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Accurate Analytical and Statistical Approaches to Reduce *O-C* Discrepancies in the Precessional Parameters

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ABSTRACT. The *Hipparcos* catalog provides a reference frame at optical wavelengths for the new International Celestial Reference System (ICRS). This new reference system was adopted following the resolution agreed at the 23rd IAU General Assembly held in Kyoto in 1997. Differences in the *Hipparcos* system of proper motions and the previous materialization of the reference frame, the FK5, are expected to be caused only by the combined effects of the motion of the equinox of the FK5 and the precession of the equator and the ecliptic. Several authors have pointed out an inconsistency between the differences in proper motion of the *Hipparcos*-FK5 and the correction of the precessional values derived from VLBI and lunar laser ranging (LLR) observations. Most of them have claimed that these discrepancies are due to slightly biased proper motions in the FK5 catalog. The different mathematical models that have been employed to explain these errors have not fully accounted for the discrepancies in the correction of the precessional parameters. Our goal here is to offer an explanation for this fact. We propose the use of independent parametric and nonparametric models. The introduction of a nonparametric model, combined with the inner product in the square integrable functions over the unitary sphere, would give us values which do not depend on the possible interdependencies existing in the data set. The evidence shows that zonal studies are needed. This would lead us to introduce a local nonparametric model. All these models will provide independent corrections to the precessional values, which could then be compared in order to study the reliability in each case. Finally, we obtain values for the precession corrections that are very consistent with those that are currently adopted.

1. INTRODUCTION

The *Hipparcos* catalog provides a reference frame at optical wavelengths for the new International Celestial Reference System (ICRS), with the accurate positions and proper motions of more than 118,000 stars (around 1 mas and 1 $\mathrm{mas\,yr^{-1}}$; ESA 1997). The decision to adopt this new reference system was taken at the IAU General Assembly held in Kyoto in 1997. Differences between the Hipparcos system of proper motions and the previous version of the reference frame, the FK5, are expected to be caused only by the combined effects of the motion of the equinox of the FK5 and the precession of the equator and the ecliptic. The IAU has recommended that the terms "precession of the equator" and "precession of the ecliptic" should be used instead of "luni-solar precession" and "planetary precession," respectively (Hilton et al. 2006), and we shall therefore use this new notation throughout this article. Previous authors, such as Walter & Hering (2005) and Zhu (2000), have used the notation Δp to denote the correction in the precession of the equator (then called the luni-solar precession), and Fricke (1977) used Δp_1, but in order to follow the notation used in the IAU 2006 resolutions, we refer to these corrections as $\Delta\psi_A$.

The usual way to represent the relationships between systems of proper motions from two catalogs mathematically is

$$\begin{aligned}\Delta\mu_\alpha\cos\delta &= -\omega_x\cos\alpha\sin\delta - \omega_y\sin\alpha\sin\delta + \omega_z\cos\delta,\\ \Delta\mu_\delta &= \omega_x\sin\alpha - \omega_y\cos\alpha,\end{aligned} \tag{1}$$

where α and δ represent the equatorial coordinates right ascension and declination, respectively, and $\Delta\mu_\alpha$, $\Delta\mu_\delta$ are corrections for the proper motions in right ascension and declination. The spins ω_x, ω_y, ω_z are related to the values of the differences in the precessional parameters. Several authors have pointed out an inconsistency in the differences in the proper motion of the *Hipparcos*-FK5 with the value corresponding to the correction in the precession of the equator and the ecliptic rate, as derived from VLBI and lunar laser ranging (LLR) observations. Among these authors, for example, Walter & Hering (2005) and Zhu (2000, 2006) provided different physical-mathematical models that were not altogether successful. They suggested that one of the reasons for these discrepancies could be the internal bias of the proper motion system of the FK5.

This is why we consider that any physical interpretation of a set of discrete data requires appropriate mathematical treatment in order to obtain parametric values that may have a physical meaning. Regardless of the physical sources of these inconsistencies, the data obtained through statistical procedures must be interpreted a posteriori and, thus, we have to be sure that the model and the methodology used do not generate errors in the final results (namely, the corrections for the precessional parameters). The model in equation (1) is unbiased and so it may give erroneous results when applied to a biased set of data (Marco et al. 2004). Our aim is to obtain the corrections to the precessional parameters using a model that does not depend on the correlation between the residuals of the random variables (namely $\Delta\mu_\alpha \cos\delta$, $\Delta\mu_\delta$).

