SESHADRI-TYPE CONSTANTS AND NEWTON-OKOUNKOV BODIES FOR NON-POSITIVE AT INFINITY VALUATIONS OF HIRZEBRUCH SURFACES

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ABSTRACT. We consider flags $E_\bullet = \{X \supset E \supset \{q\}\}$, where E is an exceptional divisor defining a non-positive at infinity divisorial valuation ν_E of a Hirzebruch surface $\mathbb{F}_\delta$, q a point in E and X the surface given by ν_E, and determine an analogue of the Seshadri constant for pairs (ν_E, D), D being a big divisor on $\mathbb{F}_\delta$. The main result is an explicit computation of the vertices of the Newton-Okounkov bodies of pairs $(E_\bullet, D)$ as above, showing that they are quadrilaterals or triangles and distinguishing one case from another.

1. INTRODUCTION

Let L be a big line bundle on a normal complex projective variety X. Consider a real valuation ν of X, that is a valuation of the function field of X centered at the local ring of a closed point in X. Assume $H^0(L) \neq 0$ and set $\hat{\mu}_L(\nu) = \lim_{m\to\infty} m^{-1} a_{\max}(mL, \nu)$, where $a_{\max}(mL, \nu)$ is the last value of the vanishing sequence of $H^0(mL)$ along ν [5]. The value $\hat{\mu}_L(\nu)$ contains, for valuations, similar information as the Seshadri constant for points; then we consider it as a Seshadri-type constant for the pair (L, ν). Seshadri constants were used in [11] for studying the Fujita conjecture and other Seshadri-type constants were introduced in [9] for ideal sheaves. The bound $\hat{\mu}_L(\nu) \geq \sqrt{L^2/\mathrm{vol}(\nu)}$, where $\mathrm{vol}(\nu)$ means volume of the valuation ν, is proved in [5] but the exact value of $\hat{\mu}_L(\nu)$ is, in general, very hard to compute.

A flag of subvarieties of a smooth irreducible complex projective variety X (of dimension n) is a sequence of smooth irreducible subvarieties Y_j, $0 \leq j \leq n$,

$$Y_\bullet := \{X = Y_0 \supset Y_1 \supset \cdots \supset Y_n\},$$

where each Y_j has codimension j in X. $Y_\bullet$ defines a rank n valuation $\nu_{Y_\bullet}$ of the function field $K(X)$ and the Newton-Okounkov body $\Delta_{\nu_{Y_\bullet}}(D)$ of a big divisor D on X with respect to $\nu_{Y_\bullet}$ (or $Y_\bullet$) is the closed convex hull of the set

$$\bigcup_{m\geq 1} \left\{ \frac{\nu_{Y_\bullet}(f)}{m} \mid f \in H^0(X, \mathcal{O}_X(mD)) \setminus \{0\} \right\}.$$

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Newton-Okounkov bodies were introduced by Okounkov [26, 27, 28] and afterwards developed by Lazarsfeld and Mustaţă [24] and Kaveh and Khovanskii [20]. These bodies allow us to study linear systems defined by the involved divisor and valuation. As in the case of $\hat{\mu}_L(\nu)$, an explicit computation of these bodies is also very difficult.

Recently there have been some advances in the study of flags $E_\bullet = \{Z \supset E \supset \{q\}\}$, where Z is the rational surface given by a divisorial valuation ν_E of the complex projective plane $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{C}}$, E is the defining divisor of ν_E and q a (closed) point in E. The valuation ν_E is centered at $\mathcal{O}_{\mathbb{P}^2,p}, p \in \mathbb{P}^2$. In this case, the rank two valuation $\nu_{E_\bullet}$ is an exceptional curve valuation of $\mathbb{P}^2$, also centered at $\mathcal{O}_{\mathbb{P}^2,p}$. Exceptional curve valuations constitute one of the five classes in the Spivakovsky classification of valuations of function fields of surfaces [29] and its denomination comes from [14]. In [19] (see also [8]) the Newton-Okounkov body of a divisor associated to the pull-back of the line bundle $L = \mathcal{O}_{\mathbb{P}^2}(1)$ with respect to $\nu_{E_\bullet}$ has been described, being the Seshadri-type constant $\hat{\mu}_L(\nu_E)$ an important ingredient. This constant has been found useful to treat other important problems. Indeed, ν_E is called *minimal* when $\hat{\mu}_L(\nu_E) = \sqrt{1/\mathrm{vol}(\nu_E)}$, and there is a valuative conjecture, strongly involving the above concept, which implies the Nagata conjecture [18] (see also [12]). Reference [18] also contains results in the direction of the above valuative conjecture. Non-positive at infinity valuations of $\mathbb{P}^2$, ν_E, constitute an interesting class of divisorial valuations. Lately, valuations in this last class have been studied and used in several contexts [6, 15, 25]. Among their important properties, one can mention that they determine those surfaces given by divisorial valuations of $\mathbb{P}^2$ whose cone of curves is finitely generated and its extremal rays are as few as possible [16]; $\hat{\mu}_L(\nu_E)$ can be explicitly obtained [18]; and the vertices of the Newton-Okounkov body with respect to any valuation $\nu_{E_\bullet}$ as above, can also be explicitly computed [19].

