Artificial Intelligence Research and Development
I. Sanz et al. (Eds.)

doi:10.3233/FAIA230688

Reversal Error in Mathematics: Training Against It Using $\overset{\curvearrowleft}{\varepsilon}$ *coach*

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Abstract. This paper presents $\overset{\curvearrowleft}{\varepsilon}$ *coach*, a serious game prototype which aims at training players to detect and avoid reversal error when representing word problems of the type *"N times as many S as P"* algebraically as an equation.

The $\overset{\curvearrowleft}{\varepsilon}$ *coach* visualizes the relative size of the entities in the relation following a qualitative model based on containers. This serious game also provides a quantitative visualization of the entities related using graphics, it helps players to resolve the formulated equation using discrete values and it shows players' progress.

Keywords. Education, reversal error, SP-type problems, qualitative representations, relative size, reasoning, serious games, serious games, AI for good.

1. Introduction

In mathematics education, the reversal error or the "students-professors problem" (*SP-type* problem), identified by Clement *et al.* [1], refers to reverse the relationship between two variables in an algebraic word problem, e.g. «Write an equation using the variables S and P to represent the following statement: *There are six times as many students as professors at this university*. Use S for the number of students and P for the number of professors». In these statements, Clement *et al.* [1] observed that most of the wrong answers were in the form of $6 \cdot S = P$, which they called reversal error because students reversed the order of the letters, compared to the correct answer $6 \cdot P = S$. And as Clement *et al.*[1] identified: *appears to be rather resilient and requires considerable attention and discussion before students can learn to overcome it* [2,1]. Factors influencing the reversal error were identified by Cohen *et al.*[3] and recently extended by Jankvist and Niss [4].

The literature reports few applications developed to study the reversal error [5]. To our knowledge, the only interactive tool used to address the reversal error is POETIC (Purdue Optimization modeling Education Tool—Interactive equation Component) [6], whose results showed that interactive visualization of equations can reduce the occurrence of this error. However, there was no substantial improvement in error elimination in their study in which the presence of syntactic obstructions in problem statements or the type of magnitude used were not taken into account. Soneira *et al.* [7] showed that there are significant differences in the production of reversal errors depending on the syntactic

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configuration as well as the type of magnitude involved in the statement. A recent study [8] investigated human brain differences during the process of problem-solving with reversal error showing significant differences between participants who committed reversal error and those who did not. The group that committed reversal error needed more resources and a greater cognitive demand, resulting in greater brain activation of the bilateral fronto-parietal network. The group that did not commit reversal error showed brain activation only in the left fronto-parietal network indicating that participants who are competent problem-solvers developed solid algebraic knowledge, in terms of the meaning of the variable and the logic of the process of constructing an equation. Ventura-Campos *et al.* [8] suggested that the reversal error may be a manifestation of a brain function resulting from inadequate learning. As studies in the literature [8,6] showed that traditional abstract teaching methods used so far do not help eliminate this type of error, interventions based on more concrete and visual methods are needed, where participants understand the relative size relation between the variables in a SP-type equation.

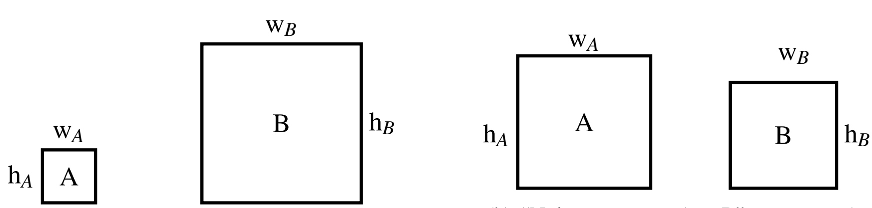
To the best of our knowledge, applications in the literature related to reversal error were designed and implemented as tests for researchers to gather data in order to conduct their research. Inspired by the interactive videogames developed in the literature to train spatial reasoning skills [9,10], we developed the $\overset{\curvearrowleft}{\varepsilon}$ *coach* serious game to train users to avoid committing reversal error by showing them the relation among the compared entities visually, in two ways: (i) qualitatively showing the containers of both entities and (ii) quantitatively by helping users resolving their written equations interactively.