Because these data come from the *Hipparcos*-FK5 comparison, they are discrete; to obtain global results such as the above-mentioned corrections to the precessional parameters, the residuals should be properly defined over the whole sphere. To achieve this, it is necessary to use a suitable method and we can choose between two options to perform the adjustments: one parametric and one nonparametric. Parametric and nonparametric models could be used separately for each random variable and they are clearly independent. A comparison of their results could be used as a measure of their reliability. The parametric models imply an a priori interpretation of the structure of the residuals, which may be incompatible with the statistical properties of the random variables that they represent ($\Delta\mu_\alpha \cos\delta$, $\Delta\mu_\delta$). These properties should be preserved when we expand them to cover the whole sphere. In particular, the existence of a bias is a traceback to the use of an unbiased geometrical model, such as the one given in equation (1). This problem disappears if we use independent analytical adjustments, some of which will be outlined later.

The kernel nonparametric models, henceforth KNP (Simonoff 1996; Wand & Jones 1995), do not make any assumptions about the dependence of the random variable residuals (nevertheless, it is possible to prove that they have a relationship of dependence and it makes sense to suppose the existence of a spin), in contrast to the case of geometrical models.

Parametric and nonparametric models can be used to compute the induced ω_y, ω_z values independently by means of a usual, but discretized, inner product in the functional space of the square integrable functions on the unitary sphere [$L^2(S^2)$]. These values do not depend on whether bias is considered or not, and their correct determination is critical in order to obtain a $\Delta\psi_A$ correction that is mathematically coherent with the data and the model used. We should also highlight the precision of the results that were obtained and the fact that we worked at two different levels: the statistical and the numerical. We have already stated that the distribution of the data on the sphere is practically homogeneous, and so the orthonormality functional properties with respect to the norm of $L^2(S^2)$ are well preserved for discretization and consequently for the orthonormality of the discretized space. The numerical methods that we will employ throughout this work (integration and numerical adjustment on the sphere) do not produce significant errors, because they only affect the second decimal place. A possible source of errors may be the procedure used for nonparametric adjustment. We took the values h_α, $h_{\sin\delta}$ (Marco et al. 2004) for the bandwidths for each random variable in equation (3) in order to minimize the asymptotic value of the mean squared error, hereafter AMISE, over the whole sphere. See the Appendix and Wand & Jones (1995) for more details on these topics. These values are a function of the variance-covariance matrix of the random variable, the size of the sample and the geometrical dimension of the random vector. The value of the AMISE is approximately 0.01 in the units employed in the study.

Our conclusions agree with those of other authors, such as Mignard & Froeschlé (2000), in the sense that some of the discrepancies found when comparing the obtained and the observed values of precession are due to zonal errors. This fact should be confirmed by means of a more specific model. The use of a local KNP, henceforth KNP$_L$, will make it possible to conduct such a study. The KNP and KNP$_L$ models will be briefly explained in the next section, while in a later section we will perform approximations to the ω_y and ω_z values (which are independent of the method) to show their stability. As the same authors indicate in their papers (Mignard & Froeschlé 2000), the explanations that have been proposed for the numerical discrepancies are not sufficient. Our explanations do account for a rather more significant part of the numerical values. The numerical results will be listed and we will see that our method explains a high percentage of the precessional values as derived from VLBI and LLR observations.

2. THE MATHEMATICAL MODELS

We have a set of discrete data on the sphere and we want to obtain some functions that represent these data on the whole sphere. In this section, we will define and apply methods that allow us to obtain such functions, taking into account that the statistical properties of the discrete sample should be preserved.

In Section 3 we will describe the next step: analysis of the data set in order to ensure that the data are spatially distributed over the sphere in a sufficiently homogeneous way, and that they are distributed as normal random variables with a nonnull mean. The first property (spatial homogeneity) is not necessary, but it is desirable, since it ensures that the results obtained from a discrete and a continuous least squares method can be compared.

