NONDISCRETE P-GROUPS CAN BE REFLEXIVE

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Abstract. We present a series of examples of nondiscrete reflexive P-groups (i.e., groups in which all G_δ-sets are open) as well as noncompact reflexive ω-bounded groups (in which the closure of every countable set is compact). Our main result implies that every product of feathered (equivalently, almost metrizable) Abelian groups equipped with the P-modified topology is a reflexive group. In particular, every compact Abelian group with the P-modified topology is reflexive. This answers a question posed by S. Hernández and P. Nickolas and solves a problem raised by Ardanza-Trevijano, Chasco, Domínguez, and Tkachenko.

AMS Subj. Class. (2000): 54H11; 22D35; 54G10; 54D30; 54B10

Keywords: Pontryagin's duality; Reflexive; P-group; ω-bounded; Product; Σ-product; Compact subset; Character depends on countably many coordinates

1. Introduction

Extending Pontryagin's duality to diverse classes of topological groups beyond locally compact ones has been the object of attention through the last 60 years. It has become patent in recent times that the duality properties of precompact groups and of projective limits of discrete groups, two otherwise well studied classes of topological groups, are poorly understood. We refer the reader to [7] for the case of precompact groups and to [14] and [15] where the need of an improved

Date: February 15, 2010.

This work was started while the second and third listed authors were visiting the Department of Mathematics of Universitat Jaume I, Spain, in 2008. They thank their hosts for hospitality and support.

(1) Research of this author supported by the Spanish Ministry of Science (including FEDER funds), grant MTM2008-04599/MTM and Fundació Caixa Castelló-Bancaixa, grant P1.1B2008-26.

(2) Research of this author supported by a grant for foreign visitors under the Promotion of Research Program of University Jaume I.

(3) This author was supported by CONACyT of Mexico, grant 000000000074468.

knowledge of the duality properties emerged in the study of the structure of projective limits of Lie groups.

The duality theory of P-groups involves immediately both classes of topological groups. A P-group is a topological group in which the intersection of any countable family of open sets is open. Some basic information about P-groups can be found in [3, Section 4.4].

It is easy to see that P-groups are projective limits of discrete groups that do not contain infinite compact subsets. Because of this latter property the character group of a P-group is always precompact. Even more, it has the far stronger property of being *ω-bounded* (the closure of any countable subset is compact), see Lemma 5.1 below. Therefore P-groups occupy the region of pro-Lie groups that is farthest to locally compact groups whereas ω-bounded groups are among the most compact-like classes of topological groups.

Despite a considerable interest to the subject, the only result in the literature concerning the duality of P-groups is Leptin's example [17], later reproduced by Noble [18] and Banaszczyk [5, Example 7.11]. This example is the inverse limit of an uncountable family of discrete groups which turns out to be a nondiscrete P-group whose second dual is discrete. Hence Leptin's group is not reflexive. This motivates the following question posed by S. Hernández and P. Nickolas at the International Workshop on Topological Groups and Dynamic Systems in Madrid, 2008:

Question 1. Must every reflexive Abelian P-group be discrete?

This problem is naturally linked with the question on whether a precompact, noncompact group can be reflexive (recall that the dual group of a P-group is ω-bounded and hence precompact). The first examples of precompact, noncompact reflexive groups have been recently obtained in [1] and [11], but the examples presented there are precompact groups with no infinite compact subsets; so they are far from being ω-bounded. Question 1 has therefore the natural accompanying question.

Question 2. How close to being compact can a reflexive precompact Abelian group be? In particular, can a noncompact ω-bounded group be reflexive?

The main objective of this paper is to answer Question 1 (in the negative) and Question 2 (in the positive). We do this in Theorem 6.8 below by proving that every product of discrete Abelian groups with the P-modified topology is reflexive. This result is then extended to products of feathered (equivalently, almost metrizable) Abelian groups. In an attempt to trace the borders between reflexive and nonreflexive P-groups we also give a number of new examples of reflexive and nonreflexive P-groups.

We also establish in Propositions 5.6 and 5.9 that the class of reflexive P-groups has unexpectedly good permanence properties—it contains quotient groups and if a reflexive P-group G is a dense subgroup of a topological group H, then H is reflexive P-group as well.

2. Notation

All groups considered here are assumed to be Abelian if otherwise is not specified explicitely. The complex plane with its usual multiplication and topology is denoted by $\mathbb{C}$. A *character* on a group G is a homomorphism of G to the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The set $\{e^{i\varphi} : -\pi/2 < \varphi < \pi/2\}$ is denoted by $\mathbb{T}_+$. Pontryagin's duality theory is based on relating a topological group G with the group $G^\wedge$ of continuous characters of G. The group $G^\wedge$ will be equipped with the topology of uniform convergence on the compact subsets of G. This topology has as a neighbourhood basis at the identity the sets

$$K^\triangleright = \{\chi \in G^\wedge : \chi(x) \in \mathbb{T}_+ \text{ for all } x \in K\},$$

where K runs over the family of compact subsets of G. If a topological group G is well-represented by its continuous characters, one can recover G from $G^\wedge$ by forming the bidual group $G^{\wedge\wedge} = (G^\wedge)^\wedge$ and then considering the canonical *evaluation homomorphism* $\alpha_G : G \to G^{\wedge\wedge}$ defined by $\alpha_G(x)(\chi) = \chi(x)$, for all $x \in G$ and $\chi \in G^\wedge$. We say accordingly that G is *reflexive* if the homomorphism $\alpha_G : G \to G^{\wedge\wedge}$ is a topological isomorphism.

We will use the concept of *nuclear group* as it appears in [4, 5]. It is worth mentioning that every closed subgroup H of a nuclear group G is *dually embedded* in G, i.e., for every $x \in G \setminus H$ there exists a continuous character $\chi: G \to \mathbb{T}$ such that $\chi(H) = \{1\}$ and $\chi(x) \neq 1$ (see [5, Corollary 8.6]). In particular, continuous characters of a nuclear group G separate points of G which in its turn implies that the evaluation homomorphism $\alpha_G : G \to G^{\wedge\wedge}$ is injective.

Given a topological group H, we denote by PH the *P-modification* of H which is the same underlying group H endowed with the finer topological group topology whose base is formed by the family of G_δ-sets in the original group H. The subgroup of H generated by a set $A \subseteq H$ is $\langle A\rangle$. Sometimes we use $\langle A, B\rangle$ for $A, B \subseteq H$ to denote the group $\langle A \cup B\rangle$.

A group G is *boolean* if every $x \in G$ satisfies $x + x = 0_G$. The group $\mathbb{Z}_2 = \{0, 1\}$ and all powers of $\mathbb{Z}_2$ are boolean groups.

Let $\{D_i\colon i \in I\}$ be a family of topological groups and $D = \prod_{i\in I} D_i$ the product group with the usual Tychonoff product topology. Elements of D will be regarded as functions $x\colon I \to \bigcup_{i\in I} D_i$ such that $x(i) \in D_i$ for all $i \in I$. If $x \in D$, we define the *support* of x as

$$\mathrm{supp}(x) = \{i \in I\colon x(i) \neq 0_i\},$$

where 0_i is the neutral element of D_i. With these notations, the subgroup

$$\Sigma D = \{x \in D\colon |\mathrm{supp}(x)| \leq \omega\}$$

of D is called the Σ-*product* of the family $\{D_i : i \in I\}$. Similarly,

$$\sigma D = \{x \in D\colon |\mathrm{supp}(x)| < \omega\}$$

is a subgroup of D which is called the σ-*product* of the family $\{D_i : i \in I\}$.

Let $[I]^{\leq\omega}$ denote the family of all countable subsets of the index set I. For every $i \in I$, we denote by $\mathcal{S}_i$ the family of all subgroups of type G_δ in D_i. It is clear that $\mathcal{S}_i$ is a base of PD_i at 0_i. The collection of sets

$$U(J, f) = \{x \in D\colon x(i) \in f(i) \text{ for all } i \in J\},$$

where $J \in [I]^{\leq\omega}$ and f a function with the domain J such that $f(i) \in \mathcal{S}_i$ for each $i \in J$, constitutes a base at the identity of PD, the P-modification of the product group D. Clearly, ΣD is a dense subgroup of PD, while σD is a dense subgroup of D.

In what follows we will also use the sets

$$U(J) = \{x \in D\colon x(i) = 0_i \text{ for all } i \in J\},$$

with $J \subset I$. Notice that if the groups D_i's are discrete, then the family $\{U(J) : J \in [I]^{\leq\omega}\}$ forms a local base at the identity of PD.

We will mainly be working with topological groups G such that

$$\Sigma D \subseteq G \subseteq PD.$$

The subgroup ΣD of PD will always carry the topology inherited from PD, i.e., ΣD is a P-group.

Let G be a subgroup of PD and $J \subset I$. We will say that a character $\chi\colon G \to \mathbb{T}$ *depends on* J if there is $x \in G$ with $\mathrm{supp}(x) \subset J$ such that $\chi(x) \neq 1$ (notice that $x \in G \cap U(I \setminus J)$).

It is easy to see (see Lemma 4.1 below) that a character $\chi\colon G \to \mathbb{T}$ is continuous if and only if there are $J \in [I]^{\leq\omega}$ and a function f such that $G \cap U(J,f) \subset \ker\chi$. Therefore, for every continuous character $\chi\colon G \to \mathbb{T}$, there is $J \in [I]^{\leq\omega}$ such that χ does not depend on $I \setminus J$. We say in this case that χ *depends (at most) on countably many coordinates*. Similarly, we say that a set $K \subset G^\wedge$ *depends (at most) on countably many coordinates* if there is $J \in [I]^{\leq\omega}$ such that every $\chi \in K$ does not depend on $I \setminus J$.

3. The results

As we mentioned above, Leptin [17] gave an example of a nonreflexive P-group which was the subgroup of $P\mathbb{Z}_2^{\omega_1}$ consisting of elements with finite support (i.e. the σ-product of ω_1 copies of the discrete two-element group $\mathbb{Z}_2$). Here we extend Leptin's argument to deduce the following result (see Proposition 7.3):

Theorem 1. *The Σ-product $\Sigma D \subset PD$ is not reflexive, where $D = \prod_{i\in I} D_i$ is the product of an uncountable family of nontrivial discrete Abelian groups.*

We also extend Theorem 1 to certain subgroups between $\Sigma = \Sigma\mathbb{Z}_2^\tau$ and $P\mathbb{Z}_2^\tau$ as follows (see Theorem 7.8):

Theorem 2. *If L is a countable subgroup of $P\mathbb{Z}_2^\tau$, for an uncountable cardinal τ, then the subgroup $G_L = \Sigma + L$ of $P\mathbb{Z}_2^\tau$ is not reflexive.*

It may be worth to observe that while the proof of Theorem 1 uses an argument close to Leptin's, this argument does not work for G_L and a different one is needed for Theorem 2.

