Mediterr. J. Math. (2023) 20:307
https://doi.org/10.1007/s00009-023-02512-0

Mediterranean Journal of Mathematics

The Constant of Interpolation in Bloch Type Spaces

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Abstract. It is known that there exists a constant $0 < \Delta_1 < 1$ such that any Δ_1-separated sequence for the pseudohyperbolic distance in the open unit disk $\mathbf{D}$ of $\mathbb{C}$ is interpolating for the classical Bloch space $\mathcal{B}$. We will prove that $0.8114 < \Delta_1 < 0.9785$ and we will also generalize this result for Bloch type spaces $\mathcal{B}_{v_p}$ for $v_p(z) = (1-|z|^2)^p$. In particular, we will provide a construction to calculate an estimate of the lower and upper bounds for the corresponding constant of separation Δ_p for these spaces. We also prove that Δ_p tends to 1 when $p \to \infty$.

Mathematics Subject Classification. Primary 30H30; Secondary 30H05.

Keywords. Bloch space, interpolating sequence.

1. Introduction and Background

Let $\mathbf{D}$ be the open unit disk of the complex plane $\mathbb{C}$. Recall that the classical Bloch space $\mathcal{B}$ is given by the space of analytic functions $f : \mathbf{D} \to \mathbb{C}$ such that $\|f\|_B = \sup_{z\in\mathbf{D}}(1 - |z|^2)|f'(z)| < \infty$. The study of interpolating sequences for the classical Bloch space $\mathcal{B}$ was started by Attele in [1] and Madigan and Matheson in [9]. The study of interpolating sequences for Bloch type spaces $\mathcal{B}_v$ was introduced in [10], where the weight $1-|z|^2$ was substituted by a more general weight $v(z)$. In this work, we will deal with weights $v_p(z) = (1-|z|^2)^p$ for $1 \leq p < \infty$.

Recall that a sequence $(z_n) \subset \mathbf{D}$ is said to be an interpolating sequence for $\mathcal{B}_v$ if for any $(a_n) \in \ell_\infty$ there exists $f \in \mathcal{B}_v$ such that $v(z_n)f'(z_n) = a_n$ for any $n \in \mathbb{N}$. The interpolating operator $T : \mathcal{B}_v \to \ell_\infty$ is defined by $T(f) = (v(z_n)f'(z_n))$. Notice that T is clearly linear and (z_n) is interpolating for $\mathcal{B}_v$ if and only if T is surjective or, equivalently, if there exists a mapping $S : \ell_\infty \to \mathcal{B}_v$ such that $T \circ S = Id_{\ell_\infty}$. Given $\delta > 0$, the sequence (z_n) is said

Supported by PID2019-106529GB-I00 (MICINN. Spain) and the predoctoral Grant CIACIF/2021/479 (Generalitat Valenciana).

Birkhäuser

Published online: 23 September 2023

to be δ-separated (or simply separated) if $\rho(z_k, z_j) \geq \delta$ for $k \neq j$, where the pseudohyperbolic distance ρ is defined by:

$$\rho(z,w) = \left| \frac{z-w}{1-\bar{z}w} \right| \quad \text{for any } z, w \in \mathbf{D}.$$

It is well known that $\rho(z,w) = |\varphi_z(w)|$ where φ_z is the automorphism from $\mathbf{D}$ onto itself given by $\varphi_z(w) = (z-w)/(1-\bar{z}w)$. For any $z \in \mathbf{D}$, we have that $\rho(\varphi_z(w_1), \varphi_z(w_2)) = \rho(w_1, w_2)$, $w_1, w_2 \in \mathbf{D}$. It is also well known that:

$$1 - \rho(z,w)^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2} \tag{1.1}$$

and these facts about ρ will be used in the sequel. The constant of separation of (z_n) is given by $S := \inf_{n \neq k} \rho(z_n, z_k)$. It is well known that if $(z_n) \subset \mathbf{D}$ is interpolating, then it is separated (see for instance Corollary 3 in [1]). On the other side, Attele (see Proposition 4 in [1]) and Madigan and Matheson (see Proposition 1 in [9]) proved that there exists a universal constant Δ_1 such that any Δ_1-separated sequence in $\mathbf{D}$ is interpolating for $\mathcal{B}$. Making calculations from these works, it is not difficult to make an upper estimate for this constant. Nevertheless, this estimate is higher than 0.99. In Theorem 5 in [4], the authors improved this upper bound until 0.9882.

In this work, we will improve the upper estimate and we will calculate a lower estimate for Δ_1, proving that $0.8114 < \Delta_1 < 0.9785$. Furthermore, we will generalize the result for Bloch type spaces $\mathcal{B}_{v_p}$: we will prove that sufficiently separated sequences on the open unit disk are interpolating for $\mathcal{B}_{v_p}$ and will provide bounds for the corresponding universal constants Δ_p.

2. Results

2.1. Lower Estimate

Let $a > 1$ and $b > 0$ and consider the sequence:

$$\Gamma(a,b) = \left\{ \frac{a^m(bn+i)-i}{a^m(bn+i)+i} \right\}_{m,n \in \mathbb{Z}}. \tag{2.1}$$

K. Seip proved that $\Gamma(a,b)$ is interpolating for $\mathcal{B}_{v_p}$ if and only if $\frac{2\pi}{b \log a} < \frac{1}{p}$. This is based on the study of the positive density D^+ of the sequence (see [11]). As Schuster observed [12], the constant of separation of $\Gamma(a,b)$ is given by:

$$S(a,b) = \min\left\{ \frac{a-1}{a+1}, \frac{b}{\sqrt{b^2+4}} \right\}.$$

Proposition 2.1. *There exists a sequence* $(z_n) \subset \mathbf{D}$ *which is* δ*-separated for* $\delta = 0.811458$ *and* (z_n) *is not interpolating for the Bloch space* $\mathcal{B}$. *Hence,* $\Delta_1 > 0.811458$.

Proof. Consider the sequence $\Gamma(a,b)$ in (2.1). Take b=$\frac{2\pi}{loga}$ for $a > 1$ and $b > 0$. Then, we have $\frac{2\pi}{b \log a} = 1$ so the sequence is not interpolating for $\mathcal{B}$ by the comments above. Notice that functions:

$$\frac{a-1}{a+1} = 1 - \frac{2}{a+1} \quad \text{and} \quad \frac{b}{\sqrt{b^2+4}} = \sqrt{1 - \frac{4}{b^2+4}} = \sqrt{1 - \frac{4}{(\frac{2\pi}{\log a})^2 + 4}}$$

are non-decreasing and non-increasing, respectively. Hence, the highest value of $S(a,b)$ will be got when:

$$\frac{a-1}{a+1} = \frac{b}{\sqrt{b^2+4}}$$

which is equivalent to:

$$g(a) := 1 - \frac{2}{a+1} - \sqrt{1 - \frac{4}{b^2+4}} = 0$$

and bearing in mind that $b = \frac{2\pi}{\log a}$, the function $g(a)$ is clearly non-decreasing continuous with respect to a. Since $\lim_{a\to 1^+} g(a) = -1$ and $\lim_{a\to+\infty} g(a) = 1$, there exists a unique $a > 1$ such that $g(a) = 0$. Indeed, $a \approx 9.60773$ and $b \approx 2.77701$, which yields:

$$S(a,b) = \min\left\{\frac{a-1}{a+1}, \frac{b}{\sqrt{b^2+4}}\right\} \approx 0.811458.$$

Hence, we have a sequence whose constant of separation is 0.811458 but fails to be interpolating for $\mathcal{B}$. □

We can easily generalize Proposition 2.1 to B_{v_p} spaces for $1 \le p < +\infty$ taking $b = 2p\pi/\log a$ and following the same pattern:

Proposition 2.2. *For any $1 \le p < +\infty$, there exist $a > 1$ and $b > 0$ such that the sequence $\Gamma(a,b)$ is separated but fails to be interpolating for $\mathcal{B}_{v_p}$. In particular, these values can be chosen such that the constant of separation tends to 1 when $p \to \infty$.*

For some particular p, it is a straightforward calculation to determine the value of a and $S(a,b)$ such that $b = \frac{2p\pi}{\log a}$ which yields examples of more and more separated sequences which are not interpolating for $\mathcal{B}_{v_p}$. For instance:

p	a	S(a,b)
1	9.60773	0.811458
2	19.7151	0.903452
3	31.707	0.938851
4	45.3776	0.956876

Remark 2.3. There are Δ_p-separated sequences $(z_n) \subset \mathbf{D}$ which fail to be interpolating for $\mathcal{B}_{v_p}$ and $\Delta_p \to 1$ when $p \to \infty$. To show it, we will prove that $S(a,b) \to 1$ when $p \to \infty$. Indeed, from:

$$1 - \frac{2}{a+1} = \sqrt{1 - \frac{4}{b^2+4}}$$

and since $p \to \infty$ then $b \to \infty$ so the right term tends to 1. In order to get this, we need that the left term also tends to 1 which is only possible if

$a \to \infty$ (since $b = 2p\pi/\log a$, we can take a bigger than p to get it and b also tends to infinity). Hence, $S(a, b) \to 1$ and we are done.

2.2. Upper Estimate

Given $a \in \mathbf{D}$ and $0 < r < 1$, the pseudohyperbolic open disks centered at a and radius r are denoted by $D_\rho(a, r) = \{z \in \mathbf{D} : \rho(z, a) < r\}$. A standard calculation (see for instance [14]) shows that $D_\rho(a, r)$ is the euclidean disk with center $\frac{1-r^2}{1-r^2|a|^2}a$ and radius $\frac{1-|a|^2}{1-|a|^2r^2}r$. The following lemma is just an easy calculation:

Lemma 2.4. *Let $z \in \mathbf{D}$ and $0 < r < 1$. If $w \in D_\rho(z, r)$ then:*

$$\frac{1-r^2}{4}(1-|z|^2) < (1-|w|^2) < \frac{4}{1-r^2}(1-|z|^2).$$

Proof. An easy calculation shows that $w \in D_\rho(z, r)$ if and only if $\rho(z, w)^2 < r^2$ if and only if $1 - r^2 < 1 - \rho(z, w)^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{w}z|^2}$ which is equivalent to:

$$\frac{1-r^2}{(1-|z|^2)(1-|w|^2)} < \frac{1}{|1-\overline{w}z|^2} \text{ if and only if}$$
$$\frac{(1-r^2)^2|1-\overline{w}z|^2}{(1-|z|^2)^2(1-|w|^2)^2} < \frac{1}{|1-\overline{w}z|^2}$$

and since $|1 - \overline{w}z| \geq (1 - |z|)$, it follows that:

$$\frac{4}{(1-|z|^2)^2} > \frac{1}{(1-|z|)^2} \geq \frac{1}{|1-\overline{w}z|^2} > \frac{(1-r^2)^2|1-\overline{w}z|^2}{(1-|z|^2)^2(1-|w|^2)^2}$$
$$> \frac{(1-r^2)^2(1-|z|)^2}{(1-|z|^2)^2(1-|w|^2)^2} = \frac{(1-r^2)^2}{(1+|z|)^2(1-|w|^2)^2} > \frac{(1-r^2)^2}{4(1-|w|^2)^2}$$

and we are done. Changing z with w and repeating the proof we obtain the second inequality. □

Lemma 2.5. *Let $a \neq b \in \mathbf{D}$, $\rho(a, b) \geq \delta$ and consider:*

$$r_\delta = \frac{\delta}{1+\sqrt{1-\delta^2}}.$$

Then, $\overline{D_\rho(a, r_\delta)} \cap \overline{D_\rho(b, r_\delta)} = \emptyset$. In addition, if $\rho(a, b) = \delta$ then the disks $\overline{D_\rho(a, r_\delta)}$ and $\overline{D_\rho(b, r_\delta)}$ are externally tangent in the complex plane $\mathbb{C}$ so the radius r_δ is optimal to separate both disks.

Proof. It is an easy calculation that r_δ is the solution of the equation:

$$\frac{2r_\delta}{1+r_\delta^2} = \delta.$$

If $w \in D_\rho(a, r_\delta) \cap D_\rho(b, r_\delta)$ then:

$$\rho(a, b) \leq \frac{\rho(a, w) + \rho(b, w)}{1 + \rho(a, w)\rho(b, w)} < \frac{2r_\delta}{1+r_\delta^2} = \delta$$

where last inequality is clear since given any $0 \le a < 1$ the real function $(x+a)/(1+ax)$ is non-decreasing for $0 \le x < 1$. This is a contradiction, so the disks are disjoint.

Now notice that $z \in \overline{D_\rho(a, r_\delta)} \cap \overline{D_\rho(b, r_\delta)}$ if and only if $\rho(a,z) \le r_\delta$ and $\rho(b,z) \le r_\delta$. Since ρ is invariant by automorphisms and $\varphi_a(a) = 0$, this is equivalent to: $\rho(0, \varphi_a(z)) \le r_\delta$ and $\rho(\varphi_a(b), \varphi_a(z)) \le r_\delta$ if and only if $\varphi_a(z) \in \overline{D_\rho(0, r_\delta)} \cap \overline{D_\rho(\varphi_a(b), r_\delta)}$. Hence z belongs or not to $\overline{D_\rho(a, r_\delta)}$ and $\overline{D_\rho(b, r_\delta)}$ if and only if $\varphi_a(z)$ belongs or not to $\overline{D_\rho(0, r_\delta)}$ and $\overline{D_\rho(\varphi_a(b), r_\delta)}$ respectively. Notice that:

$$\overline{D_\rho(0, r_\delta)} = \overline{D_{|\cdot|}(0, r_\delta)} \quad \text{and}$$
$$\overline{D_\rho(\varphi_a(b), r_\delta)} = \overline{D_{|\cdot|}\left(\frac{1-r_\delta^2}{1-r_\delta^2|\varphi_a(b)|^2}\varphi_a(b), \frac{1-|\varphi_a(b)|^2}{1-r_\delta^2|\varphi_a(b)|^2} r_\delta\right)}$$

where $D_{|\cdot|}$ denotes the corresponding euclidean disks, so we will prove that these disks are externally tangent. For this, it is sufficient to prove that that the euclidean distance between both centers equals to the sum of both radius, that is:

$$\frac{1-r_\delta^2}{1-r_\delta^2|\varphi_a(b)|^2}|\varphi_a(b)| = r_\delta + \frac{1-|\varphi_a(b)|^2}{1-r_\delta^2|\varphi_a(b)|^2} r_\delta \text{ if and only if}$$
$$\frac{1-r_\delta^2}{1-r_\delta^2|\varphi_a(b)|^2}|\varphi_a(b)| = r_\delta\left(1 + \frac{1-|\varphi_a(b)|^2}{1-r_\delta^2|\varphi_a(b)|^2}\right)$$
$$= r_\delta \frac{2 - r_\delta^2|\varphi_a(b)|^2 - |\varphi_a(b)|^2}{1-r_\delta^2|\varphi_a(b)|^2}$$

and bearing in mind that $|\varphi_a(b)| = \rho(a,b) = \delta$, this is equivalent to:

$$(1-r^2)\delta = r_\delta(2 - r_\delta^2|\varphi_a(b)|^2 - |\varphi_a(b)|^2) = r_\delta(2 - (r_\delta^2+1)\delta^2)$$

and dividing by $1 + r_\delta^2$ we have:

$$\frac{1-r_\delta^2}{1+r_\delta^2}\delta = \frac{2r_\delta}{1+r_\delta^2} - r_\delta\delta^2$$

and since $\frac{1-r_\delta^2}{1+r_\delta^2} = \sqrt{1-\delta^2}$ because $\left(\frac{2r_\delta}{1+r_\delta^2}\right)^2 + \left(\frac{1-r_\delta^2}{1+r_\delta^2}\right)^2 = 1$, the equality is equivalent to:

$$\sqrt{1-\delta^2}\delta = \delta - r_\delta\delta^2$$

and dividing by δ and solving for r_δ we obtain:

$$r_\delta = \frac{1-\sqrt{1-\delta^2}}{\delta} = \frac{\delta}{1+\sqrt{1-\delta^2}}$$

and we are done. □

The following result is an easy consequence of the mean value for complex analytic functions f on **D**:

Proposition 2.6. *If $f : \mathbf{D} \to \mathbb{C}$ is analytic then for any $a \in \mathbf{D}$ and $0 < r < 1$ we have:*

$$|f(a)|(1-|a|^2)^2 \leq \frac{1}{\pi r^2}\int_{D_\rho(a,r)} |f(z)|\mathrm{d}z.$$

Proof. By Cauchy's integral formula, we have for any $0 < s < 1$ that

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(se^{i\sigma})\mathrm{d}\sigma \Rightarrow |f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(se^{i\sigma})|\mathrm{d}\sigma$$

and using polar coordinates:

$$\frac{1}{\pi r^2}\int_{D(0,r)} |f(z)|\mathrm{d}z = \frac{1}{\pi r^2}\int_0^r\int_0^{2\pi} s|f(se^{i\sigma})|\mathrm{d}\sigma\mathrm{d}s \geq \frac{2}{r^2}|f(0)|\int_0^r s\mathrm{d}s = |f(0)|.$$

Applying this inequality to the function $h := (f \circ \varphi_a)(\varphi_a')^2$, we obtain as a consequence of the change of variable formula and the Cauchy-Riemman equations that

$$\begin{aligned}|f(a)|(1-|a|^2)^2 = |h(0)| &\leq \frac{1}{\pi r^2}\int_{D(0,r)} |h(z)|\mathrm{d}z\\ &= \frac{1}{\pi r^2}\int_{D(0,r)} |f(\varphi_a(z))||\varphi_a'(z)|^2\mathrm{d}z =\\ &= \frac{1}{\pi r^2}\int_{\varphi_a(D(0,r))} |f(z)|\mathrm{d}z = \frac{1}{\pi r^2}\int_{D_\rho(a,r)} |f(z)|\mathrm{d}z\end{aligned}$$

and we are done.

The following results are well-known (see Theorem 1.7 in [8] or Lemma 3.10 in [14]). The measure A will denote the classical Borel measure defined on $\mathbb{C}$. □

Proposition 2.7. *We have the following results:*

a) *If $u > -1$ and $v \in \mathbb{R}$ such that $v/2$ is neither 0 nor a negative integer then:*

$$\int_{\mathbf{D}} \frac{(1-|w|^2)^u}{|1-\bar{z}w|^v}\mathrm{d}A(w) = \frac{\pi\Gamma(u+1)}{\Gamma(\frac{v}{2})^2}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{v}{2})^2}{m!\Gamma(m+u+2)}|z|^{2m}.$$

b) *For any $0 \leq x < 1$ and $s \in \mathbb{R}$, s neither 0 nor a negative integer, we have:*

$$\frac{1}{(1-x)^s} = \sum_{m=0}^{\infty}\frac{\Gamma(m+s)}{m!\Gamma(s)}x^m$$

Proposition 2.8. *Let $p, q, r \in \mathbb{R}$ such that $p > 0$, $p \geq r-2 \geq \frac{q-1}{2} > -1$ and $r-2 > 0$. If $(z_n) \subset \mathbf{D}$ is a separated sequence then:*

$$E(z) = \sup_{z\in\mathbf{D}}\left\{\sum_{n=1}^{\infty}\frac{(1-|z_n|^2)^r(1-|z|^2)^p}{|1-\overline{z_n}z|^{q+1}}\right\}$$

is bounded on $\mathbf{D}$.

Proof. Suppose that $\rho(z_n, z_k) \geq \delta > 0$ for $n \neq k$ and consider the radius r_δ given in Lemma 2.5. Fix $z \in \mathbf{D}$ and define the function $f_z(w) := \frac{1}{(1-\overline{z}w)^{q+1}}$, which is holomorphic on $\mathbf{D}$. Applying Proposition 2.6 and evaluating at z_n we have:

$$\frac{(1-|z_n|^2)^2}{|1-\overline{z}z_n|^{q+1}} = |f_z(z_n)|(1-|z_n|^2)^2 \leq \frac{1}{\pi r_\delta^2}\int_{D_\rho(z_n,r_\delta)} |f_z(w)|\ \mathrm{d}w.$$

By hypothesis $D_\rho(z_n, r_\delta) \cap D_\rho(z_k, r_\delta) = \emptyset$ for $n \neq k$, so:

$$\begin{aligned}
E(z) := \sum_{n=1}^{\infty} \frac{(1-|z_n|^2)^r(1-|z|^2)^p}{|1-\overline{z}z_n|^{q+1}} &\leq (1-|z|^2)^p \sum_{n=1}^{\infty}\left(\frac{1}{\pi r_\delta^2}(1-|z_n|^2)^{r-2}\right.\\
&\quad \left.\int_{D_\rho(z_n,r_\delta)} |f_z(w)|\mathrm{d}w\right)\\
&\leq (1-|z|^2)^p \sum_{n=1}^{\infty}\left(\frac{4^{r-2}}{(1-r_\delta^2)^{r-2}\pi r_\delta^2}\int_{D_\rho(z_n,r_\delta)}(1-|w|^2)^{r-2}|f_z(w)|\mathrm{d}w\right)\\
&= \frac{4^{r-2}(1-|z|^2)^p}{(1-r_\delta^2)^{r-2}\pi r_\delta^2}\int_{\bigcup_{n=1}^{\infty} D_\rho(z_n,r_\delta)} \frac{(1-|w|^2)^{r-2}}{|1-\overline{z}w|^{q+1}}\mathrm{d}w\\
&\leq \frac{4^{r-2}(1-|z|^2)^p}{(1-r_\delta^2)^{r-2}\pi r_\delta^2}\int_{\mathbf{D}} \frac{(1-|w|^2)^{r-2}}{|1-\overline{z}w|^{q+1}}\mathrm{d}w
\end{aligned}$$

being the second inequality a consequence of Lemma 2.4. By Proposition 2.7 a), we have:

$$E(z) \leq \frac{4^{r-2}(1-|z|^2)^p}{(1-r_\delta^2)^{r-2}\pi r_\delta^2}\frac{\pi\Gamma(r-1)}{\Gamma(\frac{q+1}{2})^2}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{q+1}{2})^2}{m!\Gamma(m+r)}|z|^{2m}.$$

Therefore, for any $z \in \mathbf{D}$, we have that:

$$\begin{aligned}
E(z) \leq &\frac{4^{r-2}(1-|z|^2)^p}{(1-r_\delta^2)^{r-2}r_\delta^2}\frac{(r-2)\Gamma(r-2)^2}{\Gamma(\frac{q+1}{2})^2\Gamma(r-2)}\\
&\times\left(\sum_{m=0}^{2}\frac{\Gamma(m+\frac{q+1}{2})^2}{m!\Gamma(m+r)}|z|^{2m}\right.\\
&\quad\left.+\sum_{m=3}^{\infty}\frac{(m+\frac{q-1}{2})^2}{(m+r-1)(m+r-2)}\frac{\Gamma(m+\frac{q-1}{2})^2}{m!\Gamma(m+r-2)}|z|^{2m}\right).
\end{aligned}$$

- **Case** $r = 2$. We obtain:

$$\begin{aligned}
E(z) \leq \frac{(1-|z|^2)^p}{\pi r_\delta^2}\frac{1}{\Gamma(\frac{q+1}{2})^2}&\left(\sum_{m=0}^{2}\frac{\Gamma(m+\frac{q+1}{2})^2}{m!\Gamma(m+2)}|z|^{2m}\right.\\
&\left.+\sum_{m=3}^{\infty}\frac{(m+\frac{q-1}{2})^2}{(m+1)m}\frac{\Gamma(m+\frac{q-1}{2})^2}{m!\Gamma(m)}|z|^{2m}\right).
\end{aligned}$$

Since $0 \geq \frac{q-1}{2}$ then:

$$\frac{(m+\frac{q-1}{2})^2}{(m+1)m} \leq 1$$

and since the function $\Gamma(x)$ is non-decreasing for $x > 1.462$, we have for any $m \geq 3$:

$$\frac{\Gamma(m+\frac{q-1}{2})^2}{m!\Gamma(m)} \leq \frac{\Gamma(m)^2}{m!\Gamma(m)} = \frac{\Gamma(m)}{m!} = \frac{1}{m}$$

and bearing in mind that $\sum_{m=1}^{\infty} \frac{1}{m}|z|^{2m} = -\log(1-|z|^2)$, we obtain:

$$E(z) \leq \frac{(1-|z|^2)^p}{\pi r_\delta^2} \frac{1}{\Gamma(\frac{q+1}{2})^2}(-\log(1-|z|^2) + P_1(z))$$

where $P_1(z)$ is a complex polynomial on $|z|$. Since $(1-|z|^2)\log(1-|z|^2)$ is bounded on $\mathbf{D}$, we are done.

- **Case** $r \neq 2$. If $r-2 \geq \frac{q-1}{2}$ then:

$$\frac{(m+\frac{q-1}{2})^2}{(m+r-1)(m+r-2)} \leq 1$$

and since the function $\Gamma(x)$ is non-decreasing for $x > 1.462$, we have for any $m \geq 3$:

$$\frac{\Gamma(m+\frac{q-1}{2})^2}{\Gamma(m+r-2)} \leq \frac{\Gamma(m+r-2)^2}{\Gamma(m+r-2)} = \Gamma(m+r-2).$$

So:

$$E(z) \leq \frac{4^{r-2}(1-|z|^2)^p}{(1-r_\delta^2)^{r-2} r_\delta^2} \frac{(r-2)\Gamma(r-2)^2}{\Gamma(\frac{q+1}{2})^2} G(z)$$

where:

$$\begin{aligned} G(z) &= \sum_{m=0}^{\infty} \frac{\Gamma(m+r-2)}{m!\Gamma(r-2)}|z|^{2m} - \frac{\Gamma(r-2)}{0!\Gamma(r-2)}|z|^0 - \frac{\Gamma(1+r-2)}{1!\Gamma(r-2)}|z|^2 \\ &\quad - \frac{\Gamma(2+r-2)}{2!\Gamma(r-2)}|z|^4 + \frac{\Gamma(\frac{q+1}{2})^2}{0!\Gamma(r)\Gamma(r-2)} + \frac{\Gamma(\frac{q+3}{2})^2}{1!\Gamma(r+1)\Gamma(r-2)}|z|^2 \\ &\quad + \frac{\Gamma(\frac{q+5}{2})^2}{2!\Gamma(r+2)\Gamma(r-2)}|z|^4 \end{aligned}$$

and since $r-2 > 0$, by Proposition 2.7 b), we have:

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+r-2)}{m!\Gamma(r-2)}|z|^{2m} = \frac{1}{(1-|z|^2)^{r-2}}.$$

Considering $p \geq r-2$, we have:

$$\frac{(1-|z|^2)^p}{(1-|z|^2)^{r-2}} \leq 1$$

so:

$$E(z) \leq \frac{4^{r-2}}{(1-r_\delta^2)^{r-2} r_\delta^2} \frac{(r-2)\Gamma(r-2)^2}{\Gamma(\frac{q+1}{2})^2} \left(1 + (1-|z|^2)^p P_2(|z|)\right)$$

where $P_2(|z|)$ is a polynomial depending on $|z|$ so $E(z)$ is uniformly bounded for $z \in \mathbf{D}$. □

Hence, we obtain a well-known result about separated sequences:

Corollary 2.9. *If* $(z_n) \subset \mathbf{D}$ *is a separated sequence then for every* $r > 1$ *we have:*

$$\sum_{n=1}^{\infty} (1 - |z_n|)^r < \infty.$$

Proof. Take $r > 1$, $p > 0$ such that $p \geq r - 2 \geq \frac{q-1}{2} > -1$. Apply Proposition 2.8 evaluated at $z = 0$ and we are done. □

Our main result refines the Madigan and Matheson result to get an upper estimate for Δ_1 (see [9]). In addition, we generalize it for Bloch type spaces $\mathcal{B}_{v_p}$ and provide a procedure to estimate Δ_p for any $1 \leq p < +\infty$.

Theorem 2.1. *Let* $p \geq 1$. *There exists a constant* $0 < \Delta_p < 1$ *such that if* $(z_n) \subset \mathbf{D}$ *is* Δ_p*-separated then it is interpolating for the Bloch space* $\mathcal{B}_{v_p}$.

Proof. Let $p \geq 1$ and consider r, q such that $p \geq r - 2 \geq \frac{q-1}{2} > -1$ and $p + r = q + 1$. For any $\lambda = (\lambda_n) \in \ell_\infty$, define:

$$T_\lambda(z) := \sum_{k=1}^{\infty} \lambda_k \frac{1}{q\overline{z_k}} \frac{(1 - |z_k|^2)^r}{(1 - \overline{z_k} z)^q}.$$

If $z_{k_0} = 0$ for any k_0 (there could only be one term because of the separability) then substitute the corresponding k_0-term of the previous series by $\lambda_{k_0} z$. Notice that T_λ is well-defined since the series is absolutely and uniformly convergent on compact sets of $\mathbf{D}$. To show this, set $\eta := \inf_{k \in \mathbb{N}}\{|z_n| \mid z_k \neq 0\}$. Since $(z_k) \subset \mathbf{D}$ is separated it does not have accumulation points in $\mathbf{D}$ and thus $\eta > 0$. Now for any $|z| \leq s < 1$, we have:

$$|T_\lambda(z)| \leq \sum_{k=1}^{\infty} |\lambda_k| \frac{1}{q|z_k|} \frac{(1 - |z_k|^2)^r}{(1 - |z_k|s)^q} \leq \frac{||\lambda||_\infty}{q\eta} \frac{1}{(1 - s)^q} \sum_{k=1}^{\infty} (1 - |z_k|^2)^r < \infty$$

because of Corollary 2.9. Adding $\lambda_k z$ for a possible $z_k = 0$ does not affect to the convergence.

Thus, T_λ is holomorphic for any $\lambda \in \ell_\infty$ and $T'_\lambda(z) = \sum_{k=1}^{\infty} \lambda_k \frac{(1-|z_k|^2)^r}{(1-\overline{z_k}z)^{q+1}}$, which implies together with Proposition 2.8 that:

$$\sup_{z \in \mathbf{D}} \{(1 - |z|^2)^p |T'_\lambda(z)|\} \leq ||\lambda||_\infty \sup_{z \in \mathbf{D}} \left\{ \sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^r (1 - |z|^2)^p}{|1 - \overline{z_k} z|^{q+1}} \right\} < \infty,$$

so $T_\lambda \in \mathcal{B}_{v_p}$. Consider $T : \ell_\infty \to \mathcal{B}_{v_p}$ given by $T(\lambda) = T_\lambda(z)$. By the comments above, it is clear that T is a well-defined linear bounded operator. Let $L : \mathcal{B}_{v_p} \to \ell_\infty$ given by $L(f) = (f'(z_n)(1 - |z_n|^2)^p)$. We will prove that if Δ_p is close enough to 1, then the operator $L \circ T : \ell_\infty \to \ell_\infty$ is invertible, so in particular L is surjective. We have that:

$$\begin{aligned} L \circ T(\lambda) &= \left((1 - |z_n|^2)^p T'_\lambda(z_n)\right)_n \\ &= \left(\lambda_n (1 - |z_n|^2)^{r+p-q-1} + \sum_{k=1, k \neq n}^{\infty} \lambda_k \frac{(1 - |z_k|^2)^r (1 - |z_n|^2)^p}{(1 - \overline{z_k} z_n)^{q+1}} \right)_n \end{aligned}$$

and since $r+p=q+1$ therefore:

$$(L\circ T-Id_{\ell_\infty})(\lambda)=\left(\sum_{k=1,k\neq n}^{\infty}\lambda_k\frac{(1-|z_k|^2)^r(1-|z_n|^2)^p}{(1-\overline{z_k}z_n)^{q+1}}\right)_n=:(F_n(\lambda)).$$

Let $||\lambda||_\infty\leq 1$ and consider $\alpha\geq 0$ such that $\alpha<p$, so $p-\alpha\geq r-2-\alpha$ and $r-2-\alpha\geq\frac{q-1}{2}-\alpha$ and $r-2-\alpha$, $\frac{q+1}{2}-\alpha>1-\alpha>0$. Since $\rho(z_n,z_k)\geq\Delta_p$ for $n\neq k$ we have for any $0\leq\alpha<p$:

$$\frac{(1-|z_k|^2)^\alpha(1-|z_n|^2)^\alpha}{|1-\overline{z_k}z_n|^{2\alpha}}=(1-\rho^2(z_n,z_k))^\alpha\leq(1-\Delta_p^2)^\alpha$$

and thus:

$$\begin{aligned}|F_n(\lambda)|&=\left|\sum_{k=1,k\neq n}^{\infty}\lambda_k\frac{(1-|z_k|^2)^r(1-|z_n|^2)^p}{(1-\overline{z_k}z_n)^{q+1}}\right|\\&\leq\sum_{k=1,k\neq n}^{\infty}\frac{(1-|z_k|^2)^r(1-|z_n|^2)^p}{|1-\overline{z_k}z_n|^{q+1}}\\&\leq(1-\Delta_p^2)^\alpha\sum_{k=1,k\neq n}^{\infty}\frac{(1-|z_k|^2)^{r-\alpha}(1-|z_n|^2)^{p-\alpha}}{|1-\overline{z_k}z_n|^{q+1-2\alpha}}.\end{aligned}$$

If we apply Proposition 2.6 to the function $f_n(z):=\frac{1}{(1-\overline{z_n}z)^{q+1-2\alpha}}$ at the point $z=z_k$ for $k\neq n$ and taking $r_p=\frac{\Delta_p}{1+\sqrt{1-\Delta_p^2}}$ from Proposition 2.5 we obtain:

$$\begin{aligned}&\frac{(1-|z_k|^2)^{r-\alpha}(1-|z_n|^2)^{p-\alpha}}{|1-\overline{z_k}z_n|^{q+1-2\alpha}}\\&\quad=|f_n(z_k)|(1-|z_k|^2)^2(1-|z_k|^2)^{r-2-\alpha}(1-|z_n|^2)^{p-\alpha}\\&\quad\leq\frac{(1-|z_n|^2)^{p-\alpha}}{\pi r_p^2}\int_{D_\rho(z_k,r_p)}\frac{(1-|z_k|^2)^{r-2-\alpha}}{|1-\overline{z_n}w|^{q+1-2\alpha}}\mathrm{d}w\\&\quad\leq\frac{(1-|z_n|^2)^{p-\alpha}}{\pi r_p^2}\int_{D_\rho(z_k,r_p)}\frac{4^{r-2-\alpha}(1-|w|^2)^{r-2-\alpha}}{(1-r_p^2)^{r-2-\alpha}|1-\overline{z_n}w|^{q+1-2\alpha}}\mathrm{d}w\end{aligned}$$

the last inequality being a consequence of Lemma 2.4. Since the disks $D_\rho(r_k,r_p)$ are disjoint, we obtain:

$$\begin{aligned}S&:=\sum_{k=1,k\neq n}^{\infty}\frac{(1-|z_k|^2)^{r-\alpha}(1-|z_n|^2)^{p-\alpha}}{|1-\overline{z_k}z_n|^{q+1-2\alpha}}\\&\leq\frac{4^{r-2-\alpha}(1-|z_n|^2)^{p-\alpha}}{\pi r_p^2(1-r_p^2)^{r-2-\alpha}}\int_{\mathbf{D}}\frac{(1-|w|^2)^{r-2-\alpha}}{|1-\overline{z_n}w|^{q+1-2\alpha}}\mathrm{d}w\end{aligned}$$

where $r-2-\alpha$ and $q+1-2\alpha>0$. Bearing in mind Proposition 2.7 a) we obtain:

$$S\leq\frac{4^{r-2-\alpha}(1-|z_n|^2)^{p-\alpha}}{\pi r_p^2(1-r_p^2)^{r-2-\alpha}}\frac{\pi\Gamma(r-1-\alpha)}{\Gamma(\frac{q+1}{2}-\alpha)^2}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{q+1}{2}-\alpha)^2}{m!\Gamma(m+r-\alpha)}|z_n|^{2m}.$$

Therefore, for any $z \in \mathbf{D}$, we have that:

$$S \leq \frac{4^{r-2-\alpha}(1-|z_n|^2)^{p-\alpha}}{(1-r_p^2)^{r-2-\alpha}r_p^2}\frac{\Gamma(r-1-\alpha)}{\Gamma(\frac{q+1}{2}-\alpha)^2}\left(\sum_{m=0}^{2}\frac{\Gamma(m+\frac{q+1}{2}-\alpha)^2}{m!\Gamma(m+r-\alpha)}|z_n|^{2m}\right.$$
$$\left.+\sum_{m=3}^{\infty}\frac{(m+\frac{q-1}{2}-\alpha)^2}{(m+r-1-\alpha)(m+r-2-\alpha)}\frac{\Gamma(m+\frac{q-1}{2}-\alpha)^2}{m!\Gamma(m+r-2-\alpha)}|z_n|^{2m}\right).$$

If $r-2 \geq \frac{q-1}{2}$ and $m \geq 3$ then:

$$\frac{(m+\frac{q-1}{2}-\alpha)^2}{(m+r-1-\alpha)(m+r-2-\alpha)} \leq 1$$

and for any $m \geq 3$, we also have:

$$\frac{\Gamma(m+\frac{q-1}{2}-\alpha)^2}{\Gamma(m+r-2-\alpha)} \leq \frac{\Gamma(m+r-2-\alpha)^2}{\Gamma(m+r-2-\alpha)} = \Gamma(m+r-2-\alpha).$$

So:

$$\begin{aligned}S &\leq \frac{4^{r-2-\alpha}(1-|z_n|^2)^{p-\alpha}}{(1-r_p^2)^{r-2-\alpha}r_p^2}\frac{(r-2-\alpha)\Gamma(r-2-\alpha)}{\Gamma(\frac{q+1}{2}-\alpha)^2}\\ &\quad\times\left(\sum_{m=0}^{2}\frac{\Gamma(m+\frac{q+1}{2}-\alpha)^2}{m!\Gamma(m+r-\alpha)}|z_n|^{2m}+\sum_{m=3}^{\infty}\frac{\Gamma(m+r-2-\alpha)}{m!}|z_n|^{2m}\right)\\ &\leq \frac{4^{r-2-\alpha}(1-|z_n|^2)^{p-\alpha}}{(1-r_p^2)^{r-2-\alpha}r_p^2}\frac{(r-2-\alpha)\Gamma(r-2-\alpha)^2}{\Gamma(\frac{q+1}{2}-\alpha)^2}\\ &\quad\times\left(\sum_{m=0}^{2}\frac{\Gamma(m+\frac{q+1}{2}-\alpha)^2}{m!\Gamma(m+r-\alpha)\Gamma(r-2-\alpha)}|z_n|^{2m}\right.\\ &\quad\left.+\sum_{m=3}^{\infty}\frac{\Gamma(m+r-2-\alpha)}{m!\Gamma(r-2-\alpha)}|z_n|^{2m}\right).\end{aligned}$$

Calling:

$$M=\left(\sum_{m=0}^{2}\frac{\Gamma(m+\frac{q+1}{2}-\alpha)^2}{m!\Gamma(m+r-\alpha)\Gamma(r-2-\alpha)}|z_n|^{2m}+\sum_{m=3}^{\infty}\frac{\Gamma(m+r-2-\alpha)}{m!\Gamma(r-2-\alpha)}|z_n|^{2m}\right)$$

we have:

$$\begin{aligned}M=&\sum_{m=0}^{\infty}\frac{\Gamma(m+r-2-\alpha)}{m!\Gamma(r-2-\alpha)}|z_n|^{2m}-\frac{\Gamma(r-2-\alpha)}{0!\Gamma(r-2-\alpha)}|z_n|^{0}-\frac{\Gamma(r-1-\alpha)}{1!\Gamma(r-2-\alpha)}|z_n|^{2}\\ &-\frac{\Gamma(r-\alpha)}{2!\Gamma(r-2-\alpha)}|z_n|^4+\frac{\Gamma(\frac{q+1}{2}-\alpha)^2}{0!\Gamma(r-\alpha)\Gamma(r-2-\alpha)}\\ &+\frac{\Gamma(\frac{q+3}{2}-\alpha)^2}{1!\Gamma(r+1-\alpha)\Gamma(r-2-\alpha)}|z_n|^2+\frac{\Gamma(\frac{q+5}{2}-\alpha)^2}{2!\Gamma(r+2-\alpha)\Gamma(r-2-\alpha)}|z_n|^4\end{aligned}$$

and by Proposition 2.7 b) we have:

$$\sum_{m=0}^{\infty}\frac{\Gamma(m+r-2-\alpha)}{m!\Gamma(r-2-\alpha)}|z_n|^{2m}=\frac{1}{(1-|z|^2)^{r-2-\alpha}}$$

we have:

$$M=\frac{1}{(1-|z|^2)^{r-2-\alpha}}-\left(1-\frac{\Gamma(\frac{q+1}{2}-\alpha)^2}{0!\Gamma(r-\alpha)\Gamma(r-2-\alpha)}\right)$$
$$-\left(r-2-\alpha-\frac{\Gamma(\frac{q+3}{2}-\alpha)^2}{1!\Gamma(r+1-\alpha)\Gamma(r-2-\alpha)}\right)|z_n|^2$$
$$-\left(\frac{(r-1-\alpha)(r-2-\alpha)}{2}-\frac{\Gamma(\frac{q+5}{2}-\alpha)^2}{2!\Gamma(r+2-\alpha)\Gamma(r-2-\alpha)}\right)|z_n|^4$$

Since $p\geq r-2$ we obtain:

$$\frac{(1-|z_n|^2)^{p-\alpha}}{(1-|z_n|^2)^{r-2-\alpha}}\leq 1$$

so we conclude:

$$S\leq\frac{4^{r-2-\alpha}}{(1-r_p^2)^{r-2-\alpha}r_p^2}\frac{(r-2-\alpha)\Gamma(r-2-\alpha)^2}{\Gamma(\frac{q+1}{2}-\alpha)^2}\left(1+(1-|z_n|^2)^{p-\alpha}Q(|z_n|)\right)$$

where $Q(|z|)$ is the polynomial given by:

$$\left(-1+\frac{\Gamma(\frac{q+1}{2}-\alpha)^2}{0!\Gamma(r-\alpha)\Gamma(r-2-\alpha)}\right)$$
$$-\left(r-2-\alpha-\frac{\Gamma(\frac{q+3}{2}-\alpha)^2}{1!\Gamma(r+1-\alpha)\Gamma(r-2-\alpha)}\right)|z_n|^2$$
$$-\left(\frac{(r-1-\alpha)(r-2-\alpha)}{2}-\frac{\Gamma(\frac{q+5}{2}-\alpha)^2}{2!\Gamma(r+2-\alpha)\Gamma(r-2-\alpha)}\right)|z_n|^4$$

so the expression $1+(1-|z_n|^2)^{p-\alpha}Q(|z|)$ is uniformly bounded for $z\in\mathbf{D}$ by a constant C. Hence:

$$||Id_{\ell_\infty}-L\circ T||\leq\frac{4^{r-2-\alpha}(1-\Delta_p^2)^\alpha}{r_p^2(1-r_p^2)^{r-2-\alpha}}\frac{(r-2-\alpha)\Gamma(r-2-\alpha)^2}{\Gamma(\frac{q+1}{2}-\alpha)^2}C$$

so we will be done if we prove that:

$$\frac{4^{r-2-\alpha}(1-\Delta_p^2)^\alpha}{r_p^2(1-r_p^2)^{r-2-\alpha}}\frac{(r-2-\alpha)\Gamma(r-2-\alpha)^2}{\Gamma(\frac{q+1}{2}-\alpha)^2}\to 0$$

as $\Delta_p\to 1$ for $\alpha>\frac{r-2}{3}$. Indeed:

$$1-r_p^2=1-\frac{\Delta_p^2}{(1+\sqrt{1-\Delta_p^2})^2}=2\frac{\sqrt{1-\Delta_p^2}}{1+\sqrt{1-\Delta_p^2}}\tag{2.2}$$

and:

$$\frac{4^{r-2-\alpha}(1-\Delta_p^2)^\alpha}{r_p^2(1-r_p^2)^{r-2-\alpha}} = \frac{4^{r-2-\alpha}\left(\sqrt{1-\Delta_p^2}\right)^{2\alpha}}{r_p^2\left(\frac{2\sqrt{1-\Delta_p^2}}{1+\sqrt{1-\Delta_p^2}}\right)^{r-2-\alpha}}$$
$$= \frac{2^{r-2-\alpha}\left(\sqrt{1-\Delta_p^2}\right)^{3\alpha-r+2}\left(1+\sqrt{1-\Delta_p^2}\right)^{r-2-\alpha}}{r_p^2}$$

so the proof is completed since $r_p \to 1$ when $\Delta_p \to 1$ and $3\alpha - r + 2 > 0$. □

Remark 2.10. Theorem gives us a procedure to calculate an upper estimate for the constant of separation Δ_p for $\mathcal{B}_{v_p}$ if $p \geq 1$.

Lemma 2.11. *If $a, b, c \geq 0$ and $0 \leq \alpha < 1$, the function $f_\alpha(x) = (1 - x^2)^{1-\alpha}(-a - bx^2 - cx^4)$ satisfies:*

$$(1-x^2)^\alpha f_\alpha'(x) = 2(a(1-\alpha) - b)x + 2(b(2-\alpha) - 2c)x^3 + 2c(3-\alpha)x^5.$$

Proposition 2.12. *There exists $0.811 < \Delta_1 < 0.9785$ such that for any separated sequence $(z_n) \subset \mathbf{D}$ such that $\rho(z_k, z_j) \geq \Delta_1$, then (z_n) is interpolating for the classical Bloch space $\mathcal{B}$ and there are examples of non-interpolating sequences which are δ-separated for values $\delta < \Delta_1$.*

Proof. Inequality $0.81 < \Delta_1$ is straightforward from Proposition 2.1. For the other inequality, use proof of Theorem 2.1. Take $p = 1, r = 3$ and $q = 3$. Notice that $0 \leq |z_n| < 1$ and take $0 \leq \alpha < 1$. We have:

$$S \leq \frac{4^{1-\alpha}(1-\Delta_1^2)^\alpha}{(1-\alpha)r_1^2(1-r_1^2)^{1-\alpha}}\left(1 + (1-|z_n|^2)^{1-\alpha}Q(|z_n|)\right)$$

where the polynomial $Q(|z_n|)$ is given by:

$$-\left(1 - \frac{1-\alpha}{2-\alpha}\right) - \left(1 - \alpha - \frac{(2-\alpha)(1-\alpha)}{3-\alpha}\right)|z_n|^2$$
$$- \left(\frac{(2-\alpha)(1-\alpha)}{2} - \frac{(3-\alpha)(2-\alpha)(1-\alpha)}{2(4-\alpha)}\right)|z_n|^4$$
$$= -\frac{1}{2-\alpha} - \frac{1-\alpha}{3-\alpha}|z_n|^2 - \frac{(2-\alpha)(1-\alpha)}{2(4-\alpha)}|z_n|^4$$

so the expression $f_\alpha(|z_n|) := 1 + (1-|z_n|^2)^{1-\alpha}Q(|z_n|)$ becomes:

$$f_\alpha(|z_n|) = 1 + (1-|z_n|^2)^{1-\alpha}\left(-\frac{1}{1-\alpha} - \frac{1-\alpha}{3-\alpha}|z_n|^2 - \frac{(2-\alpha)(1-\alpha)}{2(4-\alpha)}|z_n|^4\right).$$

We will prove that for some $0 \leq \alpha < 1$, we have $f_\alpha'(|z_n|) \geq 0$ for $0 \leq |z_n| \leq 1$ so $f_\alpha(|z_n|)$ will be non-decreasing for $|z_n|$. Notice that this is equivalent to proving that $(1-|z_n|^2)^\alpha f_\alpha'(|z_n|) \geq 0$. Taking $\alpha = 4/5$ and bearing in mind Lemma 2.11 we have for any $0 \leq |z_n| \leq 1$:

$$(1-|z_n|^2)^{4/5} f'(|z_n|) = \frac{5}{33}|z_n| + \frac{3}{44}|z_n|^3 + \frac{33}{200}|z_n|^5 \geq 0.$$

Hence the expression S is a non-decreasing function of $|z_n|$ and we have:

$$\begin{aligned} S &\leq \frac{4^{1-\alpha}(1-\Delta_1^2)^\alpha}{(1-\alpha)r_1^2(1-r_1^2)^{1-\alpha}} \lim_{|z_n|\to 1^-} \left(1+(1-|z_n|^2)^{1-\alpha}Q(|z_n|)\right) \\ &= \frac{4^{1-\alpha}(1-\Delta_1^2)^\alpha}{(1-\alpha)r_1^2(1-r_1^2)^{1-\alpha}} \end{aligned}$$

so calling $t=\sqrt{1-\Delta_1^2}<1$ and bearing in mind (2.2) we have:

$$S \leq \frac{4^{1-\alpha}t^{2\alpha}}{(1-\alpha)\left(\frac{2t}{1+t}\right)^{1-\alpha}\frac{1-t}{1+t}} = \frac{2^{1-\alpha}t^{3\alpha-1}(1+t)^{2-\alpha}}{(1-\alpha)(1-t)}$$

and for $\alpha = 4/5$, we have:

$$\frac{2^{1/5}t^{7/5}(1+t)^{6/5}}{1/5(1-t)} < 1$$

if and only if $2^{1/5}t^{7/5}(1+t)^{6/5} - \frac{1}{5}(1-t) < 0$.
which clearly has real solutions on $0 < t < 1$ by Bolzano's Theorem. A root of $2^{1/5}t^{7/5}(1+t)^{6/5} - \frac{1}{5}(1-t) = 0$ is $t \approx 0.2069$ so $\Delta_1 = \sqrt{1-t^2} \approx 0.9784$ and we are done. □

Author contributions All the authors contributed equally to the manuscript text and reviewed it.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Data availability Not applicable

Declarations

Conflict of interest The authors declare no competing interests.

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Received: May 18, 2023.
Revised: September 5, 2023.
Accepted: September 6, 2023.