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Bounded below composition operators on the space of Bloch functions on the unit ball of a Hilbert space

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Abstract

Let B_E be the open unit ball of a complex finite or infinite dimensional Hilbert space E and consider the space $\mathcal{B}(B_E)$ of Bloch functions on B_E. Using Lipschitz continuity of the dilation map on B_E given by $x \mapsto (1 - \|x\|^2)\mathcal{R}f(x)$ for $x \in B_E$, where $\mathcal{R}f$ denotes the radial derivative of $f \in \mathcal{B}(B_E)$, we study when a composition operator on $\mathcal{B}(B_E)$ is bounded below.

Keywords Bloch space · Infinite dimensional space · Automorphisms · Bounded below operator

Mathematics Subject Classification 46E50 · 30H30 · 47B33 · 32A18

1 Introduction and background

Let E be a complex Hilbert space and consider its open unit ball B_E. The space of Bloch functions f on B_E will be denoted by $\mathcal{B}(B_E)$. We will study various properties of automorphisms of the unit ball B_E which will allow us to supply conditions for a composition operator to be bounded below on $\mathcal{B}(B_E)$, extending the one-dimensional results given [4]. The study of operators on Bloch spaces on an infinite dimensional setting can be found in [3], where the boundness and compactness of composition operators are studied. Hamada also deals with bounded below composition operators on the space of Bloch functions on bounded symmetric domains [10]. The author also studies properties of extended Cesàro operators on spaces of Bloch-type functions in [9]. The results given in this work are presented as a preprint in [13].

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For our purpose, it will be needed that for any $f \in \mathcal{B}(B_E)$, the map on $x \in B_E$ given by $x \mapsto (1-\|x\|^2)\mathcal{R}f(x)$ is Lipschitz with respect to ρ_E, where ρ_E denotes the pseudohyperbolic distance (see [13]) and bearing in mind that Rf is the radial derivative of the function f.

1.1 Automorphisms on B_E. The pseudohyperbolic distance

If X is a complex Banach space and we denote by B_X its open unit ball, the function $f : B_X \to \mathbb{C}$ is said to be analytic (or holomorphic) if f is Fréchet differentiable for all $x \in B_X$ (see [15] for further information). The pseudohyperbolic distance $\rho_X(x, y)$ for $x, y \in B_X$ is described by:

$$\rho_X(x, y) = \sup\{\rho(f(x), f(y)) : f \in H^\infty(B_X),\ |f| < 1 \text{ on } \mathbb{D}\}$$

where we denote by $H^\infty(B_X)$ the space of analytic functions on B_X which are bounded. This space is endowed with the sup-norm and the pseudohyperbolic distance on $\mathbb{D}$ is given by:

$$\rho(z, w) = \left|\frac{z-w}{1-\bar{z}w}\right| \quad \text{for all } z, w \in \mathbb{D}.$$

Now consider a complex Hilbert space E and denote by $\langle \cdot, \cdot \rangle$ the natural inner product of E. We will denote by $\mathrm{Aut}(B_E)$ the space of automorphisms of B_E, that is, the bijective maps $\varphi : B_E \to B_E$ which are bianalytic. We will use these automorphisms several times in this work (see [8] for further information). For every $x \in B_E$, we will denote the automorphism $\varphi_x : B_E \longrightarrow B_E$ by:

$$\varphi_x(y) = (s_x Q_x + P_x)(m_x(y)) \tag{1.1}$$

where $s_x = \sqrt{1-\|x\|^2}$, $m_x : B_E \longrightarrow B_E$ is the analytic self-map:

$$m_x(y) = \frac{x-y}{1-\langle y, x\rangle},$$

$P_x : E \longrightarrow E$ is given by:

$$P_x(y) = \frac{\langle y, x\rangle}{\langle x, x\rangle} x$$

and $Q_x : E \longrightarrow E$ is defined by $Q_x = Id_E - P_x$, where Id_E is the identity on E. Notice that $\varphi_x(0) = x$ and also $\varphi_x(x) = 0$. It is well-known that the space of automorphisms of B_E is given by compositions of φ_x for some $x \in B_E$ with unitary transformations U of E. In addition, this space acts transitively on B_E.

The pseudohyperbolic distance on B_E is given by (see [8]):

$$\rho_E(x, y) = \|\varphi_y(x)\| \text{ for any } x, y \in B_E. \tag{1.2}$$

and using the definition of φ_x it is easy to conclude that:

$$\rho_E(x, y)^2 = 1 - \frac{(1 - \|x\|^2)(1 - \|y\|^2)}{|1 - \langle x, y\rangle|^2}. \tag{1.3}$$

1.2 The space of Bloch functions

Let $\mathbb{C}$ be the space of complex numbers and $\mathbb{D}$ the open disk of radius 1 centered at 0. The classical Bloch space $\mathcal{B}$ is given by the set of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that $\|f\|_B = \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < +\infty$. Timoney extended this space by considering domains of finite dimensional Hilbert spaces (see [17]) and in [2] the authors extended these functions to an infinite dimensional context. When we deal with a complex Hilbert space E, the holomorphic function $f : B_E \to \mathbb{C}$ belongs to the Bloch space $\mathcal{B}(B_E)$ if:

$$\|f\|_{\mathcal{B}} = \sup_{x\in B_E}(1 - \|x\|^2)\|\nabla f(x)\| < +\infty,$$

where the gradient $\nabla f(x)$ denotes the Fréchet derivative $f'(x)$ of f at x or, equivalently, if:

$$\|f\|_{\mathcal{R}} = \sup_{x\in B_E}(1 - \|x\|^2)\|\mathcal{R}f(x)\| < +\infty,$$

where $\mathcal{R}f(x) = \langle x, \overline{\nabla f(x)}\rangle$. Both semi-norms are also equivalent to:

$$\|f\|_{\mathcal{I}} = \sup_{x\in B_E}\|\widetilde{\nabla} f(x)\|, \tag{1.4}$$

where $\widetilde{\nabla} f(x)$ is the invariant gradient of f at x, that is, $\widetilde{\nabla} f(x) = \nabla(f \circ \varphi_x)(0)$ and bearing in mind that φ_x is the automorphism described in (1.1).

These three semi-norms describe norms of Banach spaces which are equivalent-modulo constant functions- in $\mathcal{B}(B_E)$ [2]. Indeed, there is a positive constant A_0 satisfying:

$$\|f\|_{\mathcal{R}} \le \|f\|_{\mathcal{B}} \le \|f\|_{\mathcal{I}} \le A_0\|f\|_R, \tag{1.5}$$

so we obtain a Banach space if we endow $\mathcal{B}(B_E)$ with one of the norms $\|\cdot\|_{\mathcal{B}-\text{Bloch}} = |f(0)| + \|\cdot\|_{\mathcal{B}}$ or $\|\cdot\|_{\mathcal{R}-\text{Bloch}}$ and $\|\cdot\|_{\mathcal{I}-\text{Bloch}}$ which are defined with the corresponding semi-norms $\|\cdot\|_{\mathcal{R}}$ and $\|\cdot\|_{\mathcal{I}}$. These semi-norms will be used along our work.

1.3 The function $(1 - \|x\|^2)|\mathcal{R}f(x)|$ is Lipschitz continuous

Let $f \in \mathcal{B}(B_E)$. Recall that the function defined on $x \in B_E$ and given by $x \mapsto (1 - \|x\|^2)|\mathcal{R}f(x)|$ is Lipschitz with respect to ρ_E. We recall several results which can be found in [13].

Theorem 1.1 *If f belongs to $\mathcal{B}(B_E)$ then:*

$$|(1-\|x\|^2)\mathcal{R}f(x)-(1-\|y\|^2)\mathcal{R}f(y)| \le 14\|f\|_{\mathcal{I}}\rho_E(x,y) \text{ for all } x, y \in B_E.$$

As a consequence, we obtain a corollary which extends results given in [1] by Attele and improved in [18] by Xiong on the Bloch space $\mathcal{B}$:

Corollary 1.2 *Consider a complex Hilbert space E. The function defined for $x \in B_E$ and given by $x \mapsto (1-\|x\|^2)|\mathcal{R}f(x)|$ is Lipschitz with respect to the pseudohyperbolic distance ρ_E. In addition, we have:*

$$|(1-\|x\|^2)|\mathcal{R}f(x)|-(1-\|y\|^2)|\mathcal{R}f(y)|| \le 14\|f\|_{\mathcal{I}}\rho_E(x,y).$$

This will allow us to provide conditions for a composition operator on the Bloch space $\mathcal{B}(B_E)$ to be bounded below.

2 Composition operators on $\mathcal{B}(B_E)$ which are bounded below

Let X and Y be Banach spaces. A linear operator $T : X \to Y$ is said to be bounded below if there is a positive constant $k > 0$ satisfying $\|x\| \le k\|T(x)\|$. A linear continuous operator T is bounded below if and only if T has closed range and it is injective.

If $\varphi : \mathbb{D} \to \mathbb{D}$ denotes an analytic map, the composition operator $C_\varphi : \mathcal{B} \to \mathcal{B}$ is defined by $C_\varphi(f) = f \circ \varphi$ and it is continuous for any φ. Define:

$$\tau_\varphi(z) = \frac{1-|z|^2}{1-|\varphi(z)|^2}\varphi'(z). \tag{2.1}$$

In [7], it was investigated when φ induces a composition operator which has closed range on $\mathcal{B}$. They proved:

Proposition 2.1 *Let C_φ be bounded below. Then there are $\varepsilon, r > 0$ such that $r < 1$ satisfying that for all $z \in \mathbb{D}$ we have $\rho(\varphi(w), z) \le r$ for all $w \in \mathbb{D}$ satisfying $|\tau_\varphi(w)| > \varepsilon$.*

The authors also studied the map defined for $z \in \mathbb{D}$ by $z \mapsto (1-|z|^2)|f'(z)|$, proving that it is Lipschitz with respect to ρ if f belongs to the Bloch space $\mathcal{B}$. Indeed, for any $f \in \mathcal{B}$ and $z, w \in \mathbb{D}$ we have:

$$|(1-|z|^2)|f'(z)|-(1-|w|^2)|f'(w)|| \le 3.31\|f\|_{\mathcal{B}}\rho(z,w). \tag{2.2}$$

This result refines a result of Attele (see [1]) who provided the constant 9 instead of 3.31. Xiong improved the constant in [18], giving $3\sqrt{3}/2 \approx 2.6$. From (2.2) we have the following sufficient condition for C_φ to be bounded below (see [7]):

Theorem 2.2 *Consider an analytic self-map φ on $\mathbb{D}$ and suppose that there is $0 < r < \frac{1}{4}$ and $\varepsilon > 0$ satisfying that for any $w \in \mathbb{D}$ there exists $z_w \in \mathbb{D}$ such that $\rho(\varphi(z_w), w) < r$ and $|\tau_\varphi(z_w)| > \varepsilon$. Then the operator $C_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded below.*

F. Deng, L. Jiang and C. Ouyang [6] and H. Chen [4] considered self-maps φ on B_n, where B_n denotes the open unit ball of a finite dimensional Hilbert space, extending these results from the one-dimensional case. However, they replaced $\tau_\varphi(z)$ by:

$$\left(\frac{1-\|z\|^2}{1-\|\varphi(z)\|^2}\right)^{(n+1)/2} |det(J_\varphi(z))| \tag{2.3}$$

where $J_\varphi(z)$ is the Jacobian matrix of φ. If φ is an automorphism of B_n then it is easy that $\tau_\varphi(z) = 1$. Moreover, the proofs of these results used the definition given by Timoney of Bloch function on B_n depending on the Bergman metric [17].

To extend the results given for the classical Bloch space $\mathcal{B}$ to a more general setting (finite or infinite dimensional), we will give sufficient and necessary conditions which avoid the Bergman metric and expression (2.3). Hence, consider a complex Hilbert space E and an analytic map $\psi : B_E \to B_E$. We define for $x \in B_E$ the expressions $\tau_\psi(x)$ and $\widetilde{\tau_\psi}(x)$ which are given by:

$$\tau_\psi(x) = \frac{1-\|x\|^2}{1-\|\psi(x)\|^2}\|\psi'(x)\| \tag{2.4}$$

and:

$$\widetilde{\tau_\psi}(x) = \frac{\sqrt{1-\|x\|^2}}{1-\|\psi(x)\|^2}\|\psi'(x)\|. \tag{2.5}$$

It is easy that $\widetilde{\tau_\psi}(x) \geq \tau_\psi(x)$.

In [3] the authors studied the boundness and also the compactness of $C_\psi : \mathcal{B}(B_E) \to \mathcal{B}(B_E)$ which is the composition operator defined by $C_\psi(f) = f \circ \psi$. It was proved that for any analytic self map ψ on B_E, the operator C_ψ is bounded. Furthermore, they proved the inequality $\|f \circ \psi\|_{\mathcal{I}} \leq \|f\|_{\mathcal{I}}$ where $\|\cdot\|_{\mathcal{I}}$ is the semi-norm defined in Sect. 1.2.

This Lemma will be useful for Lemma 2.4:

Lemma 2.3 *Consider a complex Hilbert space E and $f \in \mathcal{B}(B_E)$. Then:*

$$|f(x) - f(0)| \leq \|x\| \frac{\|f\|_{\mathcal{B}}}{1-\|x\|^2} \quad \textit{for any } x \in B_E.$$

Proof Note that:

$$
\begin{aligned}
|f(x)-f(0)| &= \left|\left(\int_0^1 f'(xt)\mathrm{d}t\right)(x)\right| \le \|x\| \left\|\int_0^1 \frac{f'(xt)(1-\|tx\|^2)}{1-\|tx\|^2}\mathrm{d}t\right\| \\
&\le \|x\|\|f\|_{\mathcal{B}} \int_0^1 \left|\frac{1}{1-\|tx\|^2}\right| \mathrm{d}t \\
&\le \|x\|\|f\|_{\mathcal{B}} \int_0^1 \frac{1}{1-\|x\|^2}\mathrm{d}t = \|x\|\frac{\|f\|_{\mathcal{B}}}{1-\|x\|^2}
\end{aligned}
$$

so the result is clear. □

Recall that $\|\cdot\|_{\mathcal{R}}$, $\|\cdot\|_{\mathcal{I}}$ and $\|\cdot\|_{\mathcal{B}}$ are equivalent, so they can be used interchangeably when studying if C_ψ is bounded below.

The following Lemma was given in [10, Lemma 2.14] with a different proof for the general case when B_E is the unit ball of a JB^*-triple. For completeness, we give a direct proof:

Lemma 2.4 *Consider a complex Hilbert space E and an analytic map $\psi : B_E \to B_E$. The composition operator $C_\psi : \mathcal{B}(B_E) \to \mathcal{B}(B_E)$ is bounded below if and only if there is $k > 0$ such that:*

$$\|C_\psi(f)\|_{\mathcal{I}} \ge k\|f\|_{\mathcal{I}} \textit{ for all } f \in \mathcal{B}(B_E).$$

Proof If C_ψ is bounded below then there exists $k > 0$ such that $\|C_\psi(f)\|_{\mathcal{I}-\text{Bloch}} \ge k\|f\|_{\mathcal{I}-\text{Bloch}}$ for $f \in \mathcal{B}(B_E)$. We define $g(x) = f(x) - f(\psi(0))$ and clearly $g(\psi(0)) = 0$. We have:

$$
\begin{aligned}
\|C_\psi(f)\|_{\mathcal{I}} = \|f\circ\psi\|_{\mathcal{I}} &= \|g\circ\psi\|_{\mathcal{I}} = \|g\circ\psi\|_{\mathcal{I}-\text{Bloch}} \\
&\ge k\|g\|_{\mathcal{I}-\text{Bloch}} \ge k\|g\|_{\mathcal{I}} = k\|f\|_{\mathcal{I}}.
\end{aligned}
$$

Now consider $\|C_\psi(f)\|_{\mathcal{I}} \ge k\|f\|_{\mathcal{I}}$ for some constant $0 < k \le 1$. We will find $k' > 0$ satisfying $\|C_\psi(f)\|_{\mathcal{I}-\text{Bloch}} \ge k'\|f\|_{\mathcal{I}-\text{Bloch}}$. Using Lemma 2.3 we obtain:

$$|f(\psi(0)) - f(0)| \le \|\psi(0)\|\frac{\|f\|_{\mathcal{B}}}{1-\|\psi(0)\|^2}$$

so we have:

$$|f(\psi(0))| \ge |f(0)| - \|\psi(0)\|\frac{\|f\|_{\mathcal{B}}}{1-\|\psi(0)\|^2} \ge |f(0)| - \frac{\|f\|_{\mathcal{I}}}{1-\|\psi(0)\|^2}.$$

and we obtain:

$$|f(\psi(0))| + \frac{1}{(1-\|\psi(0)\|^2)}\|f\|_{\mathcal{I}} \ge |f(0)|.$$

Hence:

$$\begin{aligned} & k(1-\|\psi(0)\|^2)|f(\psi(0))| + \|C_\psi(f)\|_{\mathcal{I}} \\ & \quad \geq k(1-\|\psi(0)\|^2)|f(\psi(0))| + k\|f\|_{\mathcal{I}} \geq k(1-\|\psi(0)\|^2)|f(0)| \end{aligned}$$

so we have that:

$$\begin{aligned} 2(|f(\psi(0))| + \|C_\psi(f)\|_{\mathcal{I}}) &= 2|f(\psi(0))| + \|C_\psi(f)\|_{\mathcal{I}} + \|C_\psi(f)\|_{\mathcal{I}} \\ &\geq k(1-\|\psi(0)\|^2)|f(\psi(0))| + \|C_\psi(f)\|_{\mathcal{I}} + \|C_\psi(f)\|_{\mathcal{I}} \\ &\geq k(1-\|\psi(0)\|^2)|f(0)| + \|C_\psi(f)\|_{\mathcal{I}} \\ &\geq k(1-\|\psi(0)\|^2)(|f(0)| + \|f\|_{\mathcal{I}}) \end{aligned}$$

and we can conclude:

$$\|C_\psi(f))\|_{\mathcal{I}-\text{Bloch}} \geq \frac{k(1-\|\psi(0)\|^2)}{2}\|f\|_{\mathcal{I}-\text{Bloch}}$$

so taking $k' = k(1-\|\psi(0)\|^2)/2$ we obtain that C_ψ is a bounded below operator. □

2.1 The automorphisms φ_x on B_E

In this section we will give some calculations related to the automorphisms φ_x of B_E given in (1.1) which will permit us to study conditions for C_φ to be bounded below. If E is finite dimensional, then it is well-known that φ_x is an involution (see [16]). Since the proof uses the Cartan's uniqueness theorem, we first give a new proof of this assertion, extending the result for infinite dimensional spaces:

Lemma 2.5 *If E is a complex Hilbert space and $x \in B_E$, then $\varphi_x \circ \varphi_x = Id_E$, that is, φ_x is an involution.*

Proof Using (1.1), we have:

$$\varphi_x(\varphi_x(y)) = (s_x Q_x + P_x)(m_x(\varphi_x(y)) = (s_x Q_x + P_x)\left(\frac{x-\varphi_x(y)}{1-\langle \varphi_x(y), x\rangle}\right)$$

and using the following result (it can be found as Lemma 3.6 in [14]):

$$1 - \langle \varphi_x(y), x\rangle = 1 - \langle \varphi_x(y), \varphi_x(0)\rangle = \frac{1-\|x\|^2}{1-\langle y, x\rangle}$$

we obtain:

$$\begin{aligned} \varphi_x(\varphi_x(y)) &= \frac{1-\langle y, x\rangle}{1-\|x\|^2}(s_x Q_x + P_x)\,(x-\varphi_x(y)) \\ &= \frac{1-\langle y, x\rangle}{1-\|x\|^2}\left((s_x Q_x + P_x)(x) - (s_x Q_x + P_x)((s_x Q_x + P_x)(m_x(y)))\right). \end{aligned}$$

Using $P_x \circ Q_x = Q_x \circ P_x = 0$, $P_x + Q_x = Id_E$, $P_x^2 = P_x$ and $Q_x^2 = Q_x$ we have:

$$\begin{aligned}\varphi_x(\varphi_x(y)) &= \frac{1-\langle y,x\rangle}{1-\|x\|^2}\left(x-(s_x^2 Q_x+P_x)\left(\frac{x-y}{1-\langle y,x\rangle}\right)\right)\\ &= \frac{1-\langle y,x\rangle}{(1-\|x\|^2)(1-\langle y,x\rangle)}\Big((1-\langle y,x\rangle)x-(s_x^2Q_x+P_x)\,(x-y)\Big)\\ &= \frac{1}{(1-\|x\|^2)}\Big((x-\|x\|^2P_x(y)-x+(1-\|x\|^2)Q_x(y)+P_x(y)\Big)\\ &= \frac{1}{(1-\|x\|^2)}(1-\|x\|^2)(P_x(y)+Q_x(y))=y\end{aligned}$$

so we obtain the result. □

Lemma 2.6 *For any $x \in B_E$ we have that the operator $\varphi_x'(0)$ is invertible and $\varphi_x'(0)^{-1} = \varphi_x'(x)$.*

Proof Using Lemma 2.5, we have $(\varphi_x \circ \varphi_x)'(0) = Id_E'(0) = Id_E$ so:

$$\varphi_x'(\varphi_x(0)) \circ \varphi_x'(0) = \varphi_x'(x) \circ \varphi_x'(0) = Id_E$$

and we are done. □

Recall that $\|f\|_{\mathcal{I}} = \sup_{x\in B_E} \|\widetilde{\nabla} f(x)\|$ by (1.4). For all $x \in B_E$ we have:

$$\|\widetilde{\nabla} f(x)\| = \sup_{u\in\overline{B_E}} \|f'(\varphi_x(0)) \circ \varphi_x'(0)(u)\| = \sup_{w\in E\setminus\{0\}} \frac{|f'(x)(w)|}{\|\varphi_x'(0)^{-1}(w)\|} \tag{2.6}$$

and for all $w \in E$ we have that (see [2]):

$$\|\varphi_x'(0)^{-1}(w)\|^2 = \frac{(1-\|x\|^2)\|w\|^2 + |\langle w,x\rangle|^2}{(1-\|x\|^2)^2}. \tag{2.7}$$

In [2] the following equality was also given:

$$\|\widetilde{\nabla} f(x)\|^2 = (1-\|x\|^2)\left(\|\nabla f(x)\|^2 - |\mathcal{R}f(x)|^2\right). \tag{2.8}$$

For an analytic map $\psi : B_E \to B_E$, $x \in B_E$ and $w \in E$ we will use the infinitesimal Kobayashi metric described in [10]. For a complex Hilbert space E, this metric can be described in terms of the automorphisms φ_x by:

$$\kappa_E(x,w) = \|\varphi_x'(0)^{-1}(w)\| \text{ for } x \in B_E \text{ and } w \in E.$$

We will use $\kappa(x,w)$ and $\kappa(\psi(x),\psi'(x)(w))$ for an analytic self-map $\psi : B_E \to B_E$ several times in the sequel. Notice that:

$$\kappa(\psi(x),\psi'(x)(w)) = \|\varphi_{\psi(x)}'(0)^{-1}(\psi'(x)(w))\|. \tag{2.9}$$

Lemma 2.7 *If $\psi : B_E \to B_E$ is analytic and $x \in B_E$ then:*

(a) *For $w \in E$:*

$$\frac{\|w\|^2}{1-\|x\|^2} \leq \kappa(x,w)^2 \leq \frac{\|w\|^2}{(1-\|x\|^2)^2} \tag{2.10}$$

and:

$$\frac{\|\psi'(x)(w)\|^2}{1-\|\psi(x)\|^2} \leq \kappa(\psi(x),\psi'(x)(w))^2 \leq \frac{\|\psi'(x)(w)\|^2}{(1-\|\psi(x)\|^2)^2}. \tag{2.11}$$

(b) *If there is $w_x \in E$ satisfying $\psi'(x)(w_x) = \|\psi'(x)\|\psi(x)$ then:*

$$\frac{\|\psi'(x)\|\|\psi(x)\|}{1-\|\psi(x)\|^2} = \kappa(\psi(x),\psi'(x)(w_x)) \leq \frac{\|\psi'(x)\|}{1-\|\psi(x)\|^2} \tag{2.12}$$

and under the condition $w_x \neq 0$, then:

$$\frac{\kappa(\psi(x),\psi'(x)(w_x))}{\kappa(x,w_x)} \geq \tau_\psi(x)\frac{\|\psi(x)\|}{\|w_x\|}. \tag{2.13}$$

Proof We will prove a). By (2.7) and (2.9) we obtain:

$$\kappa(x,w)^2 = \frac{(1-\|x\|^2)\|w\|^2 + |\langle w,x\rangle|^2}{(1-\|x\|^2)^2}.$$

Hence:

$$\frac{\|w\|^2}{(1-\|x\|^2)} \leq \kappa(x,w)^2 \leq \frac{\|w\|^2}{(1-\|x\|^2)^2}$$

where last inequality is true because $|\langle w,x\rangle| \leq \|w\|\|x\|$, so we conclude (2.10). Following the same pattern, we obtain a proof for (2.11).

Now we prove b). We have:

$$\begin{aligned}\kappa(\psi(x),\psi'(x)(w_x))^2 &= \frac{(1-\|\psi(x)\|^2)\|\psi'(x)(w_x)\|^2 + |\langle\psi'(x)(w_x),\psi(x)\rangle|^2}{(1-\|\psi(x)\|^2)^2}\\ &= \frac{(1-\|\psi(x)\|^2)\|\psi'(x)\|^2\|\psi(x)\|^2 + \|\psi(x)\|^4\|\psi'(x)\|^2}{(1-\|\psi(x)\|^2)^2}\\ &= \frac{\|\psi'(x)\|^2\|\psi(x)\|^2}{(1-\|\psi(x)\|^2)^2}\end{aligned}$$

and we obtain inequality (2.12). Together with inequality (2.10) results in (2.13) since:

$$\frac{\kappa(\psi(x), \psi'(x)(w_x))}{\kappa(x, w_x)} \geq \frac{1-\|x\|^2}{1-\|\psi(x)\|^2} \frac{\|\psi'(x)\|\|\psi(x)\|}{\|w_x\|}$$

and we conclude the result. □

From Lemma 2.7 we have:

Lemma 2.8 *For any $x \in B_E$ and $w \in E \setminus \{0\}$:*

$$\frac{\kappa(\psi(x), \psi'(x)(w))}{\kappa(x, w)} \leq \frac{\sqrt{1-\|x\|^2}}{1-\|\psi(x)\|^2} \left\| \psi'(x)\left(\frac{w}{\|w\|}\right)\right\| \tag{2.14}$$

and:

$$\frac{\kappa(\psi(x), \psi'(x)(w))}{\kappa(x, w)} \geq \frac{1-\|x\|^2}{\sqrt{1-\|\psi(x)\|^2}} \left\| \psi'(x)\left(\frac{w}{\|w\|}\right)\right\|. \tag{2.15}$$

The following lemma is just a contractive property of the infinitesimal Kobayashi metric. We omit the proof:

Lemma 2.9 *If ψ is an analytic self-map on B_E, then for any $x \in B_E$ and $w \in E \setminus \{0\}$ we have:*

$$\frac{\kappa(\psi(x), \psi'(x)(w))}{\kappa(x, w)} = \frac{\|\varphi'_{\psi(x)}(0)^{-1}(\psi'(x)(w))\|}{\|\varphi'_x(0)^{-1}(w)\|} \leq 1.$$

The following extension of the Schwarz-Pick lemma generalizes a result of Kalaj [12] when we deal with an infinite dimensional space. The same result for bounded symmetric domains can be found in [5].

Corollary 2.10 *Consider an analytic self map ψ on B_E. Then:*

$$\frac{1-\|x\|^2}{\sqrt{1-\|\psi(x)\|^2}} \|\psi'(x)\| \leq 1 \ \textit{for all } x \in B_E.$$

Proof Applying Lemma 2.9 and using inequality (2.15) in Lemma 2.7 we are done. □

Remark 2.11 Hamada and Kohr [11] proved that Corollary 2.10 is sharp. Kalaj [12] also proved this sharpness by considering for all $t \in (0, \pi/2)$ the self-map $\psi_t : B_2 \to B_2$ defined by $\psi_t(z, w) = (z \sin t, \cos t)$.

2.2 Results on bounded below composition operators

We will apply the study on the automorphisms φ_x to study bounded below composition operators. Hamada [10] provided a necessary condition in the context of the unit ball of a JB^*-triple by considering the existence of $\varepsilon > 0$ and $0 < r < 1$ such that if $y \in B_E$ then $\rho(\psi(x_y), y) \leq r$ for any $x_y \in B_E$ satisfying $\tau^*_\psi(x_y) \geq \varepsilon$ where:

$$\tau^*_\psi(x_y) = \sup\left\{\frac{\kappa_E(\psi(x_y), \psi'(x_y)(y))}{\kappa_E(x_y, y)} : w \in E\backslash\{0\}\right\}$$

We provide a necessary condition for the Hilbert case by adapting the proof of Theorem 2 in [6] and using $\widetilde{\tau_\psi}(x_y)$ instead of $\tau^*_\psi(x_y)$:

Theorem 2.12 *Consider an analytic self map ψ on B_E and suppose that $C_\psi : \mathcal{B}(B_E) \to \mathcal{B}(B_E)$ is a bounded below operator. Then there are $\varepsilon > 0$ and $0 < r < 1$ such that if $y \in B_E$ we have $\rho(\psi(x_y), y) \leq r$ for some $x_y \in B_E$ satisfying $\widetilde{\tau_\psi}(x_y) \geq \varepsilon$.*

Proof If C_ψ is a bounded below operator, consider $y \in B_E$ and let $f : B_E \to \mathbb{C}$ be an analytic function given by $f_y(x) = 1/(1 - \langle x, y\rangle)$.
We have:

$$f'_y(x) = \frac{\langle \cdot, y\rangle}{(1 - \langle x, y\rangle)^2}$$

so we have:

$$\begin{aligned}\|f_y\|_\mathcal{B} &= \sup_{x\in B_E}(1 - \|x\|^2)\|f'_y(x)\| = \sup_{x\in B_E}(1 - \|x\|^2)\frac{\|y\|}{|1 - \langle x, y\rangle|^2}\\ &= \sup_{x\in B_E}\|y\|\frac{1 - \|\varphi_y(x)\|^2}{1 - \|y\|^2} = \frac{\|y\|}{1 - \|y\|^2}.\end{aligned}$$

Define $g_y : B_E \to \mathbb{C}$ by $g_y(x) = f_y(x)/\|f_y\|_\mathcal{B}$ which is analytic and it is satisfied that $\|g_y\|_\mathcal{I} \geq \|g_y\|_\mathcal{B} = 1$. Using Lemma 2.4, there is a positive number k satisfying $\|g_y \circ \psi\|_\mathcal{I} \geq k\|g_y\|_\mathcal{I}$ so since:

$$\|g_y \circ \psi\|_\mathcal{I} = \sup_{x\in B_E}\|\widetilde{\nabla}(g_y \circ \psi)(x)\|,$$

there exists $x_y \in B_E$ which satisfies $\|\widetilde{\nabla}(g_y \circ \psi)(x_y)\| \geq k/2$. Hence:

$$\begin{aligned}\frac{k}{2} \leq \|\widetilde{\nabla}(g_y \circ \psi)(x_y)\| &= \sup_{w\in E\backslash\{0\}}\frac{\|\widetilde{\nabla}(g_y \circ \psi)(x_y)(w)\|}{\|w\|}\\ &= \sup_{w\in E\backslash\{0\}}\frac{|g'_y(\psi(x_y))(\psi'(x_y)(w))|}{\|\varphi'_{x_y}(0)^{-1}(w)\|}\end{aligned}$$

$$
\begin{aligned}
&= \sup_{w\in E\setminus\{0\}} \frac{|g_y'(\psi(x_y))(\psi'(x_y)(w))|}{\kappa(\psi(x_y),\psi'(x_y)(w))}\frac{\kappa(\psi(x_y),\psi'(x_y)(w))}{\kappa(x_y,w)}\\
&\le \|\widetilde{\nabla} g_y(\psi(x_y))\|\widetilde{\tau_\psi}(x_y) \qquad (2.16)
\end{aligned}
$$

where using (2.6) and (2.14) in Lemma 2.8 it is clearly deduced last inequality. By (2.8) we conclude:

$$
\begin{aligned}
\|\widetilde{\nabla} g_y(\psi(x_y))\|^2 &= (1-\|\psi(x_y)\|^2)(\|\nabla g_y(\psi(x_y))\|^2 - |\mathcal{R} g_y(\psi(x_y))|^2)\\
&= (1-\|\psi(x_y)\|^2)\frac{(1-\|y\|^2)^2}{\|y\|^2}\\
&\quad\left(\frac{\|y\|^2}{|1-\langle\psi(x_y),y\rangle|^4} - \frac{|\langle\psi(x_y),y\rangle|^2}{|1-\langle\psi(x_y),y\rangle|^4}\right)\\
&= (1-\|\psi(x_y)\|^2)(1-\|y\|^2)^2\frac{1-\left|\left\langle\psi(x_y),\frac{y}{\|y\|}\right\rangle\right|^2}{|1-\langle\psi(x_y),y\rangle|^4}.
\end{aligned}
$$

The inequality $|1-\langle c,d/\|d\|\rangle| \le 2|1-\langle c,d\rangle|$ for any $c,d\in B_E$ is clear since:

$$
\begin{aligned}
|1-\langle c,d/\|d\|\rangle| &\le |1-\langle c,d\rangle| + |\langle c, d-d/\|d\|\rangle|\\
&\le |1-\langle c,d\rangle| + 1 - \|d\| \le |1-\langle c,d\rangle| + 1 - |\langle c,d\rangle|\\
&= 2|1-\langle c,d\rangle|.
\end{aligned}
$$

From:

$$
1-\left|\left\langle\psi(x_y),\frac{y}{\|y\|}\right\rangle\right|^2 \le \left(1+\left|\left\langle\psi(x_y),\frac{y}{\|y\|}\right\rangle\right|\right)\left(1-\left|\left\langle\psi(x_y),\frac{y}{\|y\|}\right\rangle\right|\right)
$$

we conclude:

$$
\begin{aligned}
\|\widetilde{\nabla} g(\psi(x_y))\|^2 &\le 4(1-\|\psi(x_y)\|^2)(1-\|y\|^2)\frac{1}{|1-\langle\psi(x_y),y\rangle|^2}\\
&= 4(1-\|\varphi_y(\psi(x_y))\|^2) = 4(1-\rho(y,\psi(x_y))^2)
\end{aligned}
$$

so:

$$
\frac{k}{2} \le 2(1-\rho(y,\psi(x_y))^2)^{1/2}\widetilde{\tau_\psi}(x_y)
$$

which is true if and only if $\dfrac{k}{4} \le (1-\rho(y,\psi(x_y))^2)^{1/2}\widetilde{\tau_\psi}(x_y)$ and we have $\widetilde{\tau_\psi}(x_y) \ge \frac{k}{4}$.

Using (2.16) we have:

$$
\frac{k}{2} \le 2(1-\rho(y,\psi(x_y))^2)^{1/2} \sup_{w\in E\setminus\{0\}} \frac{\kappa(\psi(x_y),\psi'(x_y)(w))}{\kappa(x_y,w)}
$$

so applying Lemma 2.9:

$$\sqrt{1-\rho(y,\psi(x_y))^2} \geq k/4$$

and this expression is equivalent to:

$$\rho(y,\psi(x_y)) \leq \sqrt{1-k^2/16}.$$

Taking $r = \sqrt{1-k^2/16}$ and $\varepsilon = k/4$ we conclude the result. □

Hamada [10] provided a sufficient condition for a composition operator to be bounded below when we deal with unit balls of JB^*-triples. We will provide a new condition by extending the result given in Theorem 2.2. Hence we will consider the following condition: we will suppose that $\psi(x_y)$ belongs to the range of $\psi'(x_y)$. Recall that, as we have mentioned in (1.5), there is a positive constant A_0 satisfying:

$$\|f\|_{\mathcal{R}} \leq \|f\|_{\mathcal{B}} \leq \|f\|_{\mathcal{I}} \leq A_0\|f\|_R \text{ for any } f \in \mathcal{B}(B_E).$$

Theorem 2.13 *Let ψ be an analytic self-map on B_E. Suppose there are constants r, ε satisfying $0 < r < \frac{1}{15A_0}$ and $\varepsilon > 0$ which also satisfies that for any $y \in \mathcal{B}_E$ there exists $x_y \in B_E$ such that $\rho(\psi(x_y), y) < r$ and $\tau_\psi(x_y) > \varepsilon$. Suppose also that $\psi(x_y) = \psi'(x_y)(w_{x_y})$ for some point $w_{x_y} \in E$ satisfying $\sup_{y\in B_E}\|w_{x_y}\| < +\infty$. Then we have that $C_\psi : \mathcal{B}(B_E) \to \mathcal{B}(B_E)$ is bounded below.*

Proof Consider a function $f \in \mathcal{B}(B_E)$ satisfying $\|f\|_{\mathcal{I}} = 1$. We show the existence of $k > 0$ which satisfies that $\|f\circ\psi\|_{\mathcal{I}} \geq k$. We have that $\|f\|_{\mathcal{R}} \geq \|f\|_{\mathcal{I}}/A_0$ by (1.5) so $\|f\|_{\mathcal{R}} \geq 1/A_0$. Taking $y \in B_E$ satisfying $|\mathcal{R}f(y)|(1-\|y\|^2) \geq 14/(15A_0)$, there exists $x_y \in B_E$ such that $\rho(y,\psi(x_y)) < r$ and $\tau_\psi(x_y) > \varepsilon$. Using (1.4) and (2.6) and also by (2.9), we have for any $w \in E\backslash\{0\}$:

$$\begin{aligned}\|f\circ\psi\|_{\mathcal{I}} &= \sup_{x\in B_E}\|\widetilde{\nabla}(f\circ\psi)(x)\| \\ &\geq \frac{|(f\circ\psi)'(x_y)(w)|}{\|\varphi'_{x_y}(0)^{-1}(w)\|} = \frac{|f'(\psi(x_y))(\psi'(x_y)(w))|}{\kappa(\psi(x_y),\psi'(x_y)(w))}\frac{\kappa(\psi(x_y),\psi'(x_y)(w))}{\kappa(x_y,w)}.\end{aligned}$$

Since $\psi(x_y) \in \psi'(x_y)(E)$, there exists $w_{x_y} \in E$ such that $\psi'(x_y)(w_{x_y}) = \|\psi'(x_y)\|\psi(x_y)$ so the inequality above is clearly true taking w_{x_y}. Using (2.12) from Lemma 2.7 we obtain:

$$\frac{|f'(\psi(x_y))(\psi'(x_y)(w_{x_y}))|}{\kappa(\psi(x_y), \psi'(x_y)(w_{x_y}))} = \frac{|f'(\psi(x_y))(\|\psi'(x_y)\|\psi(x_y))|}{\kappa(\psi(x_y), \psi'(x_y)(w_{x_y}))}$$
$$= \frac{\|\psi'(x_y)\||f'(\psi(x_y))(\psi(x_y))|(1-\|\psi(x_y)\|^2)}{\|\psi'(x_y)\|\|\psi(x_y)\|}$$
$$= \frac{|\mathcal{R}f(\psi(x_y))|(1-\|\psi(x_y)\|^2)}{\|\psi(x_y)\|}$$

so:

$$\|f\circ\psi\|_{\mathcal{I}} \geq \frac{\mathcal{R}f(\psi(x_y))(1-\|\psi(x_y)\|^2)}{\|\psi(x_y)\|}\frac{\kappa(\psi(x_y), \psi'(x_y)(w_{x_y}))}{\kappa(x_y, w_{x_y})}$$

and using (2.13) from Lemma 2.7 we have:

$$\|f\circ\psi\|_{\mathcal{I}} \geq \frac{|\mathcal{R}f(\psi(x_y))|(1-\|\psi(x_y)\|^2)}{\|\psi(x_y)\|}\frac{\|\psi(x_y)\|\tau_\psi(x_y)}{\|w_{x_y}\|}$$
$$\geq |\mathcal{R}f(\psi(x_y))|(1-\|\psi(x_y)\|^2)\frac{\varepsilon}{\|w_{x_y}\|}.$$

From Corollary 1.2, we obtain:

$$||\mathcal{R}f(\psi(x_y))|(1-\|\psi(x_y)\|^2) - |\mathcal{R}f(y)|(1-\|y\|^2)| \leq 14\|f\|_{\mathcal{I}}\rho_E(\psi(x_y), y)$$

and using $\|f\|_{\mathcal{I}} = 1$, we conclude:

$$\|f\circ\psi\|_{\mathcal{I}} \geq (|\mathcal{R}f(y)|(1-\|y\|^2)| - 14\rho(\psi(x_y), y))\frac{\varepsilon}{\|w_{x_y}\|}$$
$$\geq \left(\frac{14}{15A_0} - 14r\right)\frac{\varepsilon}{\sup_{y\in B_E}\|w_{x_y}\|}$$

so we can take:

$$k = 14\left(\frac{1}{15A_0} - r\right)\frac{\varepsilon}{\sup_{y\in B_E}\|w_{x_y}\|} > 0$$

and we finally conclude $\|C_\psi(f)\|_{\mathcal{I}} \geq k$. □

Now we check that the automorphism φ_a of B_E for any $a \in B_E$ satisfies the conditions of Theorem 2.13. We will need this result, which shows $\tau_{\varphi_a}(x) \geq 1$ for all $x \in B_E$.

Lemma 2.14 *For all $a \in B_E$ we have $\tau_{\varphi_a}(x) \geq 1$ if $x \in B_E$.*

Proof Notice that by (1.3) we have:

$$\frac{1-\|x\|^2}{1-\|\varphi_a(x)\|^2} = \frac{|1-\langle x, a\rangle|^2}{1-\|a\|^2}$$

and since: $\varphi_a(x) = (P_a + s_a Q_a)(m_a(x))$, then we obtain:

$$\varphi_a'(x) = (P_a + s_a Q_a)'(m_a(x)) \circ m_a'(x) = (P_a + s_a Q_a)(m_a'(x))$$

so we have:

$$\|\varphi_a'(x)\|^2 = \|P_a(m_a'(x))\|^2 + s_a^2 \|Q_a(m_a'(x))\|^2 \geq \|P_a(m_a'(x))\|^2.$$

It is easy that:

$$m_a'(x)(y) = \frac{-(1 - \langle x, a\rangle)y + \langle y, a\rangle(a - x)}{(1 - \langle x, a\rangle)^2}$$

so:

$$\begin{aligned}
\|\varphi_a'(x)\| \geq \|P_a(m_a'(x))\| &= \sup_{y\in \overline{B}_E} \|P_a(m_a'(x))(y))\| \\
&\geq \left\| P_a\left(m_a'(x)\left(\frac{a}{\|a\|}\right)\right)\right\| \\
&= \left\| P_a\left(\frac{-(1 - \langle x, a\rangle)\frac{a}{\|a\|} + \langle \frac{a}{\|a\|}, a\rangle(a - x)}{(1 - \langle x, a\rangle)^2}\right)\right\|
\end{aligned}$$

so we obtain:

$$\begin{aligned}
\tau_{\varphi_a}(x) &\geq \frac{|1 - \langle x, a\rangle|^2}{1 - \|a\|^2}\left\| P_a\left(\frac{-(1 - \langle x, a\rangle)\frac{a}{\|a\|} + \langle \frac{a}{\|a\|}, a\rangle(a - x)}{(1 - \langle x, a\rangle)^2}\right)\right\| \\
&= \frac{1}{1 - \|a\|^2}\left\| P_a\left(-(1 - \langle x, a\rangle)\frac{a}{\|a\|} + \langle \frac{a}{\|a\|}, a\rangle(a - x)\right)\right\| \\
&= \frac{1}{1 - \|a\|^2}\left\|\left(-(1 - \langle x, a\rangle)\frac{a}{\|a\|} + \|a\|a - \frac{\langle x, a\rangle}{\|a\|^2}\|a\|a\right)\right\| \\
&= \frac{1}{1 - \|a\|^2}\left\| -\frac{1 - \|a\|^2}{\|a\|}a\right\| = 1
\end{aligned}$$

and we have $\tau_{\varphi_a}(x) \geq 1$ so we are done. □

Remark 2.15 Conditions of Theorem 2.13 are satisfied by the automorphisms φ_a for any $a \in B_E$ since by Lemma 2.14 we have:

$$\frac{1 - \|x\|^2}{1 - \|\varphi_a(x)\|^2}\|\varphi_a'(x)\| \geq 1$$

so choose $\varepsilon = 1$, $r = 0$ and for any $y \in B_E$ take $x_y = \varphi_a(y)$. Furthermore, $\varphi_a(x_y) = \varphi_a(\varphi_a(y)) = y = \varphi_a'(x_y)(w_{x_y})$ for some w_{x_y} belonging to E which satisfies $\sup_{y\in B_E} \|w_{x_y}\| < +\infty$ since the operator $\varphi_a'(x_y)$ is invertible on the space E.

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