Somewhat surprisingly we also find a wealth of non-discrete reflexive P-groups (and hence reflexive noncompact ω-bounded groups). In particular, we prove the following fact in Theorem 6.8 (and Corollary 6.9):

Theorem 3. *Let $D = \prod_{i\in I} D_i$ be a product of nontrivial discrete Abelian groups, where $|I| > \omega$. Then the nondiscrete P-group $\Pi = PD$ and the noncompact ω-bounded group $\Pi^\wedge$ are reflexive.*

The non-reflexive groups in Theorems 1 and 2 are evidently non-complete, while the reflexive groups presented in Theorem 3 are complete (apply [12, Theorem 8]). One can conjecture, therefore, that reflexive P-groups are complete. We show in Theorem 6 below that this is not the case.

Theorem 3 can be given a considerably more general form. We recall that a topological group G is called *feathered* or *almost metrizable* provided there exists a nonempty compact set K in G with a countable neighbourhood base. The next result follows from Theorem 6.13:

Theorem 4. *Let $D = \prod_{i\in I} D_i$ be a product of feathered Abelian groups. Then the group PD is reflexive.*

As a step towards the proof of Theorem 4 we show that every compact Abelian group with the P-modified topology is reflexive.

The following reflexion principle turns out to be quite useful when trying to extend the class of reflexive P-groups (see Theorem 6.14):

Theorem 5. *Suppose that $D = \prod_{i\in I} D_i$ is a product of topological groups and $\Sigma D \subset G \subset PD$. Then the group G is reflexive iff the subgroup $\pi_J(G)$ of PD_J is reflexive, for every set $J \subset I$ satisfying $|J| \le \aleph_1$ (here $\pi_J \colon D \to \prod_{i\in J} D_i$ is the projection).*

Theorem 5 enables us to present examples of dense reflexive subgroups G of the groups PD (see Corollary 6.15). Clearly, G is not complete provided that $G \neq PD$:

Theorem 6. *Let $D = \prod_{i\in I} D_i$ be a product of feathered Abelian groups. Then the subgroup*

$$\Sigma_{\aleph_1} D = \{x \in PD : |\operatorname{supp}(x)| \le \aleph_1\}$$

of PD is reflexive and every subgroup G of PD containing $\Sigma_{\aleph_1} D$ is also reflexive.

Once we have established that the class of reflexive P-groups is quite wide, it is natural to clarify the permanence properties of this class. In the next result we present two of them (see Propositions 5.6 and 5.9):

Theorem 7. *Let G be a reflexive P-group.*

(a) *If G is a dense subgroup of a topological group H, then H is also a reflexive P-group.*
(b) *If $\pi \colon G \to K$ is a continuous open homomorphism of G onto K, then K is a reflexive P-group.*

Let $\tau > \omega$ be a cardinal. To extend Theorem 3 to subgroups slightly smaller than $\Pi = P\mathbb{Z}_2^\tau$ we consider an arbitrary ultrafilter ξ on τ containing all subsets of τ with countable complement.

Note that $A \in \xi$ implies $|A| > \omega$. We then define

$$G_\xi = \{x \in \Pi\colon \operatorname{supp}(x) \notin \xi\}.$$

It is straightforward to check that G_ξ is a subgroup of Π. Also, if $x \in \Sigma$, then $|\operatorname{supp}(x)| \leq \omega$. So, $\operatorname{supp}(x) \notin \xi$ and therefore $\Sigma\Pi \subset G_\xi$. It is clear that $G_\xi \neq \Pi$ because the constant function $\mathbf{1}$ is not in G_ξ. Actually this is the "most important" element absent in G_ξ, as $\Pi = \langle G_\xi, \mathbf{1}\rangle = G_\xi \oplus \langle \mathbf{1}\rangle$. We prove in Theorem 6.17 that this smaller subgroup G_ξ of Π is also reflexive:

Theorem 8. *The nondiscrete P-group G_ξ is reflexive, for every ultrafilter ξ on τ containing the complements to countable sets.*

4. Factorization of characters on subgroups of a product group

Here we collect several results of technical nature which will be used later. Throughout this section $D = \prod_{i\in I} D_i$ stands for the Tychonoff product of a family $\{D_i : i \in I\}$ of topological groups, not necessarily Abelian. For $J \subset I$, we denote by D_J the corresponding subproduct $\prod_{i\in J} D_i$ and by $\pi_J\colon D \to D_J$ the projection.

Lemma 4.1. *Let G be a subgroup of PD and $\chi\colon G \to \mathbb{T}$ be a character. The following assertions are then equivalent:*

1. *χ is continuous, i.e., $\chi \in G^\wedge$.*
2. *There are $J \in [I]^{\leq\omega}$ and a function f such that $G \cap U(J, f) \subset \ker\chi$ (in particular, χ does not depend on $I \setminus J$).*
3. *There are a countable set $J \subset I$ and a continuous character χ_J on the subgroup $\pi_J(G)$ of PD_J such that $\chi = \chi_J \circ \pi_J{\restriction}_G$.*

Proof. Suppose χ is continuous. Since G is a P-group, the kernel of χ is an open subgroup of G. It follows from the definition of the topology of PD that there exist a countable set $J \subset I$ and a function f with the domain J such that $f(i)$ is an open subgroup of PD_i for each $i \in J$ and the basic open set $U(J, f) = \{x \in D\colon x(i) \in f(i) \text{ for all } i \in J\}$ in PD satisfies $G \cap U(J, f) \subset \ker\chi$. Clearly χ does not depend on $I \setminus J$. Hence (1) implies (2).

Suppose now that $G \cap U(J, f) \subset \ker\chi$ for a countable set $J \subset I$ and a corresponding function f. We define a character χ_J on $\pi_J(G)$ by $\chi_J(\pi_J(x)) = \chi(x)$ for any element $x \in G$. This definition is correct since the equality $\pi_J(x) = \pi_J(y)$ implies that $x^{-1}y \in G \cap U(J, f)$ and $\chi(x) = \chi(y)$. By the definition of χ_J, we see that $\chi = \chi_J \circ \pi_J\restriction_G$. It also follows from the definition of χ_J that its kernel contains the set $\pi_J(G) \cap U_f$, where $U_f = \prod_{i \in J} f(i)$ and $f(i)$ is an open subgroup of PD_i for each $i \in J$. Since U_f is an open subgroup of PD_J, we conclude that χ_J is a continuous character on the group $\pi_J(G)$. So (2) implies (3).

Finally suppose that there is a countable set $J \subset I$ and a continuous character χ_J on $\pi_J(G)$ such that $\chi = \chi_J \circ \pi_J\restriction_G$. Since the projection $\pi_J \colon PD \to PD_J$ is continuous, we see that so is the character χ. Hence (3) implies (1). $\square$

The following result is close in the spirit to [8, Theorem 4.6], where the product space carries the usual Tychonoff product topology.

Lemma 4.2. *Let G be a dense subgroup of PD and $\chi \in G^\wedge$. Then χ admits a continuous extension to a character $\overline{\chi}$ on PD and, for every set $J \subset I$, $\overline{\chi}$ does not depend on $I \setminus J$ if and only if there exists a continuous character ψ on $\pi_J(G)$ such that $\chi = \psi \circ \pi_J\restriction_G$.*

Proof. Since G is dense in PD and the circle group $\mathbb{T}$ is compact (hence complete), χ extends to a continuous character $\overline{\chi}$ on the group PD. If $\overline{\chi}$ does not depend on $I \setminus J$, where $J \subset I$, there exists a character $\overline{\psi}$ on PD_J such that $\overline{\chi} = \overline{\psi} \circ \pi_J$, where $D_J = \prod_{i \in I} D_i$ and $\pi_J \colon D \to D_J$ is the projection. Since $\pi_J \colon PD \to PD_J$ is open, the character $\overline{\psi}$ on PD_J is continuous. Then $\chi = \psi \circ \pi_J\restriction_G$, where ψ is the restriction of $\overline{\psi}$ to the subgroup $\pi_J(G)$ of PD_J. Clearly, the character ψ is continuous.

Conversely, suppose that there exists a continuous character ψ on the subgroup $\pi_J(G)$ of PD_J such that $\chi = \psi \circ \pi_J\restriction_G$. Since $\pi_J(G)$ is dense in PD_J, ψ extends to a continuous character $\overline{\psi}$ on the group PD_J. Clearly, the characters $\overline{\chi}$ and $\overline{\psi} \circ \pi_J$ coincide on the dense subgroup G of PD. Since the group $\mathbb{T}$ is Hausdorff, we see that $\overline{\chi} = \overline{\psi} \circ \pi_J$. It follows that $\overline{\chi}$ does not depend on $I \setminus J$. $\square$

In general, the existence of a continuous character ψ on $\pi_J(G)$ satisfying $\chi = \psi \circ \pi_J\restriction_G$ in Lemma 4.2 cannot be weakened to a simpler condition that χ does not depend on $I \setminus J$. However, this weakening is possible for special subgroups of PD as we will see in Corollary 4.4 below. First we need a lemma.

Lemma 4.3. *If $\Sigma D \subset G \subset PD$, then the restriction of the projection $\pi_J \colon PD \to PD_J$ to G is an open homomorphism of G onto $\pi_J(G)$, for every nonempty set $J \subset I$.*

Proof. Let J be a nonempty subset of I. Since G is a subgroup of PD, the restriction of π_J to G is a continuous homomorphism. Hence it suffices to verify that the image $\pi_J(V \cap G)$ is open in $\pi_J(G)$, for every basic open neighbourhood V of the identity e in PD. In fact, we will show that $\pi_J(V \cap G) = \pi_J(V) \cap \pi_J(G)$.

Given a basic open neighbourhood V of e in PD, one can find a countable set $C \subset I$ and open subgroups V_i of D_i for $i \in C$ such that

$$V = \{x \in D : x(i) \in V_i \text{ for each } i \in C\}.$$

Let $F = C \cap J$ and $E = C \setminus J$. It is clear that F and E are disjoint countable sets and $C = F \cup E$. Take an arbitrary point $y \in \pi_J(V) \cap \pi_J(G)$. There exists an element $x \in G$ with $\pi_J(x) = y$. Clearly, $x(i) = y(i) \in V_i$ for each $i \in F$. Since E is countable, we can find an element $x_0 \in \Sigma D$ such that $\mathrm{supp}(x_0) \cap J = \emptyset$ and $x_0(i) = x(i)$ for each $i \in E$. Then the element $z = x \cdot x_0^{-1}$ of D satisfies $z(i) \in V_i$ for each $i \in C$, so $z \in V$. Since $x_0 \in \Sigma D \subset G$, we see that $z \in G$. Hence $z \in V \cap G$ and $\pi_J(z) = \pi_J(x) \cdot (\pi_J(x_0))^{-1} = y$. This implies that $\pi_J(V) \cap \pi_J(G) \subset \pi_J(V \cap G)$. The inverse inclusion is obvious, so the equality $\pi_J(V \cap G) = \pi_J(V) \cap \pi_J(G)$ is proved. Therefore the restriction of the homomorphism π_J to G is open when considered as a mapping onto its image. □

Corollary 4.4. *Let G be a group with $\Sigma D \subset G \subset PD$, and $\chi \in G^\wedge$. Then, for every set $J \subset I$, the continuous extension $\overline{\chi}$ of χ over PD does not depend on $I \setminus J$ if and only if χ does not depend on $I \setminus J$.*

Proof. By Lemma 4.2, $\overline{\chi}$ is a continuous character on PD. Hence it suffices to verify that if χ does not depend on $I \setminus J$, neither does $\overline{\chi}$. Under this assumption, there exists a character ψ on $\pi_J(G)$ such that $\chi = \psi \circ \pi_J\restriction_G$. Since the restriction to G of the projection π_J is open by Lemma 4.3, the character ψ is continuous. We apply Lemma 4.2 once again to conclude that $\overline{\chi}$ does not depend on $I \setminus J$. □

5. Reflexivity of G and compact subsets of $G^\wedge$

Here we characterize the reflexivity of a P-group G in terms of compact subsets of the dual group $G^\wedge$. First we present a simple but useful piece of information.

