Math. Nachr. **283,** No. 4, 568 – 572 (2010) / **DOI** 10.1002/mana.200610836

Bilinear isometries on subspaces of continuous functions

Juan J. Font[*1] and **M. Sanchis**[** 1]

[1] Departament de Matemàtiques, Universitat Jaume I, Campus del Riu Sec, 12071 Castelló, Spain

Received 14 December 2006, revised 24 September 2008, accepted 5 October 2008
Published online 18 March 2010

Key words Bilinear isometry, subspaces of continuous functions, generalized peak point
MSC (2000) 46A55, 46E15

Let A and B be strongly separating linear subspaces of $C_0(X)$ and $C_0(Y)$, respectively, and assume that $\partial A \neq \emptyset$ (∂A stands for the set of generalized peak points for A) and $\partial B \neq \emptyset$. Let $T : A \times B \longrightarrow C_0(Z)$ be a bilinear isometry. Then there exist a nonempty subset Z_0 of Z, a surjective continuous mapping $h : Z_0 \longrightarrow \partial A \times \partial B$ and a norm-one continuous function $a : Z_0 \longrightarrow K$ such that $T(f, g)(z) = a(z)f(\pi_x(h(z))g(\pi_y(h(z))$ for all $z \in Z_0$ and every pair $(f, g) \in A \times B$. These results can be applied, for example, to non-unital function algebras.

1 Introduction

Let X be a locally compact Hausdorff space. As usual, $C_0(X)$ (resp. $C(X)$ if X is compact) stands for the Banach space of all continuous scalar-valued functions on X which vanish at infinity, endowed with the supremum norm, $\|\cdot\|_\infty$. In [6], the authors proved the following bilinear version of the well-known Holsztyński's Theorem on non-surjective linear isometries of $C(X)$-spaces ([5]):

Let $T : C(X) \times C(Y) \longrightarrow C(Z)$ be a bilinear isometry. Then there exist a closed subset Z_0 of Z, a surjective continuous mapping $h : Z_0 \longrightarrow X \times Y$ and a norm-one continuous function $a \in C(Z)$ such that $T(f, g)(z) = a(z)f(\pi_x(h(z))g(\pi_y(h(z))$ for all $z \in Z_0$ and every pair $(f, g) \in C(X) \times C(Y)$.

The proof of this result rests heavily on the powerful Stone-Weierstrass Theorem. In this paper we extend this bilinear version of Holsztyński's Theorem to a more general context, where Stone-Weierstrass Theorem is not applicable (see Theorem 3.6). Our version is valid, among others, for completely regular (in particular, extremely regular) subspaces of $C_0(X)$ and for non-unital function algebras.

2 Preliminaries

Let X be a locally compact space and A be a linear subspace of $C_0(X)$. It is said that A is separating (resp. strongly separating ([1])) if for distinct $x, y \in X$, there exists $f \in A$ such that $f(x) \neq f(y)$ (resp. $|f(x)| \neq |f(y)|$).

Let $x_0 \in X$. It is said that x_0 is a *generalized peak point* (also called *strong boundary point* or *weak peak point*) for A if for every open neighborhood, V, of x_0 there exists $f \in A$ such that $\|f\| = |f(x_0)| = 1$ and f vanishes outside V. We shall write ∂A to denote the set of generalized peak points for A and $Ch(A)$ to denote the Choquet boundary for A, which is to say, the subspace of X consisting of the extreme points of the closed unit ball of the dual of A.

* e-mail: font@mat.uji.es
** Corresponding author: e-mail: sanchis@mat.uji.es

3 Bilinear isometries

In the sequel, we shall assume that A and B are nonvoid linear subspaces of $C_0(X)$ and $C_0(Y)$, respectively, whose sets of generalized peak points are not empty (see Remark 3.7 below). Furthermore, $T : A \times B \longrightarrow C_0(Z)$ will be a bilinear mapping which satisfies

$$\|T(f,g)\| = \|f\|\|g\|$$

for every $(f,g) \in A \times B$, which is to say that T is a *bilinear isometry*.