The rest of the paper is organized as follows. Section 2 presents a qualitative model for comparing the size of the entities in *SP-type* problems. Section 3 presents the $\overset{\curvearrowleft}{\varepsilon}$ *coach* serious game prototype. Finally a discussion is provided.

2. Modelling Qualitatively the Container Relative Size in the *SP-type* problems

In the literature, Qualitative Spatial Representations and Reasoning (QSR) [11,12] model and reason about properties of *space* (i.e. topology, location, direction, proximity, geometry, size, etc.) and their evolution between continuous neighbouring situations. In cognitive science, qualitative models [13] have also been successful to solve oddity tasks [14], Raven's Progressive Matrices [15], 3D perspective description matching [16] and paper folding-and-punching reasoning [17]. As far as we are concerned, QSR has been never applied to model the container relative size in *SP-type* problems.

In order to qualitatively visualize the relative size relation *"N times as many A as B"*, a container-box is drawn without elements whose size must represent this relation.



(a) "N times as many B as A" $\rightarrow N \cdot A = B$.

(b) "N times as many A as B" $\rightarrow N \cdot B = A$.

Figure 1. Visualizing containers: in (a) A is *smaller-than* B; whereas in (b) A is *larger-than* B.

In order to determine logically which entity, A or B, is larger, a qualitative model can be used:

$$S = \langle A_S, Q_{RS}, OP_{RS} \rangle$$

where A_S refers to the arity or the number of entities implied in each relation, that is 2 in *SP-type* problems (i.e., A and B); Q_{RS} refers to the reference system defined as the set of relations between the objects which depends on the level of granularity considered; and OP_{RS} refers to the operations that define all possible relations between the entities.

Relative size is a continuous magnitude that can be discretised using the following reference system $Q_{RS} = \{S_{Rel}, S_{LAB}, S_{INT}\}$, where S_{Rel} refers to the relation for comparing the entities; S_{LAB} refers to the set of labels for relatize size; and S_{INT} refers to the values of S_{Rel} related to each label. In general, $S_{LAB} = \{S_1, S_2, ..., S_K\}$ and $S_{INT} = \{[0, s_1], (s_1, s_2], ..., (s_{K-1}, s_K]\}$ where K is the number of concepts used for defining relative sizes. As an example,

$S_{Rel_1} = A/B$
$S_{LAB1} = \{$*smaller-than* $(<<)$, *same-as* $(==)$, *larger-than* $(>>)\}$
$L_{INT1} = \{(0, 1), 1, (1, \infty)\}$

For *SP-type* problems like *"N times as many A as B"*, we must take into account all the possible relations between entities, which take the form of equations:

$$S_{Rel_2} = Eq_{A,B,\square} = \{\ A = N \square B,\ \xrightarrow[\text{relation}]{\text{reversed}} B = N \square A, \text{ and its derivatives }\}$$

where N is the factor and takes the values: $N \in \mathbb{N}$; and $\square$ is the operator symbol and takes the values: $\square \in \{+, \cdot, -, /\}$.

$Eq_{A,B,\square}$		symmetric by $=$	symmetric by $\square$	symmetric by $=$
	$A = N \square B$	$N \square B = A$	$A = B \square N$	$B \square N = A$
reversed	$B = N \square A$	$N \square A = B$	$B = A \square N$	$A \square N = B$

Equations symmetrical by $\square$ are equivalent only if the operators are commutative:
$A = N \square B \equiv A = B \square N \iff \square = \{+, \cdot\}$
$B = N \square A \equiv B = A \square N \iff \square = \{+, \cdot\}$

In order to qualitatively visualize each entity A or B as larger or smaller in the serious game, relations between the operator symbol $\square$, the factor N, and the order of them in the equation must be taken into account. Table 1 and Table 2 shows how to infer if A is *smaller-than* B or A is *larger-than* B, or if A is the *same-as* B by analysing the size relations represented as equations in *SP-type* problems.

Note also that:

A *smaller-than* B $\xleftrightarrow{\text{involves}}$ B *larger-than* A

B *smaller-than* A $\xleftrightarrow{\text{involves}}$ A *larger-than* B

A *same-as* B $\xleftrightarrow{\text{involves}}$ B *same-as* A.

Table 1. Qualitative relative size between entities(A,B) for *SP-type* problems: factor N located after the operator $\square$.

				A *larger-than* B	A *smaller-than* B	A *same-as* B
			$\square$	$A >> B$	$A << B$	$A == B$
	$A = B \square N$	$B \square N = A$	$\cdot$	$N > 1$	–	$N = 1$
r	$B = A \square N$	$A \square N = B$		–	$N > 1$	$N = 1$
	$A = B \square N$	$B \square N = A$	$+$	$N > 0$	–	$N = 0$
r	$B = A \square N$	$A \square N = B$		–	$N > 0$	$N = 0$
	$A = B \square N$	$B \square N = A$	$-$	–	$N > 0$	$N = 0$
r	$B = A \square N$	$A \square N = B$		$N > 0$	–	$N = 0$
	$A = B \square N$	$B \square N = A$	$/$	–	$N > 1$	$N = 1; N = A/B$
r	$B = A \square N$	$A \square N = B$		$N > 1$	–	$N = 1; N = A/B$

Table 2. Qualitative relative size between entities(A,B) for *SP-type* problems: factor N located before the operator $\square$.

				A *larger-than* B	A *smaller-than* B	A *same-as* B
			$\square$	$A >> B$	$A << B$	$A == B$
	$A = N \square B$	$N \square B = A$	$\cdot$	$N > 1$	–	$N = 1$
r	$B = N \square A$	$N \square A = B$		–	$N > 1$	$N = 1$
	$A = N \square B$	$N \square B = A$	$+$	$N > 0$	–	$N = 0$
r	$B = N \square A$	$N \square A = B$		–	$N > 0$	$N = 0$
	$A = N \square B$	$N \square B = A$	$-$	$N > A+B$	$N < A+B$	$N = A+B$
r	$B = N \square A$	$N \square A = B$		$N < A+B$	$N > A+B$	$N = A+B$
	$A = N \square B$	$N \square B = A$	$/$	$N > A \cdot B$	$N < A \cdot B$	$N = A \cdot B$
r	$B = N \square A$	$N \square A = B$		$N < A \cdot B$	$N > A \cdot B$	$N = A \cdot B$

From Table 1 and Table 2 we can infer which entity A or B is larger and represent their containers visually. In order to know the exact quantitative relation between A and B, the equation relating A and B must be resolved by assigning values to A, to B, and to the operator ($\square$) and the factor (N). The minimum container area of an entity (C_0) is defined by $C_0 = w_0 \cdot h_0$ where h_0 is the height and w_0 is the width of the image representing the icon. So, the area for the container representing A is $C_A = A \cdot C_0$ and the area for the container representing B is $C_B = B \cdot C_0$, and their wide sides and high sides (as shown in Figure 1) are: $w_A = h_A = \sqrt{C_A}$ and $w_B = h_B = \sqrt{C_B}$.

3. The $\overset{\curvearrowleft}{\varepsilon}$ *coach* serious game prototype

The $\overset{\curvearrowleft}{\varepsilon}$ *coach* prototype[2] has been designed and implemented to train users to avoid reversal error. It has been developed using Unity3D 2021.3.12f1 and it is targeted to Android and iOS devices. The $\overset{\curvearrowleft}{\varepsilon}$ *coach* serious game runs on any tablet/mobile with an operating system Android 4.1 Jelly Bean (API 16) or upper version.

The $\overset{\curvearrowleft}{\varepsilon}$ *coach* targets teenagers at their last stage at the high school as users, so that they resolve their reversal error problem before starting their jobs or their studies at university.

[2] *Spatial Reasoning Games* channel: https://www.youtube.com/@spatialreasoninggames

Section 3.1 explains the gameplay at $\overset{\curvearrowleft}{\varepsilon}$ *coach* and Section 3.2 describes its design and development.

3.1. *The gameplay at $\overset{\curvearrowleft}{\varepsilon}$ coach prototype*

The $\overset{\curvearrowleft}{\varepsilon}$ *coach* prototype includes 8 levels, composed of a set of 10 exercises, which are selected and downloaded from a dataset stored in Firebase Firestore. As Figure 2 shows, the gameplay for each exercise involves the following interactive steps: **(1) Problem analysis:** users are presented with the written problem; **(2) Equation writing:** users write a equation using the calculator displayed on screen which depends on the read problem; **(3) Equation reasoning:** users must write with their own words the sentence that better represents their equation; **(4) Equation resolution:** if users do not commit reversal error, their equation is broken down so that it is resolved by providing numerical values to the variables in order to validate the equation; otherwise, if users commit reversal error, the game shows them a visual representation of both entities sizes, this representation can be qualitative (i.e. showing the relative size of both containers) or quantitative (i.e. showing the concrete images of entities inside a container).

The game starts on the path menu where users can visualize which levels are finished and which are available to complete. By touching a round button on the path, players can start solving the exercise corresponding to that level. Once players solve an exercise, the progress bar level is displayed. After users complete at least 8 exercises, they can go back to the main screen to continue the path in the map.

3.2. *Development of $\overset{\curvearrowleft}{\varepsilon}$ coach as a serious game prototype*

Figure 3 outlines the main components of the $\overset{\curvearrowleft}{\varepsilon}$ *coach* serious game prototype:

(1) Loading exercises: written exercises are imported from a file in the cloud stored in Firebase's Firestore, a flexible and scalable NoSQL data model. The data are stored in documents with fields mapping values and these documents are stored in collections, which are used to build queries. For each exercise, the data stored is: a collection with the statement, the labels corresponding to the related entities (i.e. A, B), the type of operation ($\square$), the factor N, the possible equations as well as the reversed equations which relate both entities.

(2) Relative Visual Representation: in order to help players visualize the relation between the two entities in the *SP-type* problem, the $\overset{\curvearrowleft}{\varepsilon}$ *coach* displays two containers of related size. Users who committed reversal error are shown two displays of relative sizes. Each display represents one of the entities in the *SP-type* problem and their size depend on the values given by the written statement in the exercise. An example of a *SP-type* problem statement can be: *"There are 3 times as many trees as birds"*, so the display corresponding to the entity *trees* (T) must be 3 times larger than the display representing the entity *birds* (B). This relation can be shown qualitatively by the size of the containers as explained in Section 2 or it can be shown quantitatively since these displays can also include as many graphics as mentioned in the statement (see Figure 3).

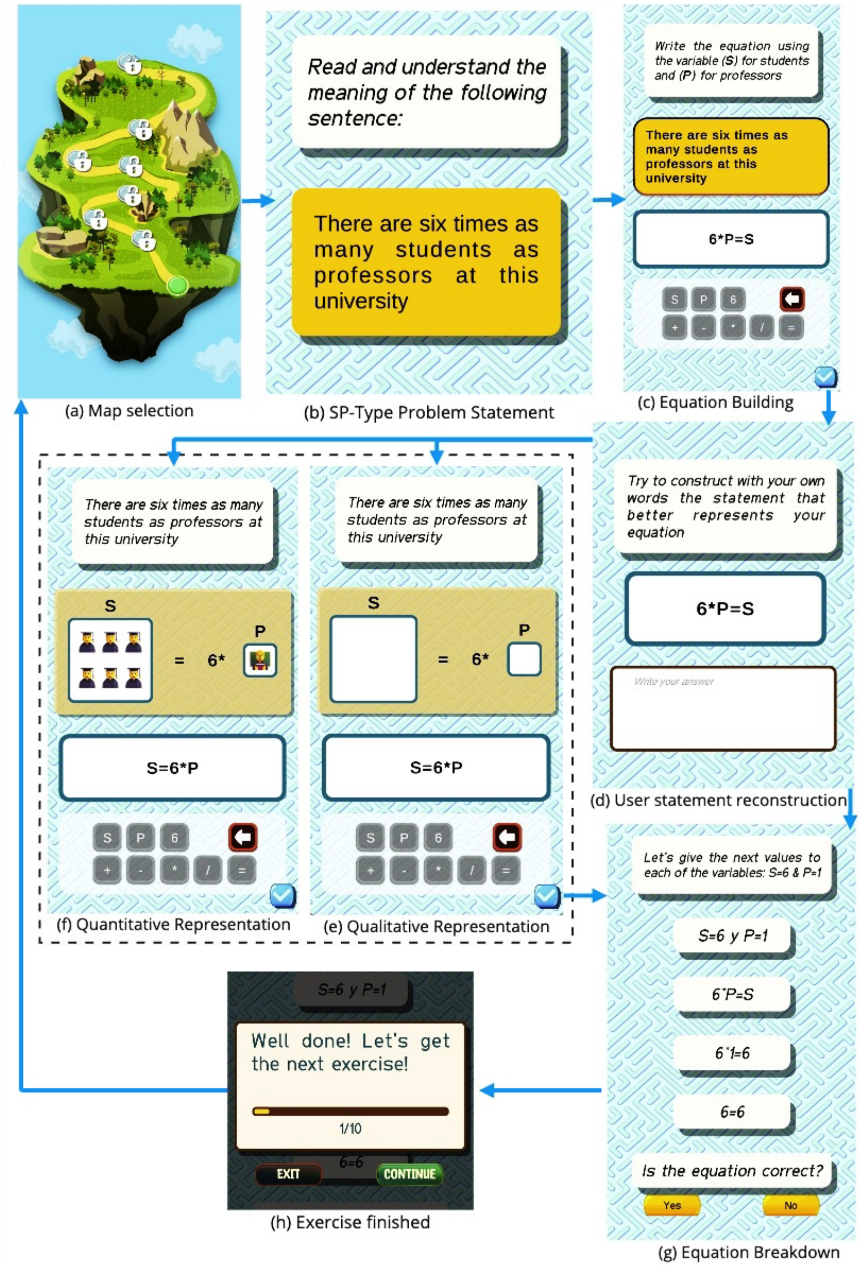


Figure 2. Gameplay at $\overset{\curvearrowleft}{\varepsilon}$ *coach*: overview of steps to complete for each exercise.

(3) User data collection: if allowed by users, the $\overset{\curvearrowleft}{\varepsilon}$ *coach* stores data about their performance at each level for generating statistics. In order to follow the rules of the European General Data Protection Regulation[3], approved and adopted by the EU Parliament

[3]General Data Protection Regulation: https://eugdpr.org/

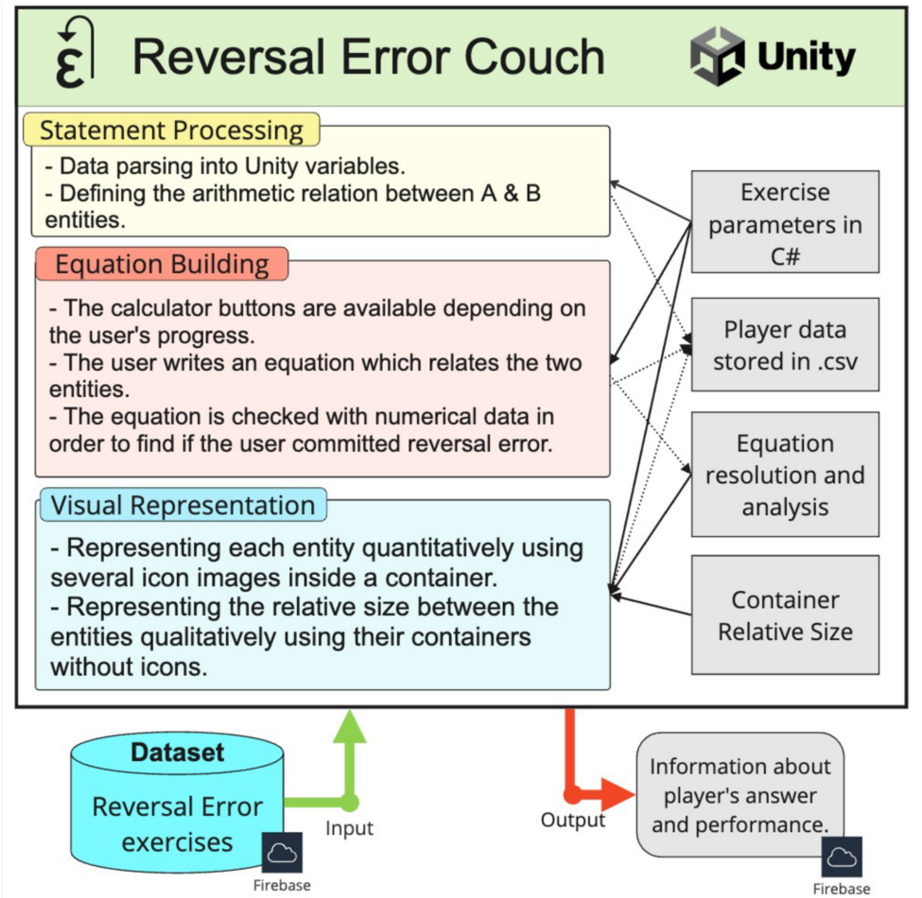


Figure 3. Diagram showing the parts composing $\overset{\curvearrowleft}{\varepsilon}$ *coach*.

in April 2016, this serious game shows initially a pop-up message that notifies the user about data storage purpose. Users can disable this option by unchecking the corresponding toggle in the settings menu.

4. Discussion

The $\overset{\curvearrowleft}{\varepsilon}$ *coach* serious game intends to train students to detect and avoid reversal error when formulating *SP-type* problems. It is based on a map and presents an intuitive and interactive gamified experience. Players are presented with algebraic word problems formulated using typical narratives and then they are assisted on finding the corresponding relation between the entities presented. For that, two approaches are proposed: (i) a qualitative visual representation of containers whose size represent the relative relation of size between the entities and (ii) reasoning behind the formulated equation by breaking it down until it is resolved.

A dataset of *SP-type* problems in Spanish has been built which is imported by the $\overset{\curvearrowleft}{\varepsilon}$ *coach* prototype. This dataset is intended to be used in a study where participants who usually commit reversal error will be trained to avoid it using the $\overset{\curvearrowleft}{\varepsilon}$ *coach*. If the results of this study show improvement in participants' performance, $\overset{\curvearrowleft}{\varepsilon}$ *coach* is intended to be transferred to society as an educational tool.

As future work, we intend to carry out controlled tests on our prototype in our lab in order to finalize the product and also to gather gameplay data to find out which approach do players prefer in order to clarify the entities relative size relation: if (i) visualization of containers or (ii) equation breakdown resolution.

Acknowledgements

This research has been partially funded by the Spanish Ministry of Science under grants PID2021-123152OB-C22 and PID2020-118763GA-100 both funded by MCIN/AEI/10.13039/501100011033 and by the European Union and FEDER/ERDF (European Regional Development Funds). The project UJI-A2021-04 financed by Universitat Jaume I and the funding by the Ministry of Labor and Social Economy under INVESTIGO Program grant INVEST/2022/308 are also acknowledged.

Z. Falomir acknowledges the Ramon y Cajal fellowship (RYC2019-027177-I/AEI/10.13039/501100011033) awarded by the Spanish Ministry of Science, Innovation and Universities.

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