In this paper, we leave $\mathbb{P}^2$ as a background surface and focus on the δth (complex) Hirzebruch surface $\mathbb{F}_\delta$, for $\delta \geq 0$. This is a novel setting in this context allowing us to obtain new results. Then, in analogy to the case of $\mathbb{P}^2$, we introduce the concept of non-positive at infinity divisorial valuation of $\mathbb{F}_\delta$ (centered at $\mathcal{O}_{\mathbb{F}_\delta,p}, p \in \mathbb{F}_\delta$). This concept depends on the value of δ, the position of the point p and certain linear systems (see Definitions 2.4 and 2.5). As for $\mathbb{P}^2$, these valuations determine those rational surfaces Z defined by divisorial valuations of Hirzebruch surfaces such that the number of generators of their cones of curves are reduced to the minimum possible [17]. Notice that although the valuations of $\mathbb{F}_\delta$ do no differ from those of $\mathbb{P}^2$ (when they are considered as local objects), the classes of non-positive at infinity valuations of $\mathbb{P}^2$ and $\mathbb{F}_\delta$ are different [17, Remark 3.10].

The goals of this paper are two fold:

(1) To introduce a *concept of minimality* for divisorial valuations of $\mathbb{F}_\delta$ (Definition 2.2) and to *compute the value* $\hat{\mu}_D(\nu)$ for non-positive at infinity divisorial valuations ν of $\mathbb{F}_\delta$ and big divisors D on $\mathbb{F}_\delta$ (Theorem 2.6). Notice that, in our context,

$$\hat{\mu}_D(\nu) = \sup\left\{t > 0 \mid D^* - tE_r \text{ is big on Z}\right\},$$

where D^* is the pull-back of D on Z and E the defining divisor of ν.

(2) To explicitly *determine the vertices* of the *Newton-Okounkov bodies* (of pull-backs) of big divisors D on $\mathbb{F}_\delta$ with respect to flags $E_\bullet = \{Z \supset E \supset \{q\}\}$, where Z is the surface defined by a non-positive at infinity divisorial valuation ν_E of $\mathbb{F}_\delta$, E the defining divisor of ν_E and q a (closed) point in E.

Our main results are Theorems 3.4, 3.10, 3.12, 3.13, 3.20 and 3.22. We prove that the vertices of the above mentioned Newton-Okounkov bodies depend only on the expression of D, the volume of ν and the values of the germs at p of the fibre and sections on $\mathbb{F}_\delta$ whose strict transforms on Z (together with those of the exceptional divisors) span the cone of curves. These values are with respect to the two divisorial valuations involved in the exceptional curve valuation $\nu_{E_\bullet}$ (see the paragraph before Definition 3.1).

This paper is structured as follows. Section 2 introduces the concepts considered in the paper: special and non-special, minimal and non-positive at infinity divisorial valuations. These concepts will be extended to exceptional curve valuations ν in Section 3. Moreover, Section 2 is devoted to determine Seshadri-type constants, while Section 3 computes Newton-Okounkov bodies. We show, in Theorem 3.4, that minimal with respect to a big divisor D exceptional curve valuations ν of $\mathbb{F}_\delta$ are those whose Newton-Okounkov body $\Delta_\nu(D)$ is a specific triangle T. T is the truncated convex cone of the (x, y)-plane generated by the value semigroup of ν and bounded by the line $x = \hat{\mu}_D(\nu_r)$, ν_r being the divisorial valuation defined by the first projection of ν. This fact also happens for valuations of $\mathbb{P}^2$. When ν_r is not minimal, in our case (ν_r is non-positive at infinity), $\Delta_\nu(D)$ is either a quadrilateral or a triangle. This last case only happens under certain conditions which depend on the divisor D and the valuation ν_r. Seshadri-type constants and Newton-Okounkov bodies with respect to non-positive at infinity valuations of $\mathbb{P}^2$ can be obtained as a particular case of the results in Sections 2 and 3. We conclude by saying that, in Subsection 3.3, we give two tables summarizing the different cases, considered in Subsections 3.1 and 3.2, corresponding to non-minimal valuations. Our tables provide the specific vertices of the Newton-Okounkov bodies in each case.