2.1. The Global and Local KNP Models over the Sphere

Nonparametric adjustments compute the mathematical expectation of a certain random variable conditioned by another one. For example, if X is the random variable (in our case,

$\Delta\mu_\alpha \cos\delta$ or $\Delta\mu_\delta$), the method consists in finding

$$m_X(\alpha,\delta) = E[X|(\alpha,\delta)] = \int_D x f(x|\alpha,\delta)dx = \int_D x \frac{f(x,\alpha,\delta)}{f_{(\alpha,\delta)}(\alpha,\delta)}dx, \quad (2)$$

where $E[X|(\alpha,\delta)]$ is the mathematical expectation of a random variable X conditioned by the value of the (α,δ) (homogeneous random variable position); D is the range of X; $f(x,\alpha,\delta)$ is the joint density function of the three variables; $f_{(\alpha,\delta)}(\alpha,\delta)$ is the marginal density; and we have the formula $f(x|(\alpha,\delta))$ $f_{(\alpha,\delta)}(\alpha,\delta) = f(x,\alpha,\delta)$ (Wand & Jones 1995). All of them might be unknown so they will have to be approximated in some way. This can be achieved with an estimator (Simonoff 1996) using a kernel K which fulfills the properties: $K \geq 0$, $\int K dx = 1$ and $\int xK dx = 0$. The expression of the estimator for the joint density is

$$\hat{f}(x,\alpha,\delta) = \frac{1}{nh_x h_\alpha h_{\sin\delta}} \times \sum_{i=1}^{n} K_x\left(\frac{x-x_i}{h_x}\right) K_\alpha\left(\frac{\alpha-\alpha_i}{h_\alpha}\right) K_\delta\left(\frac{\sin\delta - \sin\delta_i}{h_{\sin\delta}}\right). \quad (3)$$

Here, n is the size of the sample, the h-values are "discretizations" (properly named bandwidth) of X, α and δ; K refers to the Epanechnikov kernel (Simonoff 1996):

$$K(x) = \begin{cases} \frac{3}{4}(1-x^2) & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}, \quad (4)$$

with the usual condition for the density:

$$\frac{1}{4\pi\mu(D)} \int_D \int_{S^2} \hat{f}(x,\alpha,\delta)\cos\delta dx d\alpha d\delta = 1, \quad (5)$$

where $\mu(D)$ is the measure of D. Analogous conditions are given for the marginal density. If we apply equation (2) with the approximation $\hat{f} \simeq f$ (and similarly for the marginal density), consider equation (3) and apply the kernel properties, we obtain an expression like that of Nadaraya-Watson (Simonoff 1996), but for the sphere:

$$m_X(\alpha,\delta) = \sum_{i=1}^{n} w_i x_i, \quad w_i = \frac{K_\alpha(\frac{\alpha-\alpha_i}{h_\alpha})K_\delta(\frac{\sin\delta-\sin\delta_i}{h_{\sin\delta}})}{\sum_{j=1}^{n} K_\alpha(\frac{\alpha-\alpha_j}{h_\alpha})K_\delta(\frac{\sin\delta-\sin\delta_j}{h_{\sin\delta}})}. \quad (6)$$

This expression is independent of the kernel used. We have taken the kernel given in equation (4) because it fulfills the minimum value for the AMISE. A local study can be useful when there are important discrepancies among the statistical parameters of the variables determined by their zonal position. Local polynomial estimations are based on finding the solution to a natural weighted least-squares problem (Simonoff 1996):

$$\min_{\beta_j} \sum_{i=1}^{n} [y_i - \beta_0 - \beta_1(x-x_i) - \ldots - \beta_p(x-x_i)^p]^2 \times K\left(\frac{x-x_i}{h}\right), \quad (7)$$

n being the number of points considered and p the desired degree of the polynomial . Let M_x be the design matrix:

$$M_x = \begin{bmatrix} 1 & x-x_1 & & (x-x_1)^p \\ \vdots & \vdots & \ldots & \vdots \\ 1 & x-x_n & & (x-x_n)^p \end{bmatrix}, \quad (8)$$

and let W_x be the weighted matrix:

$$W_x = h^{-1}\mathrm{diag}\left[K\left(\frac{x-x_1}{h}\right), \ldots, K\left(\frac{x-x_n}{h}\right)\right]. \quad (9)$$

Then, if $M_x^t W_x M_x$ (t denotes transposition) is invertible, we obtain

$$\hat{\beta} = (M_x^t W_x M_x)^{-1} M_x^t W_x y, \quad (10)$$

and the estimator $\hat{m}_p$ for the desired random variable is given by

$$\hat{m}_p(x) = e_1^t (M_x^t W_x M_x)^{-1} M_x^t W_x y, \quad (11)$$

e_r being a $(p+1) \times 1$ vector having a value of 1 in the rth entry and zero elsewhere. We can see that the case for $p = 0$ is the KNP model.

