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A note on the mean-square solution of the hypergeometric differential equation with uncertainties

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Abstract
The Fröbenius method of power series has been applied to several linear random differential equations. The interest relies on the derivation of a closed-form mean-square solution and on the possibility of approximating statistical measures at exponential convergence rate. In this paper, we deal with the hypergeometric differential equation with random coefficients and initial conditions. On the interval (0, 1), random power series centered at the regular singular point 0 are employed, which are given in terms of the hypergeometric function. We find the stochastic basis of mean-square solutions and solve random initial-value problems. The approximation of the expectation and the variance is studied and illustrated computationally.

Keywords Hypergeometric random differential equation · Fröbenius method · Power series · Mean-square calculus · Approximation of statistics

Mathematics Subject Classification 34F05 · 34A30 · 33C05

1 Introduction

In a random differential equation, the input parameters of the differential equation (coefficients, initial or boundary conditions, etc.) are considered as random variables, with any probability distribution. The solution is, then, a stochastic process. There are many theoretical studies on this type of models, for example, the investigation of solutions in the mean-square sense (Syski 1967; Soong 1973; Neckel and Rupp 2013; Villafuerte et al. 2010; Licea et al. 2013; Cortés and Jornet 2020; Jornet 2023a). This means to work under the mean-square random calculus of second-order random variables and stochastic processes, where limits are considered in the sense of the expectation and the variance. Random differential equations constitute natural extensions of the deterministic counterpart, and they are conceptually distinct to stochastic differential equations of Itô type driven by white noise (Øksendal 2013).

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An important part of research in applied mathematics is concerned with the derivation of explicit or semi-explicit solutions to models (Area and Nieto 2021; Jornet 2021a). The calculation of power series-type solutions falls into this category. Many linear random differential equations have been studied under the context of random mean-square calculus (Airy, Hermite, Legendre, Laguerre, Bessel, etc.) (Jornet 2023a; Calatayud et al. 2018; Jornet 2021b; Cortés et al. 2017; Villafuerte 2023), by extending the Fröbenius method for power series (Birkhoff and Rota 1989). Even non-linear random models have been addressed with the use of power series (Villafuerte and Chen-Charpentier 2012; Jornet 2021c). The interest is not merely theoretical; from a computational standpoint, rapid approximations to the expectation and the variance may be obtained by considering truncated series. The convergence rate is exponential, like other techniques used in forward uncertainty quantification such as polynomial chaos or perturbation expansions (Smith 2013; Santonja and Chen-Charpentier 2012; Jornet 2021d, e), to improve the classical Monte Carlo simulation.

In this paper, the aim is to study the hypergeometric random differential equation

$$t(1-t)\ddot{x}(t)+[\gamma-(\alpha+\beta+1)t]\dot{x}(t)-\alpha\beta x(t)=0, \tag{1}$$

on the domain $I=(0,1)$. It may have initial conditions at $t_0 \in I$

$$x(t_0)=y_0, \quad \dot{x}(t_0)=y_1. \tag{2}$$

Any linear differential equation with two derivatives and at most three regular singular points is equivalent to the hypergeometric differential equation. The input parameters y_0, y_1, γ, α and β may be random variables, which implies that x is a stochastic process. The goal is to study when x is a stochastic solution, namely a mean-square solution based on random power series. This problem was posed in the conclusions of Cortés et al. (2017), which proposed to investigate other kinds of Bessel differential equations (Weber, Kelvin, Neumann, etc.) and other equations, such as Jacobi, hypergeometric, etc.

The plan of the paper is the following. In Sect. 2, preliminary results on mean-square calculus are summarized or established. In Sect. 3, the general mean-square solution to (1) is obtained. In Sect. 4, the initial-value problem (1) with (2) is solved in a mean-square sense, and the approach to approximate the expectation and the variance of the solution is described and illustrated computationally. Finally, Sect. 5 draws the main conclusions.

2 Preliminaries on mean-square calculus

We summarize the main definitions about mean-square calculus and state the results (lemmas) that will be required later, with proofs when necessary. The reader may consult the references (Soong 1973; Neckel and Rupp 2013; Villafuerte et al. 2010; Jornet 2023a).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A random variable $y:\Omega \to \mathbb{R}$ is of order p, $1 \le p < \infty$, if

$$\mathbb{E}[|y|^p]=\int_\Omega |y|^p \mathrm{d}\mathbb{P} < \infty,$$

where $\mathbb{E}$ denotes the expectation operator. The random Lebesgue space $\mathrm{L}^p(\Omega)$ is formed by the set of random variables of order p. With the norm

