A Nonparametric Approach to the Noise Density in Stochastic Volatility Models

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Abstract

We propose a nonparametric method to determine the functional form of the noise density in discrete-time stochastic volatility models of financial returns. Our approach suggests that the assumption of Gaussian noise is often adequate, but we do observe deviations from Gaussian noise for some assets, for instance gold.

Key words: Stochastic volatility, noise density, nonparametric, moment ratio

JEL codes: C14, G10

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1 Introduction

Simple nonlinear transformations of financial returns, typically interpreted as measures of volatility, exhibit significant positive autocorrelations, while autocorrelations in returns themselves are absent beyond a lag of a few minutes (see, e.g., Pagan, 1996). These statistical regularities have motivated *stochastic volatility (SV) models* that decompose returns into the product of a noise term and a slowly varying volatility factor. Since both the volatility factor and the noise term are latent variables, any assumption about the probability density of the noise term will have an influence on the modeling of the volatility factor. We propose a novel non-parametric method, in the sense that it does not depend on specifications of the volatility factor, to determine the functional form of the noise density in the SV decomposition.

2 Stochastic Volatility Decomposition

Financial returns, $r(t)$, are observed at integer multiples of a time resolution Δt. The correlation in even functions of $r(t)$, e.g. in absolute or squared returns, motivates the discrete SV decomposition (see, e.g., Cont, 2001)

$$r(t) = \sigma(t) \cdot \eta(t), \tag{1}$$

where $\eta(t)$ is an *iid* noise term with zero mean, and $\sigma(t)$ is a positive volatility factor that is independent of $\eta(t)$. Our non-parametric approach to the probability density (pdf) of $\eta(t)$ assumes that **(i)** $\sigma(t)$ is independent of η at previous times,[1] and that **(ii)** $\sigma(t)$ varies slowly relative to $\eta(t)$, i.e.

$$E[\sigma^q(t+\tau)\,\sigma^q(t)] = E[\sigma^{2q}(t)] \tag{2}$$

for $\tau \ll \tau_c$, where $\tau_c \gg \Delta t$ denotes the characteristic time scale in the dynamics of the volatility factor. Since $E[|r|^{2q}]$ may not exist for $q > 2$, we will limit q to the interval $q \in [0, 2]$. Finally, in order to estimate error bars in our empirical applications, we need the additional assumption that **(iii)** returns are independent for time separations much larger than τ_c.

The choice of the stochastic process governing the dynamics of $\sigma(t)$ is usually motivated by the desire to simultaneously obtain some degree of analytical tractability for the volatility process while reasonably describing the empirical regularities (see, e.g., Shephard, 2005). The choice of the pdf of η is arbitrary; interpreting the noise as the result of many independent changes, one would frequently assume Gaussian noise, though other functional forms have been suggested as well, for instance in the "GARCH-t" model of Bollerslev (1987).

3 Ratio of Moments

In order to obtain information about the noise density, we consider the following *observable* moment ratio of returns

$$M(q) = \frac{E[|r(t+\tau)\;r(t)|^q]}{E[|r(t)|^{2q}]} \; . \tag{3}$$

Inserting eq. (1) and utilizing assumption **(i)**, $M(q)$ can be written as

$$M(q) = \frac{E[\sigma(t+\tau)^q\,\sigma(t)^q]}{E[\sigma(t)^{2q}]} \cdot \frac{E[|\eta(t)|^q]^2}{E[|\eta(t)|^{2q}]} \; . \tag{4}$$

Under assumption **(ii)**, $M(q)$ depends only on η for small τ, but not on σ:

$$M(q) \equiv M_\eta(q) = \frac{E[|\eta(t)|^q]^2}{E[|\eta(t)|^{2q}]} \,. \tag{5}$$

If the noise is Gaussian, the ratio $M_\eta(q)$ turns out to be

$$M_\eta(q) = \frac{\Gamma^2((q+1)/2)}{\sqrt{\pi}\,\Gamma(q+1/2)}, \tag{6}$$

where $\Gamma(\cdot)$ denotes the Euler gamma function. Notice that $M_\eta(q)$ does not depend on the variance of a normally distributed noise factor. Since we are dealing with ratios of moments, $M_\eta(q)$ will be parameter-free for any pdf of η that contains a single scale parameter. Moreover, $M_\eta(2)$ is the inverse of the kurtosis of η. Thus a value of $M_\eta(2)$ smaller (larger) than 1/3 indicates that the pdf of η is leptokurtic (platykurtic).