For any $x \in X$, let

$$C_x := \{f \in A : 1 = \|f\| = |f(x)|\}.$$

Lemma 3.1 *Assume $(x,y) \in \partial A \times \partial B$. The set*

$$I_{x,y} := \{z \in Z : 1 = \|T(f,g)\| = |(T(f,g)(z)| \text{ for } (f,g) \in C_x \times C_y\}$$

is nonempty.

Proof. For any $f \in A$ and $g \in B$, let $L(f,g) := \{z \in Z : \|T(f,g)\| = |T(f,g)(z)|\}$ and let $M_{f,g} := \left\{z \in Z : |T(f,g)(z)| \geq \frac{\|T(f,g)\|}{2}\right\}$ which is compact since $T(f,g) \in C_0(Z)$. To prove that $I_{x,y}$ is nonempty, and since $I_{x,y}$ is a closed subset of $M_{f,g}$, we shall check that if $f_1, \ldots, f_n$ belong to C_x and $g_1, \ldots, g_n$ belong to C_y, then $\bigcap_{i,j} L(f_i, g_j) \neq \emptyset$. Let $f \in A$ and $g \in B$ defined as follows:

$$f := \sum_{i=1}^{n} \frac{|f_i(x)|}{f_i(x)} f_i$$

and

$$g := \sum_{j=1}^{n} \frac{|g_j(y)|}{g_i(y)} g_i.$$

It is clear that $|f(x)| = n = \|f\|$ and $|g(y)| = n = \|g\|$. Hence, $\|T(f,g)\| = \|f\|\|g\| = n^2$ since T is a bilinear isometry and there exists $z \in Z$ such that

$$|T(f,g)(z)| = n^2 = \left| \sum_{i,j} \frac{|f_i(x)|}{f_i(x)} \frac{|g_j(y)|}{g_j(y)} T(f_i, g_j)(z) \right|.$$

As $\|T(f_i, g_j)\| \leq 1$ for every i, j, we infer that $|T(f_i, g_j)(z)| = 1$, which is to say that $z \in \bigcap_{i,j} L(f_i, g_j)$, as was to be proved. □

Lemma 3.2 *Assume $(x_0, y) \in \partial A \times \partial B$. Fix $g \in C_y$ and define a linear isometry $S : A \longrightarrow C_0(Z)$ as $S(f) := T(f,g)$. If $f(x_0) = 0$, then $(Sf)(z) = 0$ for all $z \in I_{x_0,y}$.*

Proof. Assume there exists $z_0 \in I_{x_0,y}$ such that $(Sf)(z_0) \neq 0$ for some $f \in A$. Let us assume that $\|f\| = 1$ and $(Sf)(z_0) = \alpha$ with $0 < \alpha \leq 1$. Let $U = \{x \in X : |f(x)| \geq \frac{\alpha}{2}\}$. There is $f' \in A$ such that $1 = \|f'\| = |f'(x_0)|$, $|f'(x)| < 1$ for all $x \in U$ and, multiplying by a constant if necessary, $(Sf')(z_0) = 1$. Since U is compact, there exists $s := \sup_{x \in U}\{|f'(x)|\} < 1$. Then we can find a positive integer M such that $1 + Ms < \alpha + M$. If we take $x \in U$, then

$$|(f + Mf')(x)| \leq 1 + Ms.$$

If $x \notin U$, then

$$|(f + Mf')(x)| \leq \frac{\alpha}{2} + M.$$

Hence $\|f + Mf'\| < \alpha + M$, but $\alpha + M = (Sf)(z_0) + M(Sf')(z_0) \leq \|S(f + Mf')\|$, which is a contradiction. □

Lemma 3.3 *Assume $(f,g) \in A \times B$ and $(x_0, y_0) \in \partial A \times \partial B$. If $f(x_0) = g(y_0) = 0$, then $T(f,g)(z) = 0$ for all $z \in I_{x_0,y_0}$.*

Proof. Assume, without loss of generality, that $\|f\| = \|g\| = 1$ and suppose, contrary to what we claim, that $T(f,g)(z_0) = \alpha \neq 0$ for some $z_0 \in I_{x_0,y_0}$.