2. Seshadri-type constants for non-positive at infinity valuations of Hirzebruch surfaces

2.1. **Hirzebruch surfaces and valuations of Hirzebruch surfaces.** Let $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}}$ be the projective line over the complex field $\mathbb{C}$ and δ a non-negative integer. The δth *Hirzebruch surface* is the projective ruled surface over $\mathbb{P}^1$, $\mathbb{F}_\delta := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-\delta))$, together with the projection morphism $\mathrm{pr} : \mathbb{F}_\delta \to \mathbb{P}^1$. The Picard group $\mathrm{Pic}(\mathbb{F}_\delta)$ of $\mathbb{F}_\delta$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and admits as generators the class of a fiber F of pr and that of a section M of pr linearly equivalent to $\delta F + M_0$ satisfying that $M \cap M_0 = \emptyset$, where M_0 denotes, if $\delta > 0$ (respectively, $\delta = 0$) the unique section on $\mathbb{F}_\delta$ with negative self-intersection (respectively, a section); see for instance [4, Proposition IV.1]. It holds that $F^2 = 0, F \cdot M = 1$ and $M^2 = \delta$.

In the case $\delta > 0$, the section M_0 is called *special*, and a point p of $\mathbb{F}_\delta$ is *special* if $p \in M_0$ and *general* otherwise. A nef (respectively, big) divisor on $\mathbb{F}_\delta$ is linearly equivalent to $aF + bM$, where a and b are non-negative integers (respectively, a and b are integers such that $b > 0$ and $a > -\delta b$ (see [23, Remark 2.2.27]).

Let $(R, \mathfrak{m})$ be a two-dimensional local regular ring and K its quotient field. A valuation of K is a surjective map $\nu : K^*(= K \setminus \{0\}) \to G$, where G is a totally ordered commutative group, such that, for $f, g \in K^*$, it satisfies

$$\nu(f+g) \geq \min\{\nu(f), \nu(g)\} \text{ and } \nu(fg) = \nu(f) + \nu(g).$$

The local ring $R_\nu = \{f \in K \mid \nu(f) \geq 0\} \cup \{0\}$, whose maximal ideal is $\mathfrak{m}_\nu = \{f \in K \mid \nu(f) > 0\} \cup \{0\}$, is called the *valuation ring* of ν. When $R \cap \mathfrak{m}_\nu = \mathfrak{m}$, one says that ν is *centered* at R.

Valuations of K centered at R correspond one-to-one to simple sequences of point blowups

$$\pi : \cdots \to Z_n \xrightarrow{\pi_n} Z_{n-1} \to \cdots \to Z_1 \xrightarrow{\pi_1} Z_0 = \operatorname{Spec} R, \tag{2.1}$$

where the first blowup π_1 is centered at the point $p := p_1$ corresponding to the maximal ideal $\mathfrak{m}$ and the blowup π_{i+1} is centered at the unique closed point p_{i+1} which belongs to the exceptional divisor created by π_i and such that the valuation is centered at $\mathcal{O}_{Z_i,p_{i+1}}$. The set $\mathcal{C}_\nu = \{p = p_1, p_2, \ldots\}$ is called the *configuration of infinitely near points of* ν. Denote by E_i the exceptional divisor on Z_i obtained by blowing up p_i. A point p_i is *proximate* to p_j, denoted by $p_i \to p_j$, when p_i belongs to the strict transform of E_j on Z_{i-1}. The point p_i is called *satellite* when it is proximate to p_j, for some $j < i-1$; otherwise, it is named *free*. Given a divisor D on Z_i, abusing the notation, we will denote by $\tilde{D}$ and D^* the strict and total transforms of D on any surface Z_j for $j \geq i$; also the strict transforms of the exceptional divisors E_i will be written simply E_i.

The previous valuations were studied by Zariski and Abhyankar (see [1, 2, 30, 31]). Spivakovsky, in [29], classified them in five types according to their dual graphs. These dual graphs are trees whose vertices correspond 1-1 to the exceptional divisors associated with the sequence (2.1) and two vertices are joined by an edge if the corresponding exceptional divisors intersect. Each vertex of the dual graph is labelled with a positive integer i which represents E_i. We say that two vertices i and j satisfy $i \preccurlyeq j$ if the path in the dual graph joining 1 and j goes through i.