4 Empirical Application

In order to judge whether the theoretical prediction of the moment ratio in eq. (5) is a reasonable description of the empirical moment ratio in eq. (3), we perform the following procedure. First, we calculate values and errors for the empirical moment ratio by dividing each time series $r(t)$ into B blocks with size much larger than τ_c. In each block, we determine $E[|r(t+\tau)\; r(t)|^q]$ by linearly extrapolating from the first ten lags to $\tau \to 0$, and then divide by $E[|r(t)|^{2q}]$ to obtain block values $M_i(q)$ for $i = 1, \ldots, B$.[2] According to assumption **(iii)**, the $M_i(q)$ are independent, and we can estimate $M(q)$ by the sample mean $\hat{M}(q) = B^{-1} \sum_{i=1}^{B} M_i(q)$, and its 95% confidence interval from the sample variance.[3] We apply this method to the following data on logarithmic returns:

daily DAX values (01/1973–03/2007), DEM/USD exchange rates (01/1974–12/1998), gold prices (08/1976–07/2007), and various instances of individual DAX stock prices (01/1974–12/2001), all taken from Datastream. We also consider high frequency DAX values (01/1985–12/1995) taken from Lux (2001).

The value of τ_c is estimated with an exponential fit from the decay of the autocorrelation of $|r(t)|$, and turns out to be in the range $\tau_c \in [50\Delta t, 100\Delta t]$. Our choice of $B = 8$ represents a trade-off between precision, which is increasing with B, and statistical independence, which is decreasing with B.[4]

[Figure 1 here]

Figure 1 compares the prediction for a Gaussian noise factor in eq. (6) with data $\hat{M}(q)$ computed from daily DAX returns as a function of q, showing agreement within the 95% confidence level. As a graphical way of illustrating how sensitive the results are to the assumed pdf of η, we also plot the prediction for a leptokurtic noise factor with double exponential (symmetric Laplace) pdf, and a platykurtic noise factor with uniform pdf, which both fail to reproduce the moment ratio $\hat{M}(q)$. In case of Laplacian noise the predicted moment ratio is $M_L(q) = \Gamma^2(q+1)/\Gamma(2q+1)$, and for uniform noise it is $M_U(q) = (2q+1)/(q+1)^2$.

[Figure 2 here]

Figure 2 shows the theoretical moment ratio in eq. (6) and the observed moment ratio $\hat{M}(q)$ for a number of assets in our data sample. While the assumption of Gaussian noise seems reasonable for DAX data at various frequencies and for the DEM/USD exchange rate, we do observe deviations for gold and some individual shares in the DAX, for instance in the case of Siemens stock.

The deviations lie below the Gaussian curve (6), favoring a more leptokurtic pdf of the noise factor, which raises the question why the interpretation of the noise as an aggregate of many independent changes on time scales shorter than Δt fails for certain assets. Moreover, since our approach is independent of the volatility process $\sigma(t)$, the finding of a leptokurtic noise component would imply that the leptokurtic nature of returns cannot be adequately captured by embodying heavy tails in σ only.[5]

5 Conclusion

Our nonparametric method suggests that the assumption of a Gaussian noise component in SV models is a reasonable choice, at least in many cases for which we had data, for instance the DEM/USD exchange rate or the German stock index at various frequencies. Gold and some of the individual DAX stocks, on the other hand, exhibit empirical moment ratios that appear to deviate from the assumption of a Gaussian noise component. Depending on the particular goals of a financial engineer or econometrician, leptokurtic specifications of the noise component might prove useful, and the moment-ratio approach presented in this letter provides a quick way to calibrate other noise densities.

We are grateful to Thomas Lux for providing us with his cleaned set of high frequency DAX data, and to Vasyl Golosnoy for useful comments.

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Notes

[1]Notice that this assumption is violated by GARCH models.

[2]Instead of extrapolating, we also performed the analysis for the first lag, $\tau = 1$, and averages of early lags, obtaining very similar results in each case.

[3]Since we are mostly interested in a qualitative impression of the involved magnitudes, we checked that the autocorrelation of $M_i(q)$ does not show any systematic deviation from zero, justifying assumption **(iii)** at least in a first approximation.

[4]Choosing $B = 9, 10, 11$ yields very similar values for the error bars.

[5]The theoretical value of M(2) in GARCH(1,1) turns out to be larger than the empirical value. Thus our findings would in a sense corroborate the results of Bollerslev (1987), who finds that a GARCH(1,1) model often provides a better fit if the noise term is student-t rather than normally distributed.