Lemma 5.1. *If G is a P-group, then the dual group $G^\wedge$ is ω-bounded.*

Proof. Since G is a P-group, all compact subsets of G are finite. Hence $G^\wedge$ is a topological subgroup of $\mathbb{T}^G$, where $\mathbb{T}$ is the circle group with its usual compact topology inherited from the complex plane. Let C be a countable subset of $G^\wedge$. Since the circle group $\mathbb{T}$ and its power $\mathbb{T}^G$ are compact, the set $\overline{C}$ is also compact (the closure is taken in $\mathbb{T}^G$). We claim that $\overline{C} \subset G^\wedge$.

To verify this inclusion, take an arbitrary element $\varphi \in \overline{C}$. It is well known (and easy to see) that φ is a homomorphism of G to $\mathbb{T}$, so it suffices to prove the continuity of φ at the neutral element 0_G of G. Since each $\chi \in C$ is continuous at 0_G, there exists an open neighbourhood U_χ of 0_G in G such that $\chi(U_\chi) = \{1\}$; here 1 is the neutral element of $\mathbb{T}$. Then $V = \bigcap_{\chi \in C} U_\chi$ is an open neighbourhood of 0_G, and we claim that $\varphi(V) = \{1\}$. Indeed, if $x \in V$, then $\chi(x) = 1$ for each $\chi \in C$, so it follows from $\varphi \in \overline{C}$ that $\varphi(x) = 1$. Thus, φ is also continuous at 0_G, i.e., $\varphi \in G^\wedge$. Therefore the group $G^\wedge$ is ω-bounded. $\square$

Let us isolate a property that a P-group must possess in order to be reflexive. In what follows we say that a set $K \subset G^\wedge$ is *constant on a subgroup* H of G if every $\chi \in K$ is constant on H.

Lemma 5.2. *Let G be a P-group. The evaluation mapping $\alpha_G\colon G \to G^{\wedge\wedge}$ is continuous if and only if every compact set $K \subseteq G^\wedge$ is constant on an open subgroup of G.*

Proof. Necessity. Let K be a compact subset of $G^\wedge$. If α_G is continuous, there exists an open neighbourhood U of the neutral element e in G such that $\alpha_G(U) \subset K^\triangleright$. Since G is a P-group, it follows from [3, Lemma 4.4.1 a)] that there exists an open subgroup V of G such that $V \subset U$. Clearly the set $\mathbb{T}_+$ does not contain nontrivial subgroups, so $\chi(V) = \{1\}$ for each $\chi \in K$. Thus K is constant on V.

Sufficiency. It suffices to verify the continuity of the homomorphism α_G at the neutral element of G. Let $K^\triangleright$ be a basic open set in $G^{\wedge\wedge}$, with K a compact subset of $G^\wedge$. By hypothesis, there exists an open subgroup V of G such that every $\chi \in K$ is constant on V. Then $\alpha_G(V) \subset K^\triangleright$. Therefore α_G is continuous at e. $\square$

Since we are mainly concerned with (subgroups of) product groups, it is worth to reformulate the above lemma for this special case.

Corollary 5.3. *Let G be a subgroup of PD, where $D = \prod_{i \in I} D_i$ is the product of an arbitrary family of topological groups. The evaluation mapping $\alpha_G\colon G \to G^{\wedge\wedge}$ is continuous if and only if* *for every compact set $K \subseteq G^\wedge$, one can find a set $J \in [I]^{\leq\omega}$ and an open subgroup U of PD_J such that every $\chi \in K$ is constant on the set $G \cap \pi_J^{-1}(U)$.*

Proof. By Lemma 5.2, the continuity of α_G means that every compact set $K \subset G^\wedge$ is constant on an open subgroup V of G. Since the sets $G \cap \pi_J^{-1}(U)$, where $J \in [I]^{\leq\omega}$ and U is an open subgroup of PD_J, form an open basis at the neutral element of G, the required conclusion is immediate. □

Theorem 5.4. *A P-group G is reflexive if and only if every compact set $K \subset G^\wedge$ is constant on an open subgroup of G.*

Proof. By [3, Lemma 4.4.1 a)], every P-group has a base at the identity consisting of open subgroups. Hence G is a topological subgroup of a product of discrete groups. Since the class of nuclear groups contains discrete Abelian groups and is closed under taking products and arbitrary subgroups (here we apply Propositions 7.3, 7.5, and 7.6 from [5]), it follows that the group G is nuclear.

We already know that the evaluation homomorphism $\alpha_G : G \to G^{\wedge\wedge}$ is injective because G is a nuclear group. By the same reason, the mapping $\alpha_G : G \to \alpha_G(G)$ is open, where $\alpha_G(G)$ carries the topology inherited from $G^{\wedge\wedge}$ (see Theorem 8.5 and Lemma 14.3 of [5]).

Since the P-group G has no infinite compact subsets, $G^\wedge$ carries the topology of pointwise convergence on elements of G. It follows that $G^{\wedge\wedge} = \alpha_G(G)$ (see for instance [9, Theorem 1.3]). The P-group G is therefore reflexive if and only if α_G is continuous. The theorem is then a direct consequence of Lemma 5.2. □

Here is a coordinatewise form of Theorem 5.4 which is immediate after Corollary 5.3.

Theorem 5.5. *Let $D = \prod_{i\in I} D_i$ be a product of topological groups. A subgroup G of PD is reflexive if and only if for every compact set $K \subset G^\wedge$, there exist a set $J \in [I]^{\leq\omega}$ and an open subgroup U of PD_J such that K is constant on $G \cap \pi_J^{-1}(U)$.*

The following two somewhat unexpected facts fail to hold outside the class of P-groups. The first of them says that reflexivity in P-groups extends from a dense subgroup to the whole group:

Proposition 5.6. *Let G be a a dense subgroup of a topological group H. If G is a reflexive P-group, so is H.*

Proof. Suppose that G is a reflexive P-group. Since G is dense in H, it follows from [3, Lemma 4.4.1 d)] that H is a P-group. According to Theorem 5.4 it suffices to verify that every compact set $K \subset H^\wedge$ is constant on an open subgroup of H. Denote by r the natural restriction homomorphism of $H^\wedge$ to $G^\wedge$ defined by $r(\chi) = \chi\restriction_G$, for each $\chi \in H^\wedge$. Since $G \subset H$ and the dual groups $G^\wedge$ and $H^\wedge$ carry the topologies of pointwise convergence on elements of G and H, respectively, r is continuous. It also follows from the density of G in H that r is one-to-one and onto. In other words, r is a continuous isomorphism of $H^\wedge$ onto $G^\wedge$.

Since G is reflexive, Theorem 5.4 implies that the compact set $r(K) \subset G^\wedge$ is constant on an open subgroup V of G. Let U be the closure of V in H. Then U is an open subgroup of H and every $\chi \in K$ is constant on the dense subset V of U. By a continuity argument, χ is constant on U. Therefore K is constant on U, whence the reflexivity of H follows. □

The second fact establishes that the class of reflexive P-groups is stable under taking quotients. Its proof makes use of *dual homomorphisms.* Since this tool will be used several times in the article, we give a lemma explaining basic properties of dual homomorphisms.

Let us recall that a surjective mapping $f\colon X \to Y$ is *compact covering* if for every compact set $K \subset Y$, there exists a compact set $C \subset X$ such that $f(C) = K$.

Lemma 5.7. *Let $\pi\colon G \to H$ be a continuous homomorphism of topological Abelian groups. Let also $\pi^\wedge\colon H^\wedge \to G^\wedge$ be the dual homomorphism defined by $\pi^\wedge(\chi) = \chi \circ \pi$, for each $\chi \in H^\wedge$. Then:*

(a) *$\pi^\wedge$ is continuous.*
(b) *If π is compact covering, then $\pi^\wedge$ is a topological isomorphism of $H^\wedge$ onto its image $\pi^\wedge(H^\wedge)$.*
(c) *If the homomorphism π is open, then the image $\pi^\wedge(H^\wedge)$ is closed in $G^\wedge$.*

Proof. (a) is well known. Indeed, let C be a compact subset of G and $U = C^\triangleright$ a basic open neighbourhood of the neutral element in $G^\wedge$. Then $K = \pi(C)$ is a compact subset of H and $V = K^\triangleright$ is an open neighbourhood of the neutral element in $H^\wedge$. For any $\chi \in V$, we have $\pi^\wedge(\chi)(C) = (\chi \circ \pi)(C) = \chi(\pi(C)) = \chi(K) \subset \mathbb{T}_+$, that is, $\pi^\wedge(\chi) \in C^\triangleright = U$. This implies the continuity of $\pi^\wedge$ and proves (a).

(b) follows from [4, Lemma 5.17].

(c) Suppose that π is an open homomorphism of G to H and let $K = \pi(G)$. Then K is an open subgroup of H, so every continuous character on K extends to a continuous character on H. Hence $\pi^\wedge(K^\wedge) = \pi^\wedge(H^\wedge)$ and we can assume without loss of generality that $\pi(G) = H$.

Our further argument is very close to the proof of item 2) of Corollary 0.4.8 from [2]. Indeed, let ψ be in the closure of $\pi^\wedge(H^\wedge)$ in $G^\wedge$. Since finite sets are compact, the topology of $G^\wedge$ contains the topology of pointwise convergence on elements of G. Applying this fact, one easily verifies that ψ is a homomorphism of G to $\mathbb{T}$ and that ψ is constant on each fiber $\pi^{-1}(y)$, $y \in H$. Hence there exists a function $\chi\colon H \to \mathbb{T}$ such that $\psi = \chi \circ \pi$. Clearly, χ is a homomorphism. Since ψ is continuous and π is open and onto, the equality $\psi = \chi \circ \pi$ implies that χ is continuous as well. Thus, $\chi \in H^\wedge$ and $\psi = \pi^\wedge(\chi) \in \pi^\wedge(H^\wedge)$. □

Corollary 5.8. *Let $\pi\colon G \to H$ be a continuous onto homomorphism. If all compact subsets of H are finite, then $\pi^\wedge$ is a topological isomorphism of $H^\wedge$ onto the subgroup $\pi^\wedge(H^\wedge)$ of $G^\wedge$. In particular, this is the case when H is a P-group.*

Proof. Since all compact subsets of H are finite, the homomorphism π is compact covering. The required conclusion now follows from (b) of Lemma 5.7. □

Proposition 5.9. *Let $\pi\colon G \to H$ be a continuous open epimorphism of topological groups. If G is a reflexive P-group, so is H.*

Proof. The fact that the image $H = \pi(G)$ is a P-group follows from [3, Lemma 4.4.1 c)]. Let us show that H is reflexive. Take an arbitrary compact subset C of $H^\wedge$. By (a) of Lemma 5.7, the dual homomorphism $\pi^\wedge\colon H^\wedge \to G^\wedge$ is continuous, so $K = \pi^\wedge(C)$ is a compact subset of $G^\wedge$. Since the group G is reflexive, Theorem 5.4 implies that there exists an open subgroup U of G such that K is constant on U. Then C is constant on the open subgroup $V = \pi(U)$ of H. Indeed, if $\chi \in C$ and $y \in V$, take $x \in U$ with $\pi(x) = y$. Then $\chi(y) = \chi(\pi(x)) = \pi^\wedge(\chi)(x) = 1$ since $\pi^\wedge(\chi) \in K$. Applying Theorem 5.4 once again, we conclude that the group H is reflexive. □

We finish this section with two special cases of Theorem 5.5.