Let $U := \{x \in X : |f(x)| \geq \frac{\alpha}{2}\}$ and $V := \{y \in Y : |g(y)| \geq \frac{\alpha}{2}\}$.

As x_0 is a generalized peak point for A, we have $f_1 \in A$ such that $1 = \|f_1\| = |f_1(x_0)|$ and $|f_1(x)| < 1$ for all $x \in U$. Similarly, since y_0 is a generalized peak point for B, there exists $g_1 \in B$ such that $1 = \|g_1\| = |g_1(y_0)|$ and $|g_1(y)| < 1$ for all $y \in V$. It is apparent that we can assume, multiplying by a constant if necessary, that $T(f_1, g_1)(z_0) = 1$. Hence, for any positive integers M and N, we have

$$\begin{aligned}
&\|T(f + Mf_1, g + Ng_1)\| \\
&\quad \geq |T(f + Mf_1, g + Ng_1)(z_0)| \\
&\quad = |T(f,g)(z_0) + NT(f,g_1)(z_0) + MT(f_1,g)(z_0) + MNT(f_1,g_1)(z_0)| \\
&\quad = \alpha + MN.
\end{aligned}$$

On the other hand, if $x \in U$,

$$|(f + Mf_1)(x)| \leq |f(x)| + M|f_1(x)| \leq 1 + Ms,$$

where $s < 1$ stands for the maximum of f_1 on U. If $x \notin U$,

$$|(f + Mf_1)(x)| \leq |f(x)| + M|f_1(x)| \leq \frac{\alpha}{2} + M.$$

Consequently, $\|f + Mf_1\| \leq 1 + Ms$. Similarly, $\|g + Ng_1\| \leq 1 + Ns'$. Hence

$$\begin{aligned}
\alpha + MN &\leq \|T(f + Mf_1, g + Ng_1)\| \\
&= \|f + Mf_1\| \|g + Ng_1\| \\
&\leq (1 + Ms)(1 + Ns') \\
&= 1 + Ns + Ms' + MNss',
\end{aligned}$$

but it is apparent that we can choose M and N in order to have

$$1 + Ns + Ms' + MNss' < \alpha + MN,$$

which is a contradiction. □

Lemma 3.4 *If (x,y) and (x',y') belong to $\partial A \times \partial B$ and are distinct, then $I_{x,y} \cap I_{x',y'} = \emptyset$.*

Proof. Assume, contrary to what we claim, that there exists $z \in I_{x,y} \cap I_{x',y'}$. Let us suppose, without loss of generality, that $x \neq x'$.

- If $y \neq y'$, then we can choose $f \in C_x$ and $g \in C_y$ with $f(x') = g(y') = 0$. Consequently, $|T(f,g)(z)| = 1$, but, by Lemma 3.3, $|T(f,g)(z)| = 0$, which is a contradiction.
- If $y = y'$, then we can choose $f \in C_x$ and $g \in C_y$ with $f(x') = 0$. Consequently, $|T(f,g)(z)| = 1$, but, by Lemma 3.2, $|T(f,g)(z)| = 0$, which is a contradiction. □

Remark 3.5 The following result can be found in [1]:

Let A be a strongly separating linear subspace of $C_0(X)$ and assume that $\partial A \neq \emptyset$. If $S : A \longrightarrow C_0(Y)$ is a linear isometry, then there exists a subset of Y, $Y_0 := \bigcup_{x \in \partial A} I_x$, such that $(Sf)(y) = a(y) f(h(y))$ where $h : Y_0 \longrightarrow \partial A$ is a continuous surjective function and $a(y) = (Tg)(y)$ for any $g \in A$ such that $1 = \|g\| = g(h(y))$.

We are now ready to prove our main result:

Theorem 3.6 *Let A and B be strongly separating linear subspaces of $C_0(X)$ and $C_0(Y)$ respectively and assume that $\partial A \neq \emptyset$ and $\partial B \neq \emptyset$. Let $T : A \times B \longrightarrow C_0(Z)$ be a bilinear isometry. Then there exist a nonempty Z_0 of Z, a surjective continuous mapping $h : Z_0 \longrightarrow \partial A \times \partial B$ and a norm-one continuous function $a : Z_0 \longrightarrow \mathbb{K}$ such that $T(f,g)(z) = a(z) f(\pi_x(h(z)) g(\pi_y(h(z))$ for all $z \in Z_0$ and every pair $(f,g) \in A \times B$.*

Proof. Let us first define a subset Z_0 of Z as follows:

$$Z_0 := \bigcup_{(x,y)\in \partial A \times \partial B} I_{x,y}.$$

Fix $(x,y) \in \partial A \times \partial B$ and $z \in I_{x,y}$. Let us suppose that $f_1 \in C_x$ and $g_1 \in C_y$. Fix $g' \in C_y$. Then we can define the following isometries:

$$S(f) := T(f, g'),$$

$$R(h) := T(f_1, h), \quad \text{where} \quad (f,h) \in A \times B.$$

Then, by Remark 3.5,

$$\begin{aligned} T(f,g')(z) &= (Sf)(z) \\ &= S(f_1)(z) f(x) \\ &= T(f_1, g')(z) f(x) \\ &= R(g')(z) f(x) \\ &= R(g_1)(z) g'(y) f(x) \\ &= T(f_1, g_1)(z) f(x) g'(y) \\ &= a(z) f(x) g'(y). \end{aligned}$$

As $(f - f(x)f_1)(x) = 0$ and $(g - g(y)g_1)(y) = 0$, for any $(f,g) \in A \times B$ we infer, by Lemma 3.3, that

$$\begin{aligned} 0 &= T(f - f(x)f_1, g - g(y)g_1)(z) \\ &= T(f,g)(z) - f(x)T(f_1,g)(z) - g(y)T(f,g_1)(z) + f(x)g(y)T(f_1,g_1)(z) \\ &= T(f,g)(z) - f(x)a(z)f_1(x)g(y) - g(y)a(z)f(x)g_1(y) + f(x)g(y)a(z). \end{aligned}$$

Hence

$$T(f,g)(z) = a(z) f(x) g(y).$$

Let us next define a mapping $h : Z_0 \longrightarrow \partial A \times \partial B$ as $h(z) := (x,y)$ where $z \in I_{x,y}$. We claim that h is continuous. To this end, fix $z_0 \in Z_0$ and let $h(z_0) = (x_0, y_0)$. Let U be a neighborhood of x_0 and choose $f \in A$ such that $1 = \|f\| = |f(x_0)|$ and $|f| < 1$ off U. Let $s(x_0) = \sup_{x \in X \setminus U} |f(x)| = \sup_{x \in X \cup \{\infty\} \setminus U} |f(x)|$. It is apparent that $s(x_0) < 1$. Similarly, let V be a neighborhood of y_0 and choose $g \in B$ such that $1 = \|g\| = |g(y_0)|$ and $|g| < 1$ off V. Let $s(y_0) = \sup_{y \in Y \setminus U} |g(y)| = \sup_{y \in Y \cup \{\infty\} \setminus U} |g(y)|$. As above, $s(y_0) < 1$.