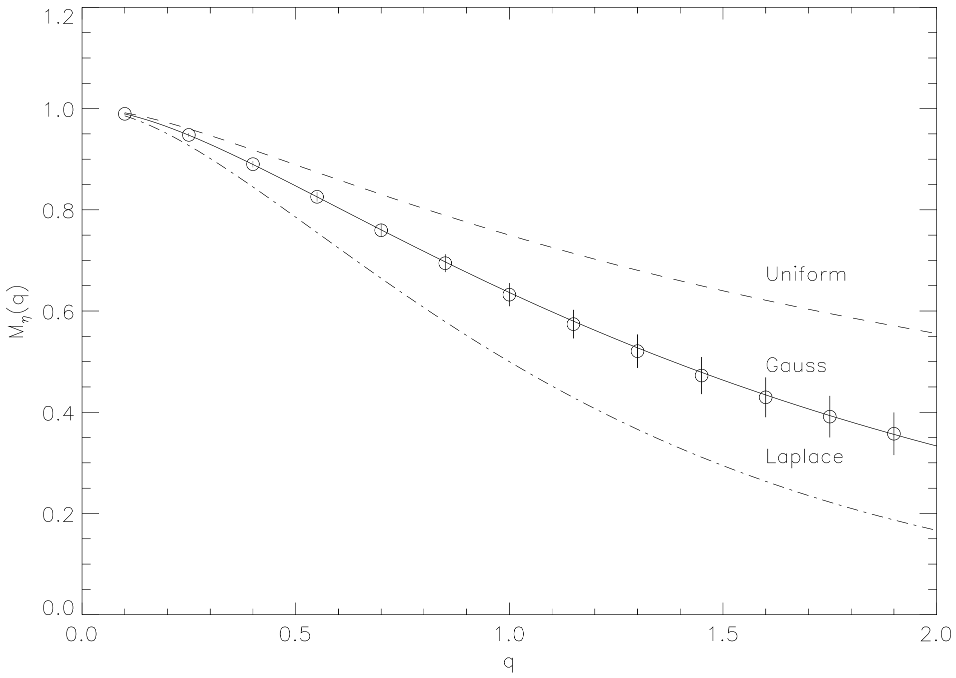

Fig. 1. Theoretical predictions for the moment ratio $M_\eta(q)$ under the assumption of a Gaussian (solid line), a uniform (dashed line), and a double-exponential (dashed dotted line) distribution of the noise factor η. The empirical moment ratio has been computed from daily DAX returns during the period 1973–2007, and the error bars correspond to 95% confidence intervals.

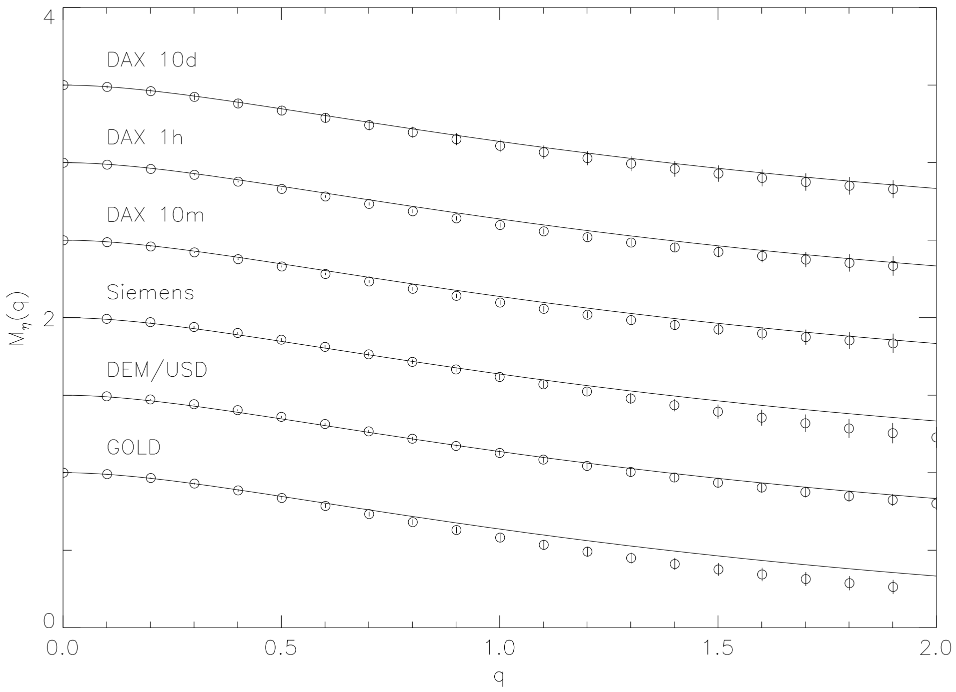

Fig. 2. Moment ratio under the assumption of Gaussian noise, $M_\eta(q)$, compared to the actual moment ratio computed from various asset returns, $\hat{M}(q)$. For better visibility, data and curves are shifted by 0.5 each time. The error bars correspond to 95% confidence intervals. The series show, from top to bottom, the moment ratio for the DAX at 10-day, hourly, and 10-minute frequencies; returns to Siemens shares, the DEM/USD exchange rate, and gold are all measured at daily frequencies.