Corollary 5.10. *Let $D = \prod_{i\in I} D_i$ be a product of topological groups. The group PD is reflexive if and only if the following hold:*

(a) *the group $PD_J = P(\prod_{i\in J} D_i)$ is reflexive for each $J \in [I]^{\leq\omega}$;*

(b) *every compact set $K \subset (PD)^\wedge$ depends at most on countably many coordinates.*

Proof. Necessity. It is easy to see that the projection $\pi_J \colon PD \to PD_J$ is open for each $J \subset I$. If the group PD is reflexive, then the reflexivity of the groups PD_J follows from Proposition 5.9. Suppose that K is a compact subset of $(PD)^\wedge$. Theorem 5.5 implies that there exist $J \in [I]^{\leq\omega}$ and an open subgroup U of PD_J such that K is constant on $\pi_J^{-1}(U)$. Then every $\chi \in K$ does not depend on $I \setminus J$, that is, K does not depend on $I \setminus J$. Hence conditions (a) and (b) hold true.

Sufficiency. Let us deduce the reflexivity of PD from (a) and (b). Take a compact set $K \subset (PD)^\wedge$. By (b), there exists a countable set $J \subset I$ such that K does not depend on $I \setminus J$.

Denote by $\varphi\colon (PD_J)^\wedge \to (PD)^\wedge$ the homomorphism dual to the projection $\pi_J \colon PD \to PD_J$. Since the projection $\pi_J \colon PD \to PD_J$ is open and all compact subsets of PD_J are finite, it follows from Corollary 5.8 that φ is a topological isomorphism of $(PD_J)^\wedge$ onto a subgroup of $(PD)^\wedge$.

We claim that $K \subset \varphi((PD_J)^\wedge)$. Indeed, take an arbitrary character $\chi \in K$. Since χ does not depend on $I \setminus J$, there exists a character ζ on PD_J such that $\chi = \zeta \circ \pi_J$. The character ζ is continuous since the projection π_J is open. Hence $\zeta \in (PD_J)^\wedge$ and $\chi = \varphi(\zeta) \in \varphi((PD_J)^\wedge)$.

Let $C = \varphi^{-1}(K)$. Then C is a compact subset of the group $(PD_J)^\wedge$. By (a), the group PD_J is reflexive. According to Theorem 5.4, PD_J contains an open subgroup U such that every character $\zeta \in C$ is constant on U. Since $\varphi(C) = K$, we see that every $\chi \in K$ is constant on the open subgroup $\pi_J^{-1}(U)$ of PD. The reflexivity of the group PD now follows from Theorem 5.5. □

We shall see in Proposition 6.7 that one can drop item (b) in the above corollary. The next result is a slight modification of Corollary 5.10, so we omit its proof.

Corollary 5.11. *Let $D = \prod_{i\in I} D_i$ be a product of topological groups such that the group PD_J is reflexive for each $J \in [I]^{\leq\omega}$. Suppose that G is a subgroup of PD satisfying $\pi_J(G) = PD_J$ for each $J \in [I]^{\leq\omega}$. Then G is reflexive if and only if every compact set $K \subset G^\wedge$ depends at most on countably many coordinates.*

6. Reflexive P-groups

We prepare here our way to show that some nondiscrete P-groups are reflexive. The lemma below is obvious and its proof is omitted.

Lemma 6.1. *Let b, t be elements of $\mathbb{T}$ and $t \neq 1$. Then there is an integer k such that $t^k \cdot b \notin \mathbb{T}_+$.*

In the following proposition and in Lemmas 6.3–6.5, $D = \prod_{i\in I} D_i$ stands for the product of an arbitrary family of topological Abelian groups.

Proposition 6.2. *Suppose that* $C = \{\chi_\eta \colon \eta < \omega_1\} \subset (PD)^\wedge$, $\mathfrak{J} = \{J_\eta \colon \eta < \omega_1\} \subset [I]^{\leq\omega}$, *and* $X = \{x_\eta \colon \eta < \omega_1\} \subset PD$ *satisfy the following conditions for each* $\eta < \omega_1$*:*

(1) χ_η *does not depend on* $I \setminus J_\eta$;

(2) $\mathrm{supp}(x_\eta) \subset J_\eta$;

(3) $\chi_\eta(x_\eta) \in \mathbb{T} \setminus \mathbb{T}_+$;

(4) *if* $\zeta < \eta$*, then* $J_\zeta \bigcap \mathrm{supp}(x_\eta) = \emptyset$.

Then every element of $\bigcap_{\gamma<\omega_1} \overline{\{\chi_\eta \colon \eta \geq \gamma\}}^{\mathbb{T}^D}$ *is discontinuous as a character on the group* PD.

Proof. For each $\gamma < \omega_1$, let $C_\gamma = \{\chi_\eta \colon \eta \geq \gamma\}$ and $K_\gamma = \overline{C_\gamma}^{\mathbb{T}^D}$. Since the family $\{K_\gamma \colon \gamma < \omega_1\}$ of compact sets is decreasing, we see that $K = \bigcap_{\gamma<\omega_1} K_\gamma$ is nonempty. Take $\rho \in K$. Suppose toward a contradiction that ρ is continuous, i.e., that $\rho \in (PD)^\wedge$. By Lemma 4.1, there exists $J \in [I]^{\leq\omega}$ such that ρ does not depend on $I \setminus J$. It follows from (2) and (4) that the family $\{\mathrm{supp}(x_\eta) \colon \eta < \omega_1\}$ is pairwise disjoint. Take a countable ordinal η_0 such that $J \bigcap \mathrm{supp}(x_\eta) = \emptyset$ for all η satisfying $\eta_0 \leq \eta < \omega_1$. It then follows that $\rho(x_\eta) = 1$ for every countable ordinal $\eta \geq \eta_0$.

Given a family $\{g_\alpha : \alpha \in A\} \subset D$ such that $\mathrm{supp}(g_\alpha) \cap \mathrm{supp}(g_\beta) = \emptyset$ for distinct $\alpha, \beta \in A$, we can define an element $g = \coprod_{\alpha\in A} g_\alpha \in D$ by the requirements that $\mathrm{supp}(g) = \bigcup_{\alpha\in A} \mathrm{supp}(g_\alpha)$ and for every $\alpha \in A$, the elements g and g_α coincide on $\mathrm{supp}(g_\alpha)$.

Now for every countable ordinal $\eta \geq \eta_0$, we define a point $g_\eta \in D$ satisfying the following two conditions:

(a) g_η is either the neutral element $\mathbf{0}$ of D or $k_\eta x_\eta$, for some $k_\eta \in \mathbb{Z}$;

(b) $\chi_\eta \left(\coprod_{\eta_0\leq\beta\leq\eta} g_\beta\right) \in \mathbb{T} \setminus \mathbb{T}_+$.

To begin, we put $g_{\eta_0} = x_{\eta_0}$. Then conditions (a) and (b) hold. Suppose now that $\eta_0 < \sigma < \omega_1$ and that g_η have been defined for all η with $\eta_0 < \eta < \sigma$ such that (a) and (b) hold. Notice that the family $\{\mathrm{supp}(g_\eta) : \eta_0 \leq \eta < \sigma\}$ is pairwise disjoint, by (a). If $\chi_\sigma \left(\coprod_{\eta_0\leq\beta<\sigma} g_\beta\right) \notin \mathbb{T}_+$, put $g_\sigma = \mathbf{0}$. If $z = \chi_\sigma \left(\coprod_{\eta_0\leq\beta<\sigma} g_\beta\right) \in \mathbb{T}_+$, we apply Lemma 6.1 to z and $t = \chi_\sigma(x_\sigma)$ to find an integer k_σ such that $z \cdot t^{k_\sigma} \notin \mathbb{T}_+$. We then put $g_\sigma = k_\sigma x_\sigma$. Since $\chi_\sigma \left(\coprod_{\eta_0\leq\beta\leq\sigma} g_\beta\right) = z \cdot t^{k_\sigma} \notin \mathbb{T}_+$, the element g_σ satisfies (a) and (b) at the stage σ. The recursive definitions are complete.

By (a), the supports of g_η's with $\eta \geq \eta_0$ are disjoint, so we can put $h_0 = \coprod_{\eta_0\leq\eta<\omega_1} g_\eta$. Again by (a), $\mathrm{supp}(h_0) \subset \bigcup\{\mathrm{supp}(x_\eta) \colon \eta_0 \leq \eta < \omega_1\}$ and, since $J \cap \mathrm{supp}(x_\eta) = \emptyset$ for every countable ordinal

$\eta \geq \eta_0$, we see that $J \cap \operatorname{supp}(h_0) = \emptyset$. Therefore $\rho(h_0) = 1$. Since $\rho \in K_{\eta_0}$ and $(\mathbb{T}_+)_{h_0} \times \mathbb{T}^{D \setminus \{h_0\}}$ is a basic open set in $\mathbb{T}^D$ containing ρ, there exists $\eta \geq \eta_0$ such that $\chi_\eta(h_0) \in \mathbb{T}_+$. However, it follows from (4) of the proposition that $J_\eta \cap \operatorname{supp}(x_\beta) = \emptyset$ if $\eta < \beta < \omega_1$, while condition (a) implies that $\operatorname{supp}(g_\beta) \subset \operatorname{supp}(x_\beta)$. Therefore, the sets J_η and $\operatorname{supp}\left(\coprod_{\eta<\beta<\omega_1} g_\beta\right)$ are disjoint and hence (1) of the proposition implies that $\chi_\eta\left(\coprod_{\eta<\beta<\omega_1} g_\beta\right) = 1$. It now follows from (b) that

$$\chi_\eta(h_0) = \chi_\eta\left(\coprod_{\eta_0 \leq \beta \leq \eta} g_\beta\right) \cdot \chi_\eta\left(\coprod_{\eta<\beta<\omega_1} g_\beta\right) = \chi_\eta\left(\coprod_{\eta_0 \leq \beta \leq \eta} g_\beta\right) \cdot 1 \notin \mathbb{T}_+.$$

This contradiction shows that ρ is discontinuous. □

Lemma 6.3. *Suppose that $\Sigma D \subset G \subset PD$, $g \in G$, $\chi \in G^\wedge$, and $\chi(g) \neq 1$. If χ does not depend on $I \setminus J$, for a countable set $J \subset I$, then there exists a point $x \in G$ with* $\operatorname{supp}(x) \subset J \cap \operatorname{supp}(g)$ *such that $\chi(x) \notin \mathbb{T}_+$.*

Proof. We can find elements $y, z \in PD$ such that $g = y + z$, $\operatorname{supp}(y) = J \cap \operatorname{supp}(g)$, and $\operatorname{supp}(z) \cap J = \emptyset$. Notice that $y \in \Sigma D \subset G$, so $z \in G$. It follows from our choice of z that $\chi(z) = 1$. Since $\chi(y) = \chi(y) \cdot \chi(z) = \chi(y+z) = \chi(g) \neq 1$, we see that $t = \chi(y) \neq 1$. Take an integer n such that $t^n \notin \mathbb{T}_+$. The point $x = ny \in G$ is as required. □

Lemma 6.4. *Suppose that $\Sigma D \subseteq G \subseteq PD$ and $K \subseteq G^\wedge$. If K depends on uncountably many coordinates, then K contains a subset $C = \{\chi_\eta \colon \eta < \omega_1\}$ and G contains a subset $X = \{x_\eta \colon \eta < \omega_1\}$ such that conditions (1)–(4) of Proposition 6.2 hold for C, X, and a suitable collection $\mathfrak{J} = \{J_\eta \colon \eta < \omega_1\}$ of countable subsets of I.*

Proof. To begin, we choose a family $\{J_\chi \colon \chi \in K\}$ of countable subsets of I such that χ does not depend on $I \setminus J_\chi$, for each $\chi \in K$.

