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NILPOTENCY OF NORMAL SUBGROUPS HAVING TWO G-CLASS SIZES

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ABSTRACT. Let G be a finite group. If N is a normal subgroup which has exactly two G-conjugacy class sizes, then N is nilpotent. In particular, we show that N is abelian or is the product of a p-group P by a central subgroup of G. Furthermore, when P is not abelian, $P/(\mathbf{Z}(G) \cap P)$ has exponent p.

1. Introduction

Let G be a finite group and N a normal subgroup of G. Since N is a union of G-conjugacy classes, it is natural to wonder what information on the structure of N can be obtained from its G-class sizes. One result of N. Itô claims that any finite group having exactly two conjugacy class sizes is nilpotent ([8]). If every G-conjugacy class contained in N has only two possible sizes, 1 or m, then is N contained in $\mathbf{F}(G)$, the Fitting subgroup of G? We remark that the fact that N could have two N-conjugacy class sizes cannot be deduced from the property that N has exactly two G-conjugacy class sizes. Some effort has been made in this direction, and in [3] the nilpotency of N is shown under the additional hypothesis that N contains some Sylow p-subgroup of G for some prime p. In this paper we extend the result with complete generality.

Theorem A. *If N is a normal subgroup of a group G and the size of any G-conjugacy class contained in N is 1 or m, for some integer m, then N is nilpotent. More precisely, N is abelian or N is the direct product of a nonabelian p-group P by a central subgroup of G. In this case, $P/(\mathbf{Z}(G) \cap P)$ has exponent p.*

The proof of the nilpotency is clearly divided into two parts. The first part is simpler and deals with the case in which N is solvable (in fact, it is enough to suppose that $\mathbf{F}(N) > \mathbf{Z}(N)$). The second part, when N is not solvable, relies on the classification of the finite simple groups by means of a result on CP-groups, that is, on groups having all elements of prime power order.

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2. Proofs

If x is any element of a group G, we denote by x^G the conjugacy class of x in G and $|x^G|$ is called the index or the class size of x in G. The rest of the notation is standard. We begin by showing the following result whose techniques are elementary.

Theorem 1. *Let N be a normal subgroup of a group G such that the size of any G-conjugacy class contained in N is 1 or m, for some integer m. Then either N is abelian or $N/(N \cap \mathbf{Z}(G))$ is a CP-group.*

Proof. We will denote by $\overline{G} = G/(\mathbf{Z}(G) \cap N)$ and we will use bars to work in this factor group. Suppose that there exists $g \in N$ such that the order of $\overline{g}$ is not a prime power and we will work to get a contradiction. Then, there exist at least two distinct primes p and q such that g_p and g_q, that is, the p-part and q-part of g respectively, are non-central elements. By hypothesis, $C_G(g) = C_G(g_p) = C_G(g_q)$. We continue by the following steps.

Step 1. $C_N(g) \leq \mathbf{Z}(C_G(g))$.

Let w be a q'-element of $C_N(g) = C_N(g_q)$ and suppose that it is noncentral in G. Then

$$C_G(g_q w) = C_G(g_q) \cap C_G(w) = C_G(g_q) = C_G(w)$$

by the hypothesis of the theorem. Hence, $w \in \mathbf{Z}(C_G(g_q)) = \mathbf{Z}(C_G(g))$. Similarly, one can obtain that if t is a p'-element of $C_N(g) = C_N(g_p)$, then $t \in \mathbf{Z}(C_G(g))$. As a consequence, we conclude that any element $z \in C_N(g)$ belongs to $\mathbf{Z}(C_G(g))$.

Step 2. If $z \in N - \mathbf{Z}(G)$ is such that $C_G(g) \neq C_G(z)$, then $C_N(g) \cap C_N(z) = Z(G) \cap N$. Furthermore, either N is abelian or $\mathbf{Z}(N) = \mathbf{Z}(G) \cap N$.

