation sets in such a way that it is suitable for functions defined on any topological group G and reduces to that of uniformly approximable Sidon sets when G is discrete and $\mathcal{A}(G) = B(G)$, and to that of R_W-sets or translation-finite sets when G is discrete and $\mathcal{A} = \mathcal{WAP}(G)$, the algebra of weakly almost periodic functions on G. Since we shall be dealing with closed subalgebras of $\ell_\infty(G)$, save some brief digressions around $B(G)$, we shall omit the adverb "uniformly" in our definition (cf. Definition 3.1).

As the reader might expect, approximable $\mathcal{A}(G)$-interpolation sets can be found in abundance if the algebra is large while they might be hard to find if the algebra is too small. As extreme cases, we could mention that all subsets of G have the property if $\mathcal{A}(G) = \ell_\infty(G)$, while non-compact locally compact groups cannot have approximable $\mathcal{AP}(G)$-interpolation sets, where $\mathcal{AP}(G)$ is the algebra of almost periodic functions on G (see below).

Our principal concern shall be with the algebras $\mathcal{LUC}(G)$, of bounded functions which are uniformly continuous with respect to the right uniformity of G, and $\mathcal{WAP}(G)$, of weakly almost periodic functions on G. A combinatorial characterization of approximable $\mathcal{LUC}(G)$- and $\mathcal{WAP}(G)$-interpolation sets will be presented in Section 4.

But beforehand, we deal in Section 3 with the more straightforward cases of the algebra $\mathcal{CB}(G)$ of bounded, continuous, scalar-valued functions and the algebra $C_0(G)$ of continuous functions vanishing at infinity on G. It turns out that for the algebras $C_0(G)$, $\mathcal{CB}(G)$ and $\mathcal{LUC}(G)$, interpolation sets and approximable interpolation sets are the same for any topological group. This is also true for the algebra $\mathcal{B}(G)$ when G is an Abelian discrete group, a fact that follows from Drury's theorem since, as we shall see in Proposition 3.5, uniformly approximable Sidon sets (in the sense of Dunkl-Ramirez [10]) are the same as our approximable $\mathcal{B}(G)$-interpolation sets when G is discrete.

We do not know if this stays true for any locally compact group, but we give a partial result towards its affirmation for locally compact metrizable groups. It should be remarked that approximability cannot be expected within the algebra $B(G)$. In fact, Dunkl and Ramirez noted in [10, Remark 5.5, page 59] that if T is a Sidon set in a discrete Abelian group G, and 1_T denotes the characteristic function of T, then $1_T \in B(G)$ if and only if T is finite. They observed also that for $G = \mathbb{Z}$, this holds for any subset of G.

It is quick to see that the closed discrete sets are $\mathcal{CB}(G)$-interpolation sets when the topological space underlying G is normal, and the finite sets are $C_0(G)$-interpolation sets (Proposition 3.3). In Section 4, we see that the right (left) uniformly discrete sets are approximable interpolation sets for the algebra $\mathcal{LUC}(G)$

($\mathcal{RUC}(G)$) for any topological group. The converse of each of these statements is proved when G is metrizable.

To study approximable $\mathcal{WAP}(G)$-interpolation sets we shall work within the frame of locally compact E-groups. This is a class of groups (introduced in [6], see Section 4 for the definitions of E-groups and E-sets) larger than that of locally compact SIN-groups whose members always admit a good supply of weakly almost periodic functions. The key concept here is that of translation-compact sets, also defined in Section 4. The main result proved in this regard is the topologized analogue of a theorem obtained by Ruppert for discrete groups (also valid for some semigroups) in [39, Theorem 7], parts of which were also proven independently by Chou in [7].

We show first that in any locally compact group G, if T is a right (or left) uniformly discrete, approximable $\mathcal{WAP}(G)$-interpolation set, then VT is translation-compact for some neighbourhood V of the identity e.

The converse is also true when G is a locally compact SIN-group or, more generally, when G is a locally compact E-group and T is an E-set.

We prove that in any locally compact E-group G, any right (or left) uniformly discrete E-set T such that VT is translation-compact, for some neighbourhood V of the identity e, is an approximable $\mathcal{WAP}(G)$-interpolation set.

We deduce that when G is in addition metrizable, right (and left) uniformly discrete sets such that VT is translation-compact determine completely the approximable $\mathcal{WAP}(G)$-interpolation sets.

Some consequences are obtained. We see first that $B(G)$-interpolation sets in a metrizable locally compact Abelian group (i.e., topological Sidon sets) are necessarily uniformly discrete with respect to some neighbourhood V of e such that VT is translation-compact. In particular, Sidon sets in a discrete Abelian group are translation-finite.

A second remarkable corollary follows. We consider the space of functions f in $\mathcal{WAP}(G)$ such $|f|$ is almost convergent to zero; that is

$$\mathcal{WAP}_0(G) = \{f \in \mathcal{WAP}(G) : \mu(|f|) = 0)\},$$

where μ is the unique invariant mean on $\mathcal{WAP}(G)$. With an additional help of Ramsey theory, we deduce that approximable $\mathcal{WAP}(G)$- and approximable $\mathcal{WAP}_0(G)$-interpolation sets are in fact the same. It follows, as a consequence of this characterization, that even under weaker conditions in Definition 3.1 (ii), there would still be no infinite subsets of a metrizable locally compact group, in particular no infinite subsets of a discrete group, which are approximable $\mathcal{AP}(G)$-interpolation sets.

Other interesting consequences follow. As in [39], approximable $\mathcal{WAP}(G)$-interpolation sets are also characterized in term of their closure in the $\mathcal{WAP}$-compactification $G^{\mathcal{WAP}}$ of G. We note that, as a consequence of these results, sets of this sort provide a combinatorial characterization of the points (called strongly primes) which do not belong to the closure of G^*G^* in the $\mathcal{WAP}$-compactification $G^{\mathcal{WAP}}$ of G, where G^* is the remainder $G^{\mathcal{WAP}} \setminus G$ .The same result is also true for the $\mathcal{LUC}$-compactification of G. Some of these facts can already be seen in Theorem 4.16, but since they go out of the scope of the present paper, we wish to develop them further in a forthcoming paper [15].

Section 5 is devoted to analyzing the behaviour of approximable $\mathcal{A}(G)$-interpolation sets under finite unions. We prove that finite unions of approximable $\mathcal{A}(G)$-interpolation sets are approximable $\mathcal{A}(G)$-interpolation sets, provided the union is uniformly discrete. We deduce a union theorem for a class of translation-compact sets (precisely those obtained from approximable $\mathcal{WAP}(G)$-interpolation sets). When G is

discrete, this was proved by Ruppert in [39] after he characterized right translation-finite sets. A direct combinatorial argument proving this fact was provided recently in [16, Lemma 5.1].

We have included in Section 6 some examples and remarks along with some questions that are left open in the paper. We, for instance, see how our characterizations of $\mathcal{A}(G)$-interpolation sets fail in the absence of metrizability. We give an example (namely the Bohr compactification $\mathbb{Z}^{\mathcal{AP}}$) of a compact non-metrizable group with an $\mathcal{AP}(G)$-interpolation set which is neither left nor right uniformly discrete. Under (CH) such an example can be found in any nonmetrizable locally compact group. We also see, by means of a simple example, why the passage from the discrete to the locally compact case needs some care: a subset $T \subseteq \mathbb{R}$ can be both an approximable $\mathcal{WAP}(\mathbb{R}_d)$-interpolation set and an approximable $\mathcal{LUC}(\mathbb{R})$-interpolation set, without being an approximable $\mathcal{WAP}(\mathbb{R})$-interpolation set.

2. Preliminaries

We list in this Section the definitions of our function algebras. We follow as much as possible notation and terminology from [2] to which the reader is directed for more details. Let $\mathcal{CB}(G)$ be the algebra of continuous, bounded, scalar-valued functions equipped with its supremum norm and $C_0(G)$ be the algebra made of continuous functions vanishing at infinity on G. For each function f defined on G, the left translate f_s of f by $s \in G$ is defined on G by $f_s(t) = f(st)$.

2.1. $\mathcal{LUC}(\mathbf{G})$, uniformly continuous functions. A bounded function f on G is *right uniformly continuous* when, for every $\epsilon > 0$, there exists a neighbourhood U of e such that
$$|f(s) - f(t)| < \epsilon \quad \text{whenever} \quad st^{-1} \in U.$$
These are functions which are left norm continuous, i.e.,
$$s \mapsto f_s : G \to \mathcal{CB}(G)$$
is continuous, and so in the literature, these functions are denoted also by $\mathcal{LC}(G)$ or $\mathcal{LUC}(G)$. In this note, we shall use the latter notation. In a like manner, we shall denote by $\mathcal{RUC}(G)$ the algebra of *left uniformly continuous* functions on G and by $\mathcal{UC}(G)$ the algebra of *right and left uniformly continuous* functions on G, hence $\mathcal{UC}(G) = \mathcal{LUC}(G) \cap \mathcal{RUC}(G)$.

2.2. $\mathcal{AP}(\mathbf{G})$, almost periodic functions. A function f is *almost periodic* when the set of all its left (equivalently, right) translates is a relatively norm compact subset in $\mathcal{CB}(G)$.

2.3. $\mathcal{WAP}(\mathbf{G})$, weakly almost periodic functions. A function $f \in \mathcal{CB}(G)$ is *weakly almost periodic* when the set of all its left (equivalently, right) translates form a relatively weakly compact subset in $\mathcal{CB}(G)$.

2.4. $\mathcal{WAP_0}(\mathbf{G})$. If μ is the unique invariant mean on $\mathcal{WAP}(G)$ (see [2], or [4]), we put
$$\mathcal{WAP}_0(G) = \{f \in \mathcal{WAP}(G) : \mu(|f|) = 0\}.$$

2.5. $\mathcal{B}(\mathbf{G})$, the uniform closure of the Fourier-Stieltjes algebra. The *Fourier-Stieltjes algebra* $B(G)$ is the space of coefficients of unitary representations of G. Equivalently, $B(G)$ is the linear span of the set of all continuous positive definite functions on G. As the Fourier-Stieltjes algebra is not uniformly closed, we will consider its uniform closure $\mathcal{B}(G)$. Hence we have the *Eberlein algebra* $\mathcal{B}(G) = \overline{B(G)}^{\|\cdot\|_\infty}$.

Recall that

$$C_0(G) \oplus \mathcal{AP}(G) \subseteq \mathcal{B}(G) \subseteq \mathcal{WAP}(G) = \mathcal{AP}(G) \oplus \mathcal{WAP}_0(G)$$
$$\subseteq \mathcal{LUC}(G) \cap \mathcal{RUC}(G) \subseteq \mathcal{LUC}(G) \subseteq \mathcal{CB}(G)$$

(see [5] for the first inclusion and [2] for the rest).

If $\mathcal{A}(G) \subseteq \ell_\infty(G)$ is a C^*-subalgebra there is a canonical morphism $\epsilon_\mathcal{A} \colon G \to G^\mathcal{A}$ of G into the spectrum (non-zero multiplicative linear functionals) $G^\mathcal{A}$ of $\mathcal{A}(G)$ that is given by evaluations:

$$\epsilon_\mathcal{A}(g)(f) = f(g), \text{ for every } f \in \mathcal{A}(G) \text{ and } g \in G.$$

This map is continuous if $\mathcal{A}(G) \subseteq \mathcal{CB}(G)$ and injective if $\mathcal{A}(G)$ separates points. In this case we will omit the function ϵ and identify G as a subgroup of $G^\mathcal{A}$.

We say that the C^*-algebra $\mathcal{A}(G)$ is admissible when it satisfies the following properties: $1 \in \mathcal{A}(G)$, $\mathcal{A}(G)$ is left translation invariant, i.e., $f_s \in \mathcal{A}(G)$ for every $f \in \mathcal{A}(G)$ and $s \in G$, and the function defined on G by $xf(s) = x(f_s)$ is in $\mathcal{A}(G)$ for every $x \in G^\mathcal{A}$ and $f \in \mathcal{A}(G)$. When $\mathcal{A}(G)$ is admissible, $G^\mathcal{A}$ becomes a semigroup compactification of the topological group G. This means that $G^\mathcal{A}$ is a compact semigroup having a dense homomorphic image of G such that the mappings

$$x \mapsto xy \colon G^\mathcal{A} \to G^\mathcal{A} \quad \text{and} \quad x \mapsto \epsilon_\mathcal{A}(s)x \colon G^\mathcal{A} \to G^\mathcal{A}$$

are continuous for every $y \in G^\mathcal{A}$ and $s \in G$. The product in $G^\mathcal{A}$ is given by

$$xy(f) = x(yf) \quad \text{for every} \quad x, y \in G^\mathcal{A} \text{ and } f \in \mathcal{A}(G).$$

When $\mathcal{A}(G) = \mathcal{LUC}(G)$, the semigroup compactification $G^{\mathcal{LUC}}$ is usually referred to as the $\mathcal{LUC}$-compactification. It is the largest semigroup compactification in the sense that any other semigroup compactification is the quotient of $G^{\mathcal{LUC}}$. When G is discrete, $G^{\mathcal{LUC}}$ and the Stone-Čech compactification βG are the same. The semigroup compactification $G^{\mathcal{WAP}}$ is referred to as the $\mathcal{WAP}$-compactification, and it is the largest semitopological semigroup compactification. The embedding $\epsilon_{\mathcal{LUC}}$ is a homeomorphism onto its image, hence G may be identified with its image in $G^{\mathcal{LUC}}$. The same is true for $G^{\mathcal{WAP}}$ when G is locally compact. The closure of a set X in $G^\mathcal{A}$ is denoted by $\overline{X}^\mathcal{A}$, while in G the closure of X is denoted as usual by $\overline{X}$.

