Ann. Funct. Anal. (2024) 15:17
https://doi.org/10.1007/s43034-024-00317-0

ORIGINAL PAPER

Lipschitz continuity of the dilation of Bloch functions on the unit ball of a Hilbert space and applications

Alejandro Miralles[1]

Received: 24 May 2023 / Accepted: 2 January 2024

Abstract
Let B_E be the open unit ball of a complex finite- or infinite-dimensional Hilbert space. If f belongs to the space $\mathcal{B}(B_E)$ of Bloch functions on B_E, we prove that the dilation map given by $x \mapsto (1-\|x\|^2)\mathcal{R}f(x)$ for $x \in B_E$, where $\mathcal{R}f$ denotes the radial derivative of f, is Lipschitz continuous with respect to the pseudohyperbolic distance ρ_E in B_E, which extends to the finite- and infinite-dimensional setting the result given for the classical Bloch space $\mathcal{B}$. To provide this result, we will need to prove that $\rho_E(zx, zy) \leq |z|\rho_E(x, y)$ for $x, y \in B_E$ under some conditions on $z \in \mathbb{C}$. Lipschitz continuity of $x \mapsto (1-\|x\|^2)\mathcal{R}f(x)$ will yield some applications on interpolating sequences for $\mathcal{B}(B_E)$ which also extends classical results from $\mathcal{B}$ to $\mathcal{B}(B_E)$. Indeed, we show that it is necessary for a sequence in B_E to be separated to be interpolating for $\mathcal{B}(B_E)$ and we also prove that any interpolating sequence for $\mathcal{B}(B_E)$ can be slightly perturbed and it remains interpolating.

Keywords Bloch space · Infinite-dimensional holomorphy · Pseudohyperbolic distance · Interpolating sequence

Mathematics Subject Classification 46E50 · 30H30 · 32A18

1 Introduction and background

Let $\mathbf{D}$ be the open unit disk of the complex plane $\mathbb{C}$ and ρ the pseudohyperbolic distance defined by $\rho(z, w) = \left|\frac{z-w}{1-z\bar{w}}\right|$. Consider a function f of the classical Bloch space $\mathcal{B}$. The study of the Lipschitz continuity of the dilation map $z \mapsto (1-|z|^2)f'(z)$ with respect to ρ was first started by Attele to study conditions for a sequence on $\mathbf{D}$

Communicated by Pedro Tradacete.

✉ Alejandro Miralles
mirallea@uji.es

[1] Departament de Matemàtiques and IMAC, Universitat Jaume I, Castelló, Spain

Published online: 13 February 2024 Birkhäuser to be interpolating for $\mathcal{B}$ (see [2]). Ghatage, Yan, and Zheng also discussed a similar result providing a different proof [8]. Then, Xiong improved this result in [14] and other authors extended it for high-dimensional holomorphic functions [6] and planar harmonic mappings [7]. For bounded symmetric domains of $\mathbb{C}^n$, see also [10].

The aim of this paper is to extend the previous result for the infinite-dimensional setting and give some applications. Other applications on bounded below composition operators can be found in [12]. Along this work, E will denote a finite- or infinite-dimensional complex Hilbert space and its open unit ball will be denoted by B_E. In Sect. 2, we will study the boundness of

$$\frac{\rho_E(zx, zy)}{|z|\rho_E(x, y)} \tag{1.1}$$

for $z \in \mathbb{C}$ and $x, y \in B_E$, such that $zx, zy \in B_E$ proving that, in general, this expression is unbounded. Nevertheless, we will show that if $|z|$ is bounded above by

$$\frac{1 + \max\{\|x\|, \|y\|\}}{2\max\{\|x\|, \|y\|\}},$$

then the expression above is bounded by 2 and this bound is the best possible. We first prove the case when we deal with $E = \mathbb{C}$ and then with any finite- or infinite-dimensional Hilbert space E. At the end of this section, we will extend this result to the case when we deal with the Banach space $C_0(S)$.

In Sect. 3, we will consider the space $\mathcal{B}(B_E)$ of Bloch functions f on B_E. Recall that for $f \in \mathcal{B}(B_E)$ and $x \in B_E$, the radial derivative $Rf(x)$ is given by $Rf(x) = \langle x, f'(x)\rangle$. As a consequence of the boundness of (1.1), we show that the dilation map $x \mapsto (1 - \|x\|^2)\mathcal{R}f(x)$ for $x \in B_E$ is Lipschitz continuous with respect to the pseudohyperbolic distance ρ_E in Sect. 3.1, extending to the finite- and infinite-dimensional setting the results mentioned above. Hence, we derive some results about interpolating sequences for $\mathcal{B}(B_E)$ in Sect. 3.2. Indeed, we supply a proof that these sequences are separated for the pseudohyperbolic distance. We also prove that interpolating sequences can be slightly perturbed and they remain interpolating, which also extends the result for $\mathcal{B}$ given in [2].

1.1 The pseudohyperbolic and hyperbolic distance

Let $\mathbf{D}$ be the open unit disk of the complex plane $\mathbb{C}$. As we have mentioned, the pseudohyperbolic distance for $z, w \in \mathbf{D}$ is given by

$$\rho(z, w) = \left|\frac{z - w}{1 - \bar{z}w}\right|.$$

If we deal with a complex Banach space X with open unit ball B_X, recall that $f : B_X \to \mathbb{C}$ is said to be holomorphic (or analytic) if it is Fréchet differentiable for any $x \in B_X$ (see [12] for further information). For any $x, y \in B_X$, the pseudohyperbolic distance $\rho_X(x, y)$ is given by

$$\rho_X(x, y) = \sup\{\rho(f(x), f(y)) : f \in H^\infty(B_X), \|f\|_\infty \leq 1\},$$

where $H^\infty(B_X)$ is the space of bounded holomorphic functions on B_X which become a Banach space (a uniform algebra, indeed) endowed with the sup-norm. The hyperbolic distance for $x, y \in B_X$ is given by

$$\beta_X(x, y) = \frac{1}{2} \log \left(\frac{1 + \rho_X(x, y)}{1 - \rho_X(x, y)} \right).$$

1.2 Automorphisms and pseudohyperbolic distance on B_E

If we deal with a complex Hilbert space E, we will denote by $\mathrm{Aut}(B_E)$ the space of automorphisms of B_E, that is, the maps $\varphi : B_E \to B_E$ which are bijective and bianalytic (see [9]). For any $x \in B_E$, the automorphism $\varphi_x : B_E \longrightarrow B_E$ is defined according to

$$\varphi_x(y) = (s_x Q_x + P_x)(m_x(y)), \tag{1.2}$$

where $s_x = \sqrt{1 - \|x\|^2}$, $m_x : B_E \longrightarrow B_E$ is the analytic self-map

$$m_x(y) = \frac{x - y}{1 - \langle y, x \rangle},$$

$P_x : E \longrightarrow E$ is the orthogonal projection along the one-dimensional subspace spanned by x, that is

$$P_x(y) = \frac{\langle y, x \rangle}{\langle x, x \rangle} x,$$

and $Q_x : E \longrightarrow E$ is the orthogonal complement $Q_x = \mathrm{Id}_E - P_x$, where Id_E denotes the identity operator on E. It is clear that $\varphi_x(0) = x$ and $\varphi_x(x) = 0$. The automorphisms of the unit ball B_E turn to be compositions of these φ_x with unitary transformations U of E.