Let $z \in N - \mathbf{Z}(G)$ be such that $C_G(g) \neq C_G(z)$. Suppose that there exists $a \in C_N(g) \cap C_N(z) - \mathbf{Z}(G)$. By Step 1, we have $a \in \mathbf{Z}(C_G(g))$. Therefore, $C_G(g) = C_G(a)$ by our assumptions. Thus, $z \in C_N(a) = C_N(g) \leq \mathbf{Z}(C_G(g))$, using Step 1 again. Consequently, $C_G(g) \leq C_G(z)$ and so $C_G(g) = C_G(z)$, which is a contradiction. Hence, $C_N(g) \cap C_N(z) = \mathbf{Z}(G) \cap N$. Suppose now that $C_G(z) = C_G(g)$ for every $z \in N - \mathbf{Z}(G)$. Then $N = C_N(z)$ for all $z \in N$; that is, N is abelian. Otherwise, there exists some $z \in N - \mathbf{Z}(G)$ such that $C_G(g) \neq C_G(z)$, and then by the first assertion $\mathbf{Z}(N) \leq C_N(g) \cap C_N(z) = \mathbf{Z}(G) \cap N$, and we deduce that $\mathbf{Z}(N) = \mathbf{Z}(G) \cap N$.

In the rest of the proof, we may assume that N is not abelian (otherwise the theorem is proved), and thus we assume that $\mathbf{Z}(N) = \mathbf{Z}(G) \cap N$.

Step 3. We have $C_{\overline{G}}(\overline{g}) = \overline{C_G(g)}$. In particular, $C_{\overline{N}}(\overline{g}) = \overline{C_N(g)}$.

We clearly have $\overline{C_G(g)} \leq C_{\overline{G}}(\overline{g}) \leq C_{\overline{G}}(\overline{g_q})$ and hence $|C_{\overline{G}}(\overline{g})|$ divides $|C_{\overline{G}}(\overline{g}_q)|$. On the other hand, if $\overline{y}$ is an r-element of $C_{\overline{G}}(\overline{g}_q)$, with $r \neq q$, then $[y, g_q] \subseteq Z(G)$. If $o(y) = k$, then $1 = [y^k, g_q] = [y, g_q^k]$. Hence, $y \in C_G(g_q^k) = C_G(g_q) = C_G(g)$ and $\overline{y} \in \overline{C_G(g)}$. Consequently,

$$|C_{\overline{G}}(\overline{g}_q)|_r \leq |\overline{C_G(g)}|_r.$$

As $|\overline{C_G(g)}|_r$ divides $|C_{\overline{G}}(\overline{g})|_r$, we conclude that $|C_{\overline{G}}(\overline{g})|_r = |\overline{C_G(g)}|_r$ for each prime $r \neq q$. Arguing similarly for the prime p, we obtain $|C_{\overline{G}}(\overline{g})|_r = |\overline{C_G(g)}|_r$ for every prime $r \neq p$. Hence, $|C_{\overline{G}}(\overline{g})| = |\overline{C_G(g)}|$ and $C_{\overline{G}}(\overline{g}) = \overline{C_G(g)}$. The second assertion is an immediate consequence.

We remark that the above three steps hold for every conjugate of g in G.

Step 4. Conclusion.

First, we claim that there exists some $x \in N$ such that $\overline{g}^{\overline{N}} \cap \overline{C_N(x)} = \emptyset$. Suppose that for every $x \in N - \mathbf{Z}(G)$, there exists some $n \in N$ such that $g^n \in C_N(x)$. Then

$$N \subseteq \bigcup_{n \in N} C_N(g)^n,$$

and this implies that $N = C_N(g)$ and so $g \in \mathbf{Z}(N) = \mathbf{Z}(G) \cap N$, a contradiction. Thus the claim is proved.

Now, the subgroup $\overline{C_N(x)}$ operates on $\overline{g}^{\overline{N}}$ by conjugation. Moreover, no element in $\overline{C_N(x)}$ distinct from 1 centralises any element in $\overline{g}^{\overline{N}}$. In fact, if there is some $1 \neq \overline{h} \in \overline{C_N(x)}$ which fixes some $\overline{g^t}$ for some $t \in N$, it follows that $\overline{h} \in \overline{C_N(g^t)} \cap \overline{C_N(x)}$, and, by applying Step 2 to g^t, we deduce that $C_G(g^t) = C_G(x)$, a contradiction. Hence, all orbits of $\overline{C_N(x)}$ on $\overline{g}^{\overline{N}}$ have the same length, that is, $|\overline{C_N(x)}|$, and this implies that $|\overline{C_N(x)}|$ divides $|\overline{g}^{\overline{N}}| = |\overline{N} : \overline{C_N(g)}|$ by applying Step 3. Therefore, $|\overline{C_N(g)}|$ divides $|\overline{N} : \overline{C_N(x)}| = |x^N|$, which implies that $|\overline{C_N(g)}|$ divides $|x^G| = |g^G| = |\overline{g}^{\overline{G}}|$ by Step 3 again.

