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Probabilistic analysis of a class of 2D-random heat equations via densities

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ABSTRACT

We give new probabilistic results for a class of random two-dimensional homogeneous heat equations with mixed homogeneous Dirichlet and Neumann boundary conditions and an arbitrary initial condition on a rectangular domain. The diffusion coefficient is assumed to be an arbitrary second-order random variable, while the initial condition is a stochastic process admitting a Karhunen-Loève expansion. We then construct pointwise convergent approximations for the main moments and the density of the solution. The theoretical results are numerically illustrated.

1. Introduction

In contrast to deterministic scenarios, random partial differential equations (RPDEs) consider coefficients, boundary values, and initial conditions as random variables or stochastic processes because, in practice, these terms are contaminated by uncertainties coming from error measurements or the partial knowledge of phenomena they try to model. The solution (in the classical sense) is then a smooth random field, and the primary objective is not only to obtain a solution, whether exact or approximate, but also to determine its main probabilistic information, such as the first moments (mean and variance) and finite distributions (fidis) [1]. To achieve this goal, a smart combination of deterministic methods for solving PDEs and probabilistic techniques is often required. As it shall be seen later, in this paper, we will take advantage of combining the method of separation of variables for solving certain PDEs with the Karhunen-Loève expansion and the Random Variable Transformation technique, which are genuine probabilistic tools,

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in order to solve the following class of 2D-random heat equation on a bounded rectangle:

$$\begin{cases} u_t = \beta \Delta u, \quad 0 < x < l_1,\, 0 < y < l_2,\, t > 0, \\ u_x(0,y,t) = u_x(l_1,y,t) = 0, \quad 0 \le y \le l_2,\, t > 0, \\ u(x,0,t) = u(x,l_2,t) = 0, \quad 0 \le x \le l_1,\, t > 0, \\ u(x,y,0) = f(x,y), \quad 0 \le x \le l_1,\, 0 \le y \le l_2. \end{cases} \tag{1.1}$$

Here, $\Delta u := u_{xx} + u_{yy}$ is the Laplacian operator. There are mixed homogeneous Dirichlet and Neumann boundary conditions and uncertain inputs. The diffusion coefficient β is a positive random variable, and the initial condition f is a random field, both defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with outcomes $\omega \in \Omega$. The solution or output, $u = u(x,y,t;\omega)$ (or $u = u(x,y,t)$ by simply hiding the ω-notation) is a smooth random field. Throughout the paper, we will work by combining the deterministic and random Lebesgue spaces, L^p, $p = 1,2$, defined on convenient sets of $\mathbb{R}^n$ or $\mathbb{R}^n \times \Omega$, $n = 1,2$, respectively, and the corresponding Lebesgue measure [2]. The paper extends previous work on the one-dimensional random heat equation and related problems to the bi-spatial dimensions [3–5]. Furthermore, the results that shall be presented in this paper can help to deal with the rigorous analysis of heat transfer in 2D (rectangular) domains when experimental data are available [6].

The paper is organized as follows. In Section 2, we analyze under which conditions the stochastic problem (1.1) has a pathwise and a mean-square solution by taking as a candidate the formal infinite series from the method of separation of variables. In Section 3, we construct approximations of the 1-PDF of the solution by representing the initial condition utilizing the Karhunen-Loève expansion. It allows us to obtain reliable approximations of its main moments. Section 4 shows an example illustrating the main theoretical findings. Conclusions are drawn in Section 5.

2. Stochastic pathwise and mean-square solution. Mean and variance

From the method of separation of variables, it can be seen, using classical techniques, that the formal series solution of problem (1.1) is given by

$$u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} a_{m,n} \mathrm{e}^{-\left((m/l_1)^2+(n/l_2)^2\right)\pi^2\beta t} \phi_{m,n}(x,y), \quad \begin{cases} a_{0,q} = \dfrac{2}{l_1 l_2}\displaystyle\int_0^{l_1}\int_0^{l_2} f(x,y)\phi_{0,q}(x,y)\,\mathrm{d}y\,\mathrm{d}x, \\ q \ge 1, \\ a_{p,q} = \dfrac{4}{l_1 l_2}\displaystyle\int_0^{l_1}\int_0^{l_2} f(x,y)\phi_{p,q}(x,y)\,\mathrm{d}y\,\mathrm{d}x, \\ p,q \ge 1, \end{cases} \tag{2.1}$$

where $\{\phi_{p,q}(x,y) = \cos(p\pi x/l_1)\sin(q\pi y/l_2) : p = 0,1,2,\ldots,\ q = 1,2,\ldots\}$.

We analyze under which conditions u is a pathwise and a mean-square solution [7,8]. Pathwise solution means that its sample-paths (i.e. the real functions obtained when fixing each $\omega \in \Omega$) are solutions in the classical sense. Mean-square solution means that limits, continuity, differentiability, etc., are considered in the topology of $\mathrm{L}^2(\Omega; \mathrm{d}\mathbb{P})$. In applications, mean-square convergence approximates the expectation and the variance of $u(x,y,t)$.

Theorem 2.1. *If $f \in \mathrm{L}^2([0,l_1]\times[0,l_2]\times\Omega)$, then the series (2.1) converges almost surely for $t > 0$ and can be differentiated termwise. For $t = 0$, it converges in $\mathrm{L}^2([0,l_1]\times[0,l_2])$ and also almost everywhere on $[0,l_1]\times[0,l_2]$, almost surely. If, in addition, $\beta(\omega) \ge \beta_{min} > 0$ almost surely, then the series (2.1) converges in the mean-square sense for $t > 0$ and can be mean-square differentiated termwise.*

Proof. By Jensen's and Hölder's inequalities [2],

$$|a_{p,q}(\omega)| \leq \frac{4}{l_1 l_2}\int_0^{l_1}\int_0^{l_2}|f(x,y;\omega)|\mathrm{d}y\,\mathrm{d}x \leq C\|f(x,y;\omega)\|_{\mathrm{L}^2([0,l_1]\times[0,l_2])}, \quad \|a_{p,q}\|_{\mathrm{L}^2(\Omega)} \leq C\|f\|_{\mathrm{L}^2([0,l_1]\times[0,l_2]\times\Omega)}, \tag{2.2}$$

for $p = 0, 1, 2, \ldots$, $q = 1, 2, \ldots$, where $C > 0$ is a constant independent of (p, q). In particular, the sequence of random coefficients is uniformly bounded, almost surely and in the mean-square sense.

Due to the exponential terms $\mathrm{e}^{-\left((m/l_1)^2+(n/l_2)^2\right)\pi^2\beta(\omega)t}$, $t > 0$, the series (2.1) converges almost surely. The series can also be differentiated termwise [9, theorem 9.14]. When $\beta(\omega) \geq \beta_{\min} > 0$ almost surely, we have

$$\mathrm{e}^{-\left((m/l_1)^2+(n/l_2)^2\right)\pi^2\beta(\omega)t} \leq \mathrm{e}^{-\left((m/l_1)^2+(n/l_2)^2\right)\pi^2\beta_{\min}t}, \quad t > 0,$$

and the series (2.1) is mean-square convergent. Also, it is mean-square differentiable termwise [3, theorem 3.1].

For $t = 0$, the Fourier series of $(x, y) \mapsto f(x, y; \omega)$ is obtained. By theory of harmonic analysis [10], it converges in $\mathrm{L}^2([0, l_1] \times [0, l_2])$ and also almost everywhere on $[0, l_1] \times [0, l_2]$, almost surely. $\square$

3. Probability density function and main statistics

The approximation of the 1-PDF of $u = u(x, y, t)$ requires a finite-dimensional random space. This may not be the case when f is a random field. A possible way of reducing dimensionality in random space is by means of an analytical representation. The Karhunen-Loève expansion, based on the spectrum of the covariance integral operator of the field, is an optimal representation in the mean-square sense. If a finite-term Karhunen-Loève expansion of f is employed, then the corresponding approximation of f will depend on a finite number of random variables [2].

Let us denote the process by $f(x, y; \omega)$, $\omega \in \Omega$. Let $\overline{f}(x, y)$ be the mean of $f(x, y; \omega)$. Let $\{\phi_i(x, y), \lambda_i\}$ be the set of eigenfunctions and (non-negative) eigenvalues of the covariance integral operator associated with f, i.e.,

$$\int_0^{l_1}\int_0^{l_2}\mathrm{Cov}[f(x_1,y_1), f(x_2,y_2)]\phi_i(x_2,y_2)\,\mathrm{d}y_2\,\mathrm{d}x_2 = \lambda_i\phi_i(x_1,y_1).$$

The set $\{\phi_i(x, y)\}$ is an orthogonal basis of $\mathrm{L}^2([0, l_1] \times [0, l_2])$. Then

$$f(x,y;\omega) = \overline{f}(x,y) + \sum_i \sqrt{\lambda_i}\,\phi_i(x,y)\xi_i(\omega), \quad \omega \in \Omega,$$

in the mean-square sense in non-random and random space, where $\xi_i(\omega)$ are zero-mean and pairwise uncorrelated random variables. If f is Gaussian, then ξ_i are normal and independent. The truncation of the infinite series to a finite partial sum is determined by the rapidity of convergence, which depends upon the decay of the eigenvalues λ_i.

Fixed $\omega \in \Omega$, the initial condition $f(x, y; \omega)$ may be expanded as a Fourier series in $\mathrm{L}^2([0, l_1] \times [0, l_2])$, in terms of $\{\phi_{p,q}(x, y) = \cos(m\pi x/l_1)\sin(n\pi y/l_2) : m = 0, 1, 2, \ldots, n = 1, 2, \ldots\}$:

$$f(x,y;\omega) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} a_{m,n}(\omega)\phi_{m,n}(x,y), \quad \omega \in \Omega.$$

This is a Karhunen-Loève expansion of f when the Fourier coefficients $\{a_{m,n}\}$ are pairwise uncorrelated random variables. For many applications, one considers random coefficients $\{a_{m,n}\}$ that are independent.

When a closed-form solution exists, with a finite number of random inputs, the 1-PDF of a random differential equation may be obtained by means of the random variable transformation technique [11,12]. In our case, we have an infinite series. The random dimensionality needs to be reduced by truncating the series. So, let

$$u_N(x,y,t) = \sum_{m=0}^{N}\sum_{n=1}^{N} a_{m,n}\mathrm{e}^{-\left((m/l_1)^2+(n/l_2)^2\right)\pi^2\beta t}\phi_{m,n}(x,y) \tag{3.1}$$

be the Nth partial sum of (2.1). Let $g_N(u;x,y,t)$ be its 1-PDF, evaluated at $u \in \mathbb{R}$. It can be computed by using the random variable transformation technique. Intuitively, $g_N(u;x,y,t) \approx g(u;x,y,t)$ for N large enough, where $g(u;x,y,t)$ is the 1-PDF of the exact solution $u(x,y,t)$ (2.1). Later, we will provide sufficient conditions to guarantee this convergence.

Lemma 3.1. Suppose that the random variable β, the random variable $a_{0,1}$ and the random vector $(a_{p,q} : (p,q) \in D_N)$ are absolutely continuous and independent, where $D_N = \{(p,q) : p = 0,1,2,\ldots,N,\ q = 1,2,\ldots,N,\ (p,q) \neq (0,1)\}$. Then

$$g_N(u;x,y,t) = \frac{1}{\sin(\pi y/l_2)}\mathbb{E}\left[g_{a_{0,1}}\left(\frac{\mathrm{e}^{\pi^2\beta t/l_2^2}}{\sin(\pi y/l_2)} \left\{ u - \sum_{(m,n)\in D_N} a_{m,n}\mathrm{e}^{-\left((m/l_1)^2+(n/l_2)^2\right)\pi^2\beta t}\phi_{m,n}(x,y) \right\} \right) \mathrm{e}^{\pi^2\beta t/l_2^2} \right], \tag{3.2}$$

for $0 < x < l_1$, $0 < y < l_2$ and $t > 0$, where $g_{a_{0,1}}$ is the PDF of $a_{0,1}$.

Proof. Let (x,y,t) be arbitrary but fixed in $(0,l_1)\times(0,l_2)\times(0,\infty)$. Then, we apply the random variable transformation technique with the following transformation mapping

$$G(a_{0,1},(a_{p,q} : (p,q)\in D_N),\beta) = (u_N(x,y,t),(a_{p,q} : (p,q)\in D_N),\beta)\,.$$

The inverse mapping, H, is given by

$$H(u,(a_{p,q} : (p,q)\in D_N),\beta) = \left(\frac{\mathrm{e}^{\pi^2\beta t/l_2^2}}{\sin(\pi y/l_2)} \left\{ u - \sum_{(m,n)\in D_N} a_{m,n}\mathrm{e}^{-\left((m/l_1)^2+(n/l_2)^2\right)\pi^2\beta t}\phi_{m,n}(x,y) \right\}, \right.$$
$$\left. (a_{p,q} : (p,q)\in D_N),\beta \right).$$

The Jacobian of the inverse mapping H is

$$J(H(u,(a_{p,q} : (p,q)\in D_N),\beta)) = \frac{\mathrm{e}^{\pi^2\beta t/l_2^2}}{\sin(\pi y/l_2)} > 0, \qquad 0 < y < l_2,$$

since the corresponding matrix of partial derivatives is triangular. Then, by the random variable transformation technique, the PDF of the random vector $(u_N, a_{p,q}, \beta)(\omega) := (u_N(x,y,t),(a_{p,q} : (p,q)\in D_N),\beta)(\omega)$ is given by

$$\begin{aligned} g_{(u_N,a_{p,q},\beta)}(u,a_{p,q},\beta) &= g_{(a_{0,1},a_{p,q},\beta)}(H(u,a_{p,q},\beta))JH(u,a_{p,q},\beta) \\ &= g_{a_{0,1}}\left(\frac{\mathrm{e}^{\pi^2\beta t/l_2^2}}{\sin(\pi y/l_2)} \left\{ u - \sum_{(m,n)\in D_N} a_{m,n}\mathrm{e}^{-\left((m/l_1)^2+(n/l_2)^2\right)\pi^2\beta t}\phi_{m,n}(x,y) \right\} \right) \\ &\quad \times\ g_{a_{p,q}}(a_{p,q})g_\beta(\beta)\frac{\mathrm{e}^{\pi^2\beta t/l_2^2}}{\sin(\pi y/l_2)}, \end{aligned}$$

where we have applied the independence assumption for the inputs $a_{0,1}$, $(a_{p,q} : (p,q)\in D_N)$ and β. By marginalizing with respect to $(a_{p,q} : (p,q)\in D_N)$ and β, the density $g_N(u;x,y,t)$ is obtained. Finally, observe that this marginal density can be written via an expectation of random variables $(a_{p,q} : (p,q)\in D_N)$ and β, by the independence. $\square$

Now, we give sufficient conditions in order to guarantee the probability density function, $g_N(u;x,y,t)$, computed in Lemma 3.1 converges pointwise and in $\mathrm{L}^1(\mathbb{R};\mathrm{d}u)$.

Theorem 3.2. *Suppose that the random variable β, the random variable $a_{0,1}$ and the random vector $(a_{p,q} : (p,q) \in D_N)$ are absolutely continuous and independent, where $D_N = \{(p,q) : p = 0,1,2,\ldots,N,\ q = 1,2,\ldots,N,\ (p,q) \neq (0,1)\}$. Assume that $f \in \mathrm{L}^2([0,l_1] \times [0,l_2] \times \Omega)$, $g_{a_{0,1}}$ is almost everywhere continuous on $\mathbb{R}$, bounded on $\mathbb{R}$, and $\mathbb{E}[\mathrm{e}^{\pi^2\beta t/l_2^2}] < \infty$. Then*

$$\lim_{N\to\infty} g_N(u;x,y,t) = g(u;x,y,t)$$

for almost every $u \in \mathbb{R}$, where $0 < x < l_1$, $0 < y < l_2$ and $t > 0$. The limit also holds in $\mathrm{L}^1(\mathbb{R};\mathrm{d}u)$.

Proof. The sequence

$$\Lambda_N(\omega) = \frac{\mathrm{e}^{\pi^2\beta(\omega)t/l_2^2}}{\sin(\pi y/l_2)} \left\{ u - \sum_{(m,n)\in D_N} a_{m,n}(\omega)\mathrm{e}^{-\left((m/l_1)^2+(n/l_2)^2\right)\pi^2\beta(\omega)t} \phi_{m,n}(x,y) \right\}$$

converges almost surely to $\Lambda(\omega) = \Lambda_{N=\infty}(\omega)$ (recall that the sequence of random coefficients, $a_{m,n}(\omega)$, is uniformly bounded, see (2.2)). Since $g_{a_{0,1}}$ is almost everywhere continuous on $\mathbb{R}$,

$$g_{a_{0,1}}(\Lambda_N(\omega)) \overset{N\to\infty}{\longrightarrow} g_{a_{0,1}}(\Lambda(\omega))$$

almost surely, by the continuous mapping theorem [13, page 7, theorem 2.3]. Since $g_{a_{0,1}}$ is bounded on $\mathbb{R}$ and $\mathbb{E}[\mathrm{e}^{\pi^2\beta t/l_1^2}] < \infty$, the dominated convergence theorem [14, result 11.32, page 321] applies:

$$\begin{aligned} g_N(u;x,y,t) &= \frac{1}{\sin(\pi y/l_2)}\mathbb{E}\left[g_{a_{0,1}}(\Lambda_N(\omega))\mathrm{e}^{\pi^2\beta t/l_2^2}\right] \quad \overset{N\to\infty}{\longrightarrow} \quad \frac{1}{\sin(\pi y/l_2)}\mathbb{E}\left[g_{a_{0,1}}(\Lambda(\omega))\mathrm{e}^{\pi^2\beta t/l_2^2}\right] \\ &= g(u;x,y,t). \end{aligned}$$

This proves the pointwise convergence. Finally, convergence in $\mathrm{L}^1(\mathbb{R};\mathrm{d}u)$ follows from Scheffé's lemma [15] (this lemma states that, for PDFs, pointwise convergence implies convergence in $\mathrm{L}^1(\mathbb{R};\mathrm{d}u)$). □

For applications, one considers the density function $g_N(u;x,y,t)$ as an approximation of $g(u;x,y,t)$. The expectation that defines $g_N(u;x,y,t)$ can be computed using

$$\mathbb{E}[r(u(x,y,t))] \approx \mathbb{E}[r(u_N(x,y,t))] = \int_{\mathbb{R}^{N^2+N}} r(u)g_N(u;x,y,t)\,\mathrm{d}\beta\,\mathrm{d}a_{p,q}, \qquad (p,q) \in D_N. \tag{3.3}$$

If $r(z) = z^i$, $i = 1,2$, one obtains the approximations of the first two moments $\mathbb{E}[(u(x,y,t))^i]$, and hence of the variance of $u(x,y,t)$.

Higher one-dimensional moments can be calculated similarly by taking $r(z) = z^l$, $l = 3,4,\ldots$. The foregoing integrals often require quadrature rules of integration. In practice, they are computable when the random dimensionality is low or moderately large. The expectation should be estimated by Monte Carlo techniques for large random dimensionality.

Remark 3.3. Here, we give an alternative approach to obtain an explicit expression for the approximation of the expectation and the covariance (hence, the variance) of $u_N(x,y,t)$ taking advantage of the representation (3.1). Indeed, using the linearity of the expectation operator and the independence between $a_{m,n}$ and β, one gets

$$\mathbb{E}[u_N(x,y,t)] = \sum_{(m,n)\in D_N} \mathbb{E}[a_{m,n}]\mathbb{E}[\mathrm{e}^{-\left((m/l_1)^2+(n/l_2)^2\right)\pi^2\beta t}]\phi_{m,n}(x,y), \tag{3.4}$$

where $\phi_{m,n}(x,y)$ is defined after (2.1). Now, recall that the covariance field is given by

$$\mathrm{Cov}[u_N(x,y,t_1)u_N(x,y,t_2)] = \mathbb{E}[u_N(x,y,t_1)u_N(x,y,t_2)] - \mathbb{E}[u_N(x,y,t_1)]\mathbb{E}[u_N(x,y,t_2)]. \tag{3.5}$$

Clearly, the two last expectations can be calculated evaluating (3.4) at $t = t_1$ and $t = t_2$, respectively, while the first expectation can be expressed as

$$\mathbb{E}[u_N(x,y,t_1)u_N(x,y,t_2)] = \sum_{(m,n),(p,q)\in D_N} \mathbb{E}[a_{m,n}a_{p,q}]\mathbb{E}[\mathrm{e}^{-\left((m/l_1)^2+(n/l_2)^2+(p/l_1)^2+(q/l_2)^2\right)\pi^2\beta t}] \times \phi_{m,n}(x,y)\phi_{p,q}(x,y),$$

where

$$\mathbb{E}[a_{m,n}a_{p,q}] = \left(\frac{4}{l_1l_2}\right)^2 \int_0^{l_1}\int_0^{l_2}\int_0^{l_1}\int_0^{l_2} \mathbb{E}[f(x,y)f(w,z)]\phi_{m,n}(x,y)\phi_{p,q}(w,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}z, \quad m,n,p,q = 1,\dots,N,$$

$$\mathbb{E}[a_{0,n}a_{p,q}] = \left(\frac{\sqrt{8}}{l_1l_2}\right)^2 \int_0^{l_1}\int_0^{l_2}\int_0^{l_1}\int_0^{l_2} \mathbb{E}[f(x,y)f(w,z)]\phi_{0,n}(x,y)\phi_{p,q}(w,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}z, \quad n,p,q = 1,\dots,N,$$

$$\mathbb{E}[a_{m,n}a_{0,q}] = \left(\frac{\sqrt{8}}{l_1l_2}\right)^2 \int_0^{l_1}\int_0^{l_2}\int_0^{l_1}\int_0^{l_2} \mathbb{E}[f(x,y)f(w,z)]\phi_{m,n}(x,y)\phi_{0,q}(w,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}z, \quad m,n,q = 1,\dots,N,$$

and

$$\mathbb{E}[a_{0,n}a_{0,q}] = \left(\frac{2}{l_1l_2}\right)^2 \int_0^{l_1}\int_0^{l_2}\int_0^{l_1}\int_0^{l_2} \mathbb{E}[f(x,y)f(w,z)]\phi_{0,n}(x,y)\phi_{0,q}(w,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}z, \quad n,q = 1,\dots,N.$$

Notice that by symmetry $\mathbb{E}[a_{m,n}a_{0,q}] = \mathbb{E}[a_{0,n}a_{p,q}]$. Consequently, putting $t_1 = t_2 = t$ in (3.5), one obtains the variance of $u_N(x,y,t)$.

4. Numerical simulations

This section is aimed at illustrating the time evolution of the mean, the variance fields, and the 1-PDF of the solution to the random heat Eq. (1.1), taking advantage of the results obtained in Section 3. All simulations and graphics have been obtained using Matlab® software.

Let $\tilde{B}(t,\omega) := B(t,\omega) - \frac{t}{T}B(T,\omega)$ denote a Brownian bridge process in the interval $[0,T]$, $T > 0$, where $B(t,\omega)$ is the Brownian motion. This process has the following properties [2, Lemma 5.2.2]

$$\mathbb{E}[\tilde{B}(t,\cdot)] = 0, \quad \mathrm{Cov}_{\tilde{B}}(t_1,t_2) = \mathbb{E}[\tilde{B}(t_1,\cdot)\tilde{B}(t_2,\cdot)] = \min(t_1,t_2) - \frac{t_1t_2}{T}.$$

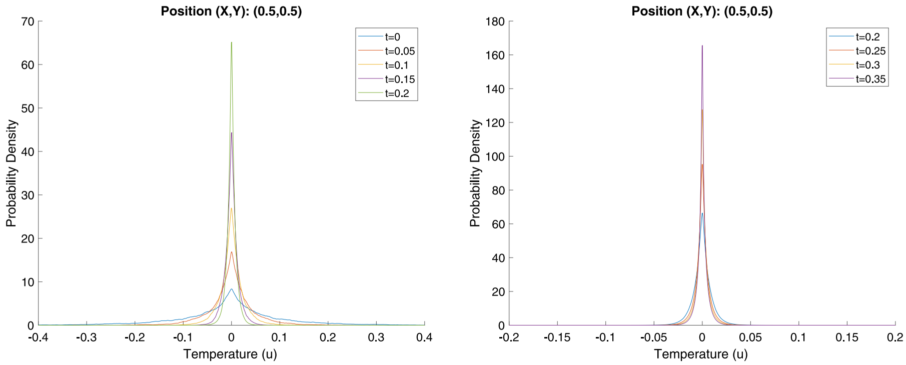
The following relation gives the Karhunen-Loève expansion of the Brownian bridge [2, Example 5.30]:

$$\tilde{B}(t,\omega) = \sum_{j=1}^{\infty} \frac{\sqrt{2T}}{j\pi}\sin\left(\frac{j\pi}{T}t\right)\xi_j(\omega), \quad \xi_j \sim \mathrm{N}(0;\,1), \tag{4.1}$$

where $\{\xi_j\}$ form an orthonormal basis of uncorrelated standard Gaussian variables in $\mathrm{L}^2(\Omega,\mathrm{d}\mathbb{P})$. However, since dealing with infinite random variables is not computationally feasible, we shall consider truncating the previous series. Specifically, we take two Brownian bridges, one in the x direction (taking $T = l_1$) and other in the y direction (taking $T = l_2$), with their corresponding truncated Karhunen-Loève expansions:

$$f(x,y) = \tilde{B}(x,\omega)\tilde{B}(y,\omega) = \sum_{i,j=1}^{N} \frac{2\sqrt{l_1l_2}}{\pi^2 ij}\sin\left(\frac{i\pi x}{l_1}\right)\sin\left(\frac{j\pi y}{l_2}\right)\xi_i\eta_j,$$

where the two Brownian bridges are assumed to be uncorrelated; that is, the basis functions $\{\xi_i\eta_j\}_{i,j}$ verify $\mathbb{E}[\xi_i\eta_j] = 0$ for any i, j, being $\xi_i,\eta_j \sim \mathrm{N}(0;1)$. For the diffusion coefficient, which must be positive, we will assume a positive truncated Gaussian distribution, $\beta \sim \mathrm{N}_{[0,+\infty)}(0.3;0.01)$.



(a) $t = 0, 0.05, 0.1, 0.15, 0.2$

(b) $t = 0.2, 0.25, 0.3, 0.35$

Fig. 1. PDF evolution of the heat distribution at the center of the plate.

Now, let us obtain the PDF of u. To this end, we will apply Lemma 3.1. Let us first observe that

$$\begin{aligned} a_{0,1}(\omega) &= \frac{2}{l_1 l_2}\int_0^{l_1}\int_0^{l_2} f(x,y,\omega)\sin\left(\frac{\pi y}{l_2}\right)\mathrm{d}y\mathrm{d}x \\ &= \frac{4}{\sqrt{l_1 l_2}\pi^2}\sum_{i,\,j=1}^{N}\xi_i(\omega)\eta_j(\omega)\frac{1}{ij}\int_0^{l_1}\sin\left(\frac{i\pi x}{l_1}\right)\mathrm{d}x\int_0^{l_2}\sin\left(\frac{j\pi y}{l_2}\right)\sin\left(\frac{\pi y}{l_2}\right)\mathrm{d}y, \end{aligned}$$

where the two above integrals can be easily calculated

$$\int_0^{l_1}\sin\left(\frac{i\pi x}{l_1}\right)\mathrm{d}x = \begin{cases} 0 & \text{if } i = 2k, \quad k = 1,2,\dots, \\ \dfrac{2l_1}{i\pi} & \text{if } i = 2k+1, \quad k = 0,1,\dots, \end{cases} \qquad \int_0^{l_2}\sin\left(\frac{j\pi y}{l_2}\right)\sin\left(\frac{\pi y}{l_2}\right)\mathrm{d}x = \begin{cases} 0 & \text{if } j \neq 1, \\ \dfrac{l_2}{2} & \text{if } j = 1. \end{cases}$$

Therefore, we obtain

$$a_{0,1}(\omega) = \frac{2}{l_1 l_2}\eta_1(\omega)\sum_{k=0}^{\lfloor\frac{N-1}{2}\rfloor}\frac{l_1 l_2}{\pi(2k+1)}\frac{2\sqrt{l_1 l_2}}{\pi^2(2k+1)}\xi_{2k+1}(\omega) = \eta_1(\omega)Z(\omega), \qquad Z \sim \mathrm{N}\left(0; \frac{16 l_1 l_2}{\pi^6}\sum_{k=0}^{\lfloor\frac{N-1}{2}\rfloor}\frac{1}{(2k+1)^4}\right).$$

Notice that to deduce the distribution of $Z(\omega)$, we have used that $\xi_1,\dots,\xi_{\lfloor\frac{N-1}{2}\rfloor} \sim \mathrm{N}(0;1)$ and mutually uncorrelated (so, independent because they are Gaussian). Now, knowing the distribution of $\eta_1(\omega)$ and $Z(\omega)$, the distribution of $a_{0,1}(\omega)$ can be obtained by applying the method of transformation of variables

$$g_{a_{0,1}}(u) = \int_{\mathbb{R}\setminus\{0\}} g_{\eta_1}\left(\frac{u}{v}\right) g_Z(v)\left|\frac{1}{v}\right|\mathrm{d}v = \mathbb{E}_Z\left[g_{\eta_1}\left(\frac{u}{Z}\right)\left|\frac{1}{Z}\right|\right].$$

Figs. 1(a) and 1(b) show the PDF of the stochastic process $\{u_{10}(t,0.5,0.5,\omega)\}_{t\geq 0}$, $\omega \in \Omega$. Note that, in this case, the temperature can be either positive or negative. However, as expected from the tensor product of two Brownian bridges, all temperature values are very close to 0, resulting in very peaked PDFs. Although it is difficult to perceive in Fig. 1(a), the PDF is slightly smoothed out as time passes by, and it becomes narrower as the variance decreases when the temperature converges to the null Dirichlet boundary condition. This latter fact is better seen in Fig. 1(b).

5. Conclusion

In this paper, we have introduced a probabilistic analysis of initial–boundary value problems for the two-dimensional heat equation on a rectangular domain. The main contributions of our study are twofold.

Firstly, we have directly considered the diffusion parameter as a random variable with an arbitrary density, allowing for a wider range of probability distributions. This approach provides more flexibility in real-world applications compared to traditional deterministic models or alternative stochastic approaches that model this parameter via perturbations driven by specific stochastic processes having nice mathematical properties, such as the white noise, which is Gaussian, so unbounded, that may be unrealistic from a practical standpoint. Secondly, the paper focuses on approximating the first probability density function of the solution, which is a significant advancement as previous works typically focused only on calculating the first moments. By extending this methodology to more complex formulations and higher dimensions, the paper may help to open new avenues and insights not only in the mathematical analysis of the heat equation but also in its real-world applications within the setting of transfer processes in Engineering.

Data availability

No data was used for the research described in the article.

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