We are only interested in divisorial and exceptional curve valuations, which are two of the types in Spivakovsky's classification. A valuation is *divisorial* when $\mathcal{C}_\nu$ is finite and it is *exceptional curve* (in the terminology of [14]) if $\mathcal{C}_\nu$ is infinite and there exists a point $p_r \in \mathcal{C}_\nu$ such that $p_i \to p_r$ for all $i > r$. The group G is isomorphic to $\mathbb{Z}$ with the usual ordering (respectively, $\mathbb{Z}^2$ with lexicographical ordering) when the valuation is divisorial (respectively, exceptional curve).

Let ν be a divisorial or exceptional curve valuation of K centered at R and $\mathcal{C}_\nu = \{p_i\}_{i \geq 1}$ its configuration of infinitely near points. For each $i \geq 1$, denote by $\mathfrak{m}_i$ the maximal ideal of the local ring $R_i = \mathcal{O}_{Z_i,p_i}$ and set $\nu(\mathfrak{m}_i) := \min\{\nu(x) \mid x \in \mathfrak{m}_i \setminus \{0\}\}$. These values satisfy the *proximity equalities* [7, Theorem 8.1.7]: $\nu(\mathfrak{m}_i) = \sum_{p_j \to p_i} \nu(\mathfrak{m}_j)$, $\; i \geq 1$, whenever the set $\{p_j \in \mathcal{C}_\nu \mid p_j \to p_i\}$ is not empty. When ν is exceptional curve and $p_i \to p_r$ for every $i > r$, then $\nu(\mathfrak{m}_r) = (a, b)$ and $\nu(\mathfrak{m}_i) = (0, c)$, for some $a, b, c \in \mathbb{Z}$, $a, c > 0$ [10].

Divisorial and exceptional curve valuations admit sets of invariants that help to study them, as the sequence of maximal contact values $\{\overline{\beta}_j(\nu)\}_{j=0}^{g+1}$ [10, (1.5.3)] and the sequence of Puiseux exponents $\{\beta'_j(\nu)\}_{j=0}^{g+1}$. Notice that both sequences can be obtained one from other [10, Theorem 1.11]. The set $\{\overline{\beta}_j(\nu)\}_{j=0}^{g+1}$ generates the semigroup of values of ν, $S_\nu := \nu(R \setminus \{0\}) \cup \{0\}$. It is a minimal generating set when ν is an exceptional curve valuation, otherwise it suffices $\{\overline{\beta}_j(\nu)\}_{j=0}^{g}$ to generate S_ν, but $\overline{\beta}_{g+1}(\nu) \in S_\nu$. The continued fraction expansions of the values $\{\beta'_j(\nu)\}_{j=0}^{g+1}$ determine (and are determined by) the dual graph of ν.

We are interested in geometric results concerning Hirzebruch surfaces and for this reason, from now on, R will be the local regular ring $\mathcal{O}_{\mathbb{F}_\delta,p}$, where $\mathbb{F}_\delta$ is a Hirzebruch surface over complex field $\mathbb{C}$ and p a closed point of $\mathbb{F}_\delta$. Along the paper, we denote by φ_C the germ at p of a curve C on $\mathbb{F}_\delta$ and by φ_i an analytically irreducible germ at p of a curve whose strict transform on Z_i is transversal to E_i at a non-singular point of the exceptional locus. In this case, *valuations of K centered at R will be called valuations of $\mathbb{F}_\delta$.*

2.2. **Seshadri-type constants for non-positive at infinity valuations of Hirzebruch surfaces.** In [5], the authors consider a vanishing sequence attached to a pair (L, ν), where L is a big line bundle on a normal projective variety X and ν is a real valuation of X, that is, a real valuation of $K(X)$ centered at the local ring of a closed point of X. The value $\lim_{m\to\infty} m^{-1} a_{\max}(mL, \nu)$, $a_{\max}(mL, \nu)$ being the last value of the above mentioned vanishing sequence, is denoted by $\hat{\mu}_L(\nu)$. When $X = \mathbb{P}^2$, this value encodes for valuations similar information as Seshadri constant for points and we say that $\hat{\mu}_L(\nu)$ is the Seshadri-type constant for the pair (L, ν). The explicit computation of these constants is a hard work. We devote this subsection to give some details on them when X is a Hirzebruch surface $\mathbb{F}_\delta$ and ν a divisorial valuation, and to provide its exact value for a large family of divisorial valuations and any big divisor on $\mathbb{F}_\delta$.