On the other hand, we claim that $\overline{C_N(g)}$ operates without fixed points on $\overline{g}^{\overline{G}} - \overline{g}^{\overline{G}} \cap \overline{C_N(g)}$ by conjugation. If some $\overline{w} \in \overline{C_N(g)}$ distinct from 1 fixes some $\overline{g^t}$ for some $t \in G$, then $\overline{w} \in C_{\overline{G}}(\overline{g}^{\overline{t}}) = \overline{C_G(g^t)}$ by applying Step 3 to g^t. Therefore, $\overline{w} \in \overline{C_N(g)} \cap \overline{C_N(g^t)}$, and $C_G(g) = C_G(g^t)$ by Step 2. Thus, $\overline{g}^{\overline{t}} = \overline{g^t} \in \overline{C_N(g)} \cap \overline{g}^{\overline{G}}$. As a consequence, $|\overline{C_N(g)}|$ divides $|\overline{g}^{\overline{G}} - \overline{g}^{\overline{G}} \cap \overline{C_N(g)}| = |\overline{g}^{\overline{G}}| - |\overline{g}^{\overline{G}} \cap \overline{C_N(g)}|$. Finally, we conclude that $\overline{C_N(g)}$ also divides $|\overline{g}^{\overline{G}} \cap \overline{C_N(g)}|$, which is not possible because

$$0 < |\overline{g}^{\overline{G}} \cap \overline{C_N(g)}| < |\overline{C_N(g)}|.$$

This contradiction shows that any element of $\overline{G}$ has prime power order. □

Corollary 2. *Let N be a normal subgroup of a group G such that the size of any G-conjugacy class contained in N is 1 or m, for some integer m. Then $N/\mathbf{Z}(N)$ is a CP-group.*

Proof. This is trivial from Theorem 1. □

In order to prove the nilpotency of N in Theorem A, we will make use of the following results.

Lemma 3. *Let G be a π-separable group. The size of the conjugacy class of every π-element of G is a π-number if and only if $G = H \times K$, where H and K are a Hall π-subgroup and a π-complement of G, respectively.*

Proof. See Lemma 8 of [1] for instance. □

Lemma 4. *Let $P \times Q$ be the direct product of a p-group P and a p'-group Q and suppose that $P \times Q$ acts on a p-group G. If $C_G(P) \subseteq C_G(Q)$, then Q acts trivially on G.*

Proof. This is Thompson's $P \times Q$-Lemma. See for instance 8.2.8 of [9]. □

Lemma 5. *Suppose that G is a solvable group and that any $x \in \mathbf{O}_q(G)$ has q-index in G for every prime q. Then G is nilpotent.*

Proof. We argue by induction on the order of G. The hypotheses are certainly inherited by normal subgroups of G, so every proper normal subgroup of G is nilpotent. If G is not nilpotent, this implies that $\mathbf{F}(G)$ is a maximal normal subgroup of G. This means that $G/\mathbf{F}(G)$ is a cyclic group of order p for some prime p, so $|G : \mathbf{F}(G)| = p$. Now, if $q \neq p$ is any prime dividing $|\mathbf{F}(G)|$, we have that any q-element of G has q-index. By Lemma 3, we have $G = Q \times H$, where H is a q-complement of G, and by induction, we conclude that H, and accordingly G, are nilpotent, which is a contradiction. □

We are going to make use of the structure of finite CP-groups. The structure of finite solvable CP-groups was given by H. Higman fifty years ago ([4]), and the structure of nonsolvable CP-groups and the classification of the simple CP-groups have been recently obtained by H. Heineken in [5].