Since K depends on uncountably many coordinates, there is a nontrivial character $\chi_0 \in K$. It follows from Lemma 6.3 that there is a point $x_0 \in G$ with $\operatorname{supp}(x_0) \subset J_{\chi_0}$ such that $\chi_0(x_0) \notin \mathbb{T}_+$. We put $J_0 = J_{\chi_0}$. Then (1)–(4) of Proposition 6.2 hold for $\eta = 0$.

Suppose now that $\sigma < \omega_1$ and that χ_η, x_η, and J_η have been defined to satisfy (1)–(4) of Proposition 6.2 for all $\eta < \sigma$. Then the set $T = \bigcup_{\eta<\sigma} J_\eta$ is countable. By assumptions of the lemma, there is a character $\chi_\sigma \in K$ depending on $I \setminus T$, i.e., there is a point $g \in G$ such that $g(i)$ is the neutral element of D_i for each $i \in T$ and $\chi_\sigma(g) \neq 1$. It now follows from Lemma 6.3 that there exists a point $x_\sigma \in G$ with $\operatorname{supp}(x_\sigma) \subset \operatorname{supp}(g) \cap J_{\chi_\sigma}$ such that $\chi_\sigma(x_\sigma) \notin \mathbb{T}_+$. Let $J_\sigma = J_{\chi_\sigma}$. It is clear that χ_η, x_η, and J_η satisfy (1)–(4) of Proposition 6.2 for all $\eta \le \sigma$. The recursive definitions are complete which finishes the proof. □

A slight modification in the above argument can be made in order to deduce the following lemma which will be applied in the proof of Proposition 6.16.

Lemma 6.5. *Suppose that* $\Sigma D \subset G \subset PD$ *and that two sets* $C = \{\chi_\eta : \eta < \omega_1\} \subset G^\wedge$ *and* $\mathfrak{J} = \{J_\eta : \eta < \omega_1\} \subset [I]^{\le\omega}$ *satisfy the following conditions for all* $\eta < \omega_1$*:*

(a) χ_η *does not depend on* $I \setminus J_\eta$*;*

(b) $J_\zeta \subset J_\eta$ *if* $\zeta < \eta$*;*

(c) χ_η *depends on* $I \setminus \bigcup_{\zeta<\eta} J_\zeta$.

Then there exists a set $X = \{x_\eta : \eta < \omega_1\} \subset G$ *such that* C*,* $\mathfrak{J}$*, and* X *satisfy conditions (1)–(4) of Proposition 6.2.*

Proposition 6.6. *Let* $D = \prod_{i\in I} D_i$ *be a product of topological groups. Then every compact set* $K \subset (PD)^\wedge$ *depends at most on countably many coordinates.*

Proof. Let $\Pi = PD$. Suppose to the contrary that a compact set $K \subset \Pi^\wedge$ depends on uncountably many coordinates. By Lemma 6.4, there exist $C \subset K$, $X \subset G$, and $\mathfrak{J} \subset [I]^{\le\omega}$ satisfying conditions (1)–(4) of Proposition 6.2. By the latter proposition, there is an element ρ in the closure of C in $\mathbb{T}^\Pi$ such that $\rho \notin \Pi^\wedge$. But K, being compact, must be closed in $\mathbb{T}^\Pi$, whence $\rho \in K \subset \Pi^\wedge$. This contradiction shows that there must exist $J \in [I]^{\le\omega}$ such that K does not depend on $I \setminus J$. □

The above proposition shows that item (b) in Corollary 5.10 can be omitted:

Proposition 6.7. *Let* $D = \prod_{i\in I} D_i$ *be a product of topological groups. Then the product group* PD *is reflexive if and only if* PD_J *is reflexive for each* $J \in [I]^{\le\omega}$.

Theorem 6.8. *Let* $D = \prod_{i\in I} D_i$ *be a product of discrete Abelian groups. Then the* P*-group* $\Pi = PD$ *and the* ω*-bounded group* $\Pi^\wedge$ *are reflexive.*

Proof. The group PD_J is discrete and hence reflexive, for every $J \in [I]^{\le\omega}$. Therefore the reflexivity of Π follows from Proposition 6.7. Hence the dual group $\Pi^\wedge$ is reflexive as well. □

In the case when the product $D = \prod_{i\in I} D_i$ in the above theorem contains uncountably many nontrivial factors, we obtain the following result that answers a question posed by S. Hernández and P. Nickolas and solves a problem raised in a comment after Proposition 2.10 in [1].

Corollary 6.9. *There exist non-discrete reflexive P-groups as well as ω-bounded noncompact reflexive groups.*

We can now establish the reflexivity of certain P-groups which are not necessarily P-modifications of products of discrete groups. A simple auxiliary lemma is in order:

Lemma 6.10. *Suppose that $\pi\colon G \to H$ is a continuous onto homomorphism of compact groups. Then the homomorphism $\pi\colon PG \to PH$ is open, where PG and PH are P-modifications of the groups G and H, respectively.*

Proof. Let e be the neutral element of G. It is clear that the sets of the form $V = \bigcap_{n\in\omega} U_n$, where U_n's are open neighbourhoods of e in G and $\overline{U}_{n+1} \subset U_n$ for each $n \in \omega$ (the closure is taken in G), constitute a base at e in PG. Therefore, it suffices to verify that every image $\pi(V)$ is open in PH. Notice that the continuous epimorphism $\pi\colon G \to H$ is open since G is compact. Using the compactness of G once again we see that $\pi(V) = \bigcap_{n\in\omega} \pi(U_n)$, so $\pi(V)$ is a G_δ-set in H. Hence $\pi(V)$ is open in PH. $\square$

Proposition 6.11. *Let H be a compact Abelian group and PH the P-modification of H. Then the group PH is reflexive.*

Proof. It is well known that one can find a compact Abelian group G of the form $G = \prod_{i\in I} G_i$, with compact metrizable factors G_i, and a continuous homomorphism π of G onto H (see [13, Lemma 1.6]). By Lemma 6.10, the homomorphism $\pi\colon PG \to PH$ is open. For every $i \in I$, let D_i be the group G_i with the discrete topology. Denote by D the product group $\prod_{i\in I} D_i$. Since the factors G_i are metrizable, the topological groups PG and PD coincide. It now follows from Theorem 6.8 that the group PG is reflexive, while Proposition 5.9 implies the reflexivity of PH. $\square$

According to [3, Section 4.3], a topological group H is *feathered* if it contains a nonempty compact set with a countable neighbourhood base in H. Let us call a topological group H *pseudo-feathered* if there exists a nonempty compact set of type G_δ in H. It is clear that every feathered group is pseudo-feathered and that H is pseudo-feathered if and only if it contains a compact subgroup of type G_δ. An Abelian group is pseudo-feathered iff it admits an open continuous homomorphism with compact kernel onto a group of countable pseudocharacter. In the following result we extend the conclusion of Proposition 6.11 to pseudo-feathered groups.

Proposition 6.12. *Let H be a pseudo-feathered Abelian group. Then the group PH is reflexive.*

Proof. Let C be a compact subgroup of type G_δ in H. Clearly, PC is then an open subgroup of PH. Since a topological group admitting an open subgroup that is reflexive is itself reflexive (Proposition 2.2 of [6]) and the group PC is reflexive by Proposition 6.11, we conclude that PH is reflexive. □

The next result is a common generalization of Theorem 6.8 and Proposition 6.12:

Theorem 6.13. *Let $H = \prod_{i\in I} H_i$ be the product of a family of pseudo-feathered Abelian groups. Then the group PH is reflexive.*

Proof. It is easy to verify that if C_n is a compact set of type G_δ in a space X_n, for each $n \in \omega$, then the compact set $C = \prod_{n\in\omega} C_n$ has type G_δ in the product space $X = \prod_{n\in\omega} X_n$. This observation implies that the group $H_J = \prod_{i\in J} H_i$ is pseudo-feathered for each $J \in [I]^{\le\omega}$. The reflexivity of PH now follows from Proposition 6.7. □

In Theorem 6.14 below we characterize the reflexivity of certain subgroups G of "big" products $PD = P\prod_{i\in I} D_i$ of topological groups in terms of projections $\pi_J(G)$ of G to relatively small subproducts $PD_J = P\prod_{i\in J} D_i$.

Theorem 6.14. *Suppose that $D = \prod_{i\in I} D_i$ is a product of topological groups and $\Sigma D \subset G \subset PD$. Then the group G is reflexive iff the subgroup $\pi_J(G)$ of PD_J is reflexive, for every set $J \subset I$ satisfying $|J| \le \aleph_1$.*

Proof. Necessity. Let G be reflexive. Take any $J \subset I$ satisfying $|J| \le \aleph_1$ and put $H = \pi_J(G)$. By Lemma 4.3, the restriction to G of the projection $\pi_J \colon PD \to PD_J$ is an open homomorphism of G onto H. Hence the reflexivity of H follows from Proposition 5.9.

Sufficiency. Suppose that $\pi_J(G)$ is reflexive, for each $J \subset I$ with $|J| \le \aleph_1$. Since $\Sigma D \subset G$, the equality $\pi_J(G) = D_J$ holds for each $J \in [I]^{\le\omega}$. Therefore, according to Corollary 5.11, it suffices to show that every compact set $K \subset G^\wedge$ depends at most on countably many coordinates. Suppose to the contrary that $G^\wedge$ contains a compact set K which depends on uncountably many coordinates. Apply Lemma 6.4 to choose families $\{\chi_\eta : \eta < \omega_1\} \subset K$, $\{x_\eta : \eta < \omega_1\} \subset G$, and $\{J_\eta : \eta < \omega_1\} \subset [I]^{\le\omega}$ satisfying conditions (1)–(4) of Proposition 6.2. Let $J = \bigcup_{\eta<\omega_1} J_\eta$. Then $J \subset I$ and $|J| \le \aleph_1$. Hence the subgroup $H = \pi_J(G)$ of PD_J is reflexive. Notice that by

Lemma 4.3, the restriction to G of the homomorphism π_J is open when considered as a mapping of G onto H. Let $\eta < \omega_1$. Since χ_η does not depend on $I \setminus J_\eta$ and $J_\eta \subset J$, there exists a continuous character ψ_η on H such that $\chi_\eta = \psi_\eta \circ \pi_J\restriction_G$. We put $\Psi = \{\psi_\eta : \eta < \omega_1\}$.

