The Journal of Geometric Analysis (2024) 34:109
https://doi.org/10.1007/s12220-024-01552-6

# Parabolicity of Invariant Surfaces

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Received: 8 December 2023 / Accepted: 7 January 2024 / Published online: 22 February 2024
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**Abstract**
We present a clear and practical way to characterize the parabolicity of a complete immersed surface that is invariant with respect to a Killing vector field of the ambient space.

**Keywords** Parabolicity · Invariant surfaces · Isometries · Killing vector fields

**Mathematics Subject Classification** Primary 58J05 · 31A05 · 53A10 · 58J65 · 58D19

## 1 Introduction

The intrinsic or extrinsic geometric conditions which guarantees that a surface is a parabolic surface has been largely studied from the beginning of the twentieth century. See for example [1, 7, 9, 10, 17]. This interplay between geometry and analytical properties of functions defined on surfaces is used to classify the surfaces in conformal types. Recall that a Riemannian manifold is said to be parabolic if every upper bounded subharmonic function is constant. A canonical reference on the parabolicity of manifolds is provided by [9]. In dimension 2, parabolicity is a conformal property. Specifically, if the surface $(M, g)$ is parabolic, then the surface $(M, e^u g)$ is also parabolic for any smooth function $u : M \to \mathbb{R}$. Parabolic manifolds satisfy a specific maximum principle, as demonstrated in [2], for example. Given a non-constant $C^2$ function $u : M \to \mathbb{R}$ with $\sup_M u = u^* < \infty$, if $M$ is a parabolic manifold, there

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exists a divergent sequence $\{x_k\}_{k\in\mathbb{N}} \subset M$ such that

$$u(x_k) > u^* - \frac{1}{k}, \quad \text{and,} \quad \Delta u(x_k) < 0.$$

Parabolicity can also be defined for manifolds $M$ with a non-empty boundary $\partial M$, as shown in [15]. A manifold with a boundary is parabolic if and only if every bounded harmonic function is determined by its values on the boundary. In other words, if $f_1, f_2 \colon M \to \mathbb{R}$ are two bounded harmonic functions with $f_1(x) = f_2(x)$ for any $x \in \partial M$, then $f_1(y) = f_2(y)$ for every $y \in M$. Given a complete Riemannian manifold $(M, g)$ and a precompact domain $\Omega \subset M$, every non-bounded connected component of $M - \Omega$ is called an end of $M$ with respect to $\Omega$. In this setting, $M$ is parabolic if and only if all the ends of $M$ with respect to $\Omega$ are parabolic.

In this article, we are interested in studying surfaces that are invariant with respect to a one-parameter group of isometries of the ambient space. For this purpose, we make use of Killing vector fields. A Killing vector field $\xi$ is defined such that $\mathcal{L}_\xi g = 0$, where $g$ represents the metric tensor. We will say that a surface is invariant with respect to a Killing field when it is invariant with respect to the one-parameter group of isometries of the ambient manifold associated with $\xi$.

The main result of this paper is the following Theorem:

**Main Theorem** *Let $\mathbb{E}$ be a n-dimensional Riemannian manifold which admits a complete Killing vector field $\xi \in \mathfrak{X}(\mathbb{E})$. Assume that an immersed complete non-compact regular surface $S \subset \mathbb{E}$ is invariant with respect to the one-parameter group of isometries of $\mathbb{E}$ associated to $\xi$. Assume that there exists $q \in S$ such that $\|\xi(q)\| \neq 0$ and denote by $\gamma(t) \subset S$ a complete curve parameterized by arc length satisfying*

$$\langle \dot{\gamma}(t), \xi \rangle = 0.$$

*Then, S is parabolic if and only if*

1. *the curve $\gamma$ is compact, or*
2. *the integral curves of $\xi$ are compact and there exists $a, b \in \mathbb{R}$, with $a < b$, such that $\int_{-\infty}^{a} \frac{1}{\|\xi(\gamma(s))\|} ds = \infty$ and $\int_{b}^{\infty} \frac{1}{\|\xi(\gamma(s))\|} ds = \infty$, or*
3. *the integral curves of $\xi$ are non-compact and $\int_{-\infty}^{0} \frac{1}{\|\xi(\gamma(s))\|} ds = \infty$ and $\int_{0}^{\infty} \frac{1}{\|\xi(\gamma(s))\|} ds = \infty$.*

***Remark 1*** We will see in Proposition 1 that the existence of a point $p \in S$, where $\xi(p) \neq 0$ allows us to prove that there exist at most two points, where the Killing vector field $\xi$ vanishes. In particular, we will prove that, if $\xi_{|S}$ vanishes at exactly two points, then $S$ is diffeomorphic to a sphere and hence it is parabolic and, if $\xi_{|S}$ vanishes at only one point, then the integral curves of $\xi$ must be compact and since $\dot{\gamma}$ is perpendicular to $\xi$, there exist $a, b \in \mathbb{R}$, with $a < b$, such that $\|\xi(\gamma(s))\| \neq 0$ in $s \in (-\infty, a) \cup (b, \infty)$ and $\int_{-\infty}^{a} \frac{1}{\|\xi(\gamma(s))\|} ds = \int_{b}^{\infty} \frac{1}{\|\xi(\gamma(s))\|} ds$.