$$\|y\|_p=(\mathbb{E}[|y|^p])^{1/p},$$

it is a Banach space. For $p=2$, it is a Hilbert space, with the inner product $\mathbb{E}[y_1y_2]$, $y_1, y_2 \in \mathrm{L}^2(\Omega)$. This inner product is related to the covariance. Notice that a random variable is of order $p = 2$ if and only if it has a well-defined expectation and a finite variance, which are the main statistical measures. On the other hand, for $p = \infty$, there is a definition of $\mathrm{L}^\infty(\Omega)$. It is the space of random variables that are bounded almost surely. The lowest bound, called essential supremum, is the norm $\|y\|_\infty$. This space is also Banach. In general, $\mathrm{L}^p(\Omega) \subseteq \mathrm{L}^q(\Omega)$ if $p > q \geq 1$ and $\|y\|_\infty = \lim_{p\to\infty} \|y\|_p$ hold.

Given a sequence $\{y_n\}_{n=1}^\infty$ in $\mathrm{L}^p(\Omega)$, $1 \leq p \leq \infty$, we say that it converges to y under $\|\cdot\|_p$ if

$$\lim_{n\to\infty} \|y_n - y\|_p = 0.$$

For $p = 2$, this convergence is referred to as mean-square convergence, and it preserves the convergence of the expectation and the variance

$$\lim_{n\to\infty} \mathbb{E}[y_n] = \mathbb{E}[y],$$
$$\lim_{n\to\infty} \mathbb{V}[y_n] = \mathbb{V}[y].$$

Mean-square convergence then provides a convenient framework, compared to other convergences such as almost surely, in probability or in distribution.

For a stochastic process $y(t)$ on an interval $J \subseteq \mathbb{R}$, it is of order $p \in [1,\infty]$ if $\mathbb{E}[|y(t)|^p] < \infty$ for all $t \in J$. It is said to be continuous at $t \in J$ in $\mathrm{L}^p(\Omega)$ if

$$\lim_{h\to 0} \|y(t+h) - y(t)\|_p = 0.$$

If there exists a stochastic process $\dot{y}(t) \in \mathrm{L}^p(\Omega)$, such that

$$\lim_{h\to 0} \left\| \frac{y(t+h) - y(t)}{h} - \dot{y}(t) \right\|_p = 0, \tag{3}$$

then y is differentiable at t in $\mathrm{L}^p(\Omega)$, with derivative $\dot{y}(t)$. In this setting, differentiability implies continuity. The stochastic process is analytic at t_1 in the $\mathrm{L}^p(\Omega)$ sense if

$$y(t) = \sum_{m=0}^{\infty} y_m (t - t_1)^m$$

for every t in a neighborhood of t_1, where $y_m \in \mathrm{L}^p(\Omega)$ and the sum converges under $\|\cdot\|_p$.

Example 1 If $a \in \mathrm{L}^p(\Omega)$ and $y(t) = at^n$ is a stochastic process, $n \geq 0$, then y is differentiable in $\mathrm{L}^p(\Omega)$. Indeed

$$\lim_{h\to 0} \left\| \frac{y(t+h) - y(t)}{h} - ant^{n-1} \right\|_p = \|a\|_p \lim_{h\to 0} \left(\frac{(t+h)^n - t^n}{h} - nt^{n-1} \right) = 0.$$

Notice that at^n is the general form of a term of a random power series.

Lemma 1 (Cortés et al. 2017, Proposition 3; Soong 1973, page 96). *Fix $1 \leq p \leq \infty$. If $y(t)$ and $z(t)$ are $\mathrm{L}^{2p}(\Omega)$-differentiable stochastic process at t, then yz is $\mathrm{L}^p(\Omega)$-differentiable at t and*

$$(yz)^{\cdot}(t) = \dot{y}(t)z(t) + y(t)\dot{z}(t).$$

If z is deterministic, then one only needs y be $\mathrm{L}^p(\Omega)$-differentiable at t to ensure that yz is $\mathrm{L}^p(\Omega)$-differentiable at t.

Lemma 2 *Fix $1 \leq p \leq \infty$. If a is a second-order random variable and $z(t)$ is an $\mathrm{L}^{\infty}(\Omega)$-differentiable stochastic process at t, then az is mean-square differentiable at t and*

$$(az)^{\cdot}(t) = a\dot{z}(t).$$

Proof We have

$$\left\| \frac{az(t+h) - az(t)}{h} - a\dot{z}(t) \right\|_2 \leq \|a\|_2 \left\| \frac{z(t+h) - z(t)}{h} - \dot{z}(t) \right\|_\infty \to 0$$

as $h \to 0$. □

Lemma 3 *If a is a random variable satisfying $\|a\|_\infty < \infty$, then t^a is $\mathrm{L}^{\infty}(\Omega)$-differentiable for $t > 0$, with derivative $(\log t)t^a$.*

Proof By the standard mean-value theorem for the trajectories

$$\frac{(t+h)^a - t^a}{h} = (\log \xi_{t,h})(\xi_{t,h})^a,$$

where the random variable $\xi_{t,h}$ lies between t and $t+h$. We want to calculate the limit as $h \to 0$, under $\|\cdot\|_\infty$. By the triangle inequality

$$\begin{aligned} \|(\log \xi_{t,h})(\xi_{t,h})^a - (\log t)t^a\|_\infty \leq \; & \|\log \xi_{t,h} - \log t\|_\infty \|t^a\|_\infty \\ & + \|\log \xi_{t,h}\|_\infty \|(\xi_{t,h})^a - t^a\|_\infty. \end{aligned}$$

We bound each one of the right-hand side terms. By the mean-value theorem for trajectories again

$$\|\log \xi_{t,h} - \log t\|_\infty \leq \left\| \frac{1}{\eta_{t,h}} \right\|_\infty \|\xi_{t,h} - t\|_\infty \leq \frac{1}{t - |h|}|h|,$$

where the random variable $\eta_{t,h}$ lies between $\xi_{t,h}$ and t. Also

$$\|\log \xi_{t,h}\|_\infty \leq \left\| \frac{1}{\eta_{t,h}} \right\|_\infty \|\xi_{t,h} - 1\|_\infty \leq \min\left\{1, \frac{1}{t - |h|}\right\}(t + |h| + 1),$$

where the new random variable $\eta_{t,h}$ lies between $\xi_{t,h}$ and 1. Another bound is

$$\begin{aligned} \|(\xi_{t,h})^a - t^a\|_\infty & \leq \|(\log \eta_{t,h})(\eta_{t,h})^a\|_\infty \|\xi_{t,h} - t\|_\infty \\ & \leq \log(t + |h|) \max\{1, (t + |h|)^{\|a\|_\infty}\}|h|. \end{aligned}$$

Finally

$$\|t^a\|_\infty \leq \max\{1, t^{\|a\|_\infty}\}.$$

As a consequence

$$\lim_{h \to 0} \|(\log \xi_{t,h})(\xi_{t,h})^a - (\log t)t^a\|_\infty = 0,$$

and we are done. □

Lemma 4 (Calatayud et al. 2018, Theorem 3.1) *Fix $p \in [1, \infty]$. Let $y(t) = \sum_{m=0}^{\infty} y_m (t - t_1)^m$ be a random power series, convergent under $\|\cdot\|_p$ on $(t_1 - \delta, t_1 + \delta)$, $\delta > 0$. Then, the random power series $\sum_{m=0}^{\infty} m y_m (t - t_1)^{m-1}$ converges under $\|\cdot\|_p$ on $(t_1 - \delta, t_1 + \delta)$, and it is equal to the $\mathrm{L}^p(\Omega)$-derivative $\dot{y}(t)$. That is, random power series can be differentiated term by term.*

A random differential equation, with initial condition, is a problem of the form

$$\dot{x} = f(t, x, \zeta),\ t \in I \subseteq \mathbb{R}; \quad x(t_0) = x_0(\zeta),$$

where ζ is a vector of input random parameters, x_0 is the initial condition at $t_0 \in I$, and x is the mean-square solution. The derivative $\dot{x}$ is interpreted under the limit (3) for $p = 2$. Mean-square limits may be useful to approximate the mean value and the standard deviation of the response (Cortés and Jornet 2020; Jornet 2021b); this is one of the main goals of uncertainty quantification. Notice that, when x and f have vector form, we are including the case of two, three, or any number of derivatives in the equation.

3 Stochastic basis for the space of mean-square solutions

Let us fix the interval $I = (0, 1)$, because $t = 0$ and $t = 1$ do not belong to the domain of the differential equation (1). We say that $\{\phi_1, \phi_2\}$ is a basis for the set of mean-square solutions to (1) on I (i.e., a fundamental set) if any mean-square solution $\phi : I \to \mathbb{R}$ to (1) is uniquely expressed as $\phi(t) = a\phi_1(t) + b\phi_2(t)$, where a and b are time-independent random variables. The goal of this section is to build such a stochastic basis.

If $\mathbb{P}[\gamma \in \mathbb{Z}] = 0$, for example, if γ has a density, then $\gamma \notin \mathbb{Z}$ almost surely. Essentially, this means that γ has a continuous random variation. In such a case, fixed any arbitrary outcome from the sample space, the path-wise solution to our problem (1) is

$$x(t) = a\phi_1(t) + b\phi_2(t),$$

where a and b are real numbers,

$$\phi_1(t) = F(\alpha, \beta, \gamma, t), \quad \phi_2(t) = t^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, t), \tag{4}$$

and $F = {}_2F_1$ is the hypergeometric series

$$F(\alpha, \beta, \gamma, t) = 1 + \sum_{n=1}^{\infty} \frac{\binom{\alpha+n-1}{n}\binom{\beta+n-1}{n}}{\binom{\gamma+n-1}{n}} t^n. \tag{5}$$

This is known from the classical, deterministic theory on Fröbenius solutions for regular singular points (Mubeen et al. 2014). When one runs over all trajectories of x, one has that a and b are random variables.

The main question is when ϕ_1 and ϕ_2 defined by (4) are mean-square solutions on I. In fact, to later solve initial-value problems, we need a stronger type of solution, under the supremum random norm $\|\cdot\|_\infty$ of the random Lebesgue space $\mathrm{L}^\infty(\Omega)$. Indeed, there are difficulties with the mean-square operational calculus of the product, that result from the fact that the mean-square norm is not multiplicative.

Proposition 1 *If the random coefficients α, β, γ belong to $\mathrm{L}^\infty(\Omega)$, then the series that defines $F(\alpha, \beta, \gamma, t)$ in (5) converges in $\mathrm{L}^\infty(\Omega)$, for $t \in (-1, 1)$. As a consequence, the series that define ϕ_1 and ϕ_2 in (4) are solutions of (1) in $\mathrm{L}^\infty(\Omega)$, on $I = (0, 1)$.*

Proof To simplify the notation, let $x_0 = 1$,

$$x_n = \frac{\binom{\alpha+n-1}{n}\binom{\beta+n-1}{n}}{\binom{\gamma+n-1}{n}}, \quad n \geq 1,$$

be the coefficients of the hypergeometric series (5). For the moment, we know that pointwise convergence holds, but not convergence under $\|\cdot\|_\infty$. We then check that

$$\sum_{n=0}^{\infty} \|x_n\|_\infty r^n < \infty,$$

for $0 < r < 1$. To control the decay of x_n, the coefficients are rewritten as

$$x_n = \frac{\Gamma(\alpha+n)\Gamma(\beta+n)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(n+1)\Gamma(\beta)\Gamma(\gamma+n)},$$

where Γ is the gamma function. By the well-known recursive property $\Gamma(u+1) = u\Gamma(u)$, we obtain

$$\frac{x_n}{x_{n-1}} = \frac{(\alpha+n-1)(\beta+n-1)}{n(\gamma+n-1)}.$$

Then, applying absolute values and the triangle inequality, we have

$$\begin{aligned}
|x_n| &= \left|\frac{(\alpha+n-1)(\beta+n-1)}{n(\gamma+n-1)}\right| |x_{n-1}| \\
&\leq \frac{(|\alpha|+n+1)(|\beta|+n+1)}{n(n-1-|\gamma|)} |x_{n-1}| \\
&\leq \frac{(|\alpha|+n+1)(|\beta|+n+1)}{n(n-1-\|\gamma\|_\infty)} |x_{n-1}| \\
&\leq \frac{(\|\alpha\|_\infty+n+1)(\|\beta\|_\infty+n+1)}{n(n-1-\|\gamma\|_\infty)} |x_{n-1}|.
\end{aligned}$$

Notice that $n-1 > \|\gamma\|_\infty$, that is, n is large enough, to ensure that the denominator is positive. This yields an inequality for $\|\cdot\|_\infty$

$$\|x_n\|_\infty \leq \frac{(\|\alpha\|_\infty+n+1)(\|\beta\|_\infty+n+1)}{n(n-1-\|\gamma\|_\infty)} \|x_{n-1}\|_\infty.$$

In consequence, for $0 < r < 1$

$$\frac{\|x_n\|_\infty r^n}{\|x_{n-1}\|_\infty r^{n-1}} \leq r\frac{(\|\alpha\|_\infty+n+1)(\|\beta\|_\infty+n+1)}{n(n-1-\|\gamma\|_\infty)} \overset{n\to\infty}{\longrightarrow} r < 1.$$

Finally, by the ratio or D'Alembert test, $\sum_{n=0}^{\infty} \|x_n\|_\infty r^n < \infty$. This convergence implies that F is $C^\infty(-1, 1)$ in the sense of $\|\cdot\|_\infty$, by Lemma 4. Hence, the same occurs for ϕ_1. On the other hand, ϕ_2 has the factor $t^{1-\gamma}$ multiplied by the hypergeometric function. By combining Lemmas 1 and 3, we deduce that ϕ_2 is $C^\infty(0, 1)$ in $\mathrm{L}^\infty(\Omega)$. □

Remark 1 Given a random variable z, it is equivalent $z \in \mathrm{L}^\infty(\Omega)$ and $\mathbb{E}[|z|^m] \leq CH^m$, $m \geq m_0$, for certain $C, H, m_0 > 0$ (Calatayud et al. 2018, Section 3.3). Indeed, it is well known that $\|z\|_\infty = \lim_{m\to\infty} \|z\|_m$, where $\|z\|_m = (\mathbb{E}[|z|^m])^{1/m}$ is the mth norm in the random Lebesgue space $\mathrm{L}^m(\Omega)$. Thus, our work on $\mathrm{L}^\infty(\Omega)$ is related to the growth conditions imposed in previous contributions. Dealing with $\|\cdot\|_\infty$ is, however, easier than dealing with moments when developing inequalities.

4 Mean-square solutions of random initial-value problems and applications

In this part, we use the stochastic basis $\{\phi_1, \phi_2\}$ built in the previous section to obtain the mean-square solution to initial-value problems (1), (2). The initial states y_0 and y_1 in (2) are random variables too. To solve an initial-value problem, we need to specify the random variables from the linear combination of ϕ_1 and ϕ_2. Recall that we work on $I = (0, 1)$.

Theorem 1 *If the initial conditions y_0, y_1 belong to $\mathrm{L}^2(\Omega)$, the coefficients α, β, γ belong to $\mathrm{L}^\infty(\Omega)$, $\exists\, \delta > 0$, such that $\mathbb{P}[\gamma \in (1-\delta, 1+\delta)] = 0$, and $t_0 \in I$, then $x(t) = a\phi_1(t)+b\phi_2(t)$ solves (1) on I in the mean-square sense, where*

$$a = \frac{y_0\dot{\phi}_2(t_0) - y_1\phi_2(t_0)}{W(\phi_1, \phi_2)(t_0)}, \quad b = \frac{y_1\phi_1(t_0) - y_0\dot{\phi}_1(t_0)}{W(\phi_1, \phi_2)(t_0)}$$

are random variables in the linear combination and

$$W(\phi_1, \phi_2)(t_0) = \phi_1(t_0)\dot{\phi}_2(t_0) - \phi_2(t_0)\dot{\phi}_1(t_0)$$

is the wronskian.

Proof We know that ϕ_1 and ϕ_2 solve (1) and (2) by samples (deterministic theory), so Liouville's identity (Chicone 2006, Proposition 2.15) gives

$$W(\phi_1, \phi_2)(t) = C_I \mathrm{e}^{\int \frac{(\alpha+\beta+1)t-\gamma}{t(1-t)}\,\mathrm{d}t} = C_I \frac{(1-t)^{\gamma-\alpha-\beta-1}}{t^\gamma}, \quad t > 0,$$

where C_I is independent of t and $W(\phi_1, \phi_2)(t) = \phi_1(t)\dot{\phi}_2(t) - \phi_2(t)\dot{\phi}_1(t)$ is the wronskian. Then, if $F \equiv F(\cdot,\cdot,\cdot,t)$ and $\partial_t F \equiv \partial_t F(\cdot,\cdot,\cdot,t)$

$$\begin{aligned}
C_I &= W(\phi_1, \phi_2)(t)\frac{t^\gamma}{(1-t)^{\gamma-\alpha-\beta-1}} \\
&= \frac{t^\gamma}{(1-t)^{\gamma-\alpha-\beta-1}}\left(\phi_1(t)\dot{\phi}_2(t) - \phi_2(t)\dot{\phi}_1(t)\right) \\
&= \frac{t^\gamma}{(1-t)^{\gamma-\alpha-\beta-1}}\left(F\left[t^{1-\gamma}\partial_t F + (1-\gamma)t^{-\gamma}F\right] - t^{1-\gamma}F\partial_t F\right) \\
&= \frac{1}{(1-t)^{\gamma-\alpha-\beta-1}}\left(F\left[t\partial_t F + (1-\gamma)F\right] - tF\partial_t F\right) \\
&\xrightarrow{t\to 0} 1-\gamma
\end{aligned}$$

almost surely, because

$$\lim_{t\to 0} F(\alpha, \beta, \gamma, t) = 1$$

and

$$\lim_{t\to 0} \partial_t F(\alpha, \beta, \gamma, t) = x_1 = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma+1)} = \frac{\alpha\beta}{\gamma}.$$

In consequence, $C_I = 1-\gamma$ and

$$W(\phi_1, \phi_2)(t) = (1-\gamma)\frac{(1-t)^{\gamma-\alpha-\beta-1}}{t^\gamma}.$$

From this equality for the wronskian, we can bound

$$|W(\phi_1, \phi_2)(t_0)| \geq \delta(1 - t_0)^{\|\gamma\|_\infty+\|\alpha\|_\infty+\|\beta\|_\infty+1} =: \Delta(t_0) > 0$$

almost surely, where $\Delta(t_0)$ is non-random. Thus

$$\|a\|_2 \leq \frac{1}{\Delta(t_0)} \left(\|y_0\|_2 \|\dot{\phi}_2(t_0)\|_\infty + \|y_1\|_2 \|\phi_2(t_0)\|_\infty\right) < \infty,$$
$$\|b\|_2 \leq \frac{1}{\Delta(t_0)} \left(\|y_1\|_2 \|\phi_1(t_0)\|_\infty + \|y_0\|_2 \|\dot{\phi}_1(t_0)\|_\infty\right) < \infty.$$

By Proposition 1, the series $\phi_1(t)$ and $\phi_2(t)$ are solutions under $\|\cdot\|_\infty$. Since $a, b \in \mathrm{L}^2(\Omega)$, we conclude that $a\phi_1(t) + b\phi_2(t)$ is the solution in $\mathrm{L}^2(\Omega)$ by Lemma 2, as wanted.

For the uniqueness of mean-square solution, we notice that the Lipschitz condition stated in Soong (1973, Theorem 5.1.2) (Picard's theorem) holds on any interval $[a, b] \subseteq (0, 1)$, due to the boundedness of α, β and γ. Indeed, rewrite (1) as

$$z(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 1 \\ \frac{\alpha\beta}{t(1-t)} & \frac{\alpha+\beta+1-\gamma}{t(1-t)} \end{pmatrix},$$
$$\dot{z}(t) = B(t)z(t).$$

We say that $z = (z_1, z_2)$ belongs to $\mathrm{L}^2(\Omega)$ if

$$\|z\|_2 = \max\{\|z_1\|_2, \|z_2\|_2\} < \infty.$$

Consider also the random matrix norm

$$\|B\| = \max_i \sum_j \|b_{ij}\|_\infty.$$

If z, z' are 2D vectors in $\mathrm{L}^2(\Omega)$, then the Lipschitz condition

$$\|B(t)z - B(t)z'\|_2 \leq \|B(t)\| \|z - z'\|_2$$

holds, where

$$\int_a^b \|B(t)\| \mathrm{d}t \leq \|\alpha\|_\infty \|\beta\|_\infty \int_a^b \frac{1}{t(1-t)} \mathrm{d}t + (b - a)$$
$$+ (\|\alpha\|_\infty + \|\beta\|_\infty + 1 + \|\gamma\|_\infty) \int_a^b \frac{1}{t(1-t)} \mathrm{d}t < \infty.$$

This proves uniqueness. □

Corollary 1 *Let ϕ_1^N and ϕ_2^N be the truncated sums of the series of ϕ_1 and ϕ_2 at the order N, respectively, with*

$$F^N(\alpha, \beta, \gamma, t) = 1 + \sum_{n=1}^{N} \frac{\binom{\alpha+n-1}{n}\binom{\beta+n-1}{n}}{\binom{\gamma+n-1}{n}} t^n.$$

Consider

$$a^N = \frac{y_0 \dot{\phi}_2^N(t_0) - y_1 \phi_2^N(t_0)}{W(\phi_1^N, \phi_2^N)(t_0)}, \quad b^N = \frac{y_1 \phi_1^N(t_0) - y_0 \dot{\phi}_1^N(t_0)}{W(\phi_1^N, \phi_2^N)(t_0)}$$

and

$$W(\phi_1^N, \phi_2^N)(t_0) = \phi_1^N(t_0)\dot{\phi}_2^N(t_0) - \phi_2^N(t_0)\dot{\phi}_1^N(t_0).$$

Let $x^N(t) = a^N\phi_1^N(t) + b^N\phi_2^N(t)$. *Under the conditions of Theorem 1, we have*

$$\lim_{n\to\infty} x^N(t) = x(t) \text{ in } \mathrm{L}^2(\Omega),$$
$$\lim_{n\to\infty} \mathbb{E}[x^N(t)] = \mathbb{E}[x(t)],$$
$$\lim_{n\to\infty} \mathbb{V}[x^N(t)] = \mathbb{V}[x(t)],$$

where $\mathbb{E}$ *and* $\mathbb{V}$ *denote the expectation and the variance, respectively.*

Proof The proof is a consequence of a probability result: If a sequence of random variables is mean-square convergent, then the expectation and the variance converge (Villafuerte et al. 2010) (other types of convergence, such as almost surely, in probability or in distribution, are not sufficient to guarantee the convergence of statistics). Indeed, if $\{y_N\}_{N=1}^{\infty}$ satisfies $\|y_N - y\|_2 \to 0$ as $N \to \infty$, then, by Jensen and Cauchy–Schwarz inequalities

$$|\mathbb{E}[y_N] - \mathbb{E}[y]| = |\mathbb{E}[y_N - y]| \leq \mathbb{E}[|y_N - y|] \leq \|y_N - y\|_2 \to 0,$$
$$|\mathbb{E}[y_N^2] - \mathbb{E}[y^2]| = |\mathbb{E}[y_N^2 - y^2]| \leq \mathbb{E}[|y_N^2 - y^2|] \leq \|y_N + y\|_2\|y_N - y\|_2 \to 0,$$

and the proof is finished. □

The expectation and the variance are the most important statistics, since they capture the mean value and the dispersion and are necessary for any statistical analysis. They are related to the norm of the Hilbert space $\mathrm{L}^2(\Omega)$. For our problem, the approximations $\mathbb{E}[x(t)] \approx \mathbb{E}[x^N(t)]$ and $\mathbb{V}[x(t)] \approx \mathbb{V}[x^N(t)]$ can be used. The finite-term series $x^N(t)$, which is a polynomial of t, can be implemented in the computer. Since $\mathbb{E}[x^N(t)]$ and $\mathbb{E}[x^N(t)^2]$ are given by $\int_\Theta x^N(t|\theta) f_\theta(\theta)\mathrm{d}\theta$ and $\int_\Theta x^N(t|\theta)^2 f_\theta(\theta)\mathrm{d}\theta$, respectively, where θ denotes the set of random parameters in $\{y_0, y_1, \alpha, \beta, \gamma\}$ with compact support $\Theta \subseteq \mathbb{R}^d$ ($d \leq 5$) and density function f_θ, then these statistics are approximated by quadrature integration with respect to the weight function f_θ (Smith 2013, Chapter 11)

$$\int_\Theta x^N(t|\theta) f_\theta(\theta)\mathrm{d}\theta \approx \sum_{k_1,\ldots,k_d} x^N(t|\theta_{k_1})\cdots x^N(t|\theta_{k_d}) w_{k_1}\cdots w_{k_d}. \quad (6)$$

For other linear random differential equations, such as Legendre Jornet (2021b), quadrature integration is not necessary by linearity and by the multivariate-polynomial form of the truncated sum with respect to the random parameters. This is, of course, an advantage in computations. For our hypergeometric random differential equation, $x^N(t)$ is not a polynomial with respect to α, β, and γ.

The convergence rate of $x^N(t)$ toward $x(t)$ as $N \to \infty$ is exponential in a mean-square sense, for each $t \in I = (0, 1)$. Indeed, if $\sum_n a_n r^n$ is a convergent real power series at $r > 0$, then $\lim_{n\to\infty} a_n r^n = 0$, so there exists $C > 0$, such that $|a_n|r^n \leq C$ for all n. If $0 < s < r$, then

$$\sum_{n=N+1}^{\infty} |a_n|s^n \leq C \sum_{n=N+1}^{\infty} \left(\frac{s}{r}\right)^n = C\left(\frac{s}{r}\right)^N = C\mathrm{e}^{N\log(s/r)},$$

and we are done. This implies that the convergence rate of $\mathbb{E}[x^N(t)]$ and $\mathbb{V}[x^N(t)]$ as $N \to \infty$ is exponential, as well. However, the convergence rate is not uniform with t. Exponential convergence may be an advantage compared to usual techniques for uncertainty quantification, such as Monte Carlo simulation, which converges at rate $1/\sqrt{M}$ approximately, where M is the number of realizations. Indeed, exponential convergence may give several significant digits of accuracy of the statistics for small truncation order N, whereas Monte Carlo sampling may need 10^6 realizations for three digits or 10^8 for four digits. Nonetheless, Monte Carlo simulation is robust (i.e., it does not need to fulfill conditions), while this power-series method depends on t and on the accuracy of the quadrature integration with the dimension d.

Let us see three simple examples, where the improvement over Monte Carlo simulation is illustrated. We fix probability distributions for the input parameters and conduct forward uncertainty quantification. The focus is not put on inverse parameter estimation from data or measurements (Jornet 2023a; Dogan 2007; Corberán-Vallet et al. 2018).

Example 2 Let $y_0, y_1, \gamma \sim \text{Uniform}(0, 1/2)$ be independent random variables, $\alpha = 0.3$, $\beta = 0.5$ and $t_0 = 0.2$. We approximate the second-order moment $\mathbb{E}[x(0.3)^2]$. We use partial sums $x^N(0.3)$ and apply Gauss–Legendre quadrature for (6), on $\Theta = [0, 1/2]^3$ with $\theta = (y_0, y_1, \gamma)$. For quadrature degree 7 (i.e., 7 nodes per dimension) and truncation order $N = 15$ for the series, the result stabilizes at six significant digits, in 0.91 s: $\mathbb{E}[x(0.3)^2] \approx 0.0981873\ldots$. Monte Carlo simulation with 100, 000 realizations and numerical resolution of the problem only gives an accuracy of two significant figures: $\mathbb{E}[x(0.3)^2] \approx 0.098\ldots$, in 161 s.

Example 3 In the previous example, if $\beta \sim \text{Uniform}(0, 1/2)$ is also random, then (6) is conducted on $\Theta = [0, 1/2]^4$ with $\theta = (y_0, y_1, \gamma, \beta)$. For quadrature degree 7 and truncation order $N = 15$ for the series, the result stabilizes at six significant digits, in 6.2 s: $\mathbb{E}[x(0.3)^2] \approx 0.0975334\ldots$. Monte Carlo simulation with 100, 000 realizations and numerical resolution of the problem only gives an accuracy of two significant figures: $\mathbb{E}[x(0.3)^2] \approx 0.097\ldots$, in 194 s.

Example 4 Finally, in the previous example, if $\alpha \sim \text{Exponential}(2)|_{[0,3/2]}$ is random too (rate parameter 2 and truncation in $[0, 3/2]$), then (6) is calculated on $\Theta = [0, 1/2]^4 \times [0, 3/2]$ with $\theta = (y_0, y_1, \gamma, \beta, \alpha)$, based on Gauss-Legendre quadrature for $x^N(t|\theta) f_\theta(\theta)$. For degree 7 and truncation order $N = 15$, the result stabilizes at six significant digits, in 49 s: $\mathbb{E}[x(0.3)^2] \approx 0.0978347\ldots$. Monte Carlo simulation with 100, 000 realizations and numerical resolution of the problem only gives an accuracy of two significant figures: $\mathbb{E}[x(0.3)^2] \approx 0.09790\ldots$, in 630 s.

If, for example, $\|\gamma\|_\infty = \infty$, one could consider a truncated random variable

$$\tilde{\gamma} = \begin{cases} \gamma, & \text{if } |\gamma| \leq R, \\ 0, & \text{if } |\gamma| > R, \end{cases}$$

where $R > 0$ is any number, such that $\mathbb{P}[|\gamma| > R] \approx 0$. Then, both $\tilde{\gamma}$ and γ are very similar and $\|\tilde{\gamma}\|_\infty \leq R < \infty$. A similar approach is followed if $\mathbb{P}[\gamma \in (1 - \delta, 1 + \delta)] > 0$ for every $\delta > 0$. One could consider

$$\tilde{\gamma} = \begin{cases} \gamma, & \text{if } |\gamma - 1| > \delta, \\ 0, & \text{if } |\gamma - 1| \leq \delta, \end{cases}$$

where $\delta > 0$ is any number such that $\mathbb{P}[\gamma \in (1 - \delta, 1 + \delta)] \approx 0$. Therefore, the hypotheses of the theorem are not restrictive in practice. Boundedness of the random coefficients is a technical assumption that is often set in random systems; see Strand (1970, example, pages 4–5) or Jornet (2023b).

5 Conclusions

For the hypergeometric random differential equation, mean-square solutions have been constructed by taking advantage of the random calculus and the Fröbenius method of power series. The results obtained extend the deterministic counterpart. The paper solves one of the problems posed in Cortés et al. (2017).

In Proposition 1, a stochastic basis has been built by proving the convergence of the hypergeometric series in a stochastic sense. Boundedness of the random coefficients has been needed, due to the limitations involved when dealing with 2-norms and the multiplicative property. In Theorem 1, initial-value problems have been solved in mean square. The series forms have been of use to approximate the expectation and the variance of the solution, with improvements over Monte Carlo simulation. Three computational examples have been provided.

Our approach may be useful to continue studying other randomized linear differential equations encountered in mathematical physics, with ordinary or regular singular points. The study of approximations of density functions, beyond moments and statistics, is also of interest.

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Data Availability No data were used for this study.

Declarations

Conflict of interest The author declares that there is no conflict of interest regarding the publication of this article.

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