Let $\mathbb{F}_\delta$ be a Hirzebruch surface and p a closed point of $\mathbb{F}_\delta$. Let ν_n be a divisorial valuation of $\mathbb{F}_\delta$ defined by a sequence as (2.1) which finishes at Z_n. That is, ν_n is the valuation of the quotient field of $R := \mathcal{O}_{\mathbb{F}_\delta,p}$ centered at R defined by the exceptional divisor E_n. Consider the surface $Z = Z_n$ defined by (2.1) when $Z_0 = \mathbb{F}_\delta$. According to [13], the volume of ν_n can be defined as

$$\mathrm{vol}(\nu_n) = \lim_{\alpha\to\infty} \frac{\dim_{\mathbb{C}}(R/\mathcal{P}_\alpha)}{\alpha^2/2},$$

where $\mathcal{P}_\alpha = \{f \in R \mid \nu_n(f) \geq \alpha\} \cup \{0\}$. In this case $1/\mathrm{vol}(\nu_n)$ coincides with the last value $\overline{\beta}_{g+1}(\nu_n)$ of the sequence of maximal contact values of ν_n (see [19, Remark 2.3]).

Now consider a pseudoeffective divisor $D \sim aF + bM$ on $\mathbb{F}_\delta$, where $\sim$ denotes linear equivalence. D admits a Zariski decomposition $D = P_D + N_D$, where P_D and N_D denote, respectively, the positive and the negative part of D [23, Theorem 2.3.19]. When D is nef, then $N_D = 0$; if $\delta > 0$ and D is big but not nef, then $P_D \sim (b + a/\delta)M$ and $N_D \sim (-a/\delta)M_0$, where $b > 0$ and $-b\delta < a < 0$. Moreover, the volume of D is defined as

$$\mathrm{vol}(D) = \mathrm{vol}_{\mathbb{F}_\delta}(D) := \limsup_{m\to\infty} \frac{h^0(\mathbb{F}_\delta, mD)}{m^2/2},$$

and D is a big divisor if and only if $\mathrm{vol}(D) > 0$. By [23, Corollary 2.3.22], it holds that $\mathrm{vol}(D) = P_D^2$.

Definition 2.1. Let ν_n be a divisorial valuation of $\mathbb{F}_\delta$ and D a big divisor on $\mathbb{F}_\delta$. Following [5] and [12], we define the values $\mu_D(\nu_n)$ and $\hat{\mu}_D(\nu_n)$ as

$$\mu_D(\nu_n) := \max\{\nu_n(\varphi_{D'}) \mid D' \in |D|\} \text{ and } \hat{\mu}_D(\nu_n) := \lim_{m\to\infty} \frac{\mu_{mD}(\nu_n)}{m},$$

where $\varphi_{D'}$ is the germ of D' at p.

By Proposition 2.9 in [5], it holds that

$$\hat{\mu}_D(\nu_n) \geq \sqrt{\frac{\mathrm{vol}(D)}{\mathrm{vol}(\nu_n)}}. \tag{2.2}$$

Definition 2.2. Let ν_n be a divisorial valuation of $\mathbb{F}_\delta$ and D a big divisor on $\mathbb{F}_\delta$. The valuation ν_n is *minimal with respect to* D if $\hat{\mu}_D(\nu_n) = \sqrt{\mathrm{vol}(D)/\mathrm{vol}(\nu_n)}$.

Remark 2.3. Let ν_n be a divisorial valuation of $\mathbb{F}_\delta$ and denote by Z the surface that it defines. Assume that D is a big divisor on $\mathbb{F}_\delta$. Then, by Theorem 6.4 of [24], it holds the equality

$$\hat{\mu}_D(\nu_n) = \sup\{t \in \mathbb{Q}_+ \mid D^* - tE_n \text{ is big on } Z\},$$

where $\mathbb{Q}_+$ is the set of non-negative rational numbers.

Our next definition divides divisorial valuations ν_n of $\mathbb{F}_\delta$ in two types according to the value δ and the point p where ν_n is centered. This classification was introduced in [17].