Observe that if $\xi$ vanishes on all $S$, we cannot find a curve $\gamma \subset S$ which generates $S$, since every curve $\Gamma \subset S$ satisfies the condition $\left\langle \dot{\Gamma}(t), \xi \right\rangle = 0$. However, this situation does not append in low dimensions. Indeed, [11, Theorem 5.3] assures that each connected component of the set of zeros of $\xi$, Zero($\xi$), is a closed totally geodesic submanifold of even codimension. In particular, if $n = 3$, then $\xi$ vanishes at isolated geodesics of $\mathbb{E}$. If $n = 4$, there could exist a surface $S$, where $\xi$ vanishes and its geometric properties does not depend on $\xi$ but only on the geometry of $\mathbb{E}$; for example, we can obtain a Killing vector field in $\mathbb{R}^4$ such that it vanishes identically at the totally geodesic submanifold $\mathbb{R}^2 \subset \mathbb{R}^4$, and similarly we can find a Killing vector field in $\mathbb{H}^4$ such that it vanishes at the totally geodesic submanifold $\mathbb{H}^2 \subset \mathbb{H}^4$ but $\mathbb{R}^2$ is parabolic and $\mathbb{H}^2$ is hyperbolic. If $n > 4$, if $\xi$ vanishes on a surface $S \subset \mathbb{E}$, then $S$ can be immersed isometrically in a manifold that does not admit any Killing field a priori.

In what follows we prove the Main Theorem and then we show a couple of applications in some simply connected homogeneous manifold.

## 2 Proof of the Main Theorem: Parabolicity of Surfaces Admitting a Killing Submersion

In order to prove the Main Theorem we need to state here that the surface only can admit finitely many points, where the Killing vector field vanishes. In fact, $\xi$ vanishes at most on two points of $S$ and in such a case $S$ is diffeomorphic to a sphere. If $\xi$ only vanishes at one point of $S$, then $S$ is diffeomorphic to a plane. In both cases the integral curves of $\xi$ are diffeomorphic to $\mathbb{S}^1$, that is, $S$ is rotationally symmetric. Let us summarize this in the following proposition:

**Proposition 1** *Let $\mathbb{E}$ be a $n$-dimensional Riemannian manifold which admits a complete Killing vector field $\xi \in \mathfrak{X}(\mathbb{E})$. Assume that an immersed complete regular surface $S \subset \mathbb{E}$ is invariant with respect to the one-parameter group of isometries of $\mathbb{E}$ associated to $\xi$. Assume that there exists $q \in S$ such that $\|\xi(q)\| \neq 0$. Then, there are only three options:*

1. *The Killing vector field $\xi$ never vanishes on $S$.*
2. *The Killing vector field $\xi$ vanishes at only one point $p \in S$. The surface $S$ is topologically a plane and the integral curves of $\xi$ are compact.*
3. *The Killing vector field vanishes at only two points $p, p' \in S$. The surface $S$ is topologically an sphere and the integral curves of $\xi$ are compact.*

***Proof*** Assume that there exists a point $p \in S$ such that $\xi(p) = 0$. First of all, we are going to prove that there are no other points with vanishing $\xi$ in a geodesic ball of $S$ centered at $p$ and radius the injectivity radius of $p$.

Let us start joining $p$, where $\xi(p) = 0$, to the point $q$ (where is assumed that $\xi(q) \neq 0$) with a minimal geodesic segment $\gamma$. Assume that there exists a point $p'$, see Fig. 1, in $\gamma$, where $\xi(p') = 0$.

Let us denote by $\{\phi_t\}$ the one-parametric group of transformations associated to $\xi$. Let $q_2 = \phi_{t_0}(q)$. Since $\xi(q) \neq 0$, we can take $t_0$ sufficiently small such that $q_2 \neq q$. Moreover as we are assuming $\xi(p) = \xi(p') = 0$, we know that $\phi_{t_0}(p) = p$ and $\phi_{t_0}(p') = p'$. Since $\xi$ is a Killing vector field, $\phi_t$ are isometries and $\gamma_1 = \phi_{t_0}\gamma$ is a minimal geodesic segment as well joining $p$, $p'$, and $q_2$. By using the uniqueness of the geodesics, taking into account that $q_2 \neq q$ we conclude $\gamma_1 \neq \gamma$ and hence we

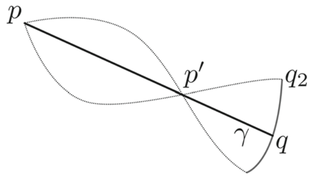


**Fig. 1** If $\xi(p) = 0$, $\xi(q) \neq 0$, and $\gamma$ is the minimal geodesic segment joining $p$ to $q$, then there is no $p' \in \gamma$ with $\xi(p') = 0$

have two different minimal geodesic segments joining $p$ and $p'$, then $p' \in \mathrm{cut}(p)$ and $\gamma$ does not realize the distance providing a contradiction. Hence there are no points $p'$ in $\gamma$ with $\xi(p') = 0$.

Therefore any point $p_1$ in $\phi_t(\gamma)$, with $0 < t < t_0$, at distance less than the injectivity radius of $p$ satisfies that $\xi(p_1) \neq 0$. So, without loss of generality, we can assume that distance between $p$ and $q$ is smaller than the injectivity radius of $p$.

Take a point $q'$ in the minimal geodesic joining $p$ and $q$ at distance less than the injectivity radius of $p$. Then, $\xi(q') \neq 0$. The orbit of $q'$ by $\phi_t$,

$$\mathcal{O}_{q'} = \left\{\phi_t(q') \,:\, t \in \mathbb{R}\right\}$$

is the geodesic sphere of radius $\mathrm{dist}(p, q')$ centered at $p$. Indeed, observe that the distance $\mathrm{dist}(p, q') = \mathrm{dist}(p, \phi_t(q'))$ since $\gamma$ and $\phi_t(\gamma)$ are minimal geodesics. Thus, $\mathcal{O}_{q'}$ is contained in the geodesic sphere centered at $p$ and radius $\mathrm{dist}(p, q)$. Let $r_0 = \mathrm{dist}(p, q') < \mathrm{inj}(p)$ and assume by contradiction that there exists $p'' \in \partial B_{r_0}(p) \setminus \mathcal{O}_{q'}$. This means that the orbit of $q'$ stops at $p' \in \partial B_{r_0}(p)$ before reaching $p''$. In particular, $\xi(p') = 0$ and $p' \in \mathrm{cut}(p)$, that is a contradiction sine $\mathrm{dist}(p, p') = r_0 < \mathrm{inj}(p)$. Therefore,

$$\mathcal{O}_{q'} = \{r \in S \,:\, \mathrm{dist}(p, r) = \mathrm{dist}(p, q')\},$$

which is compact and, as we have seen before, there are no points with $\xi = 0$ in the minimal segments joining $p$ with the points of $\mathcal{O}_{q'}$. In particular, we proved that, if $R$ is smaller than the injectivity radius of $p$, then $\xi \neq 0$ in $B_R(p) \setminus \{p\}$, where $B_R(p)$ is the geodesic ball centered at $p$ with radius $R$. Furthermore, $B_R(p)$ is foliated by integral curves of $\xi$. It can be proved that any point $p'$ with $\xi(p') = 0$ then $p' \in \mathrm{cut}(p)$, see [11, Corollary 5.2]. Moreover we can prove that if $q \in \mathrm{cut}(p)$ then $\xi(q) = 0$. Suppose, by contradiction, that $\xi(p) = 0$ and $\xi(q) \neq 0$ for some $q \in \mathrm{cut}(p)$. Consider the minimal segment

$$t \mapsto \gamma(t) = \exp_p(tu), \quad \text{with} \quad \|u\| = 1,$$

such that $\gamma(0) = p$, $\gamma(\mathrm{dist}(p, q)) = q$ and $\gamma$ minimizes distance for $t \in [0, \mathrm{dist}(p, q)]$ but it is not a minimal geodesic for $t > \mathrm{dist}(p, q)$. If we suppose that $\xi(q) \neq 0$, the orbit $\mathcal{O}_q$ is the metric sphere $\partial B_{\mathrm{dist}(p,q)}(p)$, it has positive length and it is diffeomorphic to $\mathbb{S}^1$. Moreover since $\phi_t$ acts by isometries every point $q' \in \partial B_{\mathrm{dist}(p,q)}(p)$ is a cut point of $p$ and there exists $u' \in T_pM$ such that the

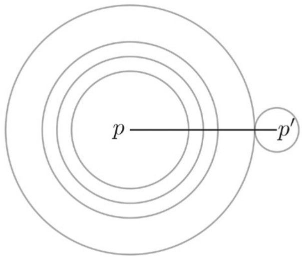


**Fig. 2** If $S$ has two points $p$ and $p'$, where $\xi$ vanish, then the points $p$ and $p'$ are antipodals points of a rotationally symmetric sphere

geodesic segment

$$t \mapsto \gamma_1(t) = \exp_p(tu'),$$

satisfies that $\gamma_1(\mathrm{dist}(p,q)) = q'$ and $\gamma_1$ minimizes distance for $t \in [0, \mathrm{dist}(p,q)]$ but it is not a minimal geodesic for $t > \mathrm{dist}(p,q)$. Thence $\mathrm{cut}(p) = \partial B_{\mathrm{dist}(p,q)}(p)$ and, see for instance [19, Lemma 4.4],

$$S = B_{\mathrm{dist}(p,q)}(p) \cup \partial B_{\mathrm{dist}(p,q)}(p).$$

But this is a contradiction with the completeness of $S$. Therefore, we can conclude that if $\xi(p) = 0$ and $q \in \mathrm{cut}(p)$, then $\xi(q) = 0$ and we can state that if $\xi(p) = 0$,

$$\mathrm{cut}(p) = \{q \in S - \{p\} \,:\, \xi(q) = 0\}.$$

Hence, if we suppose that $\xi(p) = 0$ and $\mathrm{cut}(p) = \emptyset$, $p$ is the only point in $S$, where the Killing vector field vanishes and $S$ is rotationally symmetric plane, that is there exists $\lambda \in \mathcal{C}^\infty(\mathbb{R})$ such that $S$ is isometric to $(\mathbb{R}^2, \lambda^2(x^2+y^2)(dx^2+dy^2))$.

Now suppose again that $\xi(p) = 0$ and $\mathrm{cut}(p) \neq \emptyset$. Take a point $p' \in \mathrm{cut}(p)$, and hence with $\xi(p') = 0$, and such that $\mathrm{dist}(p,p') = \mathrm{inj}(p)$. Thence for any point $q$ between $p$ and $p'$ such that $d(p',q)$ is smaller than the injectivity radius of $p'$, see Fig. 2, the orbit $\mathcal{O}_q$ need to be a geodesic sphere centered at $p$ and simultaneously a geodesic sphere centered at $p'$, namely,

$$\mathcal{O}_q = \partial B_{\mathrm{dist}(p,q)}(p) = \partial B_{\mathrm{dist}(p',q)}(p').$$

This implies that $S$ is the connected sum

$$S = \overline{B_{\mathrm{dist}(p,q)}(p)} \# \overline{B_{\mathrm{dist}(p',q)}(p')}.$$

In particular, $S$ is a sphere and there exists no other point in $S$, where $\xi$ vanishes.

Summarizing everything, if we have a point $p \in S$ with $\xi(p) = 0$ there are only two options: or $\mathrm{cut}(p) = \emptyset$ and $S$ is a rotationally symmetric plane without other points with vanishing $\xi$, or $\mathrm{cut}(p)$ contains only one point $p'$, where $\xi(p') = 0$ and $S$ is a rotationally symmetric sphere. $\square$

Using Proposition 1 we can remove a compact set $K \subset S$ so that the Killing vector field never vanishes on $S - K$. But $S$ is parabolic if and only if $S - K$ is parabolic (see for instance [9]). Then, in order to simplify the discussion of the proof we are assuming that $\xi$ never vanishes on $S$, otherwise we can do the same argument but for $S - K$.

Since the surface $S$ is invariant by the one-parameter group of isometries $G_\xi = \{\phi_t\}$ associated to the Killing vector field $\xi$, i.e., $G_\xi(S) = S$, we will also assume that $G_\xi$ acts freely and properly on $S$, otherwise $G$ is not closed in Iso$(S)$ with the compact-open topology and [14, Proposition] implies that $\xi = X_1 + X_2$, where $X_1$ and $X_2$ are two Killing vector fields with compact orbits satisfying $[X_1, X_2] = 0$, that is, $S$ is a torus and hence it is a parabolic surface. Then, $S$ admits a Killing submersion structure, that is, we can endow $S/G_\xi$ with a smooth structure and with a Riemannian metric tensor in such a way that the projection $\pi_2 : S \to S/G_\xi$, whose fibers are the integral curves of $\xi$, is a Riemannian submersion. Furthermore, since $\langle \dot{\gamma}(t), \xi \rangle = 0$, the curve $\gamma$ is diffeomorphic to the set $S/G_\xi$ and because we are using arc-length parametrization they are indeed isometric manifolds as well.

The Main Theorem is therefore equivalent to Theorem 7, where we are proving the general case on surfaces which admits a Killing submersion. In order to prove this theorem we will prove Lemmas 2, 3, 4, 5, and 6. In Lemma 2 we will prove that a conformal change on the metric tensor preserves the parabolicity for surfaces. Indeed, parabolicity is related to the conformal type in dimension 2. In Lemma 3 we will prove that given a Killing submersion we can perform a conformal change with a basic function in the total space and in the base manifold in such a way that with respect to the new metric tensors the submersion remains a Killing submersion. In Lemma 4 we will prove that a Killing submersion on a surface with constant norm of the Killing vector field has non-negative Gaussian curvature. In Lemma (5) we will prove that a complete surface with non-negative Gaussian curvature is a parabolic surface. Finally, in Lemma 6 we given an expression of the laplacian of a function of the base of the submersion. Using these lemmas we can prove Theorem 7 which is equivalent to the Main theorem.

**Lemma 2** *In dimension 2, parabolicity is preserved under conformal changes in the metric tensor.*

***Proof*** Let $(M, g)$ be a 2-dimensional Riemannian manifold and consider the conformal change of metric $\overline{g} = f^2 g$ given by the positive function $f : M \to \mathbb{R}_+$, then, the Laplacian $\overline{\Delta}$ with respect to the metric tensor $\overline{g}$ is related with the Laplacian $\Delta$ with respect to the metric tensor $g$ by (see [5] for instance)

$$\overline{\Delta} = \frac{\Delta}{f^2}.$$

Then, $(M, g)$ admits non-constant bounded subharmonic functions if and only if $(M, f^2 g)$ admits bounded subharmonic functions. □

**Lemma 3** *Let $\pi : M \to B$ be a surjective Killing submersion from to the complete Riemannian manifold $(M, g_M)$ to $(B, g_B)$ with complete Killing vector field $\xi \in$*

*$\mathfrak{X}(M)$. Let $f : B \to \mathbb{R}$ be a smooth and positive function. Then, $\pi : M \to B$ is also a Killing submersion from $(M, (f \circ \pi)^2 g_M)$ to $(B, f^2 g_B)$ with the complete Killing vector field $\xi$. Furthermore, $(M, (f \circ \pi)^2 g_M)$ is complete if $(B, f^2 g_B)$ is complete.*

***Proof*** Since $\xi$ is a Killing vector field of $g_M$, the Lie derivative of the metric tensor vanishes, i.e.,

$$\mathcal{L}_\xi g_M = 0.$$

Then

$$\mathcal{L}_\xi \left((f \circ \pi) g_M\right) = \xi(f \circ \pi) \cdot g_M + (f \circ \pi)\mathcal{L}_\xi g_M = 0,$$

and hence, $\xi$ is also a Killing vector field for $(f \circ \pi) g_M$. Likewise, since $\pi : (M, g_M) \to (B, g_B)$ is a Riemannian submersion, for any $v \in T_p M$ with $g_M(v, \xi) = 0$,

$$g_M(v, v) = g_B(d\pi(v), d\pi(v)).$$

Then, for any $v \in T_p M$ with $(f \circ \pi) g_M(v, \xi) = 0$,

$$(f \circ \pi) g_M(v, v) = f(\pi(p)) g_B(d\pi(v), d\pi(v)),$$

and hence $\pi : (M, (f \circ \pi) g_M) \to (B, f g_B)$ is a Riemannian submersion.

To prove that $M$ is complete, we consider an arbitrary Cauchy sequence $\{p_n\}_n$ in $M$ and prove that it is convergent. We consider the sequence $\{q_n = \pi(p_n)\}_n \subset B$. First notice that $\langle v, v\rangle_M \geq \langle d\pi(v), d\pi(v)\rangle_B$ for any point $p \in M$ and any tangent vector field $v \in T_p M$. Then, $\text{Length}_M(\gamma) \geq \text{Length}_B(\pi(\gamma))$ for any curve $\gamma \subset M$. It follows that $\{q_n\}_n$ is a Cauchy sequence in $B$ and, since $BM$ is complete, $\{q_n\}_n$ converges to a point $q \in B$. In particular, we can assume that $\{q_n\}_n$ is contained in a compact and simply connected subset $K \subset B$ Let $F_0 \colon K \to M$ be a local section, then, for any $n$, there exists $t_n \in \mathbb{R}$ such that $p_n = \phi_{t_n}(q_n)$. Denote by $c = \min_K \mu$. Then, for any $p \in \pi^{-1}(K)$ and any vector field $v \in T_p M$, we have $\langle v, v\rangle_M \geq c \left\langle d\pi^\perp(v), d\pi^\perp(v)\right\rangle_{\mathbb{R}}$. This implies that $\|p_i - p_j\|_M \geq c|t_i - t_j|$ for any $i, j \in \mathbb{N}$, that is, $\{t_n\}_n$ is a Cauchy sequence in $(\mathbb{R}, g_{euc})$. Since $(\mathbb{R}, g_{euc})$ is complete, we can assume that there exist $a, b \in \mathbb{R}$ such that $t_n \in [a, b]$ for any $n$. It follows that $\{p_n\}_n$ is contained in the compact subset of $\pi^{-1}(K)$ delimited by $\phi_a(F_0)$ and $\phi_b(F_0)$. Hence, $\{p_n\}_n$ is a Cauchy sequence in a compact domain, that is convergent and this completes the proof. □

**Lemma 4** *Let $\pi : S \to B$ be a Killing submersion with Killing vector field of constant norm. Then, if $\dim(S) = 2$, $S$ has non-negative Gaussian curvature.*

***Proof*** Given a point $p \in S$ and an horizontal vector $v \in \xi^\perp(p)$ with unit-length, $\|v\| = 1$, in order to obtain the Gaussian curvature, i.e., the sectional curvature $\sec(v, \xi)$ of the plane spanned by $v$ and $\xi$, let us consider a vector field $\overline{X} \in \mathfrak{X}(B)$ defined in a neighborhood $U \ni \pi(p)$, such that $\overline{X}(\pi(p)) = d\pi(v)$ and with vanishing covariant derivative $\nabla^{B}_{\overline{X}}\overline{X} = 0$ in $B$, i.e., a geodesic vector field. Then, the lift $X \in \mathfrak{X}(S)$ of $\overline{X}$ defined in $\pi^{-1}(U) \ni p$ satisfies

$$\nabla_X X = (\nabla_X X)^H + \langle \nabla_X X, \xi \rangle \xi = -\langle X, \nabla_X \xi \rangle = 0. \tag{1}$$

Here, the superscript $H$ denotes the horizontal part of a vector and we have used that since $\xi$ is a Killing vector field $\langle X, \nabla_X \xi \rangle = 0$. Then,

$$\begin{aligned} \sec(v, \xi) =& \langle R(X, \xi)X, \xi \rangle = \langle \nabla_\xi \nabla_X X - \nabla_X \nabla_\xi X + \nabla_{[X,\xi]} X, \xi \rangle \\ =& \langle -\nabla_X \nabla_\xi X + \nabla_{[X,\xi]} X, \xi \rangle = \langle -\nabla_X \left([\xi, X] + \nabla_X \xi\right) + \nabla_{[X,\xi]} X, \xi \rangle \\ =& \langle \nabla_X \left([X, \xi] - \nabla_X \xi\right) + \nabla_{[X,\xi]} X, \xi \rangle \end{aligned}$$

In order to simplify the expression let us define the following vector fields $Y := \nabla_X \xi$ and $Z := [X, \xi]$. Observe that both $X, Y$ are horizontal vector fields because $\xi$ has constant norm and thence $\langle \nabla_X \xi, \xi \rangle = \frac{1}{2} X \|\xi\| = 0$. Moreover

$$\langle Z, \xi \rangle = \langle \nabla_X \xi - \nabla_\xi X, \xi \rangle = \langle \nabla_X \xi, \xi \rangle - \langle \nabla_\xi X, \xi \rangle = \frac{1}{2} X \langle \xi, \xi \rangle = 0$$

Therefore,

$$\begin{aligned} \sec(v, \xi) =& \langle \nabla_X (Z - Y) + \nabla_Z X, \xi \rangle = \langle \nabla_X Z + \nabla_Z X, \xi \rangle - \langle \nabla_X Y, \xi \rangle \\ =& \|Y\|^2 \geq 0, \end{aligned}$$

where we have used

$$\langle \nabla_X Z, \xi \rangle = -\langle Z, \nabla_X \xi \rangle = \langle X, \nabla_Z \xi \rangle = -\langle \nabla_Z X, \xi \rangle$$

and $\langle \nabla_X Y, \xi \rangle = -\langle Y, \nabla_X \xi \rangle = -\|Y\|^2$. □

**Lemma 5** *Let $S$ be a complete surface with non-negative Gaussian curvature. Then, $S$ is a parabolic manifold.*

***Proof*** This lemma is a direct consequence of a well-known theorem due to Huber [10] which states that if the negative part of the curvature $K_- = \max\{-K, 0\}$ has finite integral, namely,

$$\int_S K_- \, dA < \infty, \tag{2}$$

then, $\int_M K \, dA \leq \chi(M)$ and $M$ is conformally equivalent to a compact Riemann surface with finitely many punctures and hence it is a parabolic surface. □

**Lemma 6** *Let $\pi : S \to \mathbb{R}$ be a Killing submersion with never vanishing Killing vector field $\xi \in \mathfrak{X}(M)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then*

$$\Delta^S (f \circ \pi)(q) = \frac{1}{\mu(x)} \frac{d}{dx} \left( \mu(x) \frac{df}{dx} \right)_{x=\pi(q)},$$

*where $\mu(x)$ is the norm of the Killing vector field $\|\xi(p)\|$ for any $p \in \pi^{-1}(x)$.*

***Proof*** Let $\pi : (M, \langle, \rangle_M) \to (B, \langle, \rangle_B)$ be a Riemannian submersion. The Laplacian of a basic function ($\widetilde{f} = f \circ \pi, \quad f : B \to \mathbb{R}$) is given by (see [3, Lemma 3.1] for instance or the computations that are made in [4])

$$(\triangle^M \widetilde{f})_q = (\triangle^B f)_x - \langle \nabla^B f_p, d\pi_q(H_q)\rangle_B, \tag{3}$$

where $x = \pi(q)$, and $H$ is the mean curvature vector field of the fiber $\pi^{-1}(p)$. In our case taking $B = \mathbb{R}$ and $M = S$ this implies

$$(\triangle^S \widetilde{f})_q = \left.\frac{d^2 f}{dx^2}\right|_{x=\pi(q)} - \left.\frac{df}{dx}\right|_{x=\pi(q)} \left\langle \frac{\partial}{\partial x}, d\pi_q(H_q)\right\rangle_{\mathbb{R}}, \tag{4}$$

where $\frac{\partial}{\partial x}$ is the unit vector tangent to $\mathbb{R}$. Taking into account that the mean curvature vector field is given by

$$H = \nabla_{\frac{\xi}{\mu}} \frac{\xi}{\mu} = \frac{1}{\mu^2} \nabla_\xi \xi.$$

Then,

$$\begin{aligned} \left\langle \frac{\partial}{\partial x}, d\pi_q(H_q)\right\rangle_{\mathbb{R}} &= \frac{1}{\mu^2}\left\langle \frac{\partial}{\partial x}, d\pi(\nabla_\xi \xi)\right\rangle_{\mathbb{R}} = \frac{1}{\mu^2}\left\langle X, \nabla_\xi \xi\right\rangle \\ &= \frac{-1}{\mu^2}\langle \xi, \nabla_X \xi\rangle = \frac{-1}{2\mu^2} X\|\xi\|^2 = \frac{-1}{2\mu^2}\frac{d}{dx}\mu^2 \end{aligned} \tag{5}$$

where $X$ is an horizontal lift of $\frac{\partial}{\partial x}$ and we have used that $\xi$ is a Killing vector field. Finally equation (4) becomes

$$(\triangle^S \widetilde{f})_q = \left.\frac{d^2 f}{dx^2}\right|_{x=\pi(q)} + \left.\frac{df}{dx}\right|_{x=\pi(q)} \frac{1}{\mu(x)} \left.\frac{d\mu}{dx}\right|_{x=\pi(q)}, \tag{6}$$

and the proposition is proved. □

By using the previous lemmas, instead of studying the parabolicity of a complete surface $(S, g)$ which admits a Killing submersion $\pi : S \to M^1$ to the connected 1-dimensional Riemannian manifold $(M, g_{\text{can}})$, we will study the parabolicity of the conformally equivalent Riemannian manifold $(S, \frac{1}{\mu^2} g)$, where $\mu(x)$ is the norm of the never vanishing Killing vector field $\|\xi(p)\|$ for any $p \in \pi^{-1}(x)$. By the lemma 2 $(S, g)$ is parabolic if and only if $(S, \frac{1}{\mu^2} g)$ is parabolic. Observe that since $S$ is complete $\pi$ needs to be a surjective map. Moreover by Lemma 3, the map $\pi$ induces a Riemannian submersion from $(S, \frac{1}{\mu^2} g)$ to $(M, \frac{1}{\mu^2} g_{\text{can}})$. Since this submersion has a Killing vector field of constant norm, $(S, \frac{1}{\mu^2}g)$ is a surface with non-negative Gaussian curvature by Lemma 4. When $M = \mathbb{S}^1(R)$ or $M = \mathbb{R}^1$ and

$$\int_{-\infty}^{0} \frac{dx}{\mu(x)} = \infty \quad \text{and} \quad \int_{0}^{\infty} \frac{dx}{\mu(x)} = \infty.$$

we have that $(M, \frac{1}{\mu^2}g_{\text{can}})$ is complete. Hence by using lemma 3 $S$ is a complete surface with non-negative Gaussian curvature and thus by Lemma 5, the surface $S$ is a parabolic manifold. On the other hand, if we assume that

$$\int_{-\infty}^{0} \frac{dx}{\mu(x)} < \infty \quad \left(\text{or} \quad \int_{0}^{\infty} \frac{dx}{\mu(x)} < \infty\right).$$

By Lemma 6 the function $F : M \to \mathbb{R}$

$$F(p) := \int_{\pi(p)}^{0} \frac{dx}{\mu(x)}$$

is a bounded and harmonic function (which implies that $S$ is a non-parabolic manifold). This can be summarized in the following theorem, which implies the Main theorem of the paper:

**Theorem 7** *Let $\pi : S \to M^1$ be a Killing Submersion from a complete and 2-dimensional Riemannian manifold $(S, g)$ to the connected 1-dimensional Riemannian manifold $(M, g_{\text{can}})$. Let us denote by $\mu(x)$ the norm of the Killing vector field $\|\xi(p)\|$ for any $p \in \pi^{-1}(x)$. Then,*

1. *If $M = \mathbb{S}^1(R)$ endowed with the canonical metric tensor, $S$ is parabolic.*
2. *If $M = \mathbb{R}$ with its canonical metric tensor, $S$ is a parabolic manifold iff*

$$\int_{-\infty}^{0} \frac{dx}{\mu(x)} = \infty \quad \text{and} \quad \int_{0}^{\infty} \frac{dx}{\mu(x)} = \infty.$$

## 3 Application of the Main Theorem: Parabolicity of Invariant Surfaces in Homogeneous 3-Manifolds

In this section we use **Main Theorem** to study the parabolicity of invariant surfaces with some geometric properties in 3-dimensional homogeneous Riemannian manifold studied in different works by many authors.

### 3.1 Parabolicity of Invariant Surfaces with Constant Mean Curvature in *Sol*$_3$

In [13], Lopez and Munteanu give a description of invariant surfaces with either constant curvature in the Thurston geometry $Sol_3$. Recall that $Sol_3$ is isometric to $\mathbb{R}^3$

endowed with the metric

$$e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$$

and the component of the identity its isometry group is generated by the following families of isometries:

$$T_1^c(x, y, z) = (x + c, y, z), \quad T_2^c(x, y, z) = (x, y + c, z), \quad T_3^c(x, y, z) = (e^{-c}x, e^c y, z + c).$$

In particular, in [13] they studied surfaces with constant mean curvature or constant Gaussian curvature invariant with respect to $T_1$, finding a profile curve $\gamma(s) = (0, y(s), z(s))$ parameterized by arc length. Notice that the Killing vector field $\partial_x$ associated to $T_1$ has norm $\|\partial_x\| = e^z$ and it is orthogonal to $\gamma'$, so we can easily apply our result just studying $\int e^{-z(s)}ds$.

The classification of minimal surfaces [13, Theorem 3.1] assures that the only $T_1$-invariant minimal surfaces of $Sol_3$ are as follows:

1. a leaf of the foliation $\{Q_t = \{(x, t, z) \mid x, z \in \mathbb{R}\}\}_{t\in\mathbb{R}}$, which are known to be isometric to the hyperbolic plane;
2. a leaf of the foliation $\{R_t = \{(x, y, t) \mid x, y \in \mathbb{R}\}\}_{t\in\mathbb{R}}$, which are known to be isometric to the Euclidean plane;
3. the surfaces $S_{\theta_0,a}$, for $\theta_0 \in \mathbb{R}\setminus\{k\pi : k \in \mathbb{Z}\}$ and $a \in \mathbb{R}$, that are generated by translating the profile curve

$$\gamma(s) = \left(0, a + e^{s\sin(\theta_0)}, \log\left(\tan(\theta_0)e^{s\sin(\theta_0)}\right)\right).$$

Obviously, the surfaces $Q_t$ are hyperbolic, while the surfaces $R_t$ are parabolic. It remains to study the parabolicity of $S_{\theta_0,a}$. Since

$$\int e^{-\log\left(\tan(\theta_0)e^{s\sin(\theta_0)}\right)}ds = e^{-s\sin(\theta_0)}\cot(\theta_0)\csc(\theta_0),$$

is a bounded function, **Main Theorem** implies that $S_{\theta_0,a}$ are hyperbolic surfaces.

For the constant mean curvature case, [13, Theorem 3.2] proves that the $z$-coordinate of the profile curve is bounded. In particular,

$$\lim_{s\to\pm\infty}\int e^{-z(s)}ds = \pm\infty$$

and **Main Theorem** implies that the respective invariant $H$-surfaces are parabolic.

### 3.2 Parabolicity of Vertical Cylinders in Killing Submersion

An easy way to study surfaces immersed in a three-manifold which are invariant with respect to a Killing vector field of the ambient space is by studying vertical cylinders in Killing submersions. We recall that a three-dimensional Killing submersion is a Riemannian submersion $\pi : \mathbb{E} \to M$ from a three-dimensional manifold $\mathbb{E}$ onto a surface $(M, g)$, both connected and orientable, such that the fibers of $\pi$ are integral curves of a Killing vector field $\xi \in \mathfrak{X}(\mathbb{E})$. These spaces have been completely classified in terms of the Riemannian surface $(M, g)$, the length of the Killing vector field $\mu = \|\xi\|$ and the so-called bundle curvature $\tau$, defined such that $\tau(p) = \frac{-1}{\mu(p)}\langle\overline{\nabla}_u\xi, v\rangle$, where $\{u, v, \xi_p/\mu(p)\}$ is a positively oriented orthonormal basis of $T_p\mathbb{E}$ (see [12, Section 2]). In this setting a vertical cylinder is a surface $S$ always tangent to $\xi$. In particular, $S$ is invariant with respect to the isometries associated to $\xi$ and it projects through $\pi$ onto a curve $\gamma \subset M$. Since $\pi$ is a Riemannian submersion, to study $\int_\Gamma \frac{1}{\mu(\Gamma)}$, where $\Gamma \subset S$ is a curve parameterized by arc length and orthogonal to $\xi$, is equivalent to study $\int_\gamma \frac{1}{(\pi_*\mu)(\gamma)}$, where $\gamma = \pi(S)$ is parameterized by arc length in $(M, g)$. In particular, in this setting, the Main Theorem, read as follows.

**Theorem 8** *Let $\pi : \mathbb{E} \to (M, g)$ be a Killing submersion with Killing length $\mu$. Then, for any complete curve $\gamma \subset M$, $\pi^{-1}(\gamma)$ is a parabolic surface if and only if $\gamma$ is complete in $(M, \frac{1}{\mu^2}g)$.*

***Remark 2*** It is interesting to notice that, given a curve $\gamma \subset M$, while the conformal metric $\frac{1}{\mu^2}g$ gives us information about the parabolicity of $S = \pi^{-1}$, the conformal metric $\mu^2 g$ gives us information about its mean curvature (see [6, Proposition 2.3]).

The parabolicity of surfaces of revolution (or surfaces with ends of revolution) in space forms of constant sectional curvature has been studied in [8]. Three-dimensional simply connected space forms of constant sectional curvature are the only manifolds with a group of isometries of dimension 6 and can be classified (up to isometries) by their sectional curvatures. We can label these spaces hence as $\mathbb{M}^3(\kappa)$. The spaces $\mathbb{M}^3(\kappa)$ includes the Euclidean space for $\kappa = 0$, the spheres for $\kappa > 0$, and the Hyperbolic space for $\kappa < 0$.

There are no three-dimensional spaces with a group of isometries of dimension 5. The simply connected 3-dimensional spaces with a group of isometries of dimension 4 are Killing submersions and they can be classified (up to isometries) by the curvature $\kappa$ of the base manifold and by the torsion $\tau$ of the fibers. We can label these spaces hence as $\mathbb{E}^3(\kappa, \tau)$ (because they are endowed with the Riemannian submersion $\pi : \mathbb{E}^3(\kappa, \tau) \to \mathbb{M}^2(\kappa)$ with constant bundle curvature $\tau$). In what follows we give a general condition to guarantee the parabolicity of rotational surfaces in the $\mathbb{E}^3(\kappa, \tau)$-spaces.

The canonical rotational model describing the $\mathbb{E}^3(\kappa, \tau)$-spaces is given by $(\Omega \times \mathbb{R}, ds^2)$, where

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 \mid \lambda(x, y) > 0\right\}, \qquad \lambda(x, y) = \frac{1}{1 + \frac{\kappa}{4}(x^2 + y^2)},$$

$$ds^2 = \lambda^2(x, y)(dx^2 + dy^2) + (\lambda(x, y)\tau(ydx - xdy) + dz)^2.$$

Since the Killing vector field $\|\partial_z\|$ has unitary norm, every surface that is invariant with respect to $\partial_z$ is parabolic. So, we focus on studying the parabolicity rotational surfaces. First notice that if $\kappa > 0$, $\mathbb{E}(\kappa, \tau)$ is a Berger sphere, in particular, since it is compact, every Killing vector field $\xi$ has bounded norm and from the Main Theorem we deduce that every surface that is invariant with respect to $\xi$ is parabolic. When $\kappa \leq 0$, we consider the Killing vector field $\xi(x, y) = -y\partial_x + x\partial_y$ generating the rotation around the $z$-axis and to describe $\mathbb{E}(\kappa, \tau)$ as a Killing submersion with respect to $\xi$, we use the cylindrical coordinates

$$x(r, \theta) = r\cos(\theta), \qquad y(r, \theta) = r\sin(\theta).$$

We obtain that the space $\mathbb{E}(\kappa, \tau)$ minus its $z$-axis is isometric to a quotient of $(\mathbb{R}^2_\kappa \times \mathbb{R}, ds^2_{rot})$, where $\mathbb{R}^2_\kappa = \left\{(r, z) \in \mathbb{R}^2 | r > 0,\ 4 + \kappa r^2 > 0\right\}$ and

$$ds^2_{rot} = \frac{16dr^2}{(4+\kappa r^2)^2} + \frac{dz^2}{(1+r^2\tau^2)} + \left(\frac{2r\sqrt{1+r^2\tau^2}}{4+\kappa r^2}\right)^2 \left(d\theta - \frac{(4+\kappa r^2)\tau}{4(1+r^2\tau^2)}dz\right)^2.$$

Here, $\xi = \partial\theta$ and the Killing submersion with respect to $\xi$ is such that

$$(M, g) = \left(\mathbb{R}^2_\kappa, \frac{16}{(4+\kappa r^2)^2}dr^2 + \frac{1}{(1+r^2\tau^2)}dz^2\right) \quad \text{and} \quad \mu(r, z) = \frac{2r\sqrt{1+r^2\tau^2}}{4+\kappa r^2}$$

and then the conformal metric tensor $g/\mu^2$ is given by

$$\frac{1}{\mu^2}g = \frac{4}{r^2(1+r^2\tau^2)}dr^2 + \frac{(4+\kappa r^2)^2}{4r^2(1+r^2\tau^2)^2}dz^2. \tag{7}$$

In particular, every complete curve $\gamma$ in $(M, g)$ generates a complete parabolic invariant surface in $\mathbb{E}^3(\kappa, \tau)$ if and only if $\gamma$ is complete with respect to the metric (7). For example, using this tool, it is easy to see that the minimal umbrellas of the Heisenberg group $\mathbb{E}^3(0, \tau)$ are hyperbolic, without computing their extrinsic area growth (see [16]). It is sufficient to consider the curve $\gamma(t) = (t, 0)$. Its norms is in the conformal metric is $\|\gamma(t)\| = \frac{2}{t\sqrt{1+t^2\tau^2}} < t^{-3/2}$ for $t > \frac{2+\sqrt{4-\tau^2}}{\tau^2}$ or $\tau > 2$, that is, $\gamma$ is not complete in the conformal metric, thus $\pi^{-1}(\gamma)$ is hyperbolic.

We can also use this tool to study the parabolicity of the rotational surfaces of constant mean curvature in $\mathbb{E}^3(-1, \tau)$ described in [18]. In particular, Peñafiel shows that a rotational surface of constant mean curvature $H \in \mathbb{R}$ is parameterized by $\gamma_d(t) = \left(\tanh\left(\frac{\sqrt{t}}{2}\right), u_d(t)\right)$, where

$$u_d(t) = \int \frac{(2H\cosh(r) + d)\sqrt{1 + 4\tau^2\tanh^2\left(\frac{r}{2}\right)}}{\sqrt{\sinh^2(r) - (2H\cosh(r) + d)^2}},$$

with $d \in \mathbb{R}$. When $d = -2H$, the rotation of $\gamma_d(t)$ generates an entire graph. When $H = 0$, the norm of $\gamma'(t)$ with respect to (7) is

$$\|\gamma'(t)\| = 2\sqrt{\frac{1}{\sinh^2(t)\left(1+\tau^2\tanh\left(\frac{t}{2}\right)\right)}} \simeq 4\sqrt{\frac{1}{1+\tau^2}}e^{-t} + o(e^{-2t}).$$

In particular, $\lim_{t\to\infty}\int \|\gamma'(t)\|$ is convergent and the rotational end generated by the rotation of $\gamma$ is hyperbolic. Furthermore, since the difference between $u_d$ and $u_{-2H}$ is bounded, the surface generated by rotating any $\gamma_d$ is hyperbolic.

On the contrary, when $H = 1/2$, we get that

$$\|\gamma'(t)\| = \left(\frac{(5+3\cosh(t))^2(1-4\tau^2+\cosh(t)+4\tau^2\cosh(t))}{8(1-\tau^2+(1+\tau^2)\cosh(t))^2} + \frac{4}{\sinh^2(t)\left(1+4\tau^2\tanh\left(\frac{t}{2}\right)\right)}\right)^{\frac{1}{2}}$$

which diverges for $t \to +\infty$. That is, the entire rotational graph with critical constant mean curvature is parabolic.

**Acknowledgements** The first author was partially supported by “INdAM - GNSAGA Project," codice CUP_E55F22000270001, and by MCIN/AEI project PID2022-142559NB-I00. The second author was partially supported by the Research Grant PID2020-115930GA-100 funded by MCIN/ AEI /10.13039/50110001103. The second author also wants to express gratitude to Irmina Gonzalbo, with whom he initially started the study of the parabolicity of surfaces of revolution. Unfortunately, her passing has now permanently separated her from us.

**Funding** Open access funding provided by Università degli Studi dell’Aquila within the CRUI-CARE Agreement.

**Data availability** No data sets were generated or analysed during the current study.

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