A recent account on semigroup compactifications is given in [19].

3. Approximable interpolation sets

We start with the main definition of the paper. We then identify the $\mathcal{A}(G)$-interpolation sets for metrizable topological groups when $\mathcal{A}(G) = C_0(G)$ and $\mathcal{CB}(G)$. They are given, respectively, by the finite sets and the closed discrete sets.

Definition 3.1. *Let G be a topological group with identity e and let $\mathcal{A}(G) \subseteq \ell_\infty(G)$. A subset $T \subseteq G$ is said to be*

    (i) *an $\mathcal{A}(G)$-interpolation set if every bounded function $f \colon T \to \mathbb{C}$ can be extended to a function $\tilde{f} \colon G \to \mathbb{C}$ such that $\tilde{f} \in \mathcal{A}(G)$.*

    (ii) *an approximable $\mathcal{A}(G)$-interpolation set if it is an $\mathcal{A}(G)$-interpolation set and for every neighbourhood U of e, there are open neighbourhoods V_1, V_2 of e with $\overline{V_1} \subseteq V_2 \subseteq U$ such that, for each $T_1 \subseteq T$ there is $h \in \mathcal{A}(G)$ with $h(V_1 T_1) = \{1\}$ and $h(G \setminus (V_2 T_1)) = \{0\}$.*

It is quick to notice that condition (ii) in Definition 3.1 already forbids any set in G to be an approximable $\mathcal{AP}(G)$-interpolation sets when G is not compact. In fact, taking in particular T_1 to be finite and U compact forces h in Definition 3.1 to have a compact support, and so h cannot be in $\mathcal{AP}(G)$. However, even if we allow only infinite subsets T_1 of T in condition (ii), approximable $\mathcal{AP}(G)$-interpolation sets still do not exist in locally compact groups. The following is true in more general

situations and in particular for any locally compact metrizable group. But since to prove this fact we need the machinery developed later in next section, we content ourselves in the present section with the following easy particular case.

Proposition 3.2. *A discrete, divisible Abelian group G does not have any proper susbet T such that its characteristic function 1_T is almost periodic.*

Proof. Let G be a discrete, divisible Abelian group. It is well-known that for Abelian groups the Bohr compactification of G can be identified with the group of *all* characters of the character group of G, i.e., $G^{\mathcal{AP}} = \left(\widehat{G}_d\right)^{\wedge}$, [29, Theorem 26.12]. Since G is divisible, $\widehat{G}$ is torsion-free and duals of torsion-free Abelian groups are connected [29, Theorem 24.23], so $G^{\mathcal{AP}}$ is connected.

Now it only remains to realize that if $T \subseteq G$ is such that its characteristic function 1_T is almost periodic, then $\overline{T}^{\mathcal{AP}}$ is closed and open in $G^{\mathcal{AP}}$, and so T must be the whole of G. $\qquad\square$

Proposition 3.3. *Let G be a topological group with identity e.*
  (i) *If $\mathcal{A}(G) \subseteq \mathcal{CB}(G)$, then the $\mathcal{A}(G)$-interpolation sets are discrete.*
  (ii) *For the algebras $C_0(G)$, $\mathcal{CB}(G)$ and $\mathcal{LUC}(G)$, every interpolation set is approximable.*
  (iii) *If the underlying topological space of G is normal, then all discrete closed subsets of G are approximable $\mathcal{CB}(G)$-interpolation sets.*
  (iv) *If G is locally compact, then all finite subsets of G are approximable $C_0(G)$-interpolation sets.*
  (v) *If G is metrizable, then the approximable $\mathcal{CB}(G)$-interpolation sets are precisely the discrete closed sets.*
  (vi) *If G is locally compact and metrizable, then the $C_0(G)$-interpolation sets are precisely the finite subsets.*

Proof. Statements (i) and (iv) are clear.

To see Statement (ii), let $T \subseteq G$, U be any neighbourhood of e and choose a neighbourhood V of e such that $V^2 \subseteq U$. Then $V(VT) \subseteq UT$, and so $V(VT)$ and $(G \setminus (UT))$ are disjoint in G. Therefore there exists $h \in \mathcal{LUC}(G)$ such that $h(\overline{VT}) = \{0\}$ and $h((G \setminus (UT))) = \{1\}$. Then the statement clearly follows for the algebras $\mathcal{CB}(G)$ and $\mathcal{LUC}(G)$.

Let now T be $C_0(G)$-interpolation set. Then it is easily seen that $\overline{T}$ is compact subset of G. For, otherwise, a nonzero constant function on T cannot be extended to a function in $C_0(G)$. It follows that if U is a compact neighbourhood of the identity, $U\overline{T}$ will be compact as well. It is therefore straightforward to find $V_1, V_2 \subset U$ and a function h with compact support that satisfies the definition of approximable interpolation sets.

Urysohn's lemma together with Statement (ii) leads immediately to Statement (iii).

As for Statement (v), suppose otherwise that T is discrete but not closed, then pick a convergent sequence (t_n) in T with its limit outside of T, and observe that the function f defined on T such that $f(t_n) = (-1)^n$ cannot be extended to a function in $\mathcal{CB}(G)$.

For the last statement, let T be a $C_0(G)$-interpolation set. By the previous statements, T must be closed and discrete. The proof of (ii) shows on the other hand that T must be relatively compact as well. Finiteness of T follows. $\qquad\square$

Remark 3.4. *In the non-metrizable situation, non-closed $\mathcal{CB}(G)$-interpolation sets exist, see Example 6.2 and Theorem 6.3.*

In [10, Corollary 2.2, page 49], it was proved that T is a Sidon set in a discrete Abelian group G (i.e., $B(G)$-interpolation set) if and only if it is a $\mathcal{B}(G)$-interpolation set. This is actually true for any discrete group, as can be deduced from a result due to Chou ([7, Lemma 3.11]). Chou's proof used direct functional analysis arguments and does not rely on harmonic analysis tools.

As noted earlier $1_T \notin B(G)$ when T is infinite. However, 1_T may be a member of $\mathcal{B}(G)$. If a Sidon set T satisfies this property, then Dunkl and Ramirez called it a uniformly approximable Sidon set ([10, Definition 5.3, page 59]). Indeed, Drury proved in [9] that Sidon sets in a discrete Abelian group are uniformly approximable Sidon sets, see also [10, Theorem 5.7, page 61]. So with our terminology, each Sidon subset of a discrete Abelian group G is an approximable $\mathcal{B}(G)$-interpolation sets. We summarize these observations in the following Proposition.

Proposition 3.5. *Let G be a discrete Abelian group and T a subset of G. Then the following statements are equivalent.*

    (i) *T is a $B(G)$-interpolation set.*
    (ii) *T is a $\mathcal{B}(G)$-interpolation set.*
    (iii) *T is an approximable $\mathcal{B}(G)$-interpolation set.*

When G is a metrizable locally compact Abelian group, the implication (i) $\Longrightarrow$ (iii) of the previous proposition is valid. For this, we need the following lemma due to Dechamps-Gondim ([8, Theorème 1.1 and Theorème 2.1]). We note that $B(G)$-interpolation sets are called *topological Sidon* sets in [8].

Lemma 3.6 (Dechamps-Gondin). *Let G be a metrizable locally compact Abelian group and T be a $B(G)$-interpolation set. Then for every $\beta > 0$ and every neighbourhood U of the identity, there exist a constant M and a compact subset $K \subseteq \widehat{G}$ such that for every finite subset X of T there exists a function $f \in L_1(\widehat{G})$ with support contained in K and*

    (i) *$\|f\|_1 \leq M$,*
    (ii) *$\widehat{f}_{|X} = 1$,*
    (iii) *$|\widehat{f}(g)| < \beta$ for every $g \notin UX$.*

Proposition 3.7. *Let G be a metrizable locally compact Abelian group. Then the $B(G)$-interpolation sets are approximable $\mathcal{B}(G)$-interpolation sets.*

Proof. Let T be a $B(G)$-interpolation set. Choose a neighbourhood U of the identity and $\beta < \frac{1}{2}$. Take from Lemma 3.6 the corresponding $K \subseteq \widehat{G}$ and $M > 0$.

We first observe that the Fourier-Stieltjes transforms $\widehat{\mu}$ of measures $\mu \in M(\widehat{G})$ with support contained in K and $\|\mu\| \leq M$ all have a common modulus of uniform continuity: indeed if $V_{K,M} = \{g \in G \colon |\chi(g) - 1| < \varepsilon/2M, \text{ for all } \chi \in K\}$, then $V_{K,M}$ is a neighbourhood of e and $gh^{-1} \in V_{K,M}$ implies that $|\widehat{\mu}(g) - \widehat{\mu}(h)| < \varepsilon$. Let $V_1 \subseteq V_{K,M}$ be a neighbourhood of e such that $\overline{V_1} \subseteq U$. We now see that V_1 and U suffice to show that T is an approximable $\mathcal{B}(G)$-interpolation set. Let to that end $T_1 \subseteq T$. By Lemma 3.6, for each finite subset $X \subseteq T_1$, there is $f_X \in L_1(\widehat{G})$ with $\|f_X\|_1 \leq M$, $\widehat{f_X}(X) = \{1\}$ and $|\widehat{f_X}(g)| < \beta$ for every $g \in G \setminus UX$. Consider the net $(f_X)_X$ where X runs over all finite subsets of T_1 ordered by inclusion and let $\mu \in M(\widehat{G})$ be the limit of some subnet of $(f_X)_X$ in the weak $\sigma(M(G), C_0(G))$-topology (recall that the ball of radius M in $M(G)$ is compact in this topology). Then $\|\mu\| \leq M$, $\widehat{\mu}(T_1) = \{1\}$ and $|\widehat{\mu}(g)| < \beta$ for every $g \in G \setminus UT_1$. Let $\psi = \widehat{\mu} \in B(G)$ for the remainder of this proof.

Extend ψ to a continuous function $\widetilde{\psi}$ on the Eberlein compactification $G^{\mathcal{B}}$ of G. Take $0 < \epsilon < 1 - \beta$ and choose a continuous function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho((-\beta, \beta)) = \{0\}$ and $\rho((1 - \epsilon, 1 + \epsilon)) = \{1\}$. Define finally $\widetilde{\phi} = \rho \circ \widetilde{\psi}$. Then

$\tilde{\phi} \in \mathcal{CB}(G^{\mathcal{B}})$, and so $\phi := \tilde{\phi}{\restriction}_{G} \in \mathcal{B}(G)$. Moreover, if $s \in V_1 T_1$ then $st^{-1} \in V_1$ for some $t \in T_1$, and so

$$|\psi(s) - 1| = |\psi(s) - \psi(t)| < \epsilon,$$

showing that $\psi(s) \in (1 - \epsilon, 1 + \epsilon)$, and therefore, $\phi(s) = \rho(\psi(s)) = 1$. If, on the other hand, $s \notin UT_1$, then $\phi(s) = \rho(\psi(s)) = 0$, since $\psi(s) \in (-\beta, \beta)$. We have thus that $\psi(V_1 T_1) = \{1\}$ and $\psi(G \setminus UT_1) = \{0\}$. This completes the proof. $\qquad\square$

4. Characterizing approximable $\mathcal{LUC}(G)$- and $\mathcal{WAP}(G)$-interpolation sets

The case of $\mathcal{LUC}(G)$ is not as straightforward as that of $C_0(G)$ or $\mathcal{CB}(G)$. But we prove in Theorem 4.9 that the $\mathcal{LUC}(G)$-interpolation sets are given by the right uniformly discrete sets when G is metrizable.

The situation becomes more delicate with the algebra $\mathcal{WAP}(G)$. Here, if G is minimally weakly almost periodic (as, for instance, the group $SL(2, \mathbb{R})$) then only finite sets are $\mathcal{WAP}(G)$-interpolation sets, for in this case $\mathcal{WAP}(G) = C_0(G) \oplus \mathbb{C}1$. So we shall work with groups which have a good supply of non-trivial weakly almost periodic functions such as SIN-groups, or more generally, E-groups (see [1] or [13]).

Theorem 4.16 concerns mainly approximable $\mathcal{WAP}(G)$-interpolation sets with G a locally compact E-group. This theorem is the topologized version of a theorem of Ruppert proved in [39] for some infinite discrete semigroups. We identify the approximable $\mathcal{WAP}(G)$-interpolation sets when G is metrizable. They are given by the left (or right) uniformly discrete sets T such that VT is translation-compact for some neighbourhood V of the identity.

Another interesting consequence is given in Theorem 4.22 which shows that left (or right) uniformly discrete, approximable $\mathcal{WAP}(G)$-interpolation sets and left (or right) uniformly discrete, approximable $\mathcal{WAP}_0(G)$-interpolations sets are in fact the same sets T, they both have the property that VT is translation-compact for some neighbourhood V of the identity. When G is metrizable, the uniform discreteness turns to be a necessary condition as well.

We should point out that part of Theorem 4.16 (and so also part of Theorem 4.22) is proved beforehand in Corollary 4.12 for any locally compact group. Theorem 4.22 together with Corollary 4.12 implies, as already promised before Proposition 3.2, that uniformly discrete sets can never be approximable $\mathcal{AP}(G)$-interpolation sets in this class of groups even under conditions weaker than those in Definition 3.1 (ii).

Definition 4.1. *Let G be a (non-compact) topological group. We say that a subset T of G is*

(i) *(Ruppert, [39]) right translation-finite if every infinite subset $L \subseteq G$ contains a finite subset F such that $\bigcap \{b^{-1}T : b \in F\}$ is finite; left translation-finite if every infinite subset $L \subseteq G$ contains a finite subset F such that $\bigcap \{Tb^{-1} : b \in F\}$ is finite; and translation-finite when it is both right and left translation-finite.*

(ii) *right translation-compact if every non-relatively compact subset $L \subseteq G$ contains a finite subset F such that $\bigcap \{b^{-1}T : b \in F\}$ is relatively compact; left translation-compact if every non-relatively compact subset $L \subseteq G$ contains a finite subset F such that $\bigcap \{Tb^{-1} : b \in F\}$ is relatively compact; and translation-compact when it is both left and right translation-compact.*

(iii) *a right t-set (left t-set) if there exists a compact subset K of G containing e such that $gT \cap T$ (respectively, $Tg \cap T$) is relatively compact for every $g \notin K$; and a t-set when it is both a right and a left t-set.*

Right t-sets were used originally by Rudin in [38] to construct weakly almost periodic functions which are not in $\mathcal{B}(G)$ for non-compact locally compact Abelian

groups with a closed discrete subgroup of unbounded order. Then they were used by Ramirez in [35] for the same purpose when G is a non-compact locally compact Abelian group. They came up again in the non-Abelian situation when Chou proved in [5] the same result if G is nilpotent or if G is an IN-group (i.e., G has an invariant compact neighbourhood of the identity). In another paper [6], Chou used these sets to construct funtions in $\mathcal{WAP}_0(G)$ which are not in $C_0(G)$.

More recently, the $\mathcal{WAP}$-functions defined with the help of right t-sets enabled Baker and Filali [1] and Filali [13] to study some algebraic properties of the $\mathcal{WAP}$-compactification $G^{\mathcal{WAP}}$ of G.

Ruppert [39, Theorem 7] and Chou [7, Proposition 2.4] proved that the translation-finite subsets of discrete groups (called R_W-sets in [7]) precisely coincide with the approximable $\mathcal{WAP}(G)$-interpolation sets. The same class of sets was used by Filali and Protasov in [16] to characterize the strongly prime ultrafilters in the Stone-Čech compactification βG of a discrete group G. They were called sparse sets.

It is evident that the right (left) t-sets are also right (left) translation-compact. Examples of right translation-finite sets which are not right t-sets are easy to construct (see also [39, Examples 11]), and even examples of right translation-finite sets which are not finite unions of right t-sets were devised in the discrete case by Chou [7, Section 3]. t-Sets having compact covering as large as that of G may be constructed by induction in any non-compact locally compact group G, see the following example taken from [13]. Recall that, for a topological space X, the compact covering of X is the minimal number $\kappa(X)$ of compact sets required to cover X.

Example 4.2. *Let G be a non-compact locally compact group and fix a compact symmetric neighbourhood V of the identity e of G. Start with $t_0 = e$, say. Let $\alpha < \kappa(G)$ and suppose that the elements t_β have been selected for every $\beta < \alpha$. Set*

$$T_\alpha = \bigcup_{\beta_1, \beta_2, \beta_3 < \alpha} V^2 t_{\beta_1}^{\epsilon_1} t_{\beta_2}^{\epsilon_2} V^2 t_{\beta_3}^{\epsilon_3},$$

where each $\epsilon_i = \pm 1$. Then $\kappa(T_\alpha) < \kappa(G)$, and so we may select an element t_α in $G \setminus T_\alpha$ for our set T. In this way, we form a set $T = \{x_\alpha : \alpha < \kappa(G)\}$. As already checked in [13], the set T is right V^2-uniformly discrete (the definition is given below), has $\kappa(T) = |T| = \kappa(G)$, $s(VT) \cap (VT)$ and $(VT)s \cap (VT)$ are relatively compact for every $s \notin V^2$, i.e., VT is a t-set.

We proceed now to prove the analogues of Statements (iii)-(vi) of Proposition 3.3 for the algebras $\mathcal{LUC}(G)$ and $\mathcal{WAP}(G)$. We shall need right (left) uniformly discrete sets and translation-compact sets instead of discrete closed sets or finite sets. The relevance of translation-compact sets in our setting is made clear in Lemma 4.3 below. We also see in Example 6.1 that translation-finite sets do not suffice to characterize approximable $\mathcal{WAP}$-interpolation sets.

A crucial tool for the rest of the paper is Grothendieck's criterion which we recall now. For the proof, see for example [2, Theorem 4.2.3], or the original paper of Grothendieck [26].

A bounded function $f : G \to \mathbb{C}$ is weakly almost periodic if and only if for every pair of sequences (s_n) and (t_m) in G,

$$\lim_n \lim_m f(s_n t_m) = \lim_m \lim_n f(s_n t_m)$$

whenever the limits exist.

The criterion may be equivalent stated in terms of ultrafilters: a bounded function $f : G \to \mathbb{C}$ is weakly almost periodic if and only if for every pair of sequences (s_n) and (t_m) in G and every pair of free ultrafilters, $\mathcal{U}$ and $\mathcal{V}$ over $\mathbb{N}$, the limits of

the function along the ultrafilters coincide, i.e.,

$$\lim_{n,\mathcal{U}} \lim_{m,\mathcal{V}} f(s_n t_m) = \lim_{m,\mathcal{V}} \lim_{n,\mathcal{U}} f(s_n t_m).$$

The proof of the following Lemma extracts the basic idea of Lemma B of [1].

Lemma 4.3. *Let G be a topological group and let $T \subseteq G$. If T is translation-compact, then every right and left uniformly continuous function supported in T is weakly almost periodic and its extension to $G^{\mathcal{WAP}}$ vanishes on $G^* G^*$.*

Proof. Let $f \colon G \to \mathbb{C}$ be in $\mathcal{UC}(G)$ and supported in T. We first prove that whenever (s_α) and (t_β) are nets in G that do not accumulate in G, then

$$\lim_{\alpha} \lim_{\beta} f(s_\alpha t_\beta) = \lim_{\beta} \lim_{\alpha} f(s_\alpha t_\beta) = 0.$$

Suppose towards a contradiction that $\lim_\alpha \lim_\beta f(s_\alpha t_\beta) \neq 0$. This means that there is α_0 such that for every $\alpha \geq \alpha_0$, we have $\lim_\beta f(s_\alpha t_\beta) \neq 0$. Since (s_α) does not cluster in G and T is translation-compact, there must exist a finite set $\{s_{\alpha_1}, \ldots, s_{\alpha_k}\}$ with $\alpha_i \geq \alpha_0$ for each $i = 1, 2, ..., k$ such that $\bigcap_{i=1}^k s_{\alpha_i}^{-1} T$ and $\bigcap_{i=1}^k T s_{\alpha_i}^{-1}$ are relatively compact. Since $\lim_\beta f(s_\alpha t_\beta) \neq 0$, there must be an index β_0 such that $f(s_{\alpha_i} t_\beta) \neq 0$ for any $\beta \geq \beta_0$ and $i = 1, 2, \ldots, k$. We deduce thus that

$$t_\beta \in \bigcap_{i=1}^k s_{\alpha_i}^{-1} T \text{ for all } \beta \geq \beta_0.$$

Since the latter set is relatively compact, we deduce that (t_β) has a cluster point in G, a contradiction showing that $\lim_\alpha \lim_\beta f(s_\alpha t_\beta) = 0$. Arguing symmetrically, we can prove that $\lim_\beta \lim_\alpha f(s_\alpha t_\beta) = 0$.

We now prove that $f \in \mathcal{WAP}(G)$. Take two sequences sequences (s_n) and (t_m) in G such that both limits $\lim_n \lim_m f(s_n t_m)$ and $\lim_m \lim_n f(s_n t_m)$ exist. If neither (s_n) nor (t_m) accumulate in G, the above argument shows that

$$\lim_{n} \lim_{m} f(s_n t_m) = \lim_{m} \lim_{n} f(s_n t_m) = 0.$$

If (s_n) has a cluster point $g \in G$, we choose a cluster point $q \in G^{\mathcal{LUC}}$ of (t_m), and taking into account that the multiplication is jointly continuous on $G \times G^{\mathcal{LUC}}$ and passing to subnets if necessary, we see that

$$\lim_{n} \lim_{m} f(s_n t_m) = f^{\mathcal{LUC}}(gq) = \lim_{m} \lim_{n} f(s_n t_m),$$

where $f^{\mathcal{LUC}}$ is the extension of f to $G^{\mathcal{LUC}}$. If, alternatively, it is (t_m) that accumulates at some point of G, we argue in the same way using $G^{\mathcal{RUC}}$ instead of $G^{\mathcal{LUC}}$ (the function f is assumed to be in $\mathcal{LUC}(G) \cap \mathcal{RUC}(G)$).

We obtain, that in any case,

$$\lim_{n} \lim_{m} f(s_n t_m) = \lim_{m} \lim_{n} f(s_n t_m),$$

and, as a consequence of Grothendiecks's criterion, $f \in \mathcal{WAP}(G)$.

Having proved that $f \in \mathcal{WAP}(G)$, the last assertion of the theorem is a straightforward consequence of the argument in the first paragraph of this proof. If $p, q \in G^* G^*$, then $p = \lim_\alpha s_\alpha$, $q = \lim_\beta t_\beta$ for some nets $(s_\alpha) \in G$ and $(t_\beta) \in G$. Since the nets (s_α) and (t_β) do not accumulate in G, the mentioned argument shows that $f^{\mathcal{WAP}}(pq) = 0$. $\square$

Definition 4.4. *Let G be a topological group with identity e, T be a subset of G and U be a neighbourhood of e. We say that T is*

(i) *right U-uniformly discrete if $Us \cap Us' = \varnothing$ for every $s \neq s' \in T$.*
(ii) *left U-uniformly discrete if $sU \cap s'U = \varnothing$ for every $s \neq s' \in T$.*

(iii) right uniformly discrete *(respectively,* left uniformly discrete*) when it is right V-uniformly discrete (respectively, left V-uniformly discrete) with respect to some neighbourhood V of e.*

(iv) *uniformly discrete if it is both left and right uniformly discrete.*

The functions we introduce below constitute an important tool to reflect combinatorial properties at the function algebra level. When f is the constant 1-function, they provide pointwise approximations to the characteristic function.

Notation 4.5. *Let T be a subset of a topological group G with identity e, and suppose that T is right (respectively, left) U-uniformly discrete with respect to some neighbourhood U of e. Let V be a symmetric neighbourhood of e such that $V^2 \subseteq U$. For each $\psi \in \mathcal{LUC}(G)$ (respectively, $\psi \in \mathcal{RUC}(G)$) with $\psi(e) = 1$ and $\psi(G \setminus V) = \{0\}$, and for each bounded function $f \colon T \to \mathbb{C}$, we define a function $f_{T,\psi}$ (respectively, $f_{\psi,T}$)$\colon G \to \mathbb{C}$ by*

$$f_{T,\psi}(s) = \sum_{t \in T} f(t)\psi\left(st^{-1}\right) \quad \left(\text{respectively, } f_{\psi,T}(s) = \sum_{t \in T} f(t)\psi\left(t^{-1}s\right) \right). \qquad (*)$$

Let G be non-compact and recall from [5], that an *E-set* is a non-relatively compact subset T of G such that for each neighbourhood U of e, the set

$$\bigcap\{t^{-1}Ut : t \in T \cup T^{-1}\}$$

is again a neighbourhood of e. When such a set exists in G, we say that G is an *E-group.* It is clear that non-compact $SIN-$groups are E-groups (with every non-relatively compact subset being an E-set). Direct products of any E-group with any topological group are again E-groups. The groups with a non-compact centre such as the matrix group $GL(n, \mathbb{R})$ belong also to this class.

Lemma 4.6. *Let G be a topological group with identity e, and let U, V be symmetric neighbourhoods of e such that $V^2 \subseteq U$ and let T be right U-uniformly discrete (respectively, left U-uniformly discrete). Consider $\psi \in \mathcal{LUC}(G)$ (respectively, $\psi \in \mathcal{RUC}(G)$) with $\psi(e) = 1$ and $\psi(G \setminus V) = \{0\}$. Then, for every bounded function $f \colon T \to \mathbb{C}$:*

(i) *$f_{T,\psi}(G \setminus VT) = \{0\}$ (respectively, $f_{\psi,T}(G \setminus TV) = \{0\}$).*

(ii) *$f_{T,\psi} \in \mathcal{LUC}(G)$ (respectively, $f_{\psi,T} \in \mathcal{RUC}(G)$) and*

$$f_{T,\psi}(vt) = f_{\psi,T}(tv) = f(t)\psi(v) \quad \text{for all } v \in V \text{ and } t \in T.$$

(iii) *If G is an E-group, T is an E-set and $\psi \in \mathcal{UC}(G)$, then $f_{T,\psi} \in \mathcal{UC}(G)$ (respectively, $f_{\psi,T} \in \mathcal{UC}(G)$). If, in addition, VT (respectively, TV) is translation-compact, then $f_{T,\psi} \in \mathcal{WAP}(G)$ (respectively, $f_{\psi,T} \in \mathcal{WAP}(G)$).*

Proof. Statement (i) follows immediately from the definition of $f_{T,\psi}$. The proof of (ii) is precisely [2, Exercise 4.4.16]). If $g, h \in G$ are such that $gh^{-1} \in V$, and $g \in Vt$ for some $t \in T$, then $h \in V^2t$, while $h \notin Vt'$ for any $t \neq t' \in T$. So for arbitrary $g, h \in G$ with $gh^{-1} \in V$, either $f_{T,\psi}(g) = f_{T,\psi}(h) = 0$ or there is $t \in T$ with

$$\left|f_{T,\psi}(g) - f_{T,\psi}(h)\right| = \left|f(t)\psi(gt^{-1}) - f(t)\psi(ht^{-1})\right|.$$

As $\psi \in \mathcal{LUC}(G)$, we see that $f_{T,\psi}$ is right uniformly continuous, i.e., $f_{T,\psi} \in \mathcal{LUC}(G)$.

To prove (iii), we check that $f_{T,\psi}$ is left uniformly continuous as well. The case of $f = 0$ is trivial, so suppose that $f \neq 0$. Given $\epsilon > 0$, choose a neighbourhood V_0 of the identity in G with $V_0 \subseteq V$ and

$$|\psi(u) - \psi(v)| < \frac{\epsilon}{\|f\|} \quad \text{whenever} \quad u^{-1}v \in V_0.$$

Let W be a neighbourhood of the identity such that $tWt^{-1} \subseteq V_0$ for every $t \in T$.

If $g \in VT$, then $g = ut$ for some $u \in V$ and $t \in T$, and so $g^{-1}h \in W$ implies that

$$h \in gW = utW \subseteq uV_0t \subseteq V^2t,$$

i.e., $h = vt$ for some $v \in U$. Now from $g^{-1}h = t^{-1}u^{-1}vt \in W$, we conclude that $u^{-1}v \in tWt^{-1} \subseteq V_0$. Thus

$$|f_{T,\psi}(g) - f_{T,\psi}(h)| = |f(t)\psi(u) - f(t)\psi(v)| \leq \|f\|\,|\psi(u) - \psi(v)| < \epsilon.$$

The argument is similar when $h \in VT$, and it is trivial when neither g nor h is in VT. Since $f_{T,\psi}$ is both left and right uniformly continuous and is supported in VT, the last part of Statement (iii) follows now from Lemma 4.3. $\square$

Remark 4.7. *Functions $\psi \in \mathcal{UC}(G)$ with $\psi(e) = 1$ and $\psi(G \setminus V) = \{0\}$ for a given neighbourhood V of the identity are always available since, by [37, Definition-proposition 2.5] the infimum of the left and right uniformities on G (the so-called lower uniformity or Roelcke uniformity) induces the original topology on G.*

Lemma 4.8. *Let G be a topological group with identity e and let $T \subseteq G$.*

(i) *If T is right uniformly discrete (respectively, left uniformly discrete), then T is an approximable $\mathcal{LUC}(G)$-interpolation set (respectively, $\mathcal{RUC}(G)$-interpolation set).*

(ii) *If G is an E-group, and T is an E-set that is right (respectively, left) U-uniformly discrete with respect some neighbourhood U of the identity and VT (respectively, TV) is translation-compact for some neighbourhood V of the identity with $V^2 \subseteq U$, then T is an approximable $\mathcal{WAP}(G)$-interpolation set.*

Proof. Suppose T is right uniformly discrete and let U be a neighbourhood of the identity with $Us \cap Us' = \varnothing$ for every $s \neq s' \in T$. By Proposition 3.3, we only need to check that T is an $\mathcal{LUC}(G)$-interpolation set. Let V be another neighbourhood of the identity with $V^2 \subseteq U$ and let $\psi \in \mathcal{LUC}(G)$ with support contained in V and $\psi(e) = 1$. If $f\colon T \to \mathbb{C}$ is any bounded function, then the function $f_{T,\psi}$ defined above in $(*)$ is an extension of f and it is in $\mathcal{LUC}(G)$ by the previous lemma.

Suppose now that G is an E-group. Choose a function $\psi \in \mathcal{UC}(G)$ with support contained in V and $\psi(e) = 1$ (see Remark 4.7). If Condition (ii) is satisfied, the above scheme will again show that T is a $\mathcal{WAP}(G)$-interpolation set using (iii) of Lemma 4.6. To see that T is an approximable $\mathcal{WAP}(G)$-interpolation, take two neighbourhoods of the identity V_1 and V_2 with $V_1^3 \subseteq V_2 \subseteq U$. Then $V_1\overline{V_1} \subseteq V_1V_1^2 \subseteq V_2$. Therefore, $\overline{V}_1$ and $G \setminus V_2$ are closed sets in G with $V_1\overline{V_1} \cap (G \setminus V_2) = \varnothing$. So we may pick $\psi \in \mathcal{LUC}(G)$ with $\psi(V_1) = 1$ and $\psi(G \setminus V_2) = 0$. If now T_1 is any subset of T, then the function $1_{T_1,\psi}$ as defined in $(*)$ with the constant function 1 on T_1 is in $\mathcal{WAP}(G)$ again by Lemma 4.6. Since the function clearly satisfies $1_{T_1,\psi}(vt) = 1$ for every $v \in V_1$ and $t \in T_1$ and $1_{T_1,\psi}(G \setminus V_2T_1) = \{0\}$, the proof is complete. $\square$

It turns out that the converses to the statements in Lemma 4.8 are valid when G is metrizable. We begin with the case of $\mathcal{LUC}(G)$.

Theorem 4.9. *Let G be a metrizable topological group and let T be a subset of G. The following assertions are equivalent.*

(i) *T is an approximable $\mathcal{LUC}(G)$- (respectively, $\mathcal{RUC}(G)$-)interpolation set.*

(ii) *T is an $\mathcal{LUC}(G)$- (respectively, $\mathcal{RUC}(G)$-)interpolation set.*

(iii) *T is right (respectively, left) uniformly discrete.*

Proof. Lemma 4.8 proves that (iii) implies (i) and that (i) implies (ii) is a matter of definition. We only have to see that (ii) implies (iii).

Suppose that T is an $\mathcal{LUC}(G)$-interpolation set that is not right uniformly discrete. Consider a neighbourhood basis at the identity $(U_n)_n$ consisting of symmetric neighbourhoods such that $U_{n+1}^2 \subseteq U_n$. Recall that T is necessarily discrete.

Since T is not right uniformly discrete, we can find $s_1, t_1 \in T$, $s_1 \neq t_1$, such that $s_1 t_1^{-1} \in U_1^2$.

Suppose now that we have chosen $2n$ different points $\{t_1, \ldots, t_n, s_1, \ldots, s_n\}$ and $1 < k_1 < k_2 < \ldots k_n$ in such a way that $s_n t_n^{-1} \in U_{k_n}^2$, but $s_n \neq t_n$.

Using the fact that T is discrete and that $t_i \neq s_j$ for all $1 \leq i, j \leq n$, we then choose $k_{n+1} > k_n$ such that:

(i) $(T \setminus \{t_i\}) \cap (U_{k_{n+1}} t_i) = (T \setminus \{s_i\}) \cap (U_{k_{n+1}} s_i) = \varnothing$ for all $i = 1, \ldots, n$

(ii) $t_i s_j^{-1} \notin U_{k_{n+1}}^2$ for all $1 \leq i, j \leq n$.

(iii) $t_i t_j^{-1} \notin U_{k_{n+1}}^2$, $s_i s_j^{-1} \notin U_{k_{n+1}}^2$ for all $1 \leq i, j \leq n, i \neq j$.

Then

$$(T \setminus \{t_1, \ldots, t_n, s_1, \ldots, s_n\}) (T \setminus \{t_1, \ldots, t_n, s_1, \ldots, s_n\})^{-1} \cap U_{k_{n+1}}^2 \neq \{e\},$$

for, otherwise, we would have $TT^{-1} \cap U_{k_{n+1}}^2 = \{e\}$ and this would imply that T is right uniformly discrete.

We now choose $t_{n+1}, s_{n+1} \in T \setminus \{t_1, \ldots, t_n, s_1, \ldots, s_n\})$ with $t_{n+1} \neq s_{n+1}$ such that $t_{n+1} s_{n+1}^{-1} \in U_{k_{n+1}}^2$.

We have constructed in this way two faithfully indexed sequences

$$T_1 = \{t_n : n \in \mathbb{N}\} \subseteq T \quad \text{and} \quad T_2 = \{s_n : n \in \mathbb{N}\} \subseteq T$$

in such a way that $T_1 \cap T_2 = \varnothing$ but $t_n s_n^{-1}$ converges to the identity.

Since T is an $\mathcal{LUC}(G)$-interpolation set, there is $f \in \mathcal{LUC}(G)$ so that $f(t_n) = 1$ and $f(s_n) = -1$. By uniform continuity, there is a neighbourhood U_0 of the identity such that $|f(x) - f(y)| < 1$ if $xy^{-1} \in U_0$. Taking then n such that $U_{k_n}^2 \subseteq U_0$, we reach a contradiction as $s_n t_n^{-1} \in U_0$, but $f(s_n) - f(t_n) = 2$. $\qquad \square$

Remark 4.10. *In the non-metrizable situation, non-uniformly discrete sets may be $\mathcal{LUC}(G)$-interpolation sets. In fact Example 6.2 gives $\mathcal{AP}(G)$-interpolation sets which are not uniformly discrete. Theorem 6.3 at the end of the paper suggests that metrizability is essential for Theorem 4.9 to be true.*

We now deal with the $\mathcal{WAP}(G)$-case. To avoid making the proof of the main theorem too cumbersome, we begin by establishing the following Lemma as well as Theorem 4.15.

Lemma 4.11. *Let G be a locally compact group with a symmetric relatively compact neighbourhood U of the identity and a right U-uniformly discrete set T. If UT is not translation-compact, then T contains a subset T_1 such that no $f \in \ell_\infty(G)$ with $f(T_1) = \{1\}$ and $f(G \setminus UT_1) = \{0\}$ is weakly almost periodic.*

Proof. We adapt the argument used by Ruppert in [39] in the discrete case. Suppose that UT is not right translation-compact. Then there exists a non-relatively compact subset L of G which contains no finite subset F for which $\bigcap_{b \in F} b^{-1} UT$ is relatively compact. Define inductively two sequences (s_n) and (t_m) in G as follows. Start with $s_1 \in L$ and let $t_1 \in s_1^{-1} T$. Suppose that $s_1, s_2, ..., s_n$ have been selected in L and $t_1, t_2, ..., t_n$ have been selected in G. Then we may choose $s_n \in L$ such that

(1) $$s_n \notin U^4 \{s_k t_l t_m^{-1} : 1 \leq k \leq l < n, \quad 1 \leq m < n\}.$$

This is possible since the latter set is relatively compact while L is not. Take then

$$t_n \in \bigcap_{m \leq n} s_m^{-1} UT,$$

but

(2) $\qquad t_n \notin s_m^{-1} U^4 \{s_l t_k : 1 \le k < l \le n\} \quad$ for all $\quad m \le n.$

This is again possible because the second set is relatively compact while the first one is not. Note that, by choice, we have $s_m t_n \in UT$ for every $m \le n$. So for every $m \le n$, there is a unique $t_{mn} \in T$ such that $s_m t_n \in U t_{mn}$. Let

$$T_1 = \{t_{mn} \in T : 1 \le m \le n < \infty\}.$$

We claim first that

(3) $\qquad U^2 \{s_m t_n : m \le n\} \cap U^2 \{s_m t_n : m > n\} = \varnothing.$

So let us consider two elements $u s_\alpha t_\beta$ and $v s_{\alpha'} t_{\beta'}$ with $\alpha \le \beta$, $\alpha' > \beta'$ and $u,$ $v \in U^2$. If $\beta \le \beta'$, put $\alpha = k$, $\beta = l$, $\alpha' = n$ and $\beta' = m$, then

$$u s_\alpha t_\beta = v s_{\alpha'} t_{\beta'}$$

implies that

$$u s_k t_l = v s_n t_m \quad \text{with} \quad 1 \le k \le l \le m < n,$$

contradicting (1). If $\beta > \beta'$, put $\alpha = m$, $\beta = n$, $\alpha' = l$ and $\beta' = k$, then

$$u s_\alpha t_\beta = v s_{\alpha'} t_{\beta'}$$

implies that

$$u s_m t_n = v s_l t_k \quad \text{with} \quad 1 \le k < l \quad \text{and} \quad m \le n.$$

If $\alpha' \le \beta$, then $l \le n$ and so this clearly contradicts (2). If $\alpha' > \beta$, then we obtain

$$v s_l t_k = u s_m t_n \quad \text{with} \quad 1 \le m \le n < l \quad \text{and} \quad 1 \le k < l,$$

contradicting (1).

   Accordingly,

(4) $\qquad U \{t_{mn} : m \le n\} \cap \{s_m t_n : m > n\} = \varnothing,$

for if $u t_{pq} = s_m t_n$ for $u \in U$, $p \le q$ and $m > n$, then $u v s_p t_q = s_m t_n$ for $u \in U$, $p \le q$, $m > n$ and some $v \in U$, which is not possible by (3).

   Let now f be any bounded function on G with $f(T_1) = \{1\}$ and $f(G \backslash UT_1) = \{0\}$. For each $m \le n$, let $u_{mn} \in U$ be such that $u_{mn} t_{mn} = s_m t_n$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and consider the limits along $\mathcal{U}$ of the sequences $\{u_{mn} : n \ge m\}$ and $\{t_{mn} : n \ge m\}$,

$$\lim_{n,\mathcal{U}} u_{mn} = u_m \in \overline{U} \quad \text{and} \quad \lim_{n,\mathcal{U}} t_{mn} = x_m \in \overline{T}^{\mathcal{LUC}}.$$

Choose another free ultrafilter $\mathcal{V}$ on $\mathbb{N}$ and the corresponding limits

$$\lim_{m,\mathcal{V}} u_m = u \quad \text{and} \quad \lim_{m,\mathcal{V}} x_m = x.$$

Then using the joint continuity property in $G^{\mathcal{LUC}}$, we see that

$$\lim_{m,\mathcal{V}} \lim_{n,\mathcal{U}} f_{u^{-1}}(s_m t_n) = \lim_{m,\mathcal{V}} \lim_{n,\mathcal{U}} f(u^{-1} u_{mn} t_{mn}) = \lim_{m,\mathcal{V}} f^{\mathcal{LUC}}(u^{-1} u_m x_m)$$

$$= f^{\mathcal{LUC}}(u^{-1} u x) = f^{\mathcal{LUC}}(x) = \lim_{m,\mathcal{V}} \lim_{n,\mathcal{U}} f(t_{mn}) = 1$$

since, once m is fixed, all but at most finitely many n's satisfy $n > m$; while

$$\lim_{n,\mathcal{U}} \lim_{m,\mathcal{V}} f_{u^{-1}}(s_m t_n) = \lim_{n,\mathcal{U}} \lim_{m,\mathcal{V}} f(u^{-1} s_m t_n) = 0$$

since, by (4), $u^{-1} s_m t_n \notin U T_1$ whenever $m > n$. Therefore, $f_{u^{-1}}$ is not weakly almost periodic and neither is f. $\square$

Corollary 4.12. *Let G be a locally compact group. Let T be a right U-uniformly discrete subset of G for some neighbourhood U of e. If T is an approximable $\mathcal{WAP}(G)$-interpolation set, then there is a relatively compact neighbourhood V of e, $V \subseteq U$ such that VT is translation-compact.*

Proof. Suppose that VT is not right translation-compact for any relatively compact neighbourhood V of e with $V \subseteq U$. To check that T is not an approximable $\mathcal{WAP}(G)$-interpolation set, let V_1, V_2 be open, relatively compact neighbourhoods of e with $\overline{V_1} \subseteq V_2 \subseteq U$. Since V_2T is not right translation-compact, we may apply Lemma 4.11 to find $T_1 \subseteq T$ such that no bounded function $f \colon G \to \mathbb{C}$ with $f(T_1) = \{1\}$ and $f(G \setminus V_2T_1) = \{0\}$ is weakly almost periodic. Therefore, T is not an approximable $\mathcal{WAP}(G)$-interpolation set.

The argument is symmetric if we suppose that VT is not left translation-compact for any neighbourhood V of e. $\qquad\square$

Corollary 4.13. *Let G be a metrizable locally compact group. If T is an approximable $\mathcal{WAP}(G)$-interpolation set, then T is right U-uniformly discrete and UT is translation-compact for some neighbourhood U of the identity.*

Proof. This is an immediate consequence of Theorem 4.9 and Corollary 4.12. $\quad\square$

The following result may be known at least in the discrete case, but we have found no reference to it in the literature. See [3, Corollary 3.5] for another important combinatorial property of Sidon sets, namely that they are finite unions of sets tending to infinity.

Corollary 4.14. *Let G be a metrizable locally compact Abelian group. If T is a $B(G)$-interpolation set (i.e., a topological Sidon set), then T is right U-uniformly discrete and UT is translation-compact for some neighbourhood U of the identity. In particular, Sidon subsets of discrete groups are translation-finite.*

Proof. $B(G)$-interpolation sets are, by Proposition 3.7, approximable $\mathcal{B}(G)$-interpolation sets, and so they are approximable $\mathcal{WAP}(G)$-interpolation sets. Corollary 4.13 finishes the proof. $\qquad\square$

Theorem 4.15. *Let G be a locally compact E-group with identity e, let T be an E-set which is right uniformly discrete with respect to some neighbourhood U of e and let V be a neighbourhood of e with $V^2 \subseteq U$. Then the following statements are equivalent.*

  (i) *Every function in $\mathcal{UC}(G)$ which is supported in VT is in $\mathcal{WAP}(G)$.*
 (ii) *For every neighbourhood W of e with $\overline{W} \subseteq V$, every function in $\mathcal{UC}(G)$ which is supported in WT is in $\mathcal{WAP}(G)$.*
(iii) *VT is right translation-compact.*
 (iv) *For every neighbourhood W of e with $\overline{W} \subseteq V$, WT is right translation-compact.*

Proof. The implication (i) $\Longrightarrow$ (ii) is clear.

For the converse, let $f \in \mathcal{UC}(G)$ be supported in VT. Since $f \in \mathcal{UC}(G)$, we can find for each $n \in \mathbb{N}$, open sets W_n and V_n such that $\overline{W_n} \subseteq V_n \subseteq \overline{V_n} \subseteq V$, and $|f(x)| \leq \frac{1}{n}$ whenever $x \in VT \setminus W_nT$. To obtain such neighbourhoods, we take for each $n \in \mathbb{N}$, a neighbourhood U_n of the identity such that $|f(x) - f(y)| < \frac{1}{n}$ whenever $xy^{-1} \in U_n$. Then let $W_n = V \setminus \overline{U_n(V^2 \setminus V)}$. It is clear that $u \in U_n$ implies $|f(uyt) - f(yt)| < \frac{1}{n}$ for every $t \in T$ and $y \in G$. But if $y \in V^2 \setminus V$ then $f(yt) = 0$ because T is V^2-uniformly discrete and f is supported in VT. It follows that $|f(ut)| < \frac{1}{n}$ whenever $u \in U_n(V^2 \setminus V)$ and $t \in T$. Now note that if $v \in V$ and $t \in T$ are such that $vt \in VT \setminus W_nT$, then $v \in \overline{U_n(V^2 \setminus V)}$, and therefore

$|f(vt)| \leq \frac{1}{n}$. Note in addition that $\overline{W_n} \subseteq U$. For if $x \in \overline{W_n}$ then $x \notin U_n(V^2 \setminus V)$, and in particular $x \notin V^2 \setminus V$. Since $x \in \overline{V} \subseteq V^2$, we conclude that $x \in V$. Now it is easy to find a neighbourhood V_n of the identity such that $\overline{W_n} \subseteq V_n \subseteq \overline{V_n} \subseteq V$.

Having obtained these two families $\{W_n\}_{n \in \mathbb{N}}$ and $\{V_n\}_{n \in \mathbb{N}}$ of neighbourhoods of e, we let for each $n \in \mathbb{N}$, $\varphi_n \in \mathcal{UC}(G)$ be such that $0 \leq \varphi_n(x) \leq 1$ for all $x \in G$, $\varphi_n = 1$ on W_n and $\varphi_n = 0$ on $G \setminus V_n$. Then, since T is an E-set, we see that for every $n \in \mathbb{N}$, the function $1_{T,\varphi_n}$ is in $\mathcal{UC}(G)$ and so is $\Phi_n = 1_{T,\varphi_n} f$. Since the latter function is supported on $V_n T$ and $\overline{V_n} \subseteq V$, we have by hypothesis that $\Phi_n \in \mathcal{WAP}(G)$. It only remains to observe that $\|f - \Phi_n\|_\infty \leq \frac{1}{n}$. To see this, note that Φ_n and f only differ on $(V \setminus W_n)T$. But, if $vt \in VT \setminus W_n T$, then $|f(vt)| \leq \frac{1}{n}$ and so

$$|f(vt) - \Phi_n(vt)| = |f(vt)||1 - \varphi_n(v)| \leq \frac{1}{n}.$$

Thus, $f \in \mathcal{WAP}(G)$ as required for Statement (i).

To prove that (i) implies (iii), assume (i) and suppose for a contradiction that VT is not translation-compact. Let then T_1 be the set provided by Lemma 4.11 for V. Then we take any right uniformly continuous function φ with support contained in V and value 1 on e and consider the function $f = 1_{T_1,\varphi}$. Then f is in $\mathcal{UC}(G)$ by (iii) of Lemma 4.6, has its support contained in VT_1 and takes the value 1 on T_1. Lemma 4.11 proves that f cannot be weakly almost periodic. This contradiction with (i) shows that VT must be translation-compact.

In the same way, we prove that (ii) implies (iv).
Lemma 4.3 proves that (iii) implies (i) and (iv) implies (ii). $\square$

Theorem 4.16. *Let G be a locally compact E-group with identity e, let T be an E-set which is right uniformly discrete with respect to some neighbourhood U of e and let V be an open, relatively compact neighbourhood of e with $V^2 \subseteq U$. Then the following statements are equivalent.*

(i) *Every function in $\mathcal{UC}(G)$ which is supported in VT is in $\mathcal{WAP}(G)$.*

(ii) *VT is right translation-compact.*

(iii) *For every $\varphi \in \mathcal{UC}(G)$ with support contained in V, the function $1_{T,\varphi}$ is in $\mathcal{WAP}(G)$ and its extension $1_{T,\varphi}^{\mathcal{WAP}}$ to $G^{\mathcal{WAP}}$ is zero on $G^* G^*$.*

(iv) *For every neighbourhood W of e with $W \subseteq V$, the set $W\overline{T}^{\mathcal{WAP}}$ is open in $G^{\mathcal{WAP}}$, and $\overline{T}^{\mathcal{WAP}} \subseteq G^{\mathcal{WAP}} \setminus G^* G^*$ (here $\overline{T}^{\mathcal{WAP}}$ is the closure of T in $G^{\mathcal{WAP}}$).*

(v) *T is a $\mathcal{WAP}(G)$-interpolation set, for every $\varphi \in \mathcal{LUC}(G)$ with support contained in $\overline{V}$ the function $1_{T,\varphi}$ is in $\mathcal{WAP}(G)$, and $\overline{VT}^{\mathcal{WAP}} = \overline{V} \times \beta T$, where βT is the Stone-Čech compactification of T.*

Moreover, T is an approximable $\mathcal{WAP}(G)$-interpolation set if and only if every neighbourhood of e contains a second relatively compact neighbourhood of e for which one (and so all) of the above statements holds.

Proof. Theorem 4.15 proves that *Statements (i) and (ii) are equivalent.*

(ii) $\Longrightarrow$ (iii). Since the function $1_{T,\varphi}$ is in $\mathcal{UC}(G)$ and is supported in VT, (iii) follows from (i) and (ii) along with Lemma 4.3.

(iii) $\Longrightarrow$ (iv) We first check that $V\overline{T}^{\mathcal{WAP}}$ is open in $G^{\mathcal{WAP}}$ (this is inspired by a proof done by Pym on $G^{\mathcal{LUC}}$ in [34]). Let vx be any point in $V\overline{T}^{\mathcal{WAP}}$, let V_0 and V_1 be neighbourhoods of e such that $V_0 v \subseteq V$ and $\overline{V_1} \subseteq V_0$. Let $\varphi : G \to [0,1]$ be a continuous function with values 1 on $\overline{V_1}$ and 0 outside of V_0, and let f denote the

right translate by v^{-1} of the function $1_{T,\varphi}$; that is,

$$f(s) = \sum_{t \in T} \varphi(st^{-1}v^{-1}).$$

Since, by assumption, $1_{T,\varphi} \in \mathcal{WAP}(G)$, $f \in \mathcal{WAP}(G)$ as well. Let then $f^{\mathcal{WAP}}$ be the continuous extension of f to $G^{\mathcal{WAP}}$, and put

$$V_2 = \{u \in V : \varphi(uv^{-1}) > 1/2\}.$$

Then $V_2\overline{T}^{\mathcal{WAP}}$ is open in $G^{\mathcal{WAP}}$ since $V_2\overline{T}^{\mathcal{WAP}} = (f^{\mathcal{WAP}})^{-1}(]1/2, 1])$. Therefore, $V_2\overline{T}^{\mathcal{WAP}}$ is a neighbourhood of vx in $G^{\mathcal{WAP}}$ which is contained in $V\overline{T}^{\mathcal{WAP}}$. This shows that $V\overline{T}^{\mathcal{WAP}}$ is open in $G^{\mathcal{WAP}}$. The same argument applies to show that $W\overline{T}^{\mathcal{WAP}}$ is open in $G^{\mathcal{WAP}}$ whenever W is a neighbourhood of e contained in V.

For the second part of Statement (iv), let V_1 and φ be again as before. Clearly, if $ut \in V_1T$, then $1_{T,\varphi}(ut) = \varphi(u) = 1$ and so $\overline{V_1T}^{\mathcal{WAP}} \subseteq G^{\mathcal{WAP}} \setminus G^*G^*$ since, by assumption, the function $1_{T,\varphi}^{\mathcal{WAP}}$ is zero on G^*G^*. It follows in particular that $\overline{T}^{\mathcal{WAP}} \subseteq G^{\mathcal{WAP}} \setminus G^*G^*$.

(iv) $\Longrightarrow$ (i). In fact, only the part $\overline{T}^{\mathcal{WAP}} \cap G^*G^* = \varnothing$ of Statement (iv) is needed to deduce Statement (i). Let $f \in \mathcal{UC}(G)$ be supported in VT and let (s_n) and (t_m) be sequences in G such that the limits

$$\lim_n \lim_m f(s_n t_m) \quad \text{and} \quad \lim_m \lim_n f(s_n t_m)$$

exist. Suppose that $\lim_n \lim_m f(s_n t_m) \neq 0$. Then there is n_0 such that for every $n \geq n_0$ there is $m(n)$ such that $f(s_n t_m) \neq 0$ for every $m \geq m(n)$, and so $s_n t_m \in VT$ for every $n \geq n_0$ and $m \geq m(n)$. It follows that if x and y are cluster points, in $G^{\mathcal{WAP}}$, of (s_n) and (t_m), respectively, then $xy \in \overline{VT}^{\mathcal{WAP}} = \overline{V}\,\overline{T}^{\mathcal{WAP}}$. Accordingly, either x or y must be in G since by assumption $\overline{T}^{\mathcal{WAP}} \cap G^*G^* = \varnothing$. However, if $x \in G$ (say), then

$$\lim_m \lim_n f(s_n t_m) = \lim_m f(xt_m) = \lim_n \lim_m f(s_n t_m).$$

If $\lim_m \lim_n f(s_n t_m) \neq 0$, we argue in the same way. The case

$$\lim_n \lim_m f(s_n t_m) = \lim_m \lim_n f(s_n t_m) = 0$$

is trivial. Therefore, $f \in \mathcal{WAP}(G)$, as required for Statement (i).

We have thus far proved that Statements (i) through (iv) are equivalent.

(i) $\Longleftrightarrow$ (v). Suppose that every function in $\mathcal{UC}(G)$ and supported in VT is weakly almost periodic.

Let $f : T \to \mathbb{C}$ be any bounded function on T and extend f to G by the function $f_{T,\varphi}$, where $\varphi \in \mathcal{LUC}(G)$, $\varphi(e) = 1$ and the support of φ is contained in V. Since $f_{T,\varphi} \in \mathcal{UC}(G)$ as seen in Lemma 4.6 (iii) and is supported in VT, $f_{T,\varphi} \in \mathcal{WAP}(G)$ by assumption. This shows that any bounded f on T extends to a weakly almost periodic function on G, and so T is a $\mathcal{WAP}(G)$-interpolation set. This also shows in particular that $1_{T,\varphi} \in \mathcal{WAP}(G)$ as required for the first part of the statement.

Since T is a $\mathcal{WAP}(G)$-interpolation set, $\overline{T}^{\mathcal{WAP}}$ is homeomorphic to βT, with the points of T fixed by the homeomorphism. So we may identify $\overline{T}^{\mathcal{WAP}}$ and βT. Then, using the joint continuity on $G \times G^{\mathcal{WAP}}$, we may consider the continuous surjection $\overline{V} \times \beta T \to \overline{VT}^{\mathcal{WAP}}$ which extends the multiplication mapping $V \times T \to VT$. We prove that this extension is injective. Let $f_{T,\varphi}^{\mathcal{WAP}}$ and $f^{\beta T}$ be the extensions of $f_{T,\varphi}$

and f to $G^{\mathcal{WAP}}$ and βT, respectively, and let $(u,x) \in U \times \overline{T}^{\mathcal{WAP}}$. Then, since T is right uniformly discrete with respect to U, we have

$$(5) \qquad f_{T,\varphi}^{\mathcal{WAP}}(ux) = \lim_\alpha \sum_{t \in T} f(t)\varphi(ut_\alpha t^{-1}) = \lim_\alpha f(t_\alpha)\varphi(u) = f^\beta(x)\varphi(u) \qquad (**)$$

(here (t_α) is a net in T converging to x in βT when x is regarded as a point in β, and converging to x in $G^{\mathcal{WAP}}$ when x is regarded as a point in $\overline{T}^{\mathcal{WAP}}$). Accordingly, if $(u,x) \neq (v,y)$ in $\overline{V} \times \beta T$, then the functions φ and f may be chosen in (5) so that $f_{T,\varphi}^{\mathcal{WAP}}$ separates ux and vy in $G^{\mathcal{WAP}}$. In fact if $x = y$ then u and v must be distinct and so we may choose $\varphi \in \mathcal{UC}(G)$ with support contained in V, $\varphi(u) = 1$ and $\varphi(v) = 0$. If $x \neq y$ we may choose $f \in \ell_\infty(T)$ such that $f^\beta(x) = 1$ and $f^\beta(y) = 0$ and $\varphi \in \mathcal{UC}(G)$ with value 1 on $\overline{V}$. We obtain in all the cases,

$$f_{T,\varphi}^{\mathcal{WAP}}(ux) = f^\beta(x)\varphi(u) = 1 \quad \text{while} \quad f_{T,\varphi}^{\mathcal{WAP}}(vy) = f^\beta(y)\varphi(v) = 0.$$

Hence, $ux \neq vy$ in $G^{\mathcal{WAP}}$. Therefore, $\overline{V} \times \beta T \to \overline{VT}^{\mathcal{WAP}}$ is injective, and so it is a homeomorphism.

For the converse, we apply Theorem 4.15. Let W be any neighbourhood of e with $\overline{W} \subseteq V$, and let f be any function in $\mathcal{UC}(G)$ supported in WT. Let $f^{\mathcal{LUC}}$ be the continuous extension of f to $G^{\mathcal{LUC}}$, and consider its restriction $f\restriction_{\overline{VT}^{\mathcal{LUC}}}$. Since $\overline{V} \times \beta T$ and $\overline{VT}^{\mathcal{LUC}}$ are homeomorphic by [34], and $\overline{V} \times \beta T$ and $\overline{VT}^{\mathcal{WAP}}$ are homeomorphic by assumption, we may regard $f\restriction_{\overline{VT}^{\mathcal{LUC}}}$ as a continuous function on $\overline{VT}^{\mathcal{WAP}}$, and extend it to a continuous function $\widetilde{f}$ on $G^{\mathcal{WAP}}$. Note now that $\overline{V}^{\mathcal{WAP}} = \overline{V}^{\mathcal{LUC}} = \overline{V}$, so

$$\widetilde{f}(vt) = f\restriction_{\overline{VT}^{\mathcal{LUC}}}(vt) = f^{\mathcal{LUC}}(vt) = f(vt) \quad \text{for every} \quad v \in \overline{V}, t \in T.$$

Taking now $\varphi \in \mathcal{LUC}(G)$ with support contained in V and $\varphi(W) = 1$, we see that $\widetilde{f}1_{T,\varphi} = f$. Indeed, both functions obviously coincide on every $s \notin VT$. If $s \in WT$, we have $1_{T,\varphi}(s) = 1$ and so $\widetilde{f}(s)1_{T,\varphi}(s) = \widetilde{f}(s) = f(s)$. Finally, if $s \in VT$ but $s \notin WT$, then $\widetilde{f}(s)1_{T,\varphi}(s) = f(s)1_{T,\varphi}(s) = 0 = f(s)$.

Since $1_{T,\varphi} \in \mathcal{WAP}(G)$ by (v), we have $f \in \mathcal{WAP}(G)$. We conclude with the help of Theorem 4.15(ii) that every function in $\mathcal{UC}(G)$ which is supported in VT is in $\mathcal{WAP}(G)$. Hence, Statement (i) holds.

Therefore, Statements (i)–(v) are equivalent.

We now prove the last statement. Necessity has been already proved in Corollary 4.12.

The converse is an easy check. Fix, to begin with, a bounded function $f: T \to \mathbb{C}$ and a neighbourhood U_0 of the identity. There is then a relatively compact neighbourhood $V_0 \subseteq U_0 \cap V$ such that any $\mathcal{UC}(G)$-function supported in $V_0 T$ is weakly almost periodic. Choose another neighbourhood V_1 of the identity with $\overline{V_1} \subseteq V_0$ and let $T_1 \subseteq T$. Consider as well a function $\psi \in \mathcal{LUC}(G)$ with $\psi(V_1) = \{1\}$ and $\psi(G \setminus V_0) = \{0\}$. By Lemma 4.6 the functions $f_{T,\psi}$ and $1_{T,\psi}$ are both in $\mathcal{UC}(G)$. Since both of them are supported in $V_0 T$, we have, by assumption, that both $f_{T,\psi}$ and $1_{T_1,\psi}$ are weakly almost periodic. Since $f_{T,\psi}$ coincides with f on T and f was arbitrary, T is a $\mathcal{WAP}(G)$-interpolation set. Since $1_{T_1,\psi}(V_1 T_1) = \{1\}$ and $1_{T_1,\psi}(G \setminus V_0 T_1) = \{0\}$, we see that T is indeed approximable. $\qquad\square$

The whole Theorem 7 of [39] and also Proposition 2.4 of [7] now follow from Theorem 4.16 when G is discrete. We emphasize here that fact for future reference.

Corollary 4.17 (Theorem 7 of [39] and Proposition 2.4 of [7]). *Let G be discrete. A subset $T \subseteq G$ is an approximable $\mathcal{WAP}(G)$-interpolation set if and only if it is translation-finite*

Corollary 4.18. *Let G be a metrizable E-group and let $T \subseteq G$ be an E-set. Then T is an approximable $\mathcal{WAP}(G)$-interpolation set if and only if T is right uniformly discrete with respect to some neighbourhood V of e such that VT is translation-compact.*

Proof. If T is an approximable $\mathcal{WAP}(G)$-interpolation set, then it is an approximable $\mathcal{LUC}(G)$-interpolation set. So by Theorem 4.9, T is right uniformly discrete with respect to some neighbourhood V of e. By Theorem 4.16, we may choose V such that VT is translation-compact.

The converse follows from Theorem 4.16. $\qquad\square$

In general when G is not an E-group, the theorem fails even if G is metrizable.

Remark 4.19. *Let $G = SL(2, \mathbb{R})$ and VT be a t-set in G as constructed in Example 4.2. Then T is right uniformly discrete, VT is translation-compact but T is not an approximable $\mathcal{WAP}(G)$-interpolation set since $SL(2, \mathbb{R})$ is minimally weakly almost periodic group, that is, $\mathcal{WAP}(G) = C_0(G) \oplus \mathbb{C}$.*

Before we prove our next main theorem in this section, we need the following lemmas, the proof of the second one relies on a Ramsey theoretic theorem of Hindman. Recall from [30, Definition 5.13] that if (x_n) is a a sequence in G then the sequence (y_n) is a *product subsystem of (x_n)* if and only if there is a sequence (H_n) of finite subsets of $\mathbb{N}$ such that for every $n \in \mathbb{N}$,

$$\max H_n < \min H_{n+1} \quad \text{and} \quad y_n = \Pi_{t \in H_n} x_t,$$

where $\Pi_{t \in H_n} x_t$ is used to denote the product in decreasing order of indices, contrarily to what is chosen in [30]. So, for instance, if $H_n = \{2, 6, 23\}$, then $\Pi_{t \in H_n} x_t = x_{23} x_6 x_2$.

We will also need to use the finite product set $FP((x_n)_n)$ associated to a sequence $(x_n)_n \subseteq G$. It is is defined as

$$FP((x_n)_n) = \left\{ \prod_{n \in F} x_n \colon F \in \mathbb{N}^{<\omega} \right\}.$$

Lemma 4.20. *If G is a non-compact locally compact group and U is a relatively compact symmetric neighbourhood of e, then G contains a sequence (x_n) such that every product subsystem of (x_n) is U-uniformly discrete.*

Proof. Start by fixing an arbitrary x_1 in G. Once $x_1, ..., x_n$ have been chosen, consider the finite set

$$F_p = \{\Pi_{t \in H} x_t : H \subseteq \{1, ..., p\}\}.$$

Then choose

$$x_{n+1} \notin F_n U^2 F_n^{-1} \cup F_n^{-1} U^2 F_n.$$

Suppose now that (y_n) is a product subsystem of (x_n). If (y_n) is not U-uniformly discrete, we can find $n < m$ such that $y_m U \cap y_n U \neq \varnothing$, that is, $y_m \in y_n U^2$. Now there exist $k(n)$ and $k(m)$ such that $k(n) < k(m)$, $y_n \in F_{k(n)}$ and $y_m \in F_{k(m)}$. We may then write $y_m = x_{k(m)} z_m$ for some $z_m \in F_{k(m)-1}$. Since $y_m \in y_n U^2$, it follows that $x_{k(m)} \in F_{k(n)} U^2 F_{k(m)-1}^{-1}$, which goes against our construction. $\qquad\square$

Lemma 4.21. *A non-compact locally compact group cannot be the union of finitely many right (respectively, left) translates of a left (respectively, right) translation-compact set.*

Proof. Suppose that $G = \bigcup_{j=1}^p Tt_j$ for some $T \subseteq G$ and $t_1, t_2, ..., t_p \in G$. Let $(x_n)_n$ be the sequence constructed in Lemma 4.20. Then

$$FP((x_n)_n) \subseteq \bigcup_{j=1}^p (FP((x_n)_n) \cap Tt_j).$$

By [30, Corollary 5.15], there exists a product subsystem $(y_n)_n$ of $(x_n)_n$ with $FP((y_n)_n)$ contained in Tt_j for some $j = 1, 2, ..., p$. Let now $L = \{y_n t_j^{-1} : n \in \mathbb{N}\}$, let $\{y_{n_1} t_j^{-1}, y_{n_2} t_j^{-1}, ..., y_{n_k} t_j^{-1}\}$ be any finite subset of L and let m be any integer greater than $\max\{n_1, n_2, ..., n_k\}$. Then, since $y_m = (y_m y_{n_i}) y_{n_i}^{-1}$ and $y_m y_{n_i} \in FP((y_n)_n)$ for each $i = 1, 2, ..., k$, we see that

$$y_m \in Tt_j y_{n_1}^{-1} \cap Tt_j y_{n_2}^{-1} \cap ... \cap Tt_j y_{n_k}^{-1}.$$

By Lemma 4.20, the set $\{y_m : m > \max\{n_1, n_2, ..., n_k\}\}$ is uniformly discrete, hence cannot be relatively compact, and so T is not left translation-compact. $\square$

Theorem 4.22. *Let G be a locally compact E-group with a neighbourhood U of the identity and a right U-uniformly discrete E-set T. Then Theorem 4.16 holds with $\mathcal{WAP}_0(G)$ replacing $\mathcal{WAP}(G)$. In particular, the following statements are equivalent.*

   (i) *T is an approximable $\mathcal{WAP}(G)$-interpolation set.*
   (ii) *T is an approximable $\mathcal{WAP}_0(G)$-interpolation set.*
   (iii) *There exists a neighbourhood V of e, $V^2 \subseteq U$ for which VT is translation-compact.*

Proof. We only need to prove the implication (iii) $\implies$ (ii). Let V be a neighbourhood of e such that $V^2 \subseteq U$ and VT is translation-compact, and let f be any function in $\mathcal{UC}(G)$ which is supported in VT. We show that $f \in \mathcal{WAP}_0(G)$, this will imply that T is an approximable $\mathcal{WAP}_0(G)$-interpolation set exactly as in Theorem 4.16. By Theorem 4.16, $f \in \mathcal{WAP}(G)$. Let $\mu \in \mathcal{WAP}(G)^*$ be the unique invariant mean on $\mathcal{WAP}(G)$. By Ryll-Nardzewski's theorem, $\mu(|f|)$ is the unique constant in the closed convex hull $co(_G|f|)$ of $_G|f|$, see [4, Theorem 1.25 and Corollary 1.26]. Here, $_sf$ is the right translate of f by s. Suppose that $\mu(|f|) > 0$, and let $\sum_{k=1}^n c_k (_{x_k}|f|) \in co(_G|f|)$ be such that

$$\| \sum_{k=1}^n c_k (_{x_k}|f|) - \mu(|f|) \| = \sup_{s \in G} | \sum_{k=1}^n c_k |f|(sx_k) - \mu(|f|) | < \frac{\mu(|f|)}{2}.$$

It follows that $G = \bigcup_{k=1}^n VT x_k^{-1}$; otherwise, if some element s of G is not in $\cup_{k=1}^n VT x_k^{-1}$, then $f(sx_k) = 0$ for every $k = 1, 2, ..., n$ since f is supported in VT, and so

$$\frac{\mu(|f|)}{2} > \| \sum_{k=1}^n c_k (_{x_k}|f|) - \mu(|f|) \| \geq | \sum_{k=1}^n c_k |f|(sx_k) - \mu(|f|) | = \mu(|f|),$$

which is absurd. Since G is not compact and VT is translation-compact, this is not possible by Lemma 4.21. This implies that $m(|f|)$ must be zero, and so $f \in \mathcal{WAP}_0(G)$, as required. $\square$

We go back to the remark given after Definition 3.1. Next corollary implies that if G is a locally compact group, then no right uniformly discrete subset of G can be an approximable $\mathcal{AP}(G)$-interpolation set even if Definition 3.1 (ii) is weakened to:

(ii)´ *T is an approximable $\mathcal{A}(G)$-interpolation set if it is an $\mathcal{A}(G)$-interpolation set and for every neighbourhood U of e, there are open neighbourhoods V_1, V_2 of*

e with $\overline{V_1} \subseteq V_2 \subseteq U$ such that, for each infinite $T_1 \subseteq T$ there is $h \in \mathcal{A}(G)$ with $h(V_1 T_1) = \{1\}$ and $h(G \setminus (V_2 T_1)) = \{0\}$.

Corollary 4.23. *Let G be a locally compact group, U a symmetric relatively compact neighbourhood of e and T an infinite right U-uniformly discrete subset of G. Then T contains an infinite subset T_1 such that no function f on G with $f(T_1) = \{1\}$ and $f(G \setminus UT_1) = \{0\}$ is in $\mathcal{AP}(G)$.*

Proof. Suppose, otherwise, that for every infinite subset T_1 of T there exists $f \in \mathcal{AP}(G)$ such that $f(T_1) = \{1\}$ and $f(G \setminus UT_1) = \{0\}$. Then this f is in $\mathcal{WAP}(G)$, and so by Theorem 4.11, UT is translation-compact. Accordingly, by Theorem 4.22, every $\mathcal{UC}(G)$-function supported on UT is in $\mathcal{WAP}_0(G)$. (Observe that the E-property is not needed to prove neither Theorem 4.11 nor the implication (iii) $\implies$ (ii) in Theorem 4.22.) Pick a nonzero $f \in \mathcal{AP}(G)$ supported on UT. Then $|f| \in \mathcal{AP}(G) \cap \mathcal{WAP}_0(G)$, but this implies $f = 0$ because μ coincides with the Haar measure on $G^{\mathcal{AP}}$ and $|f|$ would extend to a non-zero, continuous and positive function in $G^{\mathcal{AP}}$. $\square$

Corollary 4.24. *Let G be a metrizable locally compact group. Then no subset of G can be an approximable $\mathcal{AP}(G)$-interpolation set even if Definition 3.1 (ii) is weakened as above to (ii)´.*

Proof. We assume that G is not compact, as compact metrizable groups cannot contain infinite $\mathcal{AP}(G)$-interpolation sets.

Suppose T is an approximable $\mathcal{AP}(G)$-interpolation set. By Theorem 4.9, T is right U-uniformly discrete for some neighbourhood U of e. This contradicts Corollary 4.23. $\square$

5. On the union of approximable $\mathcal{A}(G)$-interpolation sets

As already noted in the introduction, in a discrete Abelian group any finite union of Sidon sets is a Sidon set [9]. A finite union of I_0-sets is however not always an I_0-set, for example the union of the two I_0-sets $\{6^n : n \in \mathbb{N}\}$ and $\{6^n + n : n \in \mathbb{N}\}$ is not an I_0-set (see for instance [31, p. 132]).

The property is not true for right t-sets either, simply take T as any right t-set, $s \neq e$ in G and consider $T \cup sT$. As we show next finite unions of right uniformly discrete sets are not in general uniformly discrete either. As a matter of fact, this will be the only obstacle towards union theorems for approximable interpolation sets, see Proposition 5.1 below.

Any finite union of right translation-finite sets stays right translation-finite, this was obtained by Ruppert as a consequence of translation-finite subsets being approximable $\mathcal{WAP}(G)$-interpolation sets for discrete G (see Theorem 4.16 and Corollary 4.17). Independently, this result was also proved directly from the definition using combinatorial arguments in [16, Lemma 5.1].

This is generalized in Corollary 5.4 below, where G is a locally compact E-group. We shall in fact deduce from Theorem 4.16 that finite unions of translation-compact sets of the form VT, where V a relatively compact neighbourhood of the identity and T is a V^2-right uniformly discrete, stay translation-compact under the condition that the union of the sets T is right (left) uniformly discrete.

Proposition 5.1. *Let G be a locally compact group. Finite unions of right (left) uniformly discrete sets are right (left) uniformly discrete if and only if G is discrete.*

Proof. Suppose that G is not discrete, and fix relatively compact neighbourhoods U and V of e with $V^2 \subseteq U$. Then by [14, Lemma 5.2], V contains a faithfully indexed sequence $S = \{s_n : n < \omega\}$ converging to the identity. Let $T_1 = \{t_n : n < \omega\}$ be a

U-right uniformly discrete set. Then T_1 and $T_2 = \{s_n t_n \colon n < \omega\}$ are both V-right uniformly discrete sets but $T_1 \cup T_2$ is not.   □

Proposition 5.2. *Let G be a topological group and let $\mathcal{A}(G)$ be a subalgebra of $\mathcal{LUC}(G)$. If T_1 and T_2 are approximable $\mathcal{A}(G)$-interpolation sets and $T_1 \cup T_2$ is right uniformly discrete, then $T_1 \cup T_2$ is an approximable $\mathcal{A}(G)$-interpolation set.*

Proof. Fix a neighbourhood W of the identity such that $T_1 \cup T_2$ is right W-uniformly discrete. Let U be an arbitrary neighbourhood of the identity.

Since T_1 and T_2 are approximable $\mathcal{A}(G)$-interpolation sets there are two pair of neighbourhoods of the identity V_{11}, V_{12}, and V_{21}, V_{22} with $\overline{V_{11}} \subseteq V_{12} \subseteq W \cap U$ and $\overline{V_{21}} \subseteq V_{22} \subseteq W \cap U$ with the properties stated in Definition 3.1 of approximable interpolation set . For each $S \subseteq T_1 \cup T_2$, we obtain from the definition two functions $h_1, h_2 \in \mathcal{A}(G)$ such that

$$h_1(V_{11}(S \cap T_1)) = \{1\}, \qquad h_2(V_{21}(S \cap (T_2 \setminus T_1))) = \{1\},$$
$$h_1\big(G \setminus V_{12}(S \cap T_1)\big) = \{0\} \quad \text{and} \quad h_2\big(G \setminus V_{22}(S \cap (T_2 \setminus T_1))\big) = \{0\}.$$

If $S \subseteq T_1$ or $S \cap T_1 = \varnothing$ (in which case $S \subseteq T_2$), we just consider one of the functions, h_2 or h_1 respectively,.

We first prove that $T_1 \cup T_2$ is an $\mathcal{A}(G)$-interpolation set. Let $f \colon T_1 \cup T_2 \to \mathbb{C}$ be a bounded function and assume that the above functions h_1 and h_2 have been constructed for $S = T_1 \cup T_2$, so that $S \cap T_1 = T_1$ and $S \cap (T_2 \setminus T_1) = T_2 \setminus T_1$. We may consider two functions $f_1, f_2 \in \mathcal{A}(G)$ such that $f_1{\restriction}_{T_1} = f{\restriction}_{T_1}$ and $f_2{\restriction}_{T_2} = f{\restriction}_{T_2}$. Since $T_1 \cup T_2$ is right U-uniformly discrete,

$$T_1 \subseteq G \setminus V_{22}(T_2 \setminus T_1) \quad \text{and} \quad T_2 \setminus T_1 \subseteq G \setminus V_{12}T_1,$$

whence it follows that f coincides with $f_1 h_1 + f_2 h_2$ on $T_1 \cup T_2$. Since $f_1 h_1 + f_2 h_2 \in \mathcal{A}(G)$, we have shown that $T_1 \cup T_2$ is an $\mathcal{A}(G)$-interpolation set.

To see that $T_1 \cup T_2$ is an approximable $\mathcal{A}(G)$-interpolation set, let S be again an arbitrary subset of $T_1 \cup T_2$. Then we only have to observe that

$$(h_1 + h_2)((V_{11} \cap V_{21})S) = \{1\} \quad \text{and} \quad (h_1 + h_2)(G \setminus (V_{12} \cup V_{22})S) = \{0\}.$$

Since $\overline{V_{11} \cap V_{21}} \subseteq V_{12} \cup V_{22} \subseteq U$, the proof is done.   □

Corollary 5.3. *Let G be a metrizable topological group, let $\mathcal{A}(G)$ be a subalgebra of $\mathcal{LUC}(G)$ and let T_1 and T_2 be approximable $\mathcal{A}(G)$-interpolation sets. Then, $T_1 \cup T_2$ is an approximable $\mathcal{A}(G)$-interpolation set if and only if $T_1 \cup T_2$ is right uniformly discrete.*

Proof. Sufficiency is proved in Proposition 5.2 above. For the necessity one only has to note that $\mathcal{A}(G)$-interpolation sets are $\mathcal{LUC}(G)$-interpolations sets, and hence they must be right uniformly discrete by Theorem 4.9.   □

The following is a generalization of the result on finite unions of translation-finite sets proved for discrete groups in [39] and [16]. Note that by Example 6.1 this cannot be proved with Corollary 4.17 alone.

Corollary 5.4. *Let G be a locally compact E-group and T_1 and T_2 be subsets of G such that $T_1 \cup T_2$ is right uniformly discrete with respect to some neighbourhood U of e. If UT_1 and UT_2 are translation-compact, then $VT_1 \cup VT_2$ is translation-compact for some neighbourhood V of e.*

Proof. By Theorem 4.16, T_1 and T_2 are approximable $\mathcal{WAP}$-interpolation sets. By Proposition 5.2, $T_1 \cup T_2$ is an approximable $\mathcal{WAP}$-interpolation set. By Theorem 4.16, $VT_1 \cup VT_2$ is translation-compact for some neighbourhood V of e.   □

6. Examples and Remarks

If G is a topological group and G_d denotes the same group equipped with the discrete topology, then $\mathcal{WAP}(G) = \mathcal{WAP}(G_d) \cap \mathcal{CB}(G)$, $\mathcal{B}(G) = \mathcal{B}(G_d) \cap \mathcal{CB}(G)$ and $\mathcal{AP}(G) = \mathcal{AP}(G_d) \cap \mathcal{CB}(G)$. It could be conjectured that whenever T is both an (approximable) $\mathcal{LUC}(G)$-interpolation set and an approximable $\mathcal{WAP}(G_d)$-interpolation, then T must be an approximable $\mathcal{WAP}(G)$-interpolation set. As we show next, it can be the case that it is not possible to get the necessary functions that are *both* continuous and weakly almost periodic.

Example 6.1. *There exists a subset $T \subseteq \mathbb{R}$ that is both uniformly discrete and translation-finite but such that UT is not translation-compact for any neighbourhood U of the identity. This provides an example of a subset $T \subseteq \mathbb{R}$ that is an approximable $\mathcal{CB}(G)$-interpolation set, an approximable $\mathcal{WAP}(G_d)$-interpolation set but yet it is not a $\mathcal{WAP}(G) = \mathcal{WAP}(G_d) \cap \mathcal{CB}(G)$-interpolation set.*

Proof. Let $(\alpha_n)_n$ be a decreasing sequence of real numbers with $\alpha_0 = 1$ such that the set $\{\alpha_n : n \geq 0\}$ is linearly independent over the rationals and $\lim_{n\to\infty} \alpha_n = 0$. We define T as $T = \{n + \alpha_n : n \in \mathbb{N}\}$. It is obvious that T is a linearly independent and uniformly discrete subset of $\mathbb{R}$. As every linearly independent subset, T is a t-set (and so translation-finite). It is actually an $\mathcal{AP}(\mathbb{R}_d)$-interpolation set. Indeed, if T_1 and T_2 are disjoint subsets of T then $\mathrm{cl}_{\mathbb{R}_d^{\mathcal{AP}}} T_1 \cap \mathrm{cl}_{G^{\mathcal{AP}}} T_2 = \varnothing$ follows simply by choosing a character $\chi \colon G \to \mathbb{T}$ sending T_1 to 1 and T_2 to -1. Then, by for example [11, Corollary 3.6.2], we see that T is an $\mathcal{AP}(\mathbb{R}_d)$-interpolation set.

Let now U be a neighbourhood of 0 in $\mathbb{R}$. If $\alpha_n - \alpha_{n+k} \in U$, then

$$n + \alpha_n \in -k + T + U.$$

Now, if $F \subseteq \mathbb{N}$ is finite, there will be an infinite subset S of $\mathbb{N}$ such that $n \in S$ implies that $\alpha_n - \alpha_{n+k} \in U$ for all $k \in F$. Then

$$\{n + \alpha_n : n \in S\} \subseteq \bigcap \{-k + U + T : k \in F\}.$$

Since $\{n + \alpha_n : n \in S\}$ is an infinite uniformly discrete set, we have that

$$\bigcap \{-k + U + T : k \in F\}$$

is not relatively compact. Therefore, $U + T$ is not translation-compact. $\square$

Next, we show that an $\mathcal{LUC}(G)$-interpolation set needs not be uniformly discrete, if G is not metrizable.

Example 6.2. *The group $K = \mathbb{Z}^{\mathcal{AP}}$, the Bohr compactification of the discrete group of the integers, contains an $\mathcal{LUC}(K)$-interpolation that is not uniformly discrete.*

Proof. Let $T \subseteq \mathbb{Z}$ be an infinite $\mathcal{AP}(\mathbb{Z})$-interpolation set (i.e., an I_0-set, for instance $T = \{2^n : n \in \mathbb{N}\}$). Then for $\mathcal{A}(K) = \mathcal{AP}(K) = \mathcal{WAP}(K) = \mathcal{LUC}(K) = \mathcal{CB}(K)$, T is an $\mathcal{A}(K)$-interpolation subset of K but, K being compact, T is not a right (left) uniformly discrete subset of K. $\square$

Next we use a argument from [36] and show that under the Continuum Hypothesis (CH) the above situation is universal among nonmetrizable Abelian groups. This indicates that, in Theorem 4.9, metrizability is a hardly avoidable hypothesis.

Theorem 6.3. *Let G be locally compact abelian group, and let $\mathcal{A}(G) = \mathcal{AP}(G)$, $\mathcal{WAP}(G)$ or $\mathcal{LUC}(G)$. Under CH, every $\mathcal{A}(G)$-interpolation set is uniformly discrete if and only if G is metrizable.*

Proof. Sufficiency is proved in Theorem 4.9 without assuming CH. Conversely, suppose that G is not metrizable and let K be a compact subgroup of G such that G/K is metrizable (this is always available, see Theorem 8.7 of [29]). Then K is not metrizable and therefore its topological weight must be at least $\mathfrak{c}$, (it is here where we use CH), therefore, by [29, Theorem 24.15], $\widehat{K}$ is an Abelian group of cardinality at least $\mathfrak{c}$.

By taking the subgroup generated by a maximal independent subset of $\widehat{K}$ containing only elements of the same order [18, Section 16], we see that $\widehat{K}$ contains a subgroup D isomorphic to a direct sum $\bigoplus_{\mathfrak{c}} H$ where H is either $\mathbb{Z}$ or a finite group. In both cases D has a quotient that admits a compact group topology (if $H = \mathbb{Z}$ we use [18, Corollary 4.13], and if H is finite then D itself works for $D \cong H^{\omega}$). Denote the discrete dual of this compact group by L. We thus have an injective group homomorphism j and a surjective group homomorphism π as follows

$$\widehat{L}_d \xleftarrow{\ \pi\ } D \xrightarrow{\ j\ } \widehat{K} \ .$$

Dualizing the above diagram, we obtain

$$\widehat{\widehat{L}_d} \xrightarrow{\ \widehat{\pi}\ } \widehat{D} \xleftarrow{\ \widehat{j}\ } \widehat{\widehat{K}} \ ,$$

where $\widehat{j}$ is a quotient homomorphism and $\widehat{\pi}$ is a topological isomorphism of $\widehat{\widehat{L}_d}$ onto a subgroup of $\widehat{D}$. By Pontryagin duality (cf. [29, Theorem 26.12]), we can identify $\widehat{\widehat{L}_d}$ with $L^{\mathcal{AP}}$ and $\widehat{\widehat{K}}$ with K. We obtain in this way

$$L^{\mathcal{AP}} \xrightarrow{\ \widehat{\pi}\ } \widehat{D} \xleftarrow{\ \widehat{j}\ } K \ .$$

Now, L, as all discrete Abelian groups, admits some infinite $\mathcal{AP}(L)$-interpolation set A (see [27] or [20]). Arguing as in Example 6.2 we see that A is an $\mathcal{AP}(L^{\mathcal{AP}})$-interpolation set and, hence, $\widehat{\pi}(A)$ is an $\mathcal{AP}(\widehat{D})$-interpolation set. Now $\widehat{\pi}(A)$ can be lifted through the quotient homomorphism $\widehat{j}$ to obtain an $\mathcal{AP}(K)$-interpolation set I. As almost periodic functions on K extend to almost periodic functions on G, we see that I is an $\mathcal{AP}(G)$-interpolation set. Being infinite and contained in a compact group it cannot be right (left) uniformly discrete. □

We would like to finish pointing some of the questions that have not been settled in the present paper and in our opinion deserve further attention.

Sidon sets (i.e. $B(G)$-interpolation sets) have been the object of serious attention in the literature, but mostly in the discrete and Abelian cases. The non-discrete, non-Abelian case have received little attention. Since $\mathcal{B}(G)$-interpolation sets have not been under the focus in the present paper, some basic questions are yet to be clarified. One may ask for instance:

Question 1. *Is Proposition 3.5 valid for all locally compact metrizable groups? In particular, are $\mathcal{B}(G)$-interpolation sets necessarily $B(G)$-interpolation sets? Another interesting question is whether all $\mathcal{B}(G)$-interpolation sets are approximable $\mathcal{B}(G)$-interpolation sets.*

Let G be a metrizable locally compact group. While $\mathcal{AP}(G)$-interpolation sets are never approximable, $\mathcal{LUC}(G)$-interpolation sets are always approximable. Sitting in between the algebras $\mathcal{LUC}(G)$ and $\mathcal{AP}(G)$ we have the algebras $\mathcal{B}(G)$ and $\mathcal{WAP}(G)$. As we have just remarked, $\mathcal{B}(G)$-interpolation sets are often approximable (at least, when G is discrete and Abelian they always are) but it is not so clear whether $\mathcal{WAP}(G)$-interpolation sets should be on the approximable side or not. So one may ask.

Question 2. *Are $\mathcal{WAP}(G)$-interpolation sets approximable? In particular, let $T \subseteq \mathbb{Z}$ be a $\mathcal{WAP}(\mathbb{Z})$-interpolation set, must T be approximable?*

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