2.2. *SH*$_2$ and the Geometrical Adjustment

A surface harmonic spherical development of order n (SH_n henceforth) is based on the hypothesis that the developed function has an integrable square on the sphere. The coefficients are found using precise formulae due to the functional orthogonality of these harmonics. The computation of the coefficients $\{c_j, j \geq 0\}$, for a given function f, and a truncation of order n ($\{Y_j, j \geq 0\}$ being the harmonic functions), enables us to verify the property of minimizing the integral:

$$\int_{S^2} \left[f(\alpha,\delta) - \sum_{j=0}^{n^2+n} c_j Y_j(\alpha,\delta)\right]^2 d\sigma, \quad (12)$$

where $d\sigma$ is the area element in the spherical domain. Considered the vector field on the unitary sphere given by $[(\Delta\alpha)\cos\delta, \Delta\delta]^t$, we know that, under certain regularity hypotheses, it is possible to obtain a development depending on the vectorial harmonic spherical functions given in Morse & Feshbach (1953):

$$\boldsymbol{R}_{n,m} = \frac{\boldsymbol{r}}{r} Y_{n,m}, \qquad \boldsymbol{S}_{n,m} = r\nabla \boldsymbol{Y}_{n,m},$$
$$\boldsymbol{T}_{n,m} = -\boldsymbol{r} \times \nabla \boldsymbol{Y}_{n,m}, \tag{13}$$

where $Y_{n,m}$ $(n \geq 0, -n \leq m \leq n)$ are the usual surface spherical harmonics. Here, we use X to denote the positional vector in Cartesian coordinates (and not a random variable). If we consider the truncated development in the following way:

$$\boldsymbol{\Delta X} = \sum_{j=-1}^{1} [c_{1,j}\boldsymbol{R}_{1,j} + d_{1,j}\boldsymbol{S}_{1,j} + e_{1,j}\boldsymbol{T}_{1,j}], \tag{14}$$

where $\boldsymbol{\Delta X} = \boldsymbol{X}(r + \Delta r, \alpha + \Delta\alpha, \delta + \Delta\delta) - \boldsymbol{X}(r, \alpha, \delta)$ is approximated in the first order, compatibility of the system must be accomplished, and this implies the functional relationships:

$$\Delta\alpha \cos\delta = [e_{1,0} \cos\delta - e_{1,1} \sin\alpha \sin\delta - e_{1,-1} \cos\alpha \sin\delta] + [d_{1,1} \cos\alpha - d_{1,-1} \sin\alpha], \tag{15}$$

$$\Delta\delta = [e_{1,-1} \sin\alpha - e_{1,1} \cos\alpha] + [-d_{1,1} \sin\alpha \sin\delta - d_{1,-1} \cos\alpha \sin\delta + d_{1,0} \cos\delta]. \tag{16}$$

In first order of time, all the parameters in equations (15) and (16) and the functions $\Delta\alpha \cos\delta$, $\Delta\delta$ have two components. The formula at the initial time is usually employed for correcting rotation and deformation in position. The velocity components stand for proper motions. It is necessary that for the proper motion components and coefficients in equations (15) and (16) $e_{1,-1} = \omega_x$, $e_{1,1} = \omega_y$, $e_{1,0} = \omega_z$, where ω_x, ω_y, and ω_z represent the components of the *Hipparcos*-FK5 spin. This model generalizes the previous geometrical one, but it does not eliminate the bias (as already pointed out with regard to equation [1]). If there is some bias in the data, the conclusions reached by applying them could be wrong. This model supposes the existence of a direct relationship between the errors in the right ascension and declination proper motions. In contrast, the SH_2 model makes no a priori assumption about the correlation of the random variables because it takes an independent development for each variable.

3. THE DATA SET: SPATIAL DISTRIBUTION AND THE RANDOM VARIABLES FOR THE PROPER MOTION

We compare the *Hipparcos* and FK5 proper motions, taking the date of our study as 1991.25 and selecting the stars that fulfill the following two conditions (Schwan 2001):

$$\sqrt{(\Delta\alpha \cos\delta)^2 + (\Delta\delta)^2} \leq 2'',$$
$$\sqrt{(\Delta\mu_\alpha \cos\delta)^2 + (\Delta\mu_\delta)^2} \leq 10\,\mathrm{mas\,yr^{-1}}. \tag{17}$$