Definition 2.4. Let ν_n be a valuation of the quotient field of $\mathcal{O}_{\mathbb{F}_\delta,p}$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$. The valuation ν_n is named to be *special* (with respect to $\mathbb{F}_\delta$ and p) when one of the following conditions holds:

(1) $\delta = 0$.
(2) $\delta > 0$ and p is a special point.
(3) $\delta > 0$, p is a general point and there is no integral curve in the complete linear system $|M|$, given by the section M, whose strict transform on Z has negative self-intersection.

The remaining valuations ν_n will be called *non-special*.

Let ν_n be a divisorial valuation of $\mathbb{F}_\delta$. We denote by F_1 the fiber which goes through the point p and, when ν_n is non-special, by M_1 the unique integral curve in $|M|$ whose strict transform on Z has negative self-intersection.

Next we introduce the so-called non-positive at infinity valuations of $\mathbb{F}_\delta$. For valuations in this family, we will be able to compute the value $\hat{\mu}_D(\nu_n)$ for any big divisor D.

Definition 2.5. Let ν_n be a special (respectively, non-special) divisorial valuation of $\mathbb{F}_\delta$. The valuation ν_n is called *non-positive at infinity* whenever $\nu_n(h) \leq 0$ for all $h \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_0))$ (respectively, $h \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_1))$).

As a consequence of [17, Theorem 3.6] (respectively, [17, Theorem 4.8]), it suffices to check the condition $2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) + \delta\nu_n(\varphi_{F_1})^2 \geq \mathrm{vol}^{-1}(\nu_n)$ (respectively, $2\nu_n(\varphi_{M_1})\nu_n(\varphi_{F_1}) - \delta\nu_n(\varphi_{F_1})^2 \geq \mathrm{vol}^{-1}(\nu_n)$) to decide whether a special (respectively, non-special) divisorial valuation ν_n of $\mathbb{F}_\delta$ is non-positive at infinity. Moreover, under this assumption, the cone of curves of the surface Z defined by ν_n is generated by the classes of the strict transforms of the divisors $F_1, M_0, E_1, \ldots, E_n$ (respectively, $F_1, M_0, M_1, E_1, \ldots, E_n$).

To conclude this section, we determine the mentioned Seshadri-type constant for any non-positive at infinity divisorial valuation and big divisor of a Hirzebruch surface $\mathbb{F}_\delta$. We also extract some consequences of this result.

Theorem 2.6. *Let ν_n be a non-positive at infinity divisorial valuation of the quotient field of $\mathcal{O}_{\mathbb{F}_\delta,p}$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$, and $D \sim aF + bM$ a big divisor on $\mathbb{F}_\delta$. Then,*

(a) *If ν_n is special, then it holds that $\hat{\mu}_D(\nu_n) = (a + b\delta)\nu_n(\varphi_{F_1}) + b\nu_n(\varphi_{M_0})$.*
(b) *Otherwise, $\hat{\mu}_D(\nu_n) = a\nu_n(\varphi_{F_1}) + b\nu_n(\varphi_{M_1})$.*

Proof. For proving Statement (a) we assume that p is a special point. When p is a point of $\mathbb{F}_0$ (respectively, p is a general point), the proof is analogous by setting $\delta = 0$ (respectively, $\nu_n(\varphi_{M_0}) = 0$). Let C be a curve on $\mathbb{F}_\delta$ such that $C \in |mD|$, where m is a positive integer, and denote by $\tilde{C}$ its strict transform on Z. By [17, Theorem 3.6], it holds that $\Lambda_n = \nu_n(\varphi_{M_0})F^* + \nu_n(\varphi_{F_1})M^* - \sum_{i=1}^n \nu_n(\mathfrak{m}_i)E_i^*$ is a nef divisor and then $\Lambda_n \cdot \tilde{C} \geq 0$. This means that

$$(a + b\delta)\nu_n(\varphi_{F_1}) + b\nu_n(\varphi_{M_0}) \geq \frac{\nu_n(\varphi_C)}{m}$$

and, so, we have found an upper bound for $\nu_n(\varphi_C)/m$, where $C \in |mD|$ and m is a positive integer. Now, consider the curve $C_1 = m(a + \delta b)F_1 + mbM_0$, then

$$C_1 \in |mD| \text{ and } \frac{\nu_n(\varphi_{C_1})}{m} = (a + \delta b)\nu_n(\varphi_{F_1}) + b\nu_n(\varphi_{M_0}),$$

which proves that the bound can be reached and Statement (a) holds.