Denote by φ the continuous homomorphism $(\pi_J\restriction_G)^\wedge$ of $H^\wedge$ to $G^\wedge$. Then $\varphi(\psi_\eta) = \chi_\eta \in K$ for each $\eta < \omega_1$, so $\varphi(\Psi) \subset K$. Since H is a P-group, all compact subsets of H are finite. Hence Corollary 5.8 implies that φ is a topological isomorphism of $H^\wedge$ onto the subgroup $\varphi(H^\wedge)$ of $G^\wedge$. Further, since the homomorphism $\pi_J\restriction_G$ of G onto H is open, it follows from item (c) of Lemma 5.7 that $\varphi(H^\wedge)$ is a closed subgroup of $G^\wedge$. Therefore, $C = K \cap \varphi(H^\wedge)$ is a compact subset of $\varphi(H^\wedge)$ and $L = \varphi^{-1}(C)$ is a compact subset of $H^\wedge$. It follows from $\Psi \subset H^\wedge$ and $\varphi(\Psi) \subset K$ that $\Psi \subset L$. The latter inclusion and the definition of the set Ψ together imply that L depends on uncountably many coordinates.

Indeed, suppose that for some countable set $A \subset J$, every element of L does not depend on $J \setminus A$. In particular, ψ_η does not depend on $J \setminus A$, for each $\eta < \omega_1$. Since $\chi_\eta = \psi_\eta \circ \pi_J\restriction_G$, we see that each χ_η does not depend on $I \setminus A$. It follows from our choice of the families $\{\chi_\eta : \eta < \omega_1\}$, $\{x_\eta : \eta < \omega_1\}$, and $\{J_\eta : \eta < \omega_1\}$ (see conditions (2)–(4) of Proposition 6.2) that $\mathrm{supp}(x_\eta) \subset J_\eta \setminus \bigcup_{\zeta<\eta} J_\zeta$ and $\chi_\eta(x_\eta) \neq 1$, for each $\eta < \omega_1$. Since the sets $A_\eta = J_\eta \setminus \bigcup_{\zeta<\eta} J_\zeta$ are pairwise disjoint, there exists $\eta < \omega_1$ such that $A \cap A_\eta = \emptyset$. Since χ_η does not depend on $I \setminus A$, this implies that $\chi_\eta(x_\eta) = 1$, which is a contradiction. We have thus proved that every compact subset K of $G^\wedge$ depends at most on countably many coordinates and, therefore, G is reflexive. □

Theorem 6.14 makes it possible to find many proper dense reflexive subgroups of big products of pseudo-feathered groups endowed with the P-modified topology:

Corollary 6.15. *Suppose that $D = \prod_{i\in I} D_i$ is a product of pseudo-feathered Abelian groups and let*

$$\Sigma_{\aleph_1} D = \{x \in PD : |\,\mathrm{supp}(x)| \leq \aleph_1\}.$$

Then every group G with $\Sigma_{\aleph_1} D \subset G \subset PD$ is reflexive.

Proof. According to Proposition 5.6 it suffices to show that $\Sigma_{\aleph_1} D$ is reflexive. It is clear that $\pi_J(\Sigma_{\aleph_1} D) = D_J = \prod_{i\in J} D_i$ for each $J \subset I$ with $|J| \leq \aleph_1$, where $\pi_J \colon D \to D_J$ is the projection. By Theorem 6.13, the groups PD_J are reflexive. One applies Theorem 6.14 to conclude that the group $\Sigma_{\aleph_1} D$ is reflexive as well. □

In what follows we identify the additive group $\mathbb{Z}_2 = \{0,1\}$ with the multiplicative subgroup $\{1,-1\}$ of $\mathbb{T}$. Hence the dual group $G^\wedge$ of every boolean P-group G is topologically isomorphic to a subgroup of $\mathbb{Z}_2^G$. We will now show that the P-group $\Pi = P\mathbb{Z}_2^{\omega_1}$ contains proper dense reflexive subgroups of the form G_ξ defined after Theorem 2.

Proposition 6.16. *Every compact set $K \subset (G_\xi)^\wedge$ depends at most on countably many coordinates, where ξ is an ultrafilter on ω_1 containing the complements to countable sets. Hence the group G_ξ is reflexive.*

Proof. On the contrary, suppose that a compact set $K \subset (G_\xi)^\wedge$ depends on uncountably many coordinates. We construct two sets $\{\chi_\eta : \eta < \omega_1\} \subset K$ and $\{J_\eta \colon \eta < \omega_1\} \subset [\omega_1]^{\leq\omega}$ satisfying the following conditions for all $\eta < \omega_1$:

(i) χ_η does not depend on $\omega_1 \setminus J_\eta$;
(ii) $J_\zeta \subset J_\eta$ if $\zeta < \eta$;
(iii) $\eta \in J_\eta$;
(iv) χ_η depends on the set $\omega_1 \setminus \left(\{\eta\} \cup \bigcup_{\zeta<\eta} J_\zeta\right)$.

Let $\chi_0 \in K$ be a nontrivial character. Take a countable set $J_0 \subset \omega_1$ such that $0 \in J_0$ and χ_0 does not depend on $\omega_1 \setminus J_0$. Suppose that for some $\eta < \omega_1$, the sequences $\{\chi_\zeta : \zeta < \eta\} \subset K$ and $\{J_\zeta : \zeta < \eta\} \subset [\omega_1]^{\leq\omega}$ have been defined to satisfy conditions (i)–(iv). Then we put $T_\eta = \bigcup_{\zeta<\eta} J_\zeta$ and choose $\chi_\eta \in K$ such that χ_η depends on the set $\omega_1 \setminus (T_\eta \cup \{\eta\})$. Such a choice of χ_η is possible since the set $T_\eta \cup \{\eta\}$ is countable and K depends on uncountably many coordinates. Let J'_η be a countable subset of ω_1 such that χ_η does not depend on $\omega_1 \setminus J'_\eta$. Then the set $J_\eta = J'_\eta \cup T_\eta \cup \{\eta\}$ is countable and χ_η does not depend on $\omega_1 \setminus J_\eta$. Therefore, the sets $\{\chi_\zeta : \zeta \leq \eta\}$ and $\{J_\zeta : \zeta \leq \eta\}$ satisfy (i)–(iv) at the step η.

For every $A \in \xi$, we put $F_A = \overline{\{\chi_\eta \colon \eta \in A\}}^{\mathbb{T}^{G_\xi}}$ and $\mathcal{C} = \{F_A \colon A \in \xi\}$. It follows from $\chi_\eta \in K$ for all $\eta < \omega_1$ and the compactness of K that $F_A \subset K$, for each $A \in \xi$. Since $\mathcal{C}$ is a family of closed subsets of the compact space $\mathbb{T}^{G_\xi}$ with the finite intersection property, $\mathcal{C}$ has non-empty intersection. Let ρ be a point in $\bigcap\{F_A \colon A \in \xi\}$. Clearly, $\rho \in K$, so ρ is continuous. Let J_ρ be a countable subset of ω_1 such that ρ does not depend on $\omega_1 \setminus J_\rho$.

Since Σ is a dense subgroup of both G_ξ and $\Pi = P\mathbb{Z}_2^{\omega_1}$, the characters ρ and χ_η admit continuous extensions $\overline{\rho} \colon \Pi \to \mathbb{T}$ and $\overline{\chi}_\eta \colon \Pi \to \mathbb{T}$, for each $\eta < \omega_1$. Again, the density of G_ξ in Π implies that $\overline{\rho}$ does not depend on $\omega_1 \setminus J_\rho$ and $\overline{\chi}_\eta$ does not depend on $\omega_1 \setminus J_\eta$.

Denote by $\mathbf{1}$ the element of Π all of whose coordinates are equal to 1. For every $\eta < \omega_1$, let $H_\eta = \{x \in \Pi\colon x(\eta) = 0\}$ and take a character ψ_η on $\Pi = \langle H_\eta, \mathbf{1}\rangle$ defined by $\psi_\eta(x) = \overline{\chi}_\eta(x)$ and $\psi_\eta(x + \mathbf{1}) = \overline{\chi}_\eta(x) + \overline{\rho}(\mathbf{1})$, for all $x \in H_\eta$. Since $\eta \in J_\eta$, we have $U(J_\eta) \subset H_\eta$. It then follows from the above definition that ψ_η does not depend on $\omega_1 \setminus J_\eta$.

Claim 1. Put $T_\eta = \bigcup_{\zeta<\eta} J_\zeta$. For every $\eta < \omega_1$, the character ψ_η depends on $\omega_1 \setminus T_\eta$.

Proof of Claim 1. Indeed, by (iv) of the recursive construction, χ_η depends on $\omega_1 \setminus (\{\eta\} \cup T_\eta)$. Hence there exists $x \in G_\xi \cap U(\{\eta\} \cup T_\eta)$ such that $\chi_\eta(x) \neq 1$. Then $x \in H_\eta$ and $\psi_\eta(x) = \overline{\chi}_\eta(x) = \chi_\eta(x) \neq 1$, and we see that ψ_η depends on $\omega_1 \setminus T_\eta$.

Claim 2. For all $\alpha < \omega_1$, $\overline{\rho} \in \overline{\{\psi_\eta\colon \alpha \le \eta < \omega_1\}}^{\mathbb{T}^\Pi}$.

Proof of Claim 2. Fix $\alpha < \omega_1$ and take $\{g_1, \ldots, g_n\} \subset \Pi$. We can assume that there exists $m \le n$ such that $\{g_1, \ldots, g_m\} \subset G_\xi$ and $\{g_{m+1}, \ldots, g_n\} \subset \Pi \setminus G_\xi$. Then $\{g_1, \ldots, g_m, g_{m+1} + \mathbf{1}, \ldots, g_n + \mathbf{1}\} \subset G_\xi$. Let $A = \omega_1 \setminus \bigcup_{i \le m} \operatorname{supp}(g_i)$, $B = \bigcap_{m<k\le n} \operatorname{supp}(g_k)$, and $C = A \cap B \cap [\alpha, \omega_1)$. It follows from our choice of $g_1, \ldots, g_n$ that $C \in \xi$. So, $\rho \in F_C$. Take $\eta \in C$ such that $\rho(g_i) = \chi_\eta(g_i)$ whenever $1 \le i \le m$ and $\rho(g_k + \mathbf{1}) = \chi_\eta(g_k + \mathbf{1})$ whenever $m < k \le n$. If $1 \le i \le m$, then $g_i(\eta) = 0$ because $\eta \in A$. So, $g_i \in H_\eta$ and $\psi_\eta(g_i) = \overline{\chi}_\eta(g_i) = \chi_\eta(g_i) = \rho(g_i) = \overline{\rho}(g_i)$. If $m < k \le n$, then $(g_k + \mathbf{1})(\eta) = 0$ because $\eta \in B$. So, $(g_k + \mathbf{1}) \in H_\eta$ and $\psi_\eta(g_k + \mathbf{1}) = \overline{\chi}_\eta(g_k + \mathbf{1}) = \chi_\eta(g_k + \mathbf{1}) = \rho(g_k + \mathbf{1}) = \overline{\rho}(g_k + \mathbf{1})$. Since ψ_η and $\overline{\rho}$ are homomorphisms and $\psi_\eta(\mathbf{1}) = \overline{\rho}(\mathbf{1})$, we see that $\psi_\eta(g_k) = \overline{\rho}(g_k)$. Therefore, $\overline{\rho} \in \overline{\{\psi_\eta\colon \alpha \le \eta < \omega_1\}}^{\mathbb{T}^\Pi}$. This completes the proof of Claim 2.