Theorem 6. *If G is a finite, nonsolvable CP-group, then there are normal subgroups B, C of G such that $1 \subseteq B \subseteq C \subseteq G$ and B is a 2-group, C/B is nonabelian and simple, and G/C is a p-group for some prime p and cyclic or generalized quaternion.*

Proof. This is the main part of Proposition 2 of [5]. □

Theorem 7. *If G is a finite nonabelian simple CP-group, then G is isomorphic to one of the following groups: $L_2(q)$, for $q = 5, 7, 8, 9, 17$, $L_3(4)$, $Sz(8)$ or $Sz(32)$.*

Proof. This is Proposition 3 of [5]. □

Theorem 8. *Suppose that N is a normal subgroup of a group G and that the size of any G-conjugacy class contained in N is 1 or m, for some integer m. Then N is nilpotent.*

Proof. We argue by induction on the order of N. Let r and q be any two primes dividing $|N|$. Let x be any r-element of N such that $x \notin \mathbf{Z}(G)$ and take Q to be a Sylow q-subgroup of $C_G(x)$. Let us consider the action of $Q \times \langle x \rangle$ on $Q_0 = \mathbf{O}_q(N)$. We claim that $C_{Q_0}(Q) \subseteq C_{Q_0}(x)$. In fact, if $z \in C_{Q_0}(Q)$ is noncentral in G, then $\langle Q, z \rangle \leq C_G(z) < G$. However, by hypothesis, $|C_G(z)|_q = |C_G(x)|_q = |Q|$, so in particular $z \in Q \cap Q_0 \subseteq C_{Q_0}(x)$. We can apply Lemma 4 and get $x \in C_N(\mathbf{O}_q(N))$. This shows that for any prime q, we have that any element lying in $\mathbf{O}_q(N)$ has q-index in N. Now, if $\mathbf{Z}(N)_q < \mathbf{O}_q(N)$ for some prime q, we take an element $w \in \mathbf{O}_q(N) - \mathbf{Z}(N)$ and we have $N = C_N(w)Q_w$ for some q-subgroup Q_w of N. We show that $C_N(w)$ is nilpotent. For any q'-element $y \in C_N(w)$, by applying the hypothesis, we have

$$C_G(yw) = C_G(w) \cap C_G(y) = C_G(w) \subseteq C_G(y).$$

In particular $y \in \mathbf{Z}(C_N(w))$, which means that $C_N(w)$ factorizes as the product of a q-group by an abelian subgroup, and hence it is nilpotent as wanted. It follows that N is solvable since it is a product of two nilpotent groups, and thus we can apply Lemma 5 to conclude that N is nilpotent, so the theorem is proved.

Therefore, we can assume for the rest of the proof that $\mathbf{F}(N) = \mathbf{Z}(N)$, so N is nonsolvable, and we will show that this leads to a contradiction. We know by Corollary 2 that $\overline{N} = N/\mathbf{Z}(N)$ is a CP-group. By Theorem 6, there exist $\overline{B}$ and $\overline{C}$

subgroups of $\overline{N}$ such that $\overline{B}$ is a 2-group, $\overline{C}/\overline{B}$ is a nonabelian simple group, and $\overline{N}/\overline{C}$ is a p-group for some prime p. As $\mathbf{F}(N/\mathbf{Z}(N)) = 1$, we certainly have $\overline{B} = 1$. On the other hand, note that $\overline{C} = \mathbf{O}^p(\overline{N})$, so in particular $\overline{C}$ is characteristic in $\overline{N}$. Furthermore, since $\mathbf{Z}(N)$ is characteristic in N, we conclude that C is normal in G. Then, by the inductive hypothesis, we can assume that $\overline{N}$ is a nonabelian simple CP-group. If $N' < N$, then N' would be nilpotent by induction, so N would be solvable, a contradiction. Hence $N' = N$, and thus N is a quasi-simple group.

Now, we claim that $|\overline{N}|$ divides m. As any element of $\overline{N}$ has prime power order, we can take a noncentral p-element $x \in N$ for any prime p dividing $|\overline{N}|$ and notice that we can factorize $C_N(x) = C_N(x)_p \times \mathbf{Z}(N)_{p'}$, where $C_N(x)_p$ is a Sylow p-subgroup of $C_N(x)$. Then

$$|x^N|_{p'} = |N : C_N(x)|_{p'} = |\overline{N}|_{p'}.$$

On the other hand, $|x^N|_{p'}$ divides $m_{p'}$, so by considering all primes we have that $|\overline{N}|$ must divide m, as claimed.