A total of 1327 stars are considered in our calculations and we need to consider the statistical distribution of the random variables $\Delta\mu_\alpha \cos\delta$, $\Delta\mu_\delta$. We compute the arithmetical mean and the standard deviation of the variables, and also the expectations and the standard deviation σ, as seen in Table 1, with

$$E[X] = \int_D x f_X(x) dx \qquad \sigma^2 = Var(X) = E[X^2] - (E[X])^2 \tag{18}$$

and

$$f_X(x) = \frac{1}{n_s h} \sum_{i=1}^{n_s} K\left(\frac{x - x_i}{h}\right), \tag{19}$$

where K given in equation (4) and n_s is the number of stars. $\Delta\mu_\alpha \cos\delta$, $\Delta\mu_\delta$ are distributed as normal random variables with a nonnull mean. The function approximated by means of kernel regression has no exact expression (the integrals were approximated using a suitable numerical integration formula). Contour levels are shown in Figures 1 and 2 for the proper motions in

TABLE 1

MEANS AND STANDARD DEVIATIONS (RESULTS IN $\mathrm{mas\,yr^{-1}}$)

	μ	σ	$E[]$	$\sqrt{Var()}$
$\Delta\mu_\alpha \cos\delta$	−0.69	2.27	−0.67	1.10
$\Delta\mu_\delta$	0.21	2.26	0.29	1.09

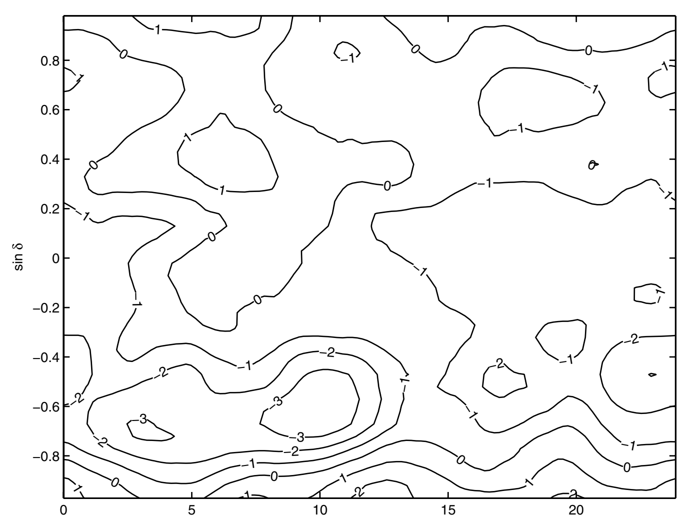

FIG. 1.—Differences *Hipparcos*-FK5 in $\Delta\mu_\alpha \cos\delta$ inducted by the KNP model ($\mathrm{mas\,yr^{-1}}$).

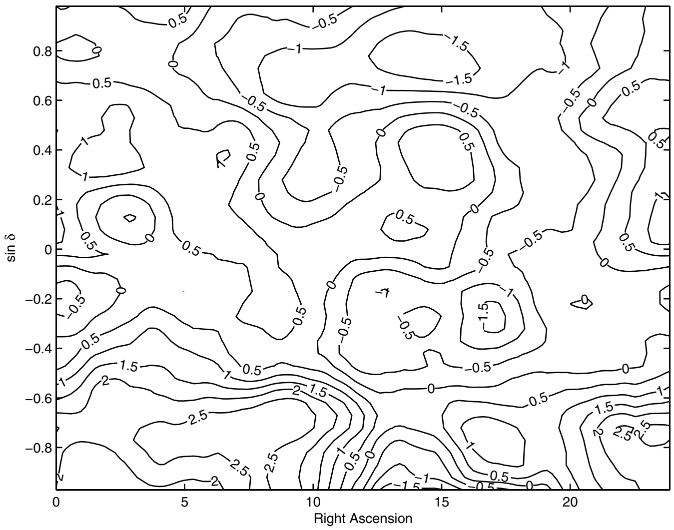

FIG. 2.—Differences *Hipparcos*-FK5 in $\Delta\mu_\delta$ inducted by the KNP model (mas yr^{-1}).

right ascension and declination, respectively, where we took $h_\alpha = 0.55$, $h_{\sin\delta} = 0.17$ as the optimal values that minimize AMISE (Simonoff 1996).

A statistical study of bands of declination zones for the proper motion is also interesting, since it allows us to consider the need for a zonal study of the declination bands. We have taken into account declination zones 15° from the equator. The statistical results are listed in Tables 2 (northern declinations) and 3 (southern declinations).

The distribution of the errors in right ascension and declination in 15° declination bands is not homogeneous, in the sense that the mean is not the same, and in some cases they do not even distribute normally.

4. CORRECTIONS TO PRECESSION FROM HIPPARCOS AND FK5 PROPER MOTIONS

Supposing that the *Hipparcos* and FK5 systems were rigid frames, the two proper motion systems would be connected using equation (1). The rotational spins ω_x, ω_y and ω_z of the FK5 system with respect to the *Hipparcos* system determine the difference in the rate of precession $\Delta\psi_A$. But the precession corrections determined from comparing the *Hipparcos* and FK5 proper motion systems show a significant discrepancy compared to the observational VLBI and LLR-determined values, $\Delta\psi_A = -3$ mas/yr and $\Delta e = -1.2$ mas/yr ($\Delta\psi_A$ is the difference in the precession of the equator rate and Δe is the fictitious motion of the equinox; Miyamoto & Soma 1993). Several authors have performed studies in order to justify these discrepancies. For example, Walter & Hering (2005) studied the *Hipparcos* proper motion system and considered that, after deducting the influence of the Oort constant B, the remaining rigid rotation component was the noninertial rotation of the *Hipparcos* system, with a value of -0.71 mas/yr. After comparing the corrected *Hipparcos* and the FK5 proper motions these authors found that they were still not coherent with the VLBI observed precession value, and concluded that an extra correction to the FK5 proper motion system was necessary. More recently, Zhu (2007) analyzed the systematic errors of PPM and ACRS (both catalogs in the FK5 system) proper motions by comparing them with the *Hipparcos* proper motions, and concluded that the existence of internal problems in the proper motions system of the FK5 were the main reason why it is not possible to use the precession correction to connect the two proper motion systems. The angular rates of rotation ω_x, ω_y and ω_z allow us to obtain information about the correction to the precession of the equator according to the following relationships (Fricke 1977):

$$\omega_x = 0, \tag{20}$$

$$\omega_y = -\Delta\psi_A \sin\varepsilon, \tag{21}$$

$$\omega_z = \Delta\psi_A \cos\varepsilon - \Delta e, \tag{22}$$

ε being the obliquity of the ecliptic, $\varepsilon = 23.4392911111°$ (J2000.0). In equation (22) we have disregarded the correction to the precession of the ecliptic because it has been proved to be negligibly small. Since the spin values represent a rigid rotation over the whole sphere, the limited number of 1327 points might provide unstable results in the numerical process (nevertheless, as we shall see, a posteriori these inaccuracies are not present, probably due to the good spatial and statistical properties of the data). We have generated a homogeneous network of points over the whole sphere, thus making it possible to combine

TABLE 2

STATISTICAL RESULTS FOR THE BANDS OF 15° IN DECLINATION (IN MAS)

δ	N° Stars	μ_α	σ_α	μ_δ	σ_δ
0° 15° ……	160	−1.11	2.42	0.05	2.16
15° 30° …..	172	0.20	1.90	0.40	1.94
30° 45° …..	147	−0.58	1.69	−0.17	2.00
45° 60° …..	107	−0.56	1.77	−1.00	1.60
60° 75° …..	80	0.42	1.31	−0.21	1.47

TABLE 3

STATISTICAL RESULTS FOR THE BANDS OF 15° OF DECLINATION (IN MAS)

δ	N° Stars	μ_α	σ_α	μ_δ	σ_δ
−15° 0° ……..	181	−0.75	2.11	−0.13	1.97
−30° −45° …..	168	−1.47	2.66	−0.15	2.26
−45° −30° …..	133	−1.96	2.48	1.35	2.90
−60° −45° …..	101	−0.95	2.42	1.92	2.46
−75° −60° …..	56	1.17	1.84	0.60	2.27

statistical and numerical methods in order to test the results. We then separately applied a SH_2 (analytical model), a KNP, and a KNP$_L$ (statistical models). It is not possible to obtain the corrections to the precessional parameters directly from these values and equations (20), (21) and (22). Instead, we need to obtain the "induced" values for these angles. Once we have defined the residuals over the whole sphere, it is possible to check the accuracy of the ω_y and ω_z values, taking into account the following points:

1. The $\Delta\psi_A$, Δe values should be obtained from ω_y and ω_z.
2. The latter values are obtained using the inner product of an adjustment function (SH_2, KNP, KNP$_L$) with the suitable orthogonal function.
3. SH_2 is an analytical model, while KNP and KNP$_L$ are statistical models. The ω_y and ω_z values are obtained by means of a numerical discretization of the functional least squares problem.