The proof of Statement (b) follows analogously by taking the divisor

$$\Delta_n = (\nu_n(\varphi_{M_1}) - \delta\nu_n(\varphi_{F_1}))F^* + \nu_n(\varphi_{F_1})M^* - \sum_{i=1}^n \nu_n(\mathfrak{m}_i)E_i^*,$$

which is nef by [17, Theorem 4.8], and the curve $C_1 = maF_1 + mbM_1$. □

Corollary 2.7. *Let ν_n be a non-positive at infinity divisorial valuation of $\mathbb{F}_\delta$ and $D \sim aF + bM$ a big and nef divisor on $\mathbb{F}_\delta$. Then,*

(a) *When ν_n is special, it is minimal with respect to D if and only if*

$$2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) + \delta\nu_n(\varphi_{F_1})^2 = \mathrm{vol}(\nu_n)^{-1}$$

and $a = b\nu_n(\varphi_{M_0})/\nu_n(\varphi_{F_1})$.

(b) *Otherwise, ν_n is minimal with respect to D if and only if*

$$2\nu_n(\varphi_{M_1})\nu_n(\varphi_{F_1}) - \delta\nu_n(\varphi_{F_1})^2 = \mathrm{vol}(\nu_n)^{-1}$$

and $a = b(\nu_n(\varphi_{M_1}) - \delta\nu_n(\varphi_{F_1}))/\nu_n(\varphi_{F_1})$.

Proof. We will prove Item (a) in the case when p_1 is a special point; when $p_1 \in \mathbb{F}_0$ (respectively, p_1 is a general point) the proof is analogous and follows from taking $\delta = 0$ (respectively, $\nu_n(\varphi_{M_0}) = 0$). A proof for Item (b) also runs similarly.

For a start, we are going to prove the minimality of ν under the conditions of the statement. Taking into account that $2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) + \delta\nu_n(\varphi_{F_1})^2 = \mathrm{vol}(\nu_n)^{-1}$, one obtains that

$$\begin{aligned}\frac{\mathrm{vol}(D)}{\mathrm{vol}(\nu_n)} =&(2ab+b^2\delta)\delta\nu_n(\varphi_{F_1})^2 + 2b(a+b\delta)\nu_n(\varphi_{F_1})\nu_n(\varphi_{M_0}) + 2ab\nu_n(\varphi_{F_1})\nu_n(\varphi_{M_0})\\ =&(a+b\delta)^2\nu_n(\varphi_{F_1})^2 + 2b(a+b\delta)\nu_n(\varphi_{F_1})\nu_n(\varphi_{M_0}) + b^2\nu_n(\varphi_{M_0})^2\\ =&\hat{\mu}_D(\nu_n)^2,\end{aligned}$$

where the second equality holds since $(a\nu_n(\varphi_{F_1}) - b\nu_n(\varphi_{M_0}))^2 = 0$. This proves that ν_n is minimal with respect to D.

Now assume that ν_n is minimal with respect to D. Then, by Theorem 2.6, it holds that

$$\left((a+b\delta)\nu_n(\varphi_{F_1}) + b\nu_n(\varphi_{M_0})\right)^2 = b(2a+\delta b)\mathrm{vol}(\nu_n)^{-1}. \tag{2.3}$$

On the other hand, one has the equality

$$\begin{aligned}\left((a+b\delta)\nu_n(\varphi_{F_1}) + b\nu_n(\varphi_{M_0})\right)^2 =& (a\nu_n(\varphi_{F_1}) - b\nu_n(\varphi_{M_0}))^2\\ &+ b(2a+\delta b)(2\nu_n(\varphi_{F_1})\nu_n(\varphi_{M_0}) + \delta\nu_n(\varphi_{F_1})^2),\end{aligned}$$

which, together with Equality (2.3), gives rise to

$$(a\nu_n(\varphi_{F_1}) - b\nu_n(\varphi_{M_0}))^2 + b(2a+\delta b)(2\nu_n(\varphi_{F_1})\nu_n(\varphi_{M_0}) + \delta\nu_n(\varphi_{F_1})^2 - \mathrm{vol}(\nu_n)^{-1}) = 0.$$

Both addends of the above expression are not negative, so they must vanish. This completes the proof.