It is well known (see [9]) that the pseudohyperbolic distance on B_E is given by

$$\rho_E(x, y) = \|\varphi_y(x)\| \text{ for any } x, y \in B_E. \tag{1.3}$$

and

$$\rho_E(x, y)^2 = 1 - \frac{(1 - \|x\|^2)(1 - \|y\|^2)}{|1 - \langle x, y \rangle|^2}. \tag{1.4}$$

1.3 The Bloch space

The classical Bloch space $\mathcal{B}$ is the set of holomorphic functions $f : \mathbf{D} \to \mathbb{C}$, such that $\|f\|_B = \sup_{z\in\mathbf{D}}(1-|z|^2)|f'(z)|$ is bounded. This supremum defines a semi-norm which becomes a norm by adding up a constant: $|f(0)| + \sup_{z\in\mathbf{D}}(1-|z|^2)|f'(z)|$. Hence, $\mathcal{B}$ becomes a complex Banach space. The semi-norm $\|\cdot\|_B$ is invariant by automorphisms, that is, $\|f \circ \varphi\|_B = \|f\|_B$ for any $f \in \mathcal{B}$ and $\varphi : \mathbf{D} \to \mathbf{D}$ an automorphism of $\mathbf{D}$.

Timoney extended Bloch functions if we deal with a finite-dimensional Hilbert space (see [13]). Blasco, Galindo, and Miralles extended them to the infinite-dimensional setting (see [5]). If we deal with a complex finite- or infinite-dimensional Hilbert space E, the analytic function $f : B_E \to \mathbb{C}$ is said to belong to the Bloch space $\mathcal{B}(B_E)$ if

$$\|f\|_{\mathcal{B}} = \sup_{x\in B_E} (1-\|x\|^2)\|\nabla f(x)\| < +\infty,$$

where it is clear that $\nabla f(x)$ is the derivative $f'(x)$ or, equivalently, if

$$\|f\|_{\mathcal{R}} = \sup_{x\in B_E} (1-\|x\|^2)\|Rf(x)\| < +\infty,$$

where $\mathcal{R}f(x)$ is the radial derivative of f at x given by $\mathcal{R}f(x) = \langle x, \overline{\nabla f(x)}\rangle$. These semi-norms are equivalent to the following one:

$$\|f\|_{\mathcal{I}} = \sup_{x\in B_E} \|\widetilde{\nabla} f(x)\|, \tag{1.5}$$

where $\widetilde{\nabla} f(x)$ denotes the invariant gradient of f at x which is given by $\widetilde{\nabla} f(x) = \nabla(f \circ \varphi_x)(0)$, where φ_x is the automorphism given in (1.2).

The three semi-norms $\|\cdot\|_{\mathcal{B}}$, $\|\cdot\|_{\mathcal{R}}$ and $\|\cdot\|_{\mathcal{I}}$ define equivalent Banach space norms-modulo the constant functions- in $\mathcal{B}(B_E)$ (see [5]). In particular, there exists a constant $A_0 > 0$, such that

$$\|f\|_{\mathcal{R}} \le \|f\|_{\mathcal{B}} \le \|f\|_{\mathcal{I}} \le A_0\|f\|_R. \tag{1.6}$$

Hence, the space $\mathcal{B}(B_E)$ can be endowed with any of the norms $\|\cdot\|_{\mathcal{B}-Bloch} = |f(0)| + \|\cdot\|_{\mathcal{B}}$ or $\|\cdot\|_{\mathcal{R}-Bloch} = |f(0)| + \|\cdot\|_{\mathcal{R}}$ or $\|\cdot\|_{\mathcal{I}-Bloch} = |f(0)| + \|\cdot\|_{\mathcal{I}}$ and $\mathcal{B}(B_E)$ becomes a Banach space. We will make use of these three semi-norms and norms along the sequel. We will also make use of this result, which states that Bloch functions on B_E are Lipschitz with respect to the hyperbolic distance (see [3]):

Proposition 1.1 *Let E be a complex Hilbert space and let $f \in \mathcal{B}(B_E)$. Then, for any $x, y \in B_E$*

$$|f(x) - f(y)| \le \|f\|_{\mathcal{I}}\beta_E(x, y).$$

2 Inequalities with the pseudohyperbolic distance

Let X be a complex Banach space. If $\varphi : B_X \to B_X$ is an analytic self-map, it is well known that $\rho_X(\varphi(x), \varphi(y)) \le \rho_X(x, y)$ for any $x, y \in B_X$ and the equality is attained if and only if φ is an automorphism of B_X. Hence, if we consider $z \in \mathbb{C}$, $|z| \le 1$ and $x, y \in B_X$, it is clear that

$$\frac{\rho_X(zx, zy)}{\rho_X(x, y)} \le 1,$$

since the map $\varphi : B_X \to B_X$ given by $\varphi(x) = zx$ is analytic on B_X. However, this situation changes dramatically if we consider the expression

$$\frac{\rho_X(zx, zy)}{|z|\rho_X(x, y)} \tag{2.1}$$

for any $z \in \mathbb{C}$, such that $zx, zy \in B_X$. We show that, in general, this expression is unbounded. Anyway, if we deal with X a complex Hilbert space or $C_0(S)$ and $z \in \mathbb{C}$ satisfies

$$|z| \le \frac{1 + \max\{\|x\|, \|y\|\}}{2 \max\{\|x\|, \|y\|\}},$$

then expression (2.1) is bounded by 2 and this will permit us to provide several applications in Sect. 3.

2.1 Unboundness

In this section, we prove that expression (2.1) is unbounded in general.

Proposition 2.1 *Let E be a complex Hilbert space. There exist a sequence $(z_n) \subset \mathbb{C}$, $x \in B_E$ and a sequence $(y_n) \subset B_E$, such that $z_n x, z_n y_n \in B_E$ but*

$$\frac{\rho_E(z_n x, z_n y_n)}{|z_n|\rho_E(x, y_n)}$$

is unbounded.

Proof We prove it for $E = \mathbb{C}$. Take for instance $x = 1/2$, $y_n = 1/2 - \frac{1}{n}$ and $z_n = 2 - \frac{1}{n}$. It is clear that $|z_n x| < 1$ and $|z_n y_n| < 1$. However

$$\rho(z_n x, z_n y_n) = \frac{\frac{1}{n}\left(2 - \frac{1}{n}\right)}{1 - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)^2} = \frac{\frac{2n-1}{n^2}}{\frac{2-9n+12n^2}{4n^3}} = \frac{4n(2n-1)}{12n^2 - 9n + 2}$$

and

$$|z_n|\rho(x, y_n) = \left(2 - \frac{1}{n}\right)\frac{\frac{1}{n}}{1 - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n}\right)} = \frac{\frac{2n-1}{n^2}}{\frac{3n+2}{4n}} = \frac{4(2n-1)}{3n^2 + 2n}.$$

Hence

$$\frac{\rho(z_n x, z_n y_n)}{|z_n|\rho(x, y_n)} = \frac{\frac{4n(2n-1)}{12n^2-9n+2}}{\frac{4(2n-1)}{3n^2+2n}} = \frac{3n^3+2n^2}{12n^2-9n+2},$$

which is clearly unbounded, since it tends to ∞ when $n \to \infty$. The result remains true if we deal with any complex Hilbert space E, since we can take $x_0 \in E$, such that $\|x_0\| = 1$ and take $u = \frac{1}{2}x_0$, $v_n = y_n x_0$. We have that

$$\begin{aligned}\rho_E(z_n u, z_n v_n)^2 &= 1 - \frac{(1-\|u_n\|^2)(1-\|v_n\|^2)}{|1-\langle u_n, v_n\rangle|^2} \\ &= 1 - \frac{(1-\|z_n x\|^2)(1-\|z_n y_n\|^2)}{|1-\langle z_n x, z_n y_n\rangle|^2} = \rho_E(z_n x, z_n y_n)^2,\end{aligned}$$

and similarly, we have $\rho_E(u, v_n) = \rho_E(x, y_n)$, so we apply the case $E = \mathbb{C}$ and we are done. □

An easy consequence is a well-known result: the pseudohyperbolic distance cannot be extended to a norm on E, since

$$\frac{\rho_E(zx, zy)}{|z|\rho_E(x, y)}$$

is unbounded, so $\rho_E(zx, zy) \neq |z|\rho_E(x, y)$.

2.2 Boundness

The main result of this section is Theorem 2.7 which states that under condition (2.1), then the expression (1.1) is bounded and the best bound possible is given by 2. The following lemma will be used to prove this result.

Lemma 2.2 *Let E be a finite- or infinite-dimensional Hilbert space, $z \in \mathbb{C}$ and $x, y \in B_E$, such that*

$$|z| \leq \frac{1+\max\{\|x\|, \|y\|\}}{2\max\{\|x\|, \|y\|\}}.$$

Then, $|1-p| \leq 2|1-|z|^2 p|$ where p denotes the scalar product $\langle x, y\rangle$.

Proof Suppose without loss of generality that $\|x\| \geq \|y\|$ and $x \neq 0$. Otherwise, the inequality is clearly true for any $z \in \mathbb{C}$. Notice that

$$|1-p| = |1-|z|^2 p + |z|^2 p - p| \leq |1-|z|^2 p| + ||z|^2-1||p|,$$

so it is sufficient to prove that $|1-|z|^2 p| + ||z|^2-1||p| \leq 2|1-|z|^2 p|$, which is equivalent to $|1-|z|^2 p| \geq ||z|^2-1||p|$. We consider two cases:

i) if $|z|^2 \le 1$, then $1 - |z|^2 \ge 0$, so we need to prove $|1 - |z|^2 p| \ge (1 - |z|^2)|p|$ which is clearly satisfied, since

$$|1 - |z|^2 p| \ge 1 - |z|^2|p| \ge |p| - |z|^2|p| = (1 - |z|^2)|p|,$$

where second inequality is true, because $|p| \le \|x\|\|y\| < 1$.

ii) On the other hand, suppose that $|z|^2 > 1$: we need to prove that $|1 - |z|^2 p| \ge (|z|^2 - 1)|p|$. Since $|1 - |z|^2 p| \ge 1 - |z|^2|p|$, it is sufficient to prove that

$$1 - |z|^2|p| \ge (|z|^2 - 1)|p|. \tag{2.2}$$

Notice that $1 - |z|^2|p| > 0$, since $|z|^2|p| \le \|zx\|\|zy\| < 1$, because $zx, zy \in B_E$. Inequality (2.2) is equivalent to $2|z|^2|p| < 1 + |p|$ which is true, since

$$2|z|^2|p| \le 2\left(\frac{1 + \|x\|}{2\|x\|}\right)^2 |p|,$$

so we need to prove that

$$2\left(\frac{1 + \|x\|}{2\|x\|}\right)^2 |p| \le 1 + |p| \leftrightarrow \left(\frac{(1 + \|x\|)^2}{2\|x\|^2} - 1\right)|p| \le 1.$$

However

$$\left(\frac{(1 + \|x\|)^2}{2\|x\|^2} - 1\right)|p| = \frac{1 + (2 - \|x\|)\|x\|}{2\|x\|^2}|p| \le \frac{1 + 1}{2\|x\|^2}\|x\|^2 = 1,$$

where last inequality is true because of the arithmetic mean-geometric mean inequality and since $|p| \le \|x\|\|y\| \le \|x\|^2$. □

2.2.1 The case $E = \mathbb{C}$

If we deal with $E = \mathbb{C}$, it is easy to prove that (2.1) is bounded:

Proposition 2.3 *Let $x, y \in \mathbf{D}$ and $z \in \mathbb{C}$, such that*

$$|z| \le \frac{1 + \max\{|x|, |y|\}}{2\max\{|x|, |y|\}}.$$

Then, $zx, zy \in \mathbf{D}$ and

$$\rho(zx, zy) \le 2|z|\rho(x, y).$$

Proof Suppose without loss of generality that $|x| \ge |y|$ and take $z \ne 0$ (otherwise, it is clear). Notice that $zx, zy \in \mathbf{D}$, since

$$|zy| \le |zx| \le \frac{1 + |x|}{2|x|}|x| = \frac{1 + |x|}{2} < 1.$$

We have

$$\rho(zx, zy) = \left|\frac{zx - zy}{1 - zx\overline{zy}}\right| = |z|\left|\frac{x - y}{1 - |z|^2 x\overline{y}}\right| = |z|\frac{|x - y|}{|1 - |z|^2 x\overline{y}|}$$

and

$$|z|\rho(x, y) = |z|\left|\frac{x - y}{1 - x\overline{y}}\right| = |z|\frac{|x - y|}{|1 - x\overline{y}|},$$

so the inequality is equivalent to

$$|z|\frac{|x - y|}{|1 - |z|^2 x\overline{y}|} \leq 2|z|\frac{|x - y|}{|1 - x\overline{y}|} \leftrightarrow |1 - x\overline{y}| \leq 2|1 - |z|^2 x\overline{y}|.$$

Calling $p = x\overline{y}$, we have to prove that $|1 - p| \leq 2|1 - |z|^2 p|$. Apply Lemma 2.2 for $E = \mathbb{C}$ and we are done. □

Remark 2.4 Notice that the bound 2 is the best possible. Indeed, take $x_n, y_n \in \mathbf{D}$, such that $x_n \to 1$ and $y_n \to -1$. It is clear that for $z_n \to 0$, the expression $|1 - \overline{x_n} y_n|$ tends to 2 when $n \to \infty$ and the expression $2|1 - |z_n|^2 \overline{x_n} y_n|$ also tends to 2, so the inequality above is sharp. □

2.2.2 The case when E is any complex Hilbert space

We will deal with $x, y \in \mathcal{B}_E$ and $z \in \mathbb{C}$, such that $zx, zy \in B_E$, we will denote by $r = \|x\|$, $s = \|y\|$ and $u = \|x\|^2 + \|y\|^2 = r^2 + s^2$. We will also denote by p the scalar product $\langle x, y\rangle$ and $m = \Re p$. This notation will be used in Lemma 2.5 and Theorem 2.7.

Lemma 2.5 *Let E be a complex Hilbert space and $x, y \in B_E$. Then*

$$\frac{\|x - y\|^2 |1 - p|^2}{\|x - y\|^2 - (\|x\|^2 \|y\|^2 - |p|^2)} \leq 4.$$

Proof The inequality is equivalent to

$$(1 - 2\Re p + |p|^2)\|x - y\|^2 \leq 4\|x - y\|^2 - 4r^2 s^2 + 4|p|^2 \text{ if and only if:}$$
$$(1 - 2\Re p + |p|^2)(r^2 + s^2 - 2\Re p) \leq 4(r^2 + s^2) - 8\Re p - 4r^2 s^2 + 4|p|^2.$$

Bearing in mind that $m = \Re p$ and $u = r^2 + s^2$, we need to prove

$$\begin{aligned}
&(1 - 2m + |p|^2)(u - 2m) \leq 4u - 8m - 4r^2 s^2 + 4|p|^2 \\
&\leftrightarrow u - 2um + u|p|^2 - 2m + 4m^2 - 2m|p|^2 \leq 4u - 8m - 4r^2 s^2 + 4|p|^2 \\
&\leftrightarrow 4u - 8m - 4r^2 s^2 + 4|p|^2 - u + 2um - u|p|^2 + 2m - 4m^2 + 2m|p|^2 \geq 0.
\end{aligned}$$

Notice that $|p| \geq |m|$ so $(4+2m-u)|p|^2 \geq (4+m-u)m^2$ since $4+2m-u \geq 0$. Hence, it is sufficient to prove

$$4u - 8m - 4r^2s^2 + (4+2m-u)m^2 - u + 2um + 2m - 4m^2 \geq 0.$$

It is also clear that $u^2 \geq 4r^2s^2$, so last inequality is equivalent to

$$\begin{aligned} &4u - 8m - u^2 + (4+2m-u)m^2 - u + 2um + 2m - 4m^2 \geq 0 \\ &\leftrightarrow 2m^3 - um^2 + 2(u-3)m + 3u - u^2 \geq 0. \end{aligned}$$

The expression at left can be easily factorized and is equal to

$$(u-2m)(3-u-m^2),$$

where both factors are clearly greater or equal to 0 and we are done. □

This lemma will be used at the end of the proof of Theorem 2.7:

Lemma 2.6 *Let* $f(a,b,c) = (3-b^2)(a-c) - (a^2-b^2)(2-c)$. *Then,* $f(a,b,c) \geq 0$ *for any* $0 \leq c \leq b \leq a \leq 1$.

Proof Notice that $f(a,b,c) = (3-b^2)a - 2(a^2-b^2) - (3-a^2)c$, so f is affine with respect to c. Hence, it is enough to prove the inequality for $c = b$. The function becomes $f(a,b,b) = (3-b^2)(a-b) - (a^2-b^2)(2-b) = (a-b)((3-b^2) - (a+b)(2-b))$, and since $a - b \geq 0$, it is enough to prove that $(3-b^2) - (a+b)(2-b) \geq 0$. The expression $g(a,b) = (3-b^2) - (a+b)(2-b)$ is affine with respect to a, so it is enough to prove it for $a = 1$. Notice that $g(1,b) = (3-b^2) - (1+b)(2-b) = 1-b$ which is clearly greater or equal to 0, so we are done. □

Theorem 2.7 *Let E be a finite- or infinite-dimensional complex Hilbert space, $z \in \mathbb{C}$ and $x, y \in B_E$. If*

$$|z| \leq \frac{1 + \max\{\|x\|, \|y\|\}}{2\max\{\|x\|, \|y\|\}},$$

then $zx, zy \in B_E$ and

$$\frac{\rho_E(zx, zy)}{|z|\rho_E(x,y)} \leq 2.$$

Proof Suppose without loss of generality that $\|x\| \geq \|y\|$ and $z \neq 0$. We will denote $\rho = \rho_E(x,y)$ and $\rho_z = \rho_E(zx, zy)$. If $\frac{1}{2} \leq |z| < 1$, then the result is clear, since

$$\rho_E(zx, zy) \leq \rho_E(x,y) \leq 2|z|\rho_E(x,y),$$

where first inequality is true because of the contractivity of the pseudohyperbolic distance for the function $g : B_E \to B_E$ given by $g(x) = zx$.

Therefore, let us prove it for $|z| < 1/2$ or $|z| \geq 1$. Taking squares, the inequality is equivalent to prove

$$\frac{\rho_z^2}{|z|^2\rho^2} \leq 4. \tag{2.3}$$

Bear in mind the expression (1.4) for the pseudohyperbolic distance and call $t = |z|^2$ which is different from 0, since $z \neq 0$. We have

$$\begin{aligned}
\frac{\rho_z^2}{|z|^2\rho^2} &= \frac{\frac{|1-tp|^2-(1-tr^2)(1-ts^2)}{|1-tp|^2}}{\frac{t(|1-p|^2-(1-r^2)(1-s^2))}{|1-p|^2}} = \frac{(|1-tp|^2-(1-tr^2)(1-ts^2))|1-p|^2}{t(|1-p|^2-(1-r^2)(1-s^2))|1-tp|^2} \\
&= \frac{(1+t^2|p|^2-2t\Re p-1-t^2r^2s^2+t(r^2+s^2))|1-p|^2}{t(1+|p|^2-2\Re p-1-r^2s^2+r^2+s^2)|1-tp|^2} \\
&= \frac{(t^2|p|^2-2t\Re p-t^2r^2s^2+t(r^2+s^2))|1-p|^2}{t(|p|^2-2\Re p-r^2s^2+r^2+s^2)|1-tp|^2} \\
&= \frac{(\|x-y\|^2-t(r^2s^2-|p|^2))|1-p|^2}{(\|x-y\|^2-(r^2s^2-|p|^2))|1-tp|^2}.
\end{aligned}$$

We will introduce the following notation:

$$A := \|x-y\|^2 = r^2+s^2-2\Re p = u-2m \tag{2.4}$$

$$B := r^2s^2-|p|^2. \tag{2.5}$$

Notice that $A - B \geq 0$, since

$$\begin{aligned}
A-B = r^2+s^2-2m-r^2s^2+|p|^2 = |1-p|^2-(1-r^2)(1-s^2) \geq 0 \\
\leftrightarrow |1-p|^2 \geq (1-r^2)(1-s^2) \leftrightarrow 1-\frac{(1-r^2)(1-s^2)}{|1-p|^2} = \rho(x,y)^2 \geq 0.
\end{aligned}$$

Using this notation, inequality (2.3) is equivalent to

$$\frac{(A-tB)|1-p|^2}{(A-B)|1-tp|^2} \leq 4, \tag{2.6}$$

so bearing in mind that $t = |z|^2$, we only need to prove (2.6) for $t \geq 1$ and $t \leq 1/4$. If $t \geq 1$, the result is clear, since

$$\frac{(A-tB)|1-p|^2}{(A-B)|1-tp|^2} \leq \left(\frac{A-B}{A-B}\right)\frac{|1-p|^2}{|1-tp|^2} = \frac{|1-p|^2}{|1-tp|^2} \leq 4,$$

where last inequality is true by Lemma 2.2. Therefore, it remains to prove it for $0 \le t \le 1/4$. Inequality 2.6 is equivalent to

$$\begin{aligned}&4(A-B)|1-tp|^2 \ge (A-tB)|1-p|^2\\ &\leftrightarrow 4(A-B)(1-2mt+t^2|p|^2) \ge (A-tB)|1-p|^2\\ &\leftrightarrow 4(A-B)|p|^2t^2 + (B|1-p|^2 - 8m(A-B))t + 4(A-B) - A|1-p|^2 \ge 0.\end{aligned}$$

Since $B, t \ge 0$ and $4(A-B) - A|1-p|^2 \ge 0$ by Lemma 2.5, this inequality is clearly true if $m < 0$. Therefore, we can suppose without loss of generality that $m \ge 0$. The inequality is equivalent to

$$4(A-B)|1-tp|^2 - (A-tB)|1-p|^2 \ge 0,$$

so we will prove last inequality. Notice that

$$\begin{aligned}&4(A-B)|1-tp|^2 - (A-tB)|1-p|^2\\ &= 4(A-B)(1-2mt+t^2|p|^2) - (A-B)(1-2m+|p|^2) - B(1-t)|1-p|^2\\ &= 4(A-B)(1-2m+|p|^2+2m(1-t)-(1-t^2)|p|^2)\\ &\quad -(A-B)(1-2m+|p|^2) - B(1-t)|1-p|^2\\ &= 3(A-B)(1-2m+|p|^2) + 8m(1-t)(A-B)\\ &\quad -4(1-t^2)|p|^2(A-B) - B(1-t)|1-p|^2.\end{aligned}$$

Since $0 \le t \le 1/4$, we have that $3/4 \le 1-t \le 1$, so

$$\begin{aligned}&4(A-B)|1-tp|^2 - (A-tB)|1-p|^2\\ &\ge 3(A-B)(1-2m+|p|^2) + 8m(1-t)(A-B)\\ &\quad -4(1-t^2)|p|^2(A-B) - B(1-t)|1-p|^2\\ &\ge 3(A-B) - 6m(A-B) + 3|p|^2(A-B) + 8m\cdot\frac{3}{4}(A-B)\\ &\quad -4|p|^2(A-B) - B|1-p|^2\\ &= 3(A-B) - 6m(A-B) + 6m(A-B) - |p|^2(A-B) - B|1-p|^2\\ &= (3-|p|^2)(A-B) - B|1-p|^2.\end{aligned}$$

Bearing in mind (2.4) and (2.5), notice that

$$\begin{aligned}&(3-|p|^2)(A-B) - B|1-p|^2 = (3-|p|^2)A - B(3-|p|^2+1-2m+|p|^2)\\ &= (3-|p|^2)A - B(4-2m) = (3-|p|^2)(r^2+s^2-2m) - (4-2m)(r^2s^2-|p|^2)\\ &\ge (3-|p|^2)(2rs-2m) - (4-2m)(r^2s^2-|p|^2)\\ &= 2(3-|p|^2)(rs-m) - 2(2-m)(r^2s^2-|p|^2) = 2f(rs,|p|,m),\end{aligned}$$

where f is the function defined in Lemma 2.6 and since $0 \leq m \leq |p| \leq rs \leq 1$, using the lemma, we are done, since

$$(3 - |p|^2)(A - B) - B|1 - p|^2 \geq 2f(rs, |p|, m) \geq 0.$$

□

2.2.3 Results for $X = C_0(S)$

Let S be a locally compact topological space and consider $X = C_0(S)$ given by the space of continuous functions $f : S \to \mathbb{C}$, such that for any $\varepsilon > 0$, there exists a closed compact subset $K \subset S$, such that $|f(x)| < \varepsilon$ for any $x \in S \setminus K$. Endowed with the sup-norm, $C_0(S)$ becomes a Banach space and the pseudohyperbolic distance for $x, y \in C_0(S)$ is well known (see [1]) and it is given by

$$\rho_X(x, y) = \sup_{t \in S} \rho(x(t), y(t)). \tag{2.7}$$

We prove that expression (2.1) is also bounded by 2 when we deal with the space $X = C_0(S)$:

Proposition 2.8 *Let $X = C_0(S)$ and $x, y \in X$. If $z \in \mathbb{C}$ satisfies*

$$|z| \leq \frac{1 + \max\{\|x\|, \|y\|\}}{2\max\{\|x\|, \|y\|\}},$$

then

$$\frac{\rho_X(zx, zy)}{|z|\rho_X(x, y)} \leq 2.$$

Proof Suppose without loss of generality that $\|x\| \geq \|y\|$. For any $t \in S$, we have that $x(t), y(t) \in \mathbf{D}$, since $\|x\| = \sup_{t \in S} |x(t)| < 1$ and $\|y\| = \sup_{t \in S} |y(t)| < 1$. The result is clear, since

$$\begin{aligned}\rho_X(zx, zy) &= \sup_{t \in S} \rho(zx(t), zy(t)) \leq \sup_{t \in S} 2|z|\rho(x(t), y(t)) \\ &= 2|z| \sup_{t \in S} \rho(x(t), y(t)) = 2|z|\rho_X(x, y),\end{aligned}$$

where first inequality is clear because of Proposition 2.3 and because for any $t \in X$, we have that

$$|z| \leq \frac{1 + \|x\|}{2\|x\|} = \frac{\frac{1}{\|x\|} + 1}{2} \leq \inf_{t \in X} \left\{ \frac{1 + |x(t)|}{2|x(t)|} \right\} \leq \frac{1 + |x(t)|}{2|x(t)|},$$

and we are done. □

3 Applications

Theorem 2.7 yields several applications. As we have mentioned, we first show that for a Bloch function $f : B_E \to \mathbb{C}$, the function $x \mapsto (1-\|x\|^2)|\mathcal{R}f(x)|$ for $x \in B_E$ is Lipschitz continuous with respect to the pseudohyperbolic distance ρ_E. Hence, we derive some results about interpolating sequences for $\mathcal{B}(B_E)$ in Sect. 3.2. Indeed, we provide a new proof that these sequences are separated for the pseudohyperbolic distance. We also prove that these sequences can be slightly perturbed and they remain interpolating.

3.1 The Lipschitz continuity of $(1-\|x\|^2)|\mathcal{R}f(x)|$

We will denote by Π the unit circle of the complex plane $\mathbb{C}$, that is, the set of complex numbers u, such that $|u| = 1$.

Lemma 3.1 *Let $f \in \mathcal{B}(B_E)$. Fix $\varepsilon > 0$ and $x, y \in B_E$. If $(1+\varepsilon u)x$ and $(1+\varepsilon u)y$ belongs to B_E for any $u \in \Pi$, then there exists $u_0 \in \Pi$, such that*

$$|\mathcal{R}f(x) - \mathcal{R}f(y)| \le \frac{1}{\varepsilon}\|f\|_{\mathcal{I}}\beta((1+\varepsilon u_0)x, (1+\varepsilon u_0)y). \tag{3.1}$$

Proof Fix $x, y \in B_E$ and $\varepsilon > 0$. Notice that the function $f(x+\varepsilon ux) - f(y+\varepsilon uy)$ defined for $u \in \Pi$ is continuous. Since Π is a compact set, there exists $u_0 \in \Pi$, such that

$$f(x+\varepsilon u_0 x) - f(y+\varepsilon u_0 y) = \max\{f(x+\varepsilon ux) - f(y+\varepsilon uy) : u \in \Pi\}.$$

Consider $g(u) = f(x+\varepsilon ux)$ for u defined on an open disk of the complex plane $\mathbb{C}$ which contains Π. It is clear that

$$g'(u) = \nabla f(x+\varepsilon ux)(\varepsilon x),$$

so $g'(0) = \varepsilon\mathcal{R}f(x)$. Similarly, if $h(u) = f(y+\varepsilon uy)$, then $h'(0) = \varepsilon Rf(y)$. By the Cauchy's integral formula, we have

$$\begin{aligned}
|\mathcal{R}f(x) - \mathcal{R}f(y)| &= |\langle x, \overline{\nabla f(x)}\rangle - \langle y, \overline{\nabla f(y)}\rangle| \\
&= \left|\frac{1}{\varepsilon}\frac{1}{2\pi i}\int_{|u|=1} f(x+\varepsilon ux) - f(y+\varepsilon uy)\frac{du}{u^2}\right| \\
&\le \frac{2\pi}{2\pi\varepsilon}|f(x+\varepsilon u_0 x) - f(y+\varepsilon u_0 y)| \\
&\le \frac{1}{\varepsilon}\|f\|_{\mathcal{I}}\beta((1+\varepsilon u_0)x, (1+\varepsilon u_0)y),
\end{aligned}$$

where last inequality is true by Proposition 1.1. □

The proof of the following lemma is an easy calculation. It will be used in Lemma 3.3.

Lemma 3.2 *For any* $0 \le t < 1$, *we have*

$$\frac{1}{2}\log\left(\frac{1+t}{1-t}\right) \le \frac{t}{1-t}.$$

Lemma 3.3 *Let* $f \in \mathcal{B}(B_E)$ *and* $x, y \in B_E$, *such that* $\|x\| \ge \|y\|$. *Then*

$$(1-\|x\|^2)|\mathcal{R}f(x) - \mathcal{R}f(y)| \le 12\|f\|_{\mathcal{I}}\rho_E(x, y).$$

Proof Take

$$\varepsilon = \frac{1-\|x\|}{2\|x\|} > 0.$$

Notice that for any $u \in \Pi$, we have that $(1+\varepsilon u)x$ and $(1+\varepsilon u)y$ belongs to B_E, since

$$(1+\varepsilon)\|x\| \le \left(1 + \frac{1-\|x\|}{2\|x\|}\right)\|x\| = \frac{1+\|x\|}{2\|x\|}\|x\| = \frac{1+\|x\|}{2} < 1,$$

so clearly $\|(1+\varepsilon u)x\| \le (1+\varepsilon)\|x\| < 1$ and since $\|y\| \le \|x\|$

$$\|(1+\varepsilon u)y\| \le (1+\varepsilon)\|y\| \le (1+\varepsilon)\|x\| < 1.$$

By Lemma 3.1, there exists $u_0 \in \Pi$, such that:

$$|\mathcal{R}f(x) - \mathcal{R}f(y)| \le \frac{2\|x\|}{1-\|x\|}\|f\|_{\mathcal{I}}\beta_E((1+\varepsilon u_0)x, (1+\varepsilon u_0)y).$$

Take $z_0 = 1 + \varepsilon u_0$ which satisfies

$$|z_0| \le 1 + \varepsilon = 1 + \frac{1-\|x\|}{2\|x\|} = \frac{1+\|x\|}{2\|x\|}.$$

By Theorem 2.7, we have that $z_0 x, z_0 y \in B_E$ and:

$$\rho_E(z_0 x, z_0 y) \le 2|z_0|\rho_E(x, y). \tag{3.2}$$

Denote

$$C := (1-\|x\|^2)|\mathcal{R}f(x) - \mathcal{R}f(y)|, \tag{3.3}$$

so we have

$$C \le (1-\|x\|^2)\frac{2\|x\|}{1-\|x\|}\|f\|_{\mathcal{I}}\beta_E(z_0 x, z_0 y).$$

Notice that using Lemma 3.2, we have

$$\beta_E(z_0x, z_0y) \le \frac{\rho_E(z_0x, z_0y)}{1-\rho_E(z_0x, z_0y)},$$

so we obtain

$$\begin{aligned} C &\le (1+\|x\|)(1-\|x\|)\frac{2\|x\|}{1-\|x\|}\|f\|_{\mathcal{I}}\frac{\rho_E(z_0x, z_0y)}{1-\rho_E(z_0x, z_0y)} \\ &\le 4\|x\|\|f\|_{\mathcal{I}}\frac{\rho_E(z_0x, z_0y)}{1-\rho_E(z_0x, z_0y)} = \frac{4\|x\|\|f\|_{\mathcal{I}}}{\frac{1}{\rho_E(z_0x,z_0y)}-1}, \end{aligned}$$

so

$$\left(\frac{1}{\rho_E(z_0x, z_0y)} - 1\right) C \le 4\|x\|\|f\|_{\mathcal{I}} \leftrightarrow \frac{C}{\rho_E(z_0x, z_0y)} - C \le 4\|x\|\|f\|_{\mathcal{I}}.$$

Bearing in mind that $\|f\|_{\mathcal{B}} \le \|f\|_{\mathcal{I}}$ (see (1.6)), we have

$$\begin{aligned} C &\le (1-\|x\|^2)|\mathcal{R}f(x)| + (1-\|x\|^2)|\mathcal{R}f(y)| \\ &\le (1-\|x\|^2)\|\nabla f(x)\|\|x\| + (1-\|y\|^2)\|\nabla f(y)\|\|y\| \le 2\|f\|_{\mathcal{B}}\|x\| \le 2\|x\|\|f\|_{\mathcal{I}}, \end{aligned}$$

so

$$\frac{C}{\rho_E(z_0x, z_0y)} \le 4\|x\|\|f\|_{\mathcal{I}} + C \le 4\|x\|\|f\|_{\mathcal{I}} + 2\|x\|\|f\|_{\mathcal{I}} = 6\|x\|\|f\|_{\mathcal{I}},$$

and we conclude $C \le 6\|x\|\|f\|_{\mathcal{I}}\rho_E(z_0x, z_0y)$.

Finally, we apply inequality (3.2), and since $|z_0| \le \frac{1+\|x\|}{2\|x\|}$, we obtain

$$\begin{aligned} C &\le 6\|x\|\|f\|_{\mathcal{I}}2|z_0|\rho_E(x, y) \le 12\|x\|\|f\|_{\mathcal{I}}\frac{1+\|x\|}{2\|x\|}\rho_E(x, y) \\ &= 12\|f\|_{\mathcal{I}}\frac{1+\|x\|}{2}\rho_E(x, y) \le 12\|f\|_{\mathcal{I}}\rho_E(x, y), \end{aligned}$$

and we are done. □

Theorem 3.4 *Let $f \in \mathcal{B}(B_E)$ and $x, y \in B_E$. Then*

$$|(1-\|x\|^2)\mathcal{R}f(x) - (1-\|y\|^2)\mathcal{R}f(y)| \le 14\|f\|_{\mathcal{I}}\rho_E(x, y).$$

Proof We call

$$F := |(1-\|x\|^2)\mathcal{R}f(x) - (1-\|y\|^2)\mathcal{R}f(y)|, \tag{3.4}$$

and suppose without loss of generality that $\|x\| \geq \|y\|$. We have that

$$\begin{aligned} F &= |(1-\|x\|^2)(\mathcal{R}f(x)-\mathcal{R}f(y)) - (\|x\|^2-\|y\|^2)\mathcal{R}f(y)| \\ &\leq (1-\|x\|^2)|\mathcal{R}f(x)-\mathcal{R}f(y)| + (\|x\|^2-\|y\|^2)|\mathcal{R}f(y)|. \end{aligned} \tag{3.5}$$

Since $\|x\|^2 - \|y\|^2 = (\|x\|+\|y\|)(\|x\|-\|y\|) \leq 2(\|x\|-\|y\|)$ and bearing in mind that $\rho_E(\|x\|, \|y\|) \leq \rho_E(x, y)$, we obtain

$$\begin{aligned} (\|x\|^2-\|y\|^2)|\mathcal{R}f(y)| &\leq 2\frac{\|x\|-\|y\|}{1-\|x\|\|y\|}(1-\|x\|\|y\|)|\mathcal{R}f(y)| \\ &\leq 2\rho_E(\|x\|, \|y\|)(1-\|y\|^2)|\mathcal{R}f(y)| \\ &\leq 2\|y\|\|f\|_{\mathcal{B}}\rho_E(x, y) \leq 2\|f\|_{\mathcal{I}}\rho_E(x, y). \end{aligned}$$

By Lemma 3.3, we know that

$$(1-\|x\|^2)|\mathcal{R}f(x)-\mathcal{R}f(y)| \leq 12\|f\|_{\mathcal{I}}\rho_E(x, y),$$

so from (3.5), we conclude

$$F \leq 12\|f\|_{\mathcal{I}}\rho_E(x, y) + 2\|f\|_{\mathcal{I}}\rho_E(x, y) = 14\|f\|_{\mathcal{I}}\rho_E(x, y),$$

and we are done. □

Hence, we obtain the result which proves the Lipschitz continuity of the mapping $x \mapsto (1-\|x\|^2)|Rf(x)|$ for $x \in B_E$.

Corollary 3.5 *Let E be a complex Hilbert space. The function $x \mapsto (1-\|x\|^2)|\mathcal{R}f(x)|$ for $x \in B_E$ is Lipschitz with respect to the pseudohyperbolic distance and the following inequality holds:*

$$|(1-\|x\|^2)|\mathcal{R}f(x)| - (1-\|y\|^2)|\mathcal{R}f(y)|| \leq 14\|f\|_{\mathcal{I}}\rho_E(x, y).$$

Proof Applying Theorem 3.4, it is clear that

$$\begin{aligned} &|(1-\|x\|^2)|\mathcal{R}f(x)| - (1-\|y\|^2)|\mathcal{R}f(y)|| \\ &\quad \leq |(1-\|x\|^2)\mathcal{R}f(x) - (1-\|y\|^2)\mathcal{R}f(y)| \leq 14\|f\|_{\mathcal{I}}\rho_E(x, y), \end{aligned}$$

and we are done. □

3.2 Results on interpolating sequences for the Bloch space

Recall that a sequence $(x_n) \subset B_E \backslash \{0\}$ is said to be interpolating for the Bloch space $\mathcal{B}(B_E)$ if, for any bounded sequence (a_n) of complex numbers, there exists $f \in \mathcal{B}(B_E)$ such that $(1-\|x_n\|^2)\mathcal{R}f(x_n) = a_n$. Attele studied in [2] this kind of interpolation for the classical Bloch space $\mathcal{B}$ and the finite- and infinite-dimensional setting was studied in [4]. We provide a new approach to prove that a necessary condition for a sequence $(x_n) \subset B_E$ to be interpolating for $\mathcal{B}(B_E)$ is to be separated for the pseudohyperbolic distance ρ_E.

Proposition 3.6 *Let E be a complex Hilbert space. If $(x_n) \subset B_E \backslash \{0\}$ is interpolating for $\mathcal{B}(B_E)$, then there exists $C > 0$, such that $\rho(x_k, x_j) \geq C$ for any $k \neq j$, $k, j \in \mathbb{N}$.*

Proof Since $(x_n) \subset B_E \backslash \{0\}$ is interpolating, there exists a sequence $(f_n) \subset \mathcal{B}(B_E)$, such that

$$(1 - \|x_n\|^2)\mathcal{R}f_n(x_n) = 1 \ \text{ and } \ (1 - \|x_k\|^2)\mathcal{R}f_n(x_k) = 0 \ \text{ if } \ k \neq n.$$

The operator $T : \mathcal{B}(B_E) \to \ell_\infty$ given by $T(f) = ((1 - \|x_n\|^2)\mathcal{R}f(x_n))$ is surjective, so by the Open Mapping Theorem, there exists $M > 0$, such that $\|f\|_R \leq M \sup_{j \in \mathbb{N}}(1 - \|x_j\|^2)|\mathcal{R}f(x_j)|$, so $\|f_n\|_R \leq M$ for any $n \in \mathbb{N}$. Applying Theorem 3.4, we have

$$\begin{aligned} |(1 - \|x_n\|^2)\mathcal{R}f_n(x_n) - (1 - \|x_k\|^2)\mathcal{R}f_n(x_k)| &\leq 14\|f_n\|_{\mathcal{I}}\rho_E(x_k, x_j) \\ &\leq 14A_0\|f_n\|_{\mathcal{R}}\rho_E(x_k, x_j) \leq 14A_0 M\rho_E(x_k, x_j). \end{aligned}$$

Hence, $1 - 0 \leq 14A_0 M\rho_E(x_k, x_j)$, and we conclude that

$$\rho_E(x_k, x_j) \geq \frac{1}{14A_0 M},$$

so we are done. □

Attele (see [2]) also proved that any interpolating sequence $(z_n) \subset \mathbf{D}$ for $\mathcal{B}$ can be slightly perturbed and the sequence remains interpolating. By means of Theorem 3.4, we adapt his proof and generalize the result to the case when we deal with any complex Hilbert space E.

Theorem 3.7 *If $(x_n) \subset B_E \backslash \{0\}$ is an interpolating sequence for $\mathcal{B}(B_E)$, then there exists $\delta > 0$, such that if $(y_n) \subset B_E$ satisfies that $\sup_{n \in \mathbb{N}} \rho_E(x_n, y_n) < \delta$, then (y_n) is also an interpolating sequence for $\mathcal{B}(B_E)$.*

Proof Since (x_n) is interpolating, the operator $T : \mathcal{B}(B_E) \to \ell_\infty$ given by $T(f) = ((1 - \|x_n\|^2)\mathcal{R}f(x_n))$ is surjective. Hence, its adjoint $T^* : \ell_\infty^* \to (\mathcal{B}(B_E))^*$ is injective and it has closed range. In particular, T^* is left-invertible. The set of left-invertible elements is open in the Banach algebra of linear operators from ℓ_∞^* to $(\mathcal{B}(B_E))^*$. Therefore, there exists δ, such that if $\|T^* - R\| < 14A_0\delta$, then R is left-invertible. If we consider $S(f) = ((1 - \|y_n\|^2)\mathcal{R}f(y_n))$, then by Theorem 3.4

$$\begin{aligned} \|(T - S)(f)\|_\infty &= \sup_{n \in \mathbb{N}} |(1 - \|x_n\|^2)\mathcal{R}f(x_n) - (1 - \|y_n\|^2)\mathcal{R}f(y_n)| \\ &\leq 14\rho_E(x_n, y_n)\|f\|_{\mathcal{I}} \leq 14A_0\|f\|_{\mathcal{R}}\rho_E(x_n, y_n) < 14A_0\delta\|f\|_{\mathcal{R}}, \end{aligned}$$

so $\|T - S\| < 14A_0\delta$, and hence, $\|T^* - S^*\| = \|T - S\| < 14A_0\delta$. We conclude that S^* is left-invertible, and hence, S is surjective, as we wanted. □

Acknowledgements This work was supported by PID2019-106529GB-I00 (MICINN, Spain).

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Data availability Not applicable.

Declarations

Conflict of interest The author has not competing interests to declare that are relevant to the content of the article.

References

1. Aron, R.M., Galindo, P., Lindström, M.: Connected components in the space of composition operators of H^∞ functions of many variables. Integr. Equ. Oper. Theory **45**, 1–14 (2003)
2. Attele, K.R.M.: Interpolating sequences for the derivatives of the Bloch functions. Glasgow Math. J. **34**, 35–41 (1992)
3. Blasco, O., Galindo, P., Lindström, M., Miralles, A.: Composition operators on the Bloch space of the unit ball of a Hilbert space. Banach J. Math. Anal. **11**(2), 311–334 (2017)
4. Blasco, O., Galindo, P., Lindström, M., Miralles, A.: Interpolating sequences for weighted spaces of analytic functions on the unit ball of a Hilbert space. Rev. Mat. Complut. **32**(1), 115–139 (2019)
5. Blasco, O., Galindo, P., Miralles, A.: Bloch functions on the unit ball of an infinite dimensional Hilbert space. J. Funct. Anal. **267**, 1188–1204 (2014)
6. Chen, S.L., Kalaj, D.: Lipschitz continuity of Bloch type mappings with respect to Bergman metric. Ann. Acad. Sci. Fenn. Math. **43**, 239–246 (2018)
7. Chen, S.L., Hamada, H., Zhu, J.-F.: Composition operators on Bloch and Hardy type spaces. Math. Z. **301**, 3939–3957 (2022)
8. Ghatage, P., Yan, J., Zheng, D.: Composition operators with closed range on the Bloch space. Proc. Am. Math. Soc. **129**(7), 2039–2044 (2000)
9. Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker Inc, New York (1984)
10. Hamada, H.: Distortion theorems, Lipschitz continuity and their applications for Bloch type mappings on bounded symmetric domains in $\mathbb{C}^n$. Ann. Acad. Sci. Fenn. Math. **44**, 1003–1014 (2019)
11. Miralles, A.: Bounded below composition operators on the space of Bloch functions on the unit ball of a Hilbert space. Banach J. Math. Anal. **17**, 73 (2023)
12. Mujica, J.: Complex Analysis in Banach Spaces, Math. Studies, vol. 120. North-Holland, Amsterdam (1986)
13. Timoney, R.M.: Bloch functions in several complex variables I. Bull. Lond. Math. Soc. **12**, 241–267 (1980)
14. Xiong, C.: On the Lipschitz continuity of the dilation of Bloch functions. Period. Math. Hung. **47**(1–2), 233–238 (2003)