Now, we have a character $\overline{\rho} \in \Pi^\wedge$, a family of characters $\{\psi_\eta\colon \eta < \omega_1\} \subset \Pi^\wedge$, and a family $\{J_\eta : \eta < \omega_1\}$ of countable subsets of ω_1. If follows from our definition of the characters ψ_η's and the above conditions (i), (ii), and (v) that $\{\psi_\eta\colon \eta < \omega_1\}$ and $\{J_\eta : \eta < \omega_1\}$ satisfy (a)–(c) of Lemma 6.5 (with ψ_η's in place of χ_η's). Since $\overline{\rho} \in \bigcap_{\alpha<\omega_1} \overline{\{\psi_\eta\colon \alpha \le \eta < \omega_1\}}^{\mathbb{T}^\Pi}$, we are in position to use Proposition 6.2 to obtain a contradiction with the fact that $\overline{\rho}$ is continuous. This contradiction shows that the compact set K depends at most on countably many coordinates. The reflexivity of G_ξ now follows from Theorem 5.5. □

To finish this section, we extend the conclusion of Proposition 6.16 to subgroups G_ξ of the group $P\mathbb{Z}_2^\tau$, for any uncountable cardinal τ.

Theorem 6.17. *Let $\tau > \omega$ be a cardinal and ξ an ultrafilter on τ containing the complements to countable sets. Then subgroup G_ξ of $P\mathbb{Z}_2^\tau$ is reflexive.*

Proof. According to Theorem 6.14 it suffices to verify that the subgroup $\pi_J(G_\xi)$ of the group $P\mathbb{Z}_2^J$ is reflexive, for every set $J \subset \tau$ satisfying $|J| \le \aleph_1$. Let us consider two possible cases.

Case 1. $J \notin \xi$. Then the definition of G_ξ implies that $\pi_J(G_\xi) = \mathbb{Z}_2^J$, so the reflexivity of $\pi_J(G_\xi)$ is immediate from Theorem 6.8.

Case 2. $J \in \xi$. Put $\eta = \{J \cap A : A \in \xi\}$. Then η is an ultrafilter on J containing the complements to countable sets. Further, the definition of G_ξ implies that the projection $\pi_J(G_\xi)$ of G_ξ coincides with the subgroup G_η of $P\mathbb{Z}_2^J$. Identifying J and ω_1 and applying Proposition 6.16, we see that the group $\pi_J(G_\xi)$ is again reflexive. □

7. Non-reflexive P-groups

We would like to trace the border between reflexivity and non-reflexivity for P-groups G such that $\Sigma D \subset G \subset PD$, where $D = \prod_{i\in I} D_i$ is a product of discrete groups. Recall that by an old result of Leptin in [17] (see also [5, Example 17.11]), the subgroup

$$\sigma\mathbb{Z}_2^{\omega_1} = \{x \in \mathbb{Z}_2^{\omega_1} : \operatorname{supp}(x) \text{ is finite}\}$$

of $P\mathbb{Z}_2^{\omega_1}$ is not reflexive. We now extend this fact to some dense subgroups of the groups of the form PD.

Let G be a subgroup of PD containing ΣD. For each $i \in I$, let $\pi_i : G \to D_i$ be the projection, $\pi_i(x) = x(i)$. For a set $J \subset I$, we also put $F_J = \overline{\{\varphi \circ \pi_i : i \in J,\ \varphi \in (D_i)^\wedge\}}^{\mathbb{T}^G} \bigcup \{\mathbf{1}\}$, where $\mathbf{1}$ is the identity of $G^\wedge$.

Lemma 7.1. *For every $J \in [I]^{\le\omega}$, $F_J \subset G^\wedge$.*

Proof. Suppose that $J \in [I]^{\le\omega}$ and take any $\rho \in F_J$. Then $U(J)$ is an open set in PD containing 0_G such that $\rho(U(J) \cap G) = \{1\}$. Therefore ρ is continuous. □

Lemma 7.2. *Let J be a nonempty subset of I and $\psi \in F_J$, $\psi \ne \mathbf{1}$. Then ψ is continuous as a character on G if and only if there exists $x \in \Sigma D$ such that $\psi(x) \ne 1$.*

Proof. Suppose that ψ is continuous. Since $\psi \ne \mathbf{1}$ and ΣD is dense in G, there is a point $x \in \Sigma D$ such that $\psi(x) \ne 1$.

Conversely, take $x \in \Sigma D$ such that $\psi(x) \ne 1$ and write $F_J = F_{J\cap \operatorname{supp}(x)} \bigcup F_{J\setminus \operatorname{supp}(x)}$. Since $(\chi \circ \pi_i)(x) = 1$ for all $i \in J \setminus \operatorname{supp}(x)$ and all $\chi \in (D_i)^\wedge$, we have that $\psi \notin F_{J\setminus \operatorname{supp}(x)}$. Then $\psi \in F_{J\cap \operatorname{supp}(x)}$. It now follows from Lemma 7.1 that ψ is continuous. □

Proposition 7.3. *Let $D = \prod_{i\in I} D_i$ be a product of nontrivial discrete Abelian groups. Then the subgroup $\Sigma = \Sigma D$ of PD is not reflexive provided that $|I| > \omega$. Furthermore, the bidual group $\Sigma^{\wedge\wedge}$ is discrete.*

Proof. Given $\psi \in F_I$, $\psi \neq \mathbf{1}$, there exists $x \in \Sigma$ such that $\psi(x) \neq 1$. It now follows from Lemma 7.2 that $F_I \subset \Sigma^\wedge$. The set F_I is compact as a closed subset of $\mathbb{T}^\Sigma$. Since F_I does not depend on countably many coordinates (it actually depends on every index $i \in I$), the group Σ is not reflexive by Theorem 5.5. It is easy to see that F_I generates a dense subgroup of $\Sigma^\wedge$, so $(F_I)^\triangleright$ contains only the neutral element of $G^{\wedge\wedge}$. Hence the bidual group $\Sigma^{\wedge\wedge}$ is discrete. □

Sets of the form F_J, with $J \neq I$, can also be used to show that some subgroups larger than $\Sigma = \Sigma D$ are not reflexive. This is done in Lemma 7.5 for groups of the form $G_L = \langle \Sigma, L\rangle$ with $L \subset PD$ satisfying $|I \setminus \bigcup_{x\in L} \operatorname{supp}(x)| \geq \omega_1$.

Lemma 7.4. *Let L be a subset of PD and $J = I \setminus \bigcup_{x\in L} \operatorname{supp}(x)$. If $|J| > \omega$, then $F_J = \overline{\{\varphi\circ\pi_i \colon i \in J,\ \varphi \in (D_i)^\wedge\}}^{\mathbb{T}^{G_L}} \bigcup \{\mathbf{1}\}$ is a compact subset of $(G_L)^\wedge$.*

Proof. Take $\psi \in F_J$, $\psi \neq \mathbf{1}$. Assume that $\psi(z) = 1$ for all $z \in \Sigma$. Take $g \in G$ such that $\psi(g) \neq 1$ and write $g = z + n_1x_1 + \ldots + n_kx_k$, where $z \in \Sigma$, $x_1, \ldots, x_k \in L$, and $n_1, \ldots, n_k \in \mathbb{Z}$. Since $\psi(g) \neq 1$ and $U = \{y \in \mathbb{T}^G : y(g) \neq 1\}$ is an open neighbourhood of ψ, there are $j \in J$ and $\varphi \in (D_j)^\wedge$ such that $(\varphi\circ\pi_j)(g) \in U$. Then $(\varphi\circ\pi_j)(g) \neq 1$ which is a contradiction because

$$(\varphi\circ\pi_j)(g) = (\varphi\circ\pi_j)(n_1x_1 + \cdots + n_kx_k) = \varphi(n_1x_1(j) + \cdots + n_kx_k(j)) = \varphi(0_G) = 1.$$

So, there is $z \in \Sigma$ such that $\psi(z) \neq 1$. Now the continuity of ψ follows from Lemma 7.2. □

Theorem 7.5. *Let L be a subset of PD such that the set $J = I \setminus \bigcup_{x\in L} \operatorname{supp}(x)$ is uncountable. Then the subgroup $G = \langle \Sigma, L\rangle$ of PD is not reflexive.*

Proof. Put $F_J = \overline{\{\varphi\circ\pi_i \colon i \in J,\ \varphi \in (D_i)^\wedge\}}^{\mathbb{T}^G}$. It follows from Lemma 7.4 that $F_J \subset G^\wedge$. Since F_J depends on every coordinate $i \in J$, we conclude that G is not reflexive in view of Theorem 5.5. □

The preceding results make use of sets of the form F_J to see that some subgroups G with $\Sigma D \subseteq G \subset PD$ are not reflexive. Sets of this sort were already used by Leptin [17] to evidence the nonreflexivity of $\sigma\mathbb{Z}_2^{\omega_1}$. We see next that F_J may not be contained in $G^\wedge$, for some $G = \langle \Sigma D, a\rangle$ and $J \subset I$. This makes necessary a different approach to show, as we do in Theorem 7.8, that these groups are not reflexive.

Let $\Pi = PD$, $\Sigma = \Sigma D$, and a be a point in $\Pi \backslash \Sigma$ such that $na \in \Sigma$ for some integer $n > 1$. We put $G = \langle \Sigma, a \rangle$, $J = \operatorname{supp}(a)$, and $F_J = \overline{\{\varphi \circ \pi_i \colon i \in J,\ \varphi \in (D_i)^\wedge\}}^{\mathbb{T}^G}$. Note that $G = \{g + ka \colon g \in \Sigma,\ 0 \le k < n\}$.

Lemma 7.6. *If $G = \langle \Sigma, a \rangle$, then F_J is not contained in $G^\wedge$.*

Proof. Since $na \in \Sigma$, there exists a countable set $C \subset I$ such that $na(i) = 0_i$ for each $i \in I \setminus C$. Therefore, we can find a divisor m of n with $m > 1$ and an uncountable set $A \subset J \setminus C$ such that the order of $a(i)$ equals m for each $i \in A$.

Let us define ψ in $\operatorname{Hom}(G, \mathbb{T})$ as follows: $\psi(g + la) = e^{(2\pi l/m)i}$ for all $g \in \Sigma$ and $l \in \mathbb{Z}$. Since $\psi(g) = 1$ for each $g \in \Sigma$ and $\psi(l_1 a) = \psi(l_2 a)$ whenever m divides $l_1 - l_2$, our definition of ψ is correct. We now show that $\psi \in F_A$. For this, let g_k be points in Σ, l_k be integers, and V_k open sets in $\mathbb{T}$ such that $\psi(g_k + l_k a) \in V_k$ for all $k \le N$, where $N \in \mathbb{N}$. Since $|\bigcup_{0 \le k \le N} \operatorname{supp}(g_k)| \le \omega$ and $|A| \ge \omega_1$, we can choose $j \in A \backslash \bigcup_{0 \le k \le N} \operatorname{supp}(g_k)$. Since the order of $a(j)$ equals m, there is a character $\rho \in (D_j)^\wedge$ such that $\rho(a(j)) = e^{(2\pi/m)i}$. Now it is easy to see that $(\rho \circ \pi_j)(g_k + l_k a) = \psi(g_k + l_k a) \in V_k$ whenever $0 \le k \le N$. Therefore $\psi \in F_A \subset F_J$. The discontinuity of ψ follows from Lemma 7.2 because $\psi \ne \mathbf{1}$ and $\psi(g) = 1$ for all $g \in \Sigma$. This proves that $\psi \in F_J \setminus G^\wedge \ne \emptyset$. $\square$

By Lemma 7.6, the argument used in Proposition 7.3 for ΣD does not work for $\langle \Sigma, \overline{1} \rangle$, where $\Sigma = \Sigma \mathbb{Z}_2^{\omega_1}$ and $\overline{1}$ is the element of $\mathbb{Z}_2^{\omega_1}$ all of whose coordinates are equal to 1. However, we show in Theorem 7.8 that the group $\langle \Sigma, \overline{1} \rangle$ is not reflexive either.

Lemma 7.7. *Let $\Pi = P\mathbb{Z}_2^{\omega_1}$ and $\Sigma = \Sigma\Pi$. Suppose that a subgroup L of Π has the property that for each $\alpha < \omega_1$, there exists an ordinal $\beta(\alpha)$ such that $\alpha < \beta(\alpha) < \omega_1$ and the restriction to L of the projection $p_{J(\alpha)} \colon \Pi \to \mathbb{Z}_2^{J(\alpha)}$ is injective, where $J(\alpha) = \beta(\alpha) \setminus \alpha$. Then the subgroup $G = \Sigma + L$ of Π is not reflexive.*

Proof. For every $\alpha < \omega_1$, let $\gamma(\alpha) = \beta(\alpha) + 1$ and decompose $\mathbb{Z}_2^{\gamma(\alpha)} = H_\alpha \oplus L_\alpha$, with $L_\alpha = p_{\gamma(\alpha)}(L)$, where $p_{\gamma(\alpha)} \colon \Pi \to \mathbb{Z}_2^{\gamma(\alpha)}$ is the projection. Observe that we can (and so we will) choose H_α to satisfy the following condition:

$$\left\{x \in \mathbb{Z}_2^{\gamma(\alpha)} \colon x(\eta) = 0 \text{ for all } \eta \in J(\alpha)\right\} \subset H_\alpha. \qquad (1)$$

We also choose, for every $\alpha < \omega_1$, a homomorphism $\psi_\alpha \colon \mathbb{Z}_2^{\gamma(\alpha)} \to \mathbb{Z}_2$ such that $\psi_\alpha(L_\alpha) = \{0\}$ and $\psi_\alpha \restriction_{H_\alpha} = \pi_{\beta(\alpha)} \restriction_{H_\alpha}$, where $\pi_{\beta(\alpha)}$ is the projection of $\mathbb{Z}_2^{\gamma(\alpha)}$ to the $\beta(\alpha)$th factor $(\mathbb{Z}_2)_{\beta(\alpha)}$. Let us put $\chi_\alpha = \psi_\alpha \circ p_{\gamma(\alpha)}$. Since both homomorphisms ψ_α and $p_{\gamma(\alpha)}$ are continuous, so is χ_α.

For every $\alpha \le \omega_1$, let $E_\alpha = \overline{\{\chi_\sigma \colon \sigma < \alpha\}}^{\mathbb{Z}_2^G}$. We claim that $E_{\omega_1} \subset G^\wedge$. Indeed, let $\rho \in E_{\omega_1}$. If $\rho \ne \mathbf{0}$, there is $g \in G$ with $\rho(g) = 1$. Take $g_1 \in \Sigma$ and $g_2 \in L$ such that $g = g_1 + g_2$. If $\rho(g_2) = 1$, then there exists $\alpha < \omega_1$ such that $\chi_\alpha(g_2) = 1$ which is not possible because $\chi_\alpha(g_2) = \psi_\alpha(p_{\gamma(\alpha)}(g_2)) = 0$. We thus have $\rho(g_1) = 1$. Now, mimicking the proof of Lemma 7.2, we may take $\alpha < \omega_1$ such that $\operatorname{supp}(g_1) \subset \alpha$, and write

$$E_{\omega_1} = E_\alpha \bigcup \overline{\{\chi_\sigma \colon \sigma \ge \alpha\}}^{\mathbb{Z}_2^G}.$$

Take any $\sigma \ge \alpha$. It follows from (1) that $p_{\gamma(\sigma)}(g_1) \in H_\sigma$ and hence $\chi_\sigma(g_1) = \psi_\sigma(p_{\gamma(\sigma)}(g_1)) = \pi_{\beta(\sigma)}(p_{\gamma(\sigma)}(g_1)) = 0$. Since $\sigma \ge \alpha$ is arbitrary, we conclude that $\rho \notin \overline{\{\chi_\sigma \colon \sigma \ge \alpha\}}^{\mathbb{Z}_2^G}$ and, therefore, $\rho \in E_\alpha$. We have thus that $E_{\omega_1} = \bigcup_{\alpha<\omega_1} E_\alpha$. Since all elements of E_α are continuous (they all do not depend on $\omega_1 \setminus \gamma(\alpha)$), we see that E_{ω_1} is a compact subset of $G^\wedge$.

Given $\alpha < \omega_1$, we define $b_\alpha \in \Sigma$ by $b_\alpha(\gamma) = 0$ if $\gamma \ne \beta(\alpha)$ and $b_\alpha(\beta(\alpha)) = 1$. This implies that $p_{\gamma(\alpha)}(b_\alpha) \in H_\alpha$ and, therefore, $\chi_\alpha(b_\alpha) = \psi_\alpha(p_{\gamma(\alpha)}(b_\alpha)) = \pi_{\beta(\alpha)}(p_{\gamma(\alpha)}(b_\alpha)) = b_\alpha(\beta(\alpha)) = 1$. It follows that χ_α depends on the index $\beta(\alpha)$ which in its turn implies that E_{ω_1} depends on uncountably many coordinates. By Theorem 5.5, the group G is not reflexive. $\square$

Theorem 7.8. *Let τ be an uncountable cardinal, $\Pi = P\mathbb{Z}_2^\tau$, and $\Sigma = \Sigma\Pi$. Then, for every countable subgroup L of Π, the group $G = \Sigma + L \subset \Pi$ is not reflexive.*

Proof. Let $L_0 = L \cap \Sigma$. Since every subgroup of the boolean group L is a direct summand, there exists a subgroup L_1 of L such that $L = L_0 \oplus L_1$. Since $L_0 \subset \Sigma$, we see that $G = \Sigma + L = \Sigma + L_0 + L_1 = \Sigma + L_1$. It follows from our definition of L_1 that the intersection $\Sigma \cap L_1$ is trivial, so the set $\operatorname{supp}(x)$ is uncountable, for each $x \in L_1$ distinct from 0_G.

Take a subset J of τ such that $|J| = \aleph_1$ and $|J \cap \operatorname{supp}(x)| = \aleph_1$ for each $x \in L_1$, $x \ne 0_G$. By Theorem 6.14, it suffices to show that the subgroup $\pi_J(G)$ of $P\mathbb{Z}_2^J$ is not reflexive. Since $\pi_J(G) = \pi_J(\Sigma) + \pi_J(L_1)$ and $\pi_J(\Sigma) = \Sigma\mathbb{Z}_2^J$, we can assume without loss of generality that $\tau = J = \omega_1$. Hence G is a subgroup of $\Pi = P\mathbb{Z}_2^{\omega_1}$.

In view of Lemma 7.7 it suffices to show that for every $\alpha < \omega_1$, there is a countable ordinal $\beta(\alpha) > \alpha$ such that the restriction to L_1 of the projection $p_{[\alpha,\beta(\alpha))}$ is injective. Given an ordinal $\alpha < \omega_1$, we take an element $\beta(x) \in \operatorname{supp}(x) \setminus \alpha$, for each $x \in L_1 \setminus \{0_G\}$, and put

$$B = \{\beta(x) : x \in L_1,\ x \ne 0_G\}.$$

Then $|B| \leq |L_1| \leq \omega$, so there exists a countable ordinal $\beta(\alpha) > \alpha$ such that $B \subseteq \beta(\alpha)$. Put $J(\alpha) = \beta(\alpha) \setminus \alpha$. It is clear that the restriction to L_1 of the projection $p_{J(\alpha)}$ is one-to-one since $B \subset J(\alpha)$ and B intersects the set $\mathrm{supp}(x) \setminus \alpha$, for each $x \in L_1$ distinct from 0_G. This finishes the proof of the theorem. □

8. Problem section

Here we present several problems whose solutions can substantially improve our understanding of the duality theory for P-groups. The first of them arises in an attempt to extend Theorem 6.8 to arbitrary products of reflexive P-groups:

Problem 1. Let $\Pi = \prod_{i \in I} G_i$ be the product of a family of reflexive P-groups. Is the group $P\Pi$ reflexive?

According to Proposition 6.7 it suffices to consider the case when the index set I in the above problem is countable. One can try to prove (or refute) a more general form of the above problem inspired by Proposition 6.11:

Problem 2. Let G be a reflexive topological group. Is the group PG then reflexive?

A direct verification shows that every reflexive P-group constructed so far contains a discrete (hence closed) subgroup of cardinality 2^ω. This explains the origin of the following problem:

Problem 3. Does there exist a nondiscrete reflexive Lindelöf P-group?

In the next problem we pretend to generalize Theorem 6.17:

Problem 4. Let τ be an uncountable cardinal and G be a subgroup of $\Pi = P\mathbb{Z}_2^\tau$ such that $|\Pi/G| < \omega$ (or $|\Pi/G| \leq \omega$). Is G reflexive? What if, additionally, G contains $\Sigma\Pi$?

We do not know whether Theorem 7.8 extends to bigger subgroups of $P\mathbb{Z}_2^{\omega_1}$:

Problem 5. Is is true that the subgroup $\Sigma + L$ of $\Pi = P\mathbb{Z}_2^{\omega_1}$ fails to be reflexive, for any subgroup L of Π satisfying $|L| \leq \aleph_1$?

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