□

Corollary 2.8. *Let ν_n be a non-positive at infinity divisorial valuation of $\mathbb{F}_\delta$. Then, ν_n is non-minimal with respect to any big divisor D on $\mathbb{F}_\delta$ whenever some of the following conditions holds:*

(a) *ν_n is special and $2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) + \delta\nu_n(\varphi_{F_1})^2 > \mathrm{vol}(\nu_n)^{-1}$.*
(b) *ν_n is non-special and $2\nu_n(\varphi_{M_1})\nu_n(\varphi_{F_1}) - \delta\nu_n(\varphi_{F_1})^2 > \mathrm{vol}(\nu_n)^{-1}$.*

Proof. We begin by proving Item (a). We only need to show that

$$\hat{\mu}_D(\nu_n)^2/P_D^2 > \overline{\beta}_{g+1}(\nu_n)$$

holds for any big divisor $D \sim aF + bM$, P_D being its positive part in the Zariski decomposition. Firstly, assume that $\delta > 0$. Let $q : (-\delta, \infty) \cap \mathbb{Q} \to \mathbb{Q}_+$ be the map

$$q(x) := \begin{cases} \dfrac{((x+\delta)\nu_n(\varphi_{F_1}) + \nu_n(\varphi_{M_0}))^2}{((1/\delta)x+1)^2\delta} & \text{if } x \in (-\delta, 0)\cap\mathbb{Q},\\ \dfrac{((x+\delta)\nu_n(\varphi_{F_1}) + \nu_n(\varphi_{M_0}))^2}{2x+\delta} & \text{if } x \in [0,\infty)\cap\mathbb{Q}.\end{cases}$$

Notice that q has an absolute minimum at the point $(x_1, q(x_1))$, where $x_1 = \nu_n(\varphi_{M_0})/\nu_n(\varphi_{F_1})$ and

$$q(x_1) = 2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) + \nu_n(\varphi_{F_1})^2\delta.$$

Since $q(a/b) = \hat{\mu}_D(\nu_n)^2/P_D^2$ (by Theorem 2.6) we have that

$$\hat{\mu}_D(\nu_n)^2/P_D^2 \geq 2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) + \nu_n(\varphi_{F_1})^2\delta > \mathrm{vol}(\nu_n)^{-1} = \overline{\beta}_{g+1}(\nu_n).$$

If $\delta = 0$, by Theorem 2.6 it holds that

$$\hat{\mu}_D(\nu_n)^2/P_D^2 - 2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) = (a\nu_n(\varphi_{F_1}) - b\nu_n(\varphi_{M_0}))^2/P_D^2 \geq 0.$$

Hence $\hat{\mu}_D(\nu_n)^2/P_D^2 \geq 2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) > \mathrm{vol}(\nu_n)^{-1} = \overline{\beta}_{g+1}(\nu_n)$.

To conclude, we notice that Item (b) can be proved following the same reasoning of the proof of Item (a), but considering the map $q_1 : (-\delta, \infty) \cap \mathbb{Q} \to \mathbb{Q}_+$,

$$q_1(x) := \begin{cases} \dfrac{(\nu_n(\varphi_{F_1})x + \nu_n(\varphi_{M_1}))^2}{((1/\delta)x + 1)^2\delta} & \text{if } x \in (-\delta, 0) \cap \mathbb{Q}, \\ \dfrac{(\nu_n(\varphi_{F_1})x + \nu_n(\varphi_{M_1}))^2}{2x + \delta} & \text{if } x \in [0, \infty) \cap \mathbb{Q}, \end{cases}$$

instead of q. □

3. Newton-Okounkov bodies of non-positive at infinity valuations

Let X be a smooth complex projective surface. A sequence

$$C_\bullet := \{X \supset C \supset \{q\}\},$$

where C is a smooth irreducible curve on X and q a closed point of C, is called a *flag* of X. The point q is the *center* of $C_\bullet$.

In this section, we study the Newton-Okounkov bodies with respect to a flag

$$E_\bullet := \{Z = Z_r \supset E_r \supset \{p_{r+1}\}\}, \tag{3.1}$$

where $Z = Z_r$ is the surface defined by a finite simple sequence of blowups as in (2.1) with $Z_0 = \mathbb{F}_\delta$ and E_r the last exceptional divisor created. We denote by p_{r+1} the center of $E_\bullet$.

Flags of smooth varieties (not only surfaces) define and are defined by discrete valuations whose rank coincides with the dimension of the variety. In our case, they correspond one-to-one to exceptional curve valuations ν (up to equivalence). The configuration of infinitely near points $\mathcal{C}_\nu = \{p_i\}_{i=1}^\infty$ of ν satisfies that the points $\{p_i\}_{i=1}^r$ are given by the divisorial valuation ν_r defined by E_r and the remaining points p_i, for $i > r$, are proximate to p_r. If the point p_{r+1} is satellite, then there exists an exceptional divisor E_η such that $\eta \neