SCHUR SPACES AND WEIGHTED SPACES OF TYPE H^∞

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Abstract. We extend some results related to composition operators on $H_v(G)$ to arbitrary linear operators on $H_{v_0}(G)$ and $H_v(G)$. We also give examples of rank-one operators on $H_v(G)$ which cannot be approximated by composition operators.

1. Introduction and Preliminaries

The aim of this paper is extending some results related to composition operators C_φ, $C_\varphi(f) = f \circ \varphi$, acting on the weighted Banach spaces of analytic functions $H_{v_0}(G)$ and $H_v(G)$, where G is an open set of $\mathbb{C}^n$, φ is an analytic map from G into G and the *weighted spaces of type* H^∞ are given by

$$H_v(G) = \left\{ f \in H(G) : \|f\|_v := \sup_{z\in G} v(z)|f(z)| < \infty \right\} \text{ and}$$

$$H_{v_0}(G) = \left\{ f \in H_v(G) : \lim_{z\to\partial G} v(z)|f(z)| = 0 \right\},$$

where $H(G)$ is the space of analytic functions on G and $v : G \to \mathbb{R}^+$ is a bounded, continuous and strictly positive function, which will be called *weight*. These spaces are Banach spaces endowed with the norm $\|\cdot\|_v$ and they appear in the study of growth conditions of analytic functions. We refer to [1], [3], [4], [5], [16], [17], [23], [25], and others for further information about these spaces. The study of composition operators on these spaces can be found in [7], [8] and [10]. See also the references therein.

We denote by $\mathbb{C}^*$ the extended complex plane. Let G_1 and G_2 be open connected domains in $\mathbb{C}$ such that $\mathbb{C}^* \setminus G_1$ has no one-point component. Let $\varphi : G_2 \longrightarrow G_1$ be an analytic map and v, w arbitrary weights on G_1 and G_2 respectively. In [7], the authors proved the following result:

Theorem 1.1. *The composition operator $C_\varphi : H_v(G_1) \longrightarrow H_v(G_2)$ is either compact or it is an isomorphism when restricted to some subspace isomorphic to ℓ_∞. In particular, every weakly compact or Rosenthal or strictly singular or strictly cosingular composition operator C_φ is automatically compact.*

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We will improve this result dealing with general linear operators $T : H_{v_0}(G) \to Y$ for any Banach space Y and considering c_0 instead of ℓ_∞. We will also study the general case when we deal with $T : H_v(G) \to Y$, proving in particular Theorem 1.1.

Recall that a Banach space X is said to have the *Dunford-Pettis property (DPP)* if any weakly compact operator $T : X \to Y$ is completely continuous for any Banach space Y. A survey on the DPP can be found in [15]. The space X is said to enjoy the *Schur property* (or X is called a *Schur space*) if any weakly convergent sequence in X is norm convergent.

The Banach space X is a *Grothendieck space* if every linear operator $T : X \longrightarrow Y$ is weakly compact for any separable Banach space Y and X is said to enjoy the *Pelczyński's property (V)* if every linear operator $T : X \longrightarrow Y$ which is not weakly compact is an isomorphism on a copy of c_0. Recall that an operator is *unconditionally converging* if it does not fix a copy of c_0. It is well-known that X enjoys the property (V) if and only if any unconditionally converging operator is weakly compact. Recall also that if X has the Pelczyński's property (V), then X^* is weakly sequentially complete (see Proposition 1.1 in [24]).

A linear operator $T : X \to Y$ is *Rosenthal* if for any bounded sequence $(x_n) \subset X$, the sequence $(T(x_n))$ has a weakly Cauchy subsequence and T is said to be *completely continuous* if it maps weakly convergent sequences into norm convergent ones. The operator T is said to be *strictly singular* if, for each infinite dimensional subspace $X_1 \subset X$, T is not an isomorphism on X_1. The operator T is *strictly cosingular* if a closed subspace $Y_1 \subset Y$ has finite codimension whenever $Q \circ T$ is a surjection, where Q is the quotient map from Y onto Y/Y_1. It is well-known that the set of compact, weakly compact, Rosenthal, strictly singular and strictly cosingular operators form ideals of operators.

2. Results

In this section, we study some properties of operators on $H_{v_0}(G)$ and $H_v(G)$, as weak-compactness, compactness and complete continuity. These notions are closely related to the Grothendieck property, the Pelczyński's property (V) and the Dunford-Pettis property.

First we recall the following result given by Bonet and Wolf (see Theorem 1 in [9]).

Theorem 2.1 (Bonet and Wolf)**.** *Let G be an open subset of C^N, $N \geq 1$, and let v be a strictly positive and continuous weight on G. Then the space $H_{v_0}(G)$ is isomorphic to a closed subspace of c_0. In fact, $H_{v_0}(G)$ embeds almost isometrically into c_0.*

This will allow us to prove that spaces $H_{v_0}(G)$ enjoy the DPP.

Proposition 2.2. *Let G be an open set of $\mathbb{C}^n$. For any weight $v : G \longrightarrow [0, +\infty[$, the space $H_{v_0}(G)$ has the Dunford-Pettis property.*

Proof. By Theorem 2.1, the space $H_{v_0}(G)$ is always a closed subspace of c_0. Since c_0 is hereditarily Dunford-Pettis (see Theorem 4 in [15]), we have that $H_{v_0}(G)$ has the Dunford-Pettis property itself. $\square$

If X is a Banach space enjoying the Dunford-Pettis property which does not have copies of ℓ_1, then X^* is a Schur space (see Theorem 3 in [15]). Since c_0 does not contain ℓ_1, we conclude

Proposition 2.3. *Let G be an open subset of $\mathbb{C}^n$, $n \in \mathbb{N}$ and v a weight on G. Then the Banach space $H_{v_0}(G)^*$ has the Schur property.*

There are many examples of weights v and open sets $G \subset \mathbb{C}$, such that $H_{v_0}(G)^{**} = H_v(G)$. For instance, if we consider G to be the open unit disk $\mathbf{D}$ and v is a non-increasing radial weight on $\mathbf{D}$ such that $\lim_{|z|\to 1^-} v(z) = 0$. This is also true if we deal with $G = \mathbb{C}$ and we consider any radial weight which is rapidly decreasing at infinity (i.e., $H_v(\mathbb{C})$ contains the polynomials). These results can be found in [5] and some extensions dealing with domains $G \subset \mathbb{C}^n$, $n > 1$, can be found in [2]. Boyd and Rueda have studied this problem recently connecting it with the study of M-ideals in weighted spaces of holomorphic functions (see [13]).

Now we investigate operators on $H_{v_0}(G)$. We start with the following result which characterizes Schur dual spaces (see [6]). We give a proof for the sake of completeness:

Proposition 2.4. *Let X be a Banach space. The following assertions are equivalent:*

a) *X^* is Schur.*
b) *Any weakly compact operator $T : X \longrightarrow Y$ is compact for every Banach space Y.*

Proof. Let X^* be a Schur space and let $T : X \longrightarrow Y$ be a weakly compact operator. Then, $T^* : Y^* \longrightarrow X^*$ is weakly compact, so it is also compact since X^* is Schur. Hence, T must be compact.

Conversely, suppose that each weakly compact operator on X is compact and suppose that X^* is not Schur. Then, there exists a sequence $(x_n^*) \subset X^*$ such that $x_n^* \overset{w}{\to} 0$ but $\|x_n^*\| = 1$ for any $n \in \mathbb{N}$. Consider the operator $T : X \longrightarrow c_0$ given by $T(x) = (x_n^*(x))_n$, which is clearly weakly compact, so compact by hypothesis. However, for any n there exists $x_n \in S_X$ such that $|x_n^*(x_n)| \to 1$. Since the sequence (x_n) is bounded, there exists a subsequence, that we denote by (x_n) without loss of generality, such that $T((x_n))$ is convergent to $y \in c_0$. But $\|T(x_n) - y\|_\infty = \sup_{k\in\mathbb{N}} |x_k^*(x_n) - y(k)| \geq |x_n^*(x_n) - y(n)| \geq |x_n^*(x_n)| - |y(n)|$. The first term tends to 1 but the second one tends to 0 since $y \in c_0$. Hence, $T(x_n)$ cannot converge to any element when $n \to \infty$ and T results to be a weakly compact operator which is non-compact. $\square$

Corollary 2.5. *Let G be an open set of $\mathbb{C}^n$, v a weight on G and Y a Banach space. The linear operator $T : H_{v_0}(G) \to Y$ is compact if and only if T is weakly compact.*

Proof. The result follows from Proposition 2.4 since the dual of $H_{v_0}(G)$ is a Schur space by Proposition 2.3. □

Let X be a Banach space having an unconditional basis. In [20], Pełczyński proved that each closed subspace Y of X has property (V) (hereditary) if and only if ℓ_1 is not isomorphic to any closed subspace of Y.

Proposition 2.6. *Let G be an open set of $\mathbb{C}^n$, v be any weight on G and Y be a Banach space. Then, $H_{v_0}(G)$ has property (V).*

Proof. Since c_0 has an unconditional basis, it has property (V) (hereditary) by the comments above. Since $H_{v_0}(G)$ is a closed subspace of c_0 by Theorem 2.1, it has property (V) (hereditary). □

Proposition 2.7. *Let G be an open set of $\mathbb{C}^n$, v a weight on G and Y a Banach space. The linear operator $T : H_{v_0}(G) \longrightarrow Y$ is either compact or it is an isomorphism when restricted to some subspace isomorphic to c_0.*

Proof. If T is not compact, then it is not weakly compact by Corollary 2.5. Since $H_{v_0}(G)$ has property (V) by Proposition 2.6, we conclude the result. □

Corollary 2.8. *Let G be an open set of $\mathbb{C}^n$, v a weight on G and Y a Banach space which does not contain c_0. Then, every linear operator $T : H_{v_0}(G) \longrightarrow Y$ is compact.*

Proposition 2.9. *Let G be an open set of $\mathbb{C}^n$, v a weight on G and Y a Banach space. The following assertions are equivalent for any linear operator $T : H_{v_0}(G) \longrightarrow Y$:*

a) *T is compact.*
b) *T is completely continuous.*
c) *T is weakly compact.*
d) *T is Rosenthal.*
e) *T is strictly singular.*
f) *T is strictly cosingular.*
g) *T is unconditionally converging.*
h) *$T|_Z$ is not an isomorphism whenever Z is a subspace isomorphic to c_0.*

Proof. It is clear that a) $\leftrightarrow$ h) by Proposition 2.7. In addition, a) implies b), c), d), e), f) and g). By Proposition 2.7, we conclude c), d), e), f), h) $\to$ a). Moreover, b) $\to$ a) since $H_{v_0}(G)$ does not contain an isomorphic copy of ℓ_1 by Theorem 2.1, so any completely continuous operator on $H_{v_0}(G)$ is compact by Theorem 1 in [22]. Finally, g) $\to$ c) is true because $H_{v_0}(G)$

enjoys property (V), so any unconditionally convergent operator is weakly compact. □

Now we consider operators on $H_v(G)$. Lusky proved the following result in [18]:

Theorem 2.10 (Lusky)**.** *Let G be the open unit disk $\mathbf{D}$ or the complex plane. For each radial weight v on G, either $H_v(G)$ is isomorphic to H^∞ or it is isomorphic to ℓ_∞.*

Corollary 2.11. *Let G be the open unit disk $\mathbf{D}$ or the complex plane. If v is a radial weight on G, then*

a) *$H_v(G)$ has the Dunford-Pettis property.*
b) *For any Banach space Y, every linear operator $T : H_v(G) \longrightarrow Y$ is weakly compact or it is an isomorphism restricted to a copy of ℓ_∞.*
c) *$H_v(G)$ has the Grothendieck property.*
d) *$H_v(G)$ has the Pelczyński property (V). In particular, $H_v(G)^*$ is weakly sequentially complete.*
e) *$H_v(G)$ has no Schauder decomposition.*

Proof. We use Theorem 2.10. a) It is clear that ℓ_∞ has the DPP since it is a $C(K)$ space. J. Bourgain proved that H^∞ has the DPP in [12], so we are done.

b) By Theorem 1 of [11], this is true for H^∞. Moreover, ℓ_∞ is an injective Banach space, so we apply Corollary 1.4 in [21] and the same is true for this space.

c) and d) are clear by b). The space $H_v(G)^*$ is weakly sequentially complete since $H_v(G)$ has the (V) property by Proposition 1.1 in [24].

e) D. W. Dean showed that ℓ_∞ has no Schauder decomposition by proving that weak-star convergence in ℓ_∞^* implies weak convergence and proving that for weakly compact operators $T : Z \to \ell_\infty$ and $S : \ell_\infty \to Y$, the composition $S \circ T$ is compact. As Dean remarks, this result can be extended for any space X satisfying these conditions. First condition is satisfied since $H_v(G)$ is a Grothendieck space by c). The other one is true since $H_v(G)$ has the Dunford-Pettis property by a), so we are done. □

Proposition 2.12. *Let G be an open set of $\mathbb{C}^n$, v a weight on G and Y a separable Banach space. If $H_v(G) = H_{v_0}(G)^{**}$ and $H_v(G)$ is a Grothendieck space, then any linear operator $T : H_{v_0}(G) \longrightarrow Y$ can be extended to $\tilde{T} : H_v(G) \longrightarrow Y$ if and only if T is compact.*

Proof. If T is compact then T is weakly compact, so $T^{**}(X^{**}) \subset Y$ and we are done by taking $\tilde{T} := T^{**}$.

Conversely, suppose that T can be extended to $\tilde{T}$. Since Y is separable and $H_v(G)$ is a Grothendieck space, then $\tilde{T}$ is weakly compact, so T will be also weakly compact, hence compact by Proposition 2.9 and we are done. □

Corollary 2.13. *For every radial weight v on $G = \mathbf{D}$ or $G = \mathbb{C}$ and every separable Banach space Y, a linear operator $T : H_{v_0}(G) \longrightarrow Y$ can be extended to $\tilde{T} : H_v(G) \longrightarrow Y$ if and only if T is compact.*

In [19], Lusky and Taskinen have studied the isomorphism classes of $H_v(G)$ when we deal with domains G of $\mathbb{C}^n$, $n > 1$. Under some conditions on the weight v, they conclude that $H_v(G)$ is isomorphic to ℓ_∞ if $G = \mathbb{C}^n$ or $G = \mathbb{B}^n$. However, they prove that $H_{v_0}(\mathbb{B}^n)$ is never isomorphic to c_0 and $H_v(\mathbb{B}^n)$ is never isomorphic to ℓ_∞ if $n > 1$.

Proposition 2.14. *Let G be an open set of $\mathbb{C}^n$ and v a weight on G. Suppose that $H_{v_0}(G)^{**} = H_v(G)$. Then,*

a) *Let Y be a Banach space. If $c_0 \subset Y$, then there exist non-compact weakly compact operators $T : H_v(G) \longrightarrow Y$. In particular, these operators $T : H_v(G) \longrightarrow Y$ are neither compact nor an isomorphism when restricted to a copy of ℓ_∞.*

b) *Let Y be a Banach space. If $T : H_v(G) \to Y^*$ is w^*-w^*-continuous, then T is weakly compact if and only if T is compact.*

Proof. a) If $c_0 \subset Y$, since $H_v(G)^*$ is not a Schur space, there exists a non-compact weakly compact operator $T : H_v(G) \to Y$ (see proof of Proposition 2.4). A weakly compact operator cannot be an isomorphism when restricted to a copy of ℓ_∞ and we are done.

b) If $T : H_v(G) \longrightarrow Y^*$ is $w^* - w^*-$continuous, then $T = S^*$, and it is clear that T is weakly compact if and only if $S : Y \to H_{v_0}(G)^*$ is weakly compact. But this is true if and only if S is compact since $H_{v_0}(G)^*$ is Schur, so T is compact and we are done. $\square$

We finish this work proving that there are rank-one operators on $H_v(G)$ which cannot be approximated by composition operators.

If we denote by $i : X \to X^{**}$ the natural inclusion $i(x)(x^*) = x^*(x)$ of X into its bidual X^{**}, X is said to be *reflexive* if i is onto. Recall that for any Banach space X, every $x^{**} \in X^{**}$ is the weak-star limit of a net of elements $i(x) \in i(X)$ by the Goldstine's Theorem. Moreover, there are many examples of Banach spaces X such that any $x^{**} \in X^{**}$ is the sequential weak-star limit of elements of $i(X)$. The trivial ones are reflexive Banach spaces since $i(X) = X^{**}$. If X^* is separable, it is also true since bounded sets of X^{**} are $w(X^{**}, X^*)-$metrizable. However, if X is non-reflexive and weakly sequentially complete, then it is clear that only elements $i(x) \in i(X)$ of X^{**} can be weak-star limits of sequences of elements of $i(X)$. Hence, if X is weakly sequentially complete (in particular, if X is a Schur space), then X^* is non-separable.

Suppose that $H_{v_0}(G)^{**} = H_v(G)$. Notice that for any $z_0 \in G$, evaluations $\delta_{z_0} : H_v(G) \longrightarrow \mathbb{C}$ given by $\delta_{z_0}(f) = f(z_0)$ belongs to $(H_v(G))^*$ since $|f(z_0)| \leq \frac{1}{|v(z_0)|}\|f\|_v$ and $\|\delta_{z_0}\| = \frac{1}{|v(z_0)|}$. Since they are weak-star continuous, we have that $\delta_{z_0} \in i(H_{v_0}(G))^*$ for any $z_0 \in G$.

Proposition 2.15. *Let v be a weight on G and suppose that $H_{v_0}(G)^{**} = H_v(G)$. Then, for any $u \in H_v(G)^* \setminus i(H_{v_0}(G))^*$, the rank-one operator $R : H_v(G) \longrightarrow H_v(G)$ given by $R(f) = u(f).1$ cannot be pointwise sequentially approximated by composition operators.*

Proof. If R was approximated by composition operators, set a sequence of analytic functions (φ_n) such that $C_{\varphi_n} \xrightarrow{\tau_p} R$. For any $f \in H_v(G)$, we have that $C_{\varphi_n}(f) = f \circ \varphi_n \to u(f).1$, so for any $z \in G$, $f \circ \varphi_n(z) = \delta_{\varphi_n(z)}(f) \to u(f)$. Since $\varphi_n(z) \in G$ for any n, then $(\delta_{\varphi_n(z)}) \subset i(H_{v_0}(G))^*$ and it is clearly a weakly Cauchy sequence there. Since $H_{v_0}(G)^*$ is a Schur space, then it is weakly sequentially complete, so $u \in i(H_{v_0}(G))^*$, a contradiction. □

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