NON EQUIVALENCE OF NMS FLOWS ON S^3

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Abstract. We built the flows of Non singular Morse-Smale systems on the 3-sphere from its round handle decomposition. We show the existence of flows corresponding to the same link of periodic orbits that are nonequivalent. So, the link of periodic orbits is not in 1-1 correspondence with this type of flows and we search for other topological invariants such as the associated dual graph.

Keywords: Non singular Morse-Smale flows, round handle decomposition, links.

MSC 2000: 37D15

1. Introduction

Morse-Smale flows constitute a class of the simplest flows in the set of the structurally stable ones. This kind of flows are characterized by their non-wandering set consisting of a finite number of closed hyperbolic orbits and the transversal intersections of their stable and unstable manifolds.

From the results of Asimov [1] and Morgan [5] it is known that orientable, simple, compact, 3-dimensional manifolds admit a round handle decomposition with the closed orbits of the flow in the core of the round handles.

A topological characterization of the set of the periodic orbits of Non Singular Morse-Smale (NMS) flows on the 3-sphere has been made by M. Wada [7] in terms of knots and links, using a generator, the Hopf link, and six operations. The Hopf link consists in two linked trivial knots corresponding to one attractive and one repulsive periodic orbit; the six operations are basically split sums, connected sums and cabling. M. Wada uses the different attachments of round handles to prove that the set of periodic orbits of a NMS system on S^3 comes from the application of these operations on Hopf links. Conversely, he also proves that every link obtained from Hopf links by applying these operations corresponds to the set of periodic orbits of a NMS system on S^3.

Nevertheless, there exist topologically non-equivalent flows with the same link of periodic orbits, as we show in this work. This result motivates the search of new invariants being in 1-1 correspondence with the topological equivalence class of the flow ([3] and [4]).

1.1. **Round handle decomposition of S^3 and phase portraits.** The notion of the round handle decomposition (or RHD for short) was introduced by Asimov and modified by Morgan. This decomposition provides a connection between NMS flows and the topology of the underlying manifold X.

Definition 1.1. *A pair $(M, \partial_- M)$ of a manifold M and a compact submanifold $\partial_- M$ of ∂M, or by abuse of notation, a manifold M is called*
a round 0-handle if $(M, \partial_- M) \cong \left(D^2 \times S^1, \emptyset\right)$.
a round 1-handle if $(M, \partial_- M) \cong \left(D^1 \times D^1 \times S^1, D^1 \times \partial D^1 \times S^1\right)$.
a round 2-handle if $(M, \partial_- M) \cong \left(D^2 \times S^1, \partial D^2 \times S^1\right)$.

So, the round handles are diffeomorphic to tori and they correspond to a 0-handle when there is a repulsive periodic orbit in the core, a 1-handle if there is a saddle orbit in the core and a 2-handle if there is an attractive periodic orbit in the core; 0, 1 and 2 are the indices of the periodic orbits.

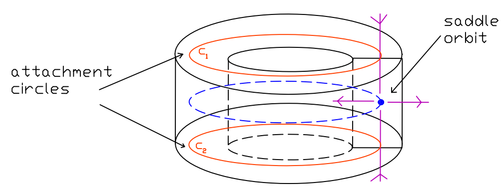

FIGURE 1. Round 1-handle

Definition 1.2. *A round handle decomposition for $(X, \partial_- X)$ is a filtration*

$$\partial_- X \times I = X_0 \subset X_1 \subset X_2 \subset ... \subset X$$

where each X_i is obtained from X_{i-1} by attaching a round handle.

The fattened round 1-handles are obtained when 1-handles are attached to repulsive tori (or attractive tori). A fattened round handle is a 3-manifold X obtained from a manifold Y by attaching a round 1-handle by means of two attaching circles c_1 and c_2, or one circle if the 1-handle is twisted.

Each fattened round handle C has the form

$$C = A \times [0,1] \bigcup_{\varphi} B_s \bigoplus_{S^1} B_u$$

where A can be a torus or two tori , $B_s \bigoplus_{S^1} B_u$ is the Whitney sum of disk bundles B_s and B_u over S^1, the image of $\varphi : (\partial B_s) \bigoplus B_u \to A \times \{1\}$ intersects every component of $A \times \{1\}$ and $\partial_- C = A \times \{0\}$.

The round 1-handle is attached by means of an attaching map defined on a regular neighbourhood of the attaching circles. Depending on these attaching circles, de 1-handle can be essentially or non essentially attached and it can be attached to one or two tori.

The different ways of attachment yield to the different types of fattened round handles with separating tori as boundary components. In order to obtain S^3, the filtration is completed with 2-handles, i.e, with solid attractive tori. If 1-handles are attached to attractive tori, the filtration is completed with solid repulsive tori. So, the filtration defines the corresponding link of periodic orbits.

The 3-sphere S^3 is a compact connected 3-dimensional manifold without boundary. It is defined by $\sum_{i=1}^{4} x_i^2 = 1$ in $\mathbb{R}^4$ and it can be represented by two solid cones with their boundaries identified. In this representation, a periodic orbit can also be sketched as a vertical line from upper boundary to the lower boundary (see figure 2).

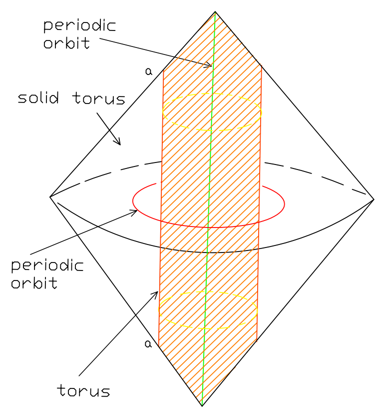

FIGURE 2. Representation of S^3

We build the phase portrait of a NMS flow using Wada operations with their corresponding round handle attachments. The simplest NMS flow consist of two periodic orbits: one repulsive and one attractive. These periodic orbits can be represented as the core of two complementary tori and the flow lines go from the sink to the source. The corresponding link is called a *hopf link.*

Wada's theorem characterizes the set of the periodic orbits of NMS flows on S^3 in terms of knots and indexed links, using a generator, the hopf link, and six operations, basically split sums and cabling. All possible links are obtained by applying Wada operations on hopf links [7].

When the two attaching circles are on one torus, we get operation II of Wada if both circles are inessential (attachment 1a), if one of them is essential and the other is inessential the result corresponds to operation III (attachment 1b) and when both of them are essential operation V is obtained (attachment 1c).

On the other hand, if the two attaching circles are on different components, operation I is obtained if both circles are inessential (attachment 2a), operation II is also obtained if one of the circles is essential and the other one is inessential (attachment 2b) and when both of them are essential we get operation IV, or V with $(1, 0)$-cables (attachment 2c).

Finally, operation VI corresponds to the case when the round 1-handle is twisted and the attaching circle is one essential circle on one torus.

We refer to operations I, II and III as type A operations, and operations IV, V and VI are called type B operations.

Following these results we study the complete flow corresponding to each of these operations in order to obtain the phase portrait of NMS flows on S^3. We draw the corresponding stable and unstable manifolds of the saddle orbit that appears in each Wada operation, corresponding to an unknot and represented by u.

We also obtain the canonical regions for the different cases, i.e., the connected components left when periodic orbits and invariant manifolds of saddle orbits are removed. We observe that regions with different topology appear depending on the attachment of the round 1-handle.

Let us consider the flow given by operation II applied on two hopf links: $II\,(h, h)$. The corresponding link of periodic orbits is $h \cdot d \cdot u$, where $\cdot$ denotes the split sum of a hopf link h, a trivial knot d, corresponding to the separated repulsive orbit, and the trivial knot u, corresponding to the saddle orbit (see figure 3). In this case the round 1-handle has been attached to one torus by means of an inessential circle and to the other one by means of an essential circle.

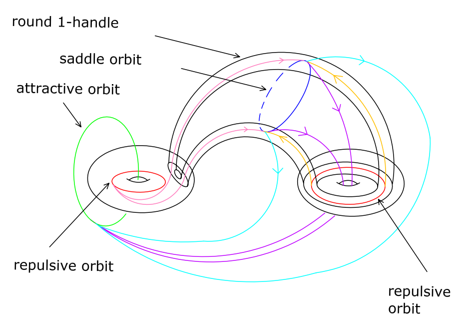

FIGURE 3. A round 1-handle attached on two tori defining operation II

The repulsive orbits correspond to the core of the repulsive tori where the 1-handle has been attached; the saddle orbit is in the core of the round 1-handle and the attractive orbit corresponds to the core of the 2-handle attached in order to obtain S^3. From the phase portrait in S^3, it can be observed that the invariant manifolds of the saddle orbit delimit three canonical regions (see figure 4).

2. Non equivalent flows with the same link of periodic orbits

Definition 2.1. *Two flows are topologically equivalent if there is a homeomorphism taking orbits to orbits preserving orientation.*

The flow of a NMS vector field is defined by the sequence of attachments of the round handles. These sequences are described by means of Wada operations. As it is shown in [2], a link can be written in terms of different Wada operations. In some

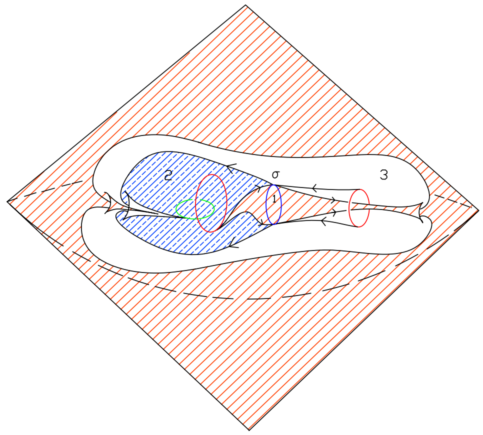

FIGURE 4. Phase portrait of the flow of $II(h,h)$ in S^3

cases this link can describe different flows depending on the type of attachment of the round handles implied. In fact, this happens for given combinations of the Wada operations, as we see in the following proposition:

Proposition 2.1. *Let F_1 and F_2 be two NMS flows on S^3 with the same link of periodic orbits l. If l is written in terms of type A Wada operations for describing F_1 and in terms of type A and B Wada operations for describing F_2, then F_1 and F_2 are not topologically equivalent.*

Proof. From relations of sets of links given in [2, Corollary 2] we consider the following ones:

- $\langle I\,[III(l_1,l_2),l_3]\rangle \subseteq \langle II\,[IV(l_1,l_2),l_3]\rangle = (l_1-d_1)\cdot(l_2-d_2)\cdot l_3\cdot u\cdot u,$
- $\langle II\,[II(l_1,l_2),l_3]\rangle \cap \langle II\,[IV(l_1,l_2),l_3]\rangle = (l_1-d_1)\cdot(l_2-d_2)\cdot l_3\cdot u\cdot u$
- $\langle II[III(l_1,l_2),l_3]\rangle \cap \langle III[IV(l_{\sigma(1)},l_{\sigma(2)}),l_{\sigma(3)}]\rangle = (l_1-d_1)\cdot(l_2-d_2)\cdot(l_3-d_3)\cdot u\cdot u$

when the indices of the components d_1 and d_2 are different.

Here, l_i are links, d_i are components of the links with index 0 or 2, $\sigma(i)$ is a permutation of $1,2,3$ and $\langle\,\rangle$ denotes the set of all the links written in terms of operations inside. In these equivalences operation IV can be replaced by operation V when $(1,0)$ cables are implied.

In the three cases, we have links that can be written using different operations. Let us see that these different operations yield to non equivalent flows.

Consider two flows F_1 and F_2 with the same link of periodic orbits included in one of the cases above. The flow F_1 coming only from type A operations and F_2 implying type A and B operations. We know that type B operations generate canonical regions that are trumpet or toroidal regions (see [4]) and these type of regions do not appear when only type A operations are implied.

These equivalences of links are satisfied when the ringed component generated by operation IV is removed by operations II or III.

Depending on the stability of the ringed component we have to consider the attachment of the round 1-handle to two linked or two separated tori.

When operation IV implies the attachment of the round 1-handle on two linked tori, toroidal canonical regions appear; these regions are preserved after applying operations II or III, so the flow F_2 has canonical regions that do not appear in F_1.

When operation IV implies the attachment of the round 1-handle to two separated tori, the saddle orbit in the core of this round 1-handle connects essentially the orbits in the cores of both tori for the flow F_2. This connections do not occur for the flow F_1 coming from type A operation, where at least one connections must be non essential.

So, there are not an homeomorphism taking orbits from the flow F_1 to the flow F_2. Then, F_1 and F_2 are not topologically equivalent. □

Let us see the following examples with more details.

(1) Consider the flows defined by $I(III(h,h),h)$ and $II(IV(h,h),h)$.

In the flow $I(III(h,h),h)$, the corresponding sequence of attachments is formed by an attachment of type 1b followed by an attachment of type 2a and finally a 2-handle is attached in order to obtain the manifold S^3. Let us observe that attachments 1b and 2a commute.

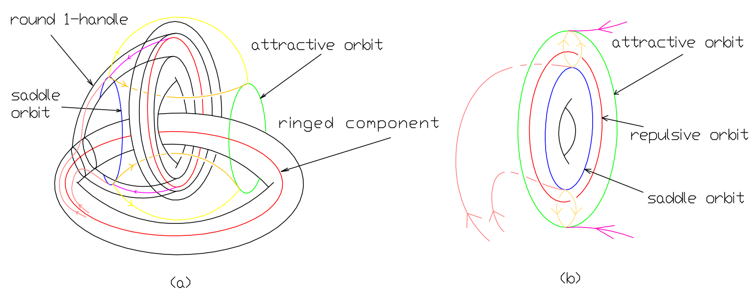

FIGURE 5. Attachment of type 2c on two linked tori

For the flow $II(IV(h,h),h)$ the sequence consists of one attachment of type 2c on two linked tori giving a link formed by four orbits (see figure 5a) and one attachment of type 2b on two attractive tori, one of them corresponding to the three orbits $d \cdot d \cdot u$ that remain when the ringed component is removed (see figure 5b); finally, a 0-handle is attached in order to obtain the manifold S^3. Taking into account the way the round handles have been attached we draw the corresponding phase portraits (see figures 6 and 7, both embedded in S^3).

From these two figures, we observe that canonical regions are different; in fact, in the second flow canonical region 5 is toroidal; then, these flows are not topologically equivalent although the link of periodic orbits is the same:

$$h \cdot d \cdot d \cdot u \cdot u$$

where one of the orbits denoted by d has index 0 and the other has index 2.

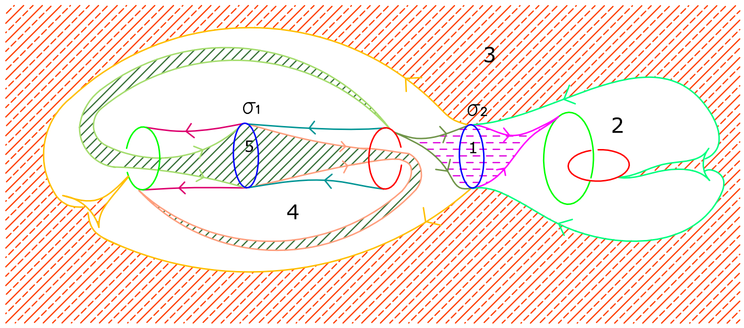

FIGURE 6. Phase portrait of the flow of $I(III(h,h),h)$

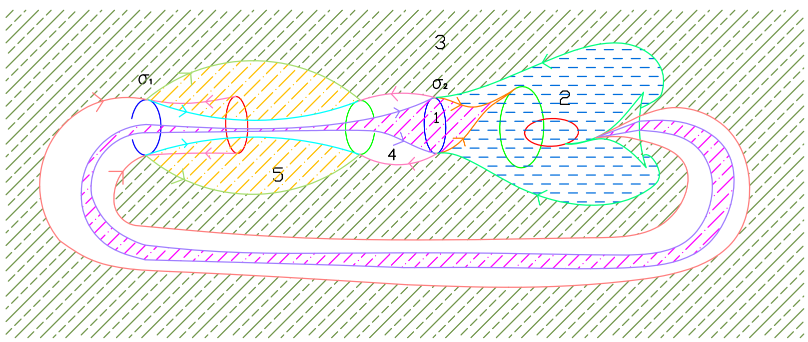

FIGURE 7. Phase portrait of the flow of $II(IV(h,h),h)$

(2) Consider now the flows defined by $II(III(h,h),h)$ and $III(IV(h,h),h)$.

For the flow $II(III(h,h),h)$, the sequence of the attachments corresponds to an attachment of type 1b followed by an attachment of type 2b and finally a 2-handle is attached in order to obtain the manifold S^3. Let us observe that attachments 1b and 2b conmute.

The flow $III(IV(h,h),h)$ is obtained from an attachment of type 2c followed by an attachment of type 1b and finally a 2-handle is attached in order to obtain the manifold S^3.

The corresponding phase portraits are depicted in figures 8 and 9.

From these two figures where canonical regions and periodic orbits can be easily observed, we see that the two essential connections to the same saddle orbit appear in the second flow, but not in the first. Then, these flows are not topologically equivalent although the link of periodic orbits is the same:

$$d \cdot d \cdot d \cdot u \cdot u$$

where one of the orbits denoted by d has index 2 and the other two have index 0.

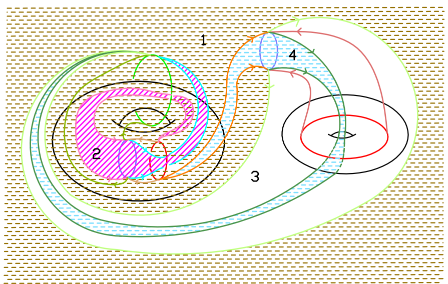

FIGURE 8: $II(III(h,h),h)$

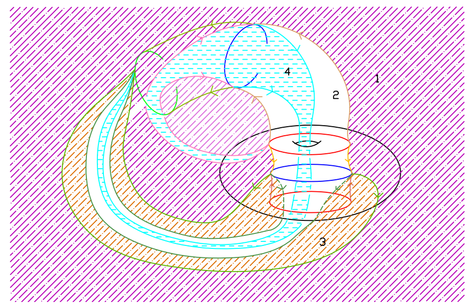

FIGURE 9: $III(IV(h,h),h)$

As we have proved in this paper, there exist non equivalent flows with the same link of periodic orbits. Our goal is to find new invariants being in 1-1 correspondence with the topological equivalence class of the flow. To achieve this aim, we consider dual graphs.

In [3] and [4] we obtain dual graphs for NMS flows on S^3 with no heteroclinic trajectories connecting two saddle orbits; moreover, we show that these duals graphs are in in 1-1 correspondence with these kind of NMS flows on S^3.

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