Since $h(z_0) = (x_0, y_0)$, then $|T(f,g)(z_0)| = \|T(f,g)\| = 1$. Let $s := \max\{s(x_0), s(y_0)\}$ and define the following open neighborhood of z_0:

$$W := \{z \in Z_0 : |T(f,g)(z_0)| > s\}.$$

Fix $z \in W$ and suppose that $h(z) := (x,y)$. Then, by the above weighted composition representation of T,

$$s < |T(f,g)(z)| = |f(x)||g(y)|,$$

and, consequently, $|f(x)| > s \geq s(x_0)$ and $|g(y)| > s \geq s(y_0)$. This yields $x \in U$ and $y \in V$, which is to say that $h(W) \subseteq U \times V$ and the proof is done. □

Remark 3.7 The set of generalized peak points, ∂A, for a separating linear subspace, A, of $C_0(X)$ could be empty (see, e.g., [4] or [3]). However, this is not the case for a wide range of subspaces of $C_0(X)$ including, for example, extremely regular, or more generally, complete regular subspaces of $C_0(X)$ and, above all, function algebras.

Corollary 3.8 *Let A and B be completely regular subspaces of $C_0(X)$ and $C_0(Y)$ respectively. Let $T : A \times B \longrightarrow C_0(Z)$ be a bilinear isometry. Then there exist a nonempty Z_0 of Z, a surjective continuous mapping $h : Z_0 \longrightarrow X \times Y$ and a norm-one continuous function $a : Z_0 \longrightarrow \mathbb{K}$ such that $T(f,g)(z) = a(z)f(\pi_x(h(z))g(\pi_y(h(z))$ for all $z \in Z_0$ and every pair $(f,g) \in A \times B$.*

Proof. It is a straightforward consequence of Theorem 3.6 since the set of generalized peak points of a completely regular subspace of $C_0(X)$ coincides with X ([2]). □

Corollary 3.9 *Let A and B be closed separating subalgebras of $C_0(X)$ and $C_0(Y)$ respectively, which is to say, non-unital function algebras. Let $T : A \times B \longrightarrow C_0(Z)$ be a bilinear isometry. Then there exist a nonempty Z_0 of Z, a surjective continuous mapping $h : Z_0 \longrightarrow Ch(A) \times Ch(B)$ and a norm-one continuous function $a : Z_0 \longrightarrow \mathbb{K}$ such that $T(f,g)(z) = a(z)f(\pi_x(h(z))g(\pi_y(h(z))$ for all $z \in Z_0$ and every pair $(f,g) \in A \times B$.*

Proof. By Theorem 6.1 in [1], we know that A is a strongly separating subspace of $C_0(X)$. Furthermore, by Theorem 2.1 in [7], ∂A coincides with the Choquet boundary for A, which is to say that ∂A is a nonempty boundary for A. Hence the proof of this corollary is again a straightforward consequence of Theorem 3.6. □

Acknowledgements Research partially supported by Spanish Ministery of Science and Technology (MTM2008-04599) and Bancaixa (P1-1B2008-26).

References

[1] J. Araujo and J. J. Font, Linear isometries between subspaces of continuous functions, Trans. Amer. Math. Soc. **349**, No. 1, 413–428 (1997).
[2] B. Cengiz, On extremely regular function spaces, Pacific J. Math. **49**, 335–338 (1973).
[3] T. Cho, On the Choquet boundary for nonclosed subspaces of $C(S)$, Pacific J. Math. **35**, 575–580 (1970).
[4] H. G. Dales, Boundaries and peak points for Banach function algebras, Proc. London Math. Soc. **22**, 121–136 (1971).
[5] H. Holsztyński, Continuous mappings induced by isometries of spaces of continuous functions, Studia Math. **26**, 133–136 (1966).
[6] A. Moreno and A. Rodríguez, A bilinear version of Holsztyński's theorem on isometries of $C(X)$-spaces, Studia Math. **166**, 83–91 (2005).
[7] N. V. Rao and A. K. Roy, Multiplicatively spectrum-preserving maps of function algebras. II, Proc. Edinb. Math. Soc. (2) **48**, 219–229 (2005).