Now, the fact that every element of N is central in G or lies in a G-conjugacy class of size m implies that

$$|N| = |\mathbf{Z}(G) \cap N| + mk,$$

for some integer k. By the above paragraph, we deduce that $|\overline{N}|$ divides $|\mathbf{Z}(G) \cap N|$, so in particular, it divides $|\mathbf{Z}(N)|$. As N is a quasi-simple group, if S is the associated simple group to N, then it can be assumed that $\mathbf{Z}(N) \subseteq M(S)$, where $M(S)$ is the Schur multiplier of S. One can easily check (for instance in [2]) that $M(S)$ has order 1, 2, 6 or 48 for the simple groups S appearing in the list of Theorem 7. In all cases, the fact that $|\overline{N}| = |S|$ divides $|M(S)|$ provides the final contradiction. □

Corollary 9. *Suppose that N is a normal subgroup of a group G such that the size of any G-conjugacy class contained in N is 1 or m, for some integer m. Then N is abelian or $N = P \times A$, with P a p-group and A central in G.*

Proof. We know that N is nilpotent by Theorem 8. If N is not abelian, then by applying Theorem 1, we have that $N/(\mathbf{Z}(G) \cap N)$ is a p-group for some prime p, and then the result follows. □

Examples for the two cases appearing in the above corollary can be easily constructed. Let N be an abelian group of odd order and let α be the involutory automorphism of N. Then N is an abelian normal subgroup of $G = N\langle\alpha\rangle$ such that the G-classes contained in N have size 1 or 2. On the other hand, let Q be the quaternion group of order 8 and $\beta \in \mathrm{Aut}(Q)$ of order 3. If $G = Q\langle\beta\rangle$, then the G-classes contained in Q have size 1 or 6. This is an example of a nonabelian normal p-subgroup of G with exactly two G-class sizes.

Several authors, first Isaacs ([7]) and later A. Mann ([10]) or L. Verardi ([11]), have independently proved that if G is a p-group with exactly two class sizes, then the exponent of $G/\mathbf{Z}(G)$ is p. We are going to extend this result for a normal p-subgroup P with two G-class sizes, and in particular we provide an alternative proof for the case $P = G$. The approach consists in defining an appropriate normal abelian subgroup contained in P which satisfies certain properties. This construction is inspired by the proof of Proposition 2.2 in [8]. We will also need the following recent result due to Isaacs.

Lemma 10. *Let $K \trianglelefteq G$, where G is an arbitrary finite group and K is abelian. Let x be a noncentral element of G, and let $y = [t, x]$ for some element $t \in K$. Then $|C_G(y)| > |C_G(x)|$, and so the G-class of y is smaller than that of x.*

Proof. This is exactly Lemma 1 of [6]. □

Theorem 11. *Suppose that P is a nonabelian normal p-subgroup of a group G such that P has only two G-conjugacy class sizes. Then $P/(\mathbf{Z}(G) \cap P)$, and in particular $P/\mathbf{Z}(P)$, has exponent p.*

Proof. We assume that the theorem is untrue and fix some element $x \in P$ such that $x^p \notin Z(G)$. Write $Z_1 = \mathbf{Z}(P)$. Since P is nonabelian, we define $Z_2/Z_1 = \mathbf{Z}(P/Z_1)$. Notice that $Z_1 < Z_2$ and that $Z_2 \trianglelefteq G$. Let $T_2 = C_P(Z_2) < C_P(Z_1) = P$ and observe also that $T_2 \trianglelefteq G$. The proof has been divided into several steps.

Step 1. If $z \in P - T_2$, then $z^p \in \mathbf{Z}(G)$. Consequently, $x \in T_2$.

By hypothesis $z \notin C_P(Z_2)$, so there exists some $y \in Z_2$ such that $1 \neq [y, z] \in Z_1$. Now, if we consider z^p, by the hypotheses of the theorem we have two possibilities: either $z^p \in \mathbf{Z}(G)$ or $C_G(z^p) = C_G(z)$. We show that the second case is not possible. Let p^a be the order of $[y, z]$. As $[y, z]$ is central in P, we have

$$1 = [y, z]^{p^a} = [y^{p^{a-1}}, z^p],$$

so $y^{p^{a-1}} \in C_G(z^p) = C_G(z)$, which yields to $1 = [y^{p^{a-1}}, z] = [y, z]^{p^{a-1}}$, and this is the required contradiction. Then $z^p \in \mathbf{Z}(G)$ and in particular, x must lie in T_2.

Step 2. There exists an abelian subgroup $T \trianglelefteq G$ with $Z_1 \subseteq T \subseteq P$ such that if $z \in P - T$, then $z^p \in \mathbf{Z}(G)$. As a consequence, $x \in T$.

If T_2 is abelian, we can take $T = T_2$ and the step is proved. So we assume that T_2 is not abelian, and we may define Z_3 by means of $Z_3/\mathbf{Z}(T_2) = \mathbf{Z}(T_2/\mathbf{Z}(T_2)) \neq 1$ and also define $T_3 = C_{T_2}(Z_3) < T_2$. Notice that $Z_1 \subseteq T_3$. Furthermore, as $Z_3/\mathbf{Z}(T_2)$ is characteristic in $T_2/\mathbf{Z}(T_2)$, and $\mathbf{Z}(T_2)$ is characteristic in T_2, we deduce that Z_3 is characteristic in T_2, so Z_3 and accordingly T_3 are normal subgroups in G.

Now, we show that if $z \in T_2 - T_3$, then $z^p \in \mathbf{Z}(G)$. As $z \notin C_P(Z_3)$, there exists some $y \in Z_3$ such that $1 \neq [y, z] \in \mathbf{Z}(T_2)$. Arguing as in Step 1, if $C_G(z^p) = C_G(z)$ we get a contradiction, so $z^p \in \mathbf{Z}(G)$. Therefore, for any $z \in P - T_3$, we have $z \in P - T_2$ or $z \in T_2 - T_3$, and both cases yield to $z^p \in \mathbf{Z}(G)$.

Thus, if T_3 is abelian, we take $T = T_3$ and the step is proved. Otherwise, we can argue as we have done with T_2 and construct from Z_3 the subgroups Z_4 and $T_4 = C_P(Z_4)$, which satisfy that $z^p \in \mathbf{Z}(G)$ for any $z \in P - T_4$. This method provides a properly descendant series of subgroups $T_i \trianglelefteq G$, with $Z_1 \subseteq T_i \subseteq P$, and satisfying the property that $z^p \in \mathbf{Z}(G)$ for any $z \in P - T_i$. As Z_1 is abelian, we can get an abelian T_i for some i, and thus $T = T_i$ is the desired subgroup.

Step 3. If $z \in T$ and $z^p \notin \mathbf{Z}(G)$, then $C_P(z) = T$. In particular $C_P(x) = T$.

As T is abelian, we have $T \subseteq C_P(z)$. Suppose that $y \in C_P(z) - T$. By Step 2, we know that $y^p \in \mathbf{Z}(G)$, so $(zy)^p = z^p y^p \notin \mathbf{Z}(G)$. Again by Step 2, we have $zy \in T$, so $y \in T$, a contradiction.

Step 4. If $\overline{G}$ denotes $G/(\mathbf{Z}(G) \cap P)$, then $\overline{C_P(x)} = C_{\overline{P}}(\overline{x})$.

Let $y \in P$ such that $\overline{y} \in C_{\overline{P}}(\overline{x})$. If $y^p \notin Z(G)$, then by Step 2, $y \in T$, and by Step 3, we have $C_P(x) = T$, so $y \in C_P(x)$. If $y^p \in \mathbf{Z}(G)$, then $(xy)^p = x^p y^p \notin \mathbf{Z}(G)$. By Step 2, we have $xy \in T$, which implies again $y \in T$. This proves that $C_{\overline{P}}(\overline{x}) \subseteq \overline{C_P(x)}$, and the other containment is obvious.

Step 5. Conclusion.

Let $g \in P - T$ and consider $y = [x, g]$. Since T is abelian, we can apply Lemma 10 and get that the G-class size of $y \in P$ is smaller than that of g. This forces y to be central in G, and as a consequence $\overline{g} \in C_{\overline{P}}(\overline{x}) = \overline{C_P(x)}$ by Step 4. Therefore, $g \in C_P(x) = T$, which is a contradiction. This shows that the element x cannot exist, so $\overline{P}$ has exponent p, and the proof finishes. $\square$

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