Since the models applied are independent of any other adjustment that may be used, each particular procedure will be as follows:

1. By using KNP and $\Phi_y = -\sin\alpha\sin\delta$ and taking into account that $(\mathrm{KNP}, Y_{21}) = \omega_y(\Phi_y, Y_{21})$ and $(\mathrm{KNP}, Y_{20}) = \omega_z$ $(\cos\delta, Y_{20})$ (see Marco et al. 2004), where parentheses represent the usual inner product and $Y_{21} = 3\sin(\alpha)\sin(\delta)\cos(\delta)$, $Y_{20} = 3/2\sin(\delta)^2 - 1/2$ are spherical surface harmonics, which are elements of the functional basis of $L^2(S^2)$ functions. Applies analogously to the KNP$_L$ model.
2. By using SH_2 (SH_2 denotes the development in equation (12) in spherical harmonics truncated at the second order) and Φ_y, and taking into account that $c_{21}(Y_{21}, Y_{21}) = \omega_y(\Phi_y, Y_{21})$, $c_{20}(Y_{20}, Y_{20}) = \omega_z(\cos\delta, Y_{20})$ (see Marco et al. 2004), where c_{21} and c_{20} represent the coefficients for the Y_{21} and Y_{20} functions in development (12), respectively.

From the ω_y and ω_z values, we can obtain $\Delta\psi_A$ using equation (21). In this case, computing Δe using equation (22) is not recommended for the following reasons:

1. There is a bias and this bias is not properly a spin, (see Fricke 1977). This is related to the next point, together with the way ω_x, ω_y and ω_z are obtained.
2. There are three correlated variables at the equator: Δe, ω_z, and the RA proper motion bias ($\overline{\Delta\mu_\alpha\cos\delta}$).

By direct application of the definition of bias of a random variable,

$$\frac{1}{4\pi}\int_{S^2}\Delta\mu_\alpha\cos\delta d\sigma = \overline{\Delta\mu_\alpha\cos\delta}. \qquad (23)$$

The correlation among the three parameters Δe, ω_z and the RA proper motion bias is true for the whole sphere, but it is particularly interesting to consider it at the equator, where we have $\delta = 0$, and the resulting relationships are $\Delta e = \overline{\Delta\mu_\alpha\cos\delta} - \omega_z$.

TABLE 4

Parameter Values for the Correction of the Precession (in mas yr^{-1}), Explained in Text

	ω_y	ω_z	$\Delta\psi_A$	Δe
Walter(Nom) ……	1.2	−1.5	−3	−1.2
KNP ……………	1.04	1.73	−2.61	−1.06
KNP$_L$ …………	0.98	1.83	−2.45	−1.16
SH_2 ……………	1.03	1.69	−2.58	−1.02
MF ……………	0.60	0.70	−1.51	−2.10
Walter …………	0.65	0.77	−1.66	−2.30

To compute Δe, the only unknown value is the bias, which provides the value $\overline{\Delta\mu_\alpha\cos\delta} = 0.67$ mas/yr The resulting values are listed in Table 4, where they can also be compared with those obtained by other authors. The values denoted as Walter(nom) are the nominal values as given by Walter & Hering (2005), corresponding to the accepted parameters of precession from the P03 precession theory of Capitaine et al. (2003).

We can see that the results obtained when applying SH_2, KNP and KNP$_L$ are in strong agreement with the precession correction derived from VLBI and LLR observations, and they are also very close to the optimum value.

5. CONCLUSIONS

1. We have a discrete set of data with some specific characteristics and we search for a generalization of these data over the whole sphere which preserves these characteristics. Some of them refer to the spatial distribution, which may be homogeneous or not, while others refer to statistical properties (in particular, mean and variance) and kind of distribution.
2. The functional and statistical adjustment must be coherent. There are several possible methods of analysis that result in different models.
3. Spherical harmonics are used in the SH_2 model, and as they include the constant function in the functional basis, their use could result in a bias for a function (we are now considering RA and DEC proper motions separately). The models coming from developments in vectorial harmonics (with only rotations or rotations plus deformation) do not admit bias.
4. The Gauss-Markov theorem states that the minimum-quadratic estimator is the best among those that appear when the residuals are distributed normally with null mean and variance σ^2. So, the application of a mathematical model to a particular case in which these hypotheses are not verified cannot be justified at all. If the data are not unbiased, then the model cannot be either. Suitable mathematical treatments are the developments in spherical harmonics (see point 3 in this list) and the nonparametric models.
5. The values for the correction of precession obtained from the global and local KNP model are in very good agreement with the optimum values as obtained from VLBI and LLR. With respect to the precision of the results obtained in the study, there are two possible sources of error: The KNP approximation; in

this case, we must take into account (Marco 2004) that the values h_α, $h_{\sin\delta}$ were taken in such a way that they minimize the AMISE. These values are a function of the variance-covariance matrix of the random variable, the size of the sample, and the geometrical dimension of the random vector. In this case, the value of the AMISE is approximately 0.01 in the units of the work. Secondly, there is the question of discretization. The reliability of the results that were obtained is based on the existence of the $L^2(S^2)$ inner product, where a numerical method of integration compatible with the desired precision was used. These two sources of error are not additive. In conclusion, both precisions are coherent with the parameters that we want to evaluate.

6. The given (statistical-analytical-numerical) procedure is linear, in the sense that if we suppose the $\Delta\mu_\alpha \cos\delta$, $\Delta\mu_\delta$ discrepancies are due to a particular cause j, the computation of the corresponding ω_{1j}, ω_{2j}, ω_{3j} contributions to ω_1, ω_2 and ω_3 would be computed following the same procedure as the one we have applied globally.

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APPENDIX

SOME REMARKS ABOUT AMISE

Let X be a random variable with density function $f(x)$. The mathematical expectation is computed by means of the expression $E[X] = \int_{-\infty}^{+\infty} x f(x) dx$ and it is an obvious generalization of the concept *arithmetic mean*.

If we want to approach a density function by means of an estimator $\hat{f}$ and with a bandwidth h, we can obtain a measure of this approach in the L^2 norm. Thus, we get the ISE (integrated squared error): $ISE = \int_{-\infty}^{+\infty} \{f(x) - \hat{f}(x;h)\}^2 dx$. Usually, the estimator depends on a sample, so this quantity becomes another random variable. Its mathematical expectation is the MISE value but it is not suitable to practical purposes. This value depends on the kind of approximation $\hat{f}$, the size of the sample n, and the bandwidth selected value h. It is possible to obtain an asymptotic expression to the MISE value in increasing powers of h. The leading terms are the AMISE value, $(1/[nh])R(K) + (1/4)h^4\mu_2(K)^2R(f'')$, where $R(K) = \int K(x)^2 dx$ and $\mu_2(K) = \int x^2 K(x) dx$

The optimization of this expression with respect to the kernel K is not direct due to the coupling between K and h, but it can be achieved rescaling K in the form $K_\delta(z) = \frac{1}{\delta}K(\frac{z}{\delta})$ and then selecting δ_0 such that $R(K_\delta) = \mu_2(K_\delta)^2$. We obtain the solution $\delta_0 = \{R(K)/\mu_2(K)^2\}^{1/5}$ and the corresponding AMISE expression becomes

$$\mathrm{AMISE}\{\hat{f}(.;h)\} = C(K_{\delta_0})\left\{\frac{1}{nh} + \frac{1}{4}h^4 R(f'')\right\} \quad \text{(A1)}$$

where $C(K) = \{R(K)^4\mu_2(K)^2\}^{1/5}$

The problem of finding a kernel K minimizing $C(K)$ subject to the conditions of the kernel, i.e., $\int K(x)dx = 1, \int xK(x)d x = 0$ and $\int x^2K(x)dx = \sigma_K^2 < \infty$, provides a family of kernels depending on this bound σ_K (Hodges & Lehmann 1956). A particular case provides the Epanechnikov kernel; see equation (4).

The optimum value for minimizing AMISE is $h_0 = \{[R(K)]/[\sigma_K^4 R(f'')]\}^{1/5} n^{-1/5}$. Selecting a reference density (such as the Gaussian density; in other cases, we should introduce a conversion factor) for f and denoting σ as the standard deviation of the data set, one can obtain an expression depending on n, the size of the sample, $h_0 \approx c_K \sigma n^{-1/5}$; c_K being a constant which depends only on the kernel K.

If we consider a random two-dimensional vector, the different optimum h values are given by the vector $H = \sum^{1/2} n^{-1/6}$ where $\sum$ is the matrix of variance-covariance. These expressions are equally true when applied to a regression function problems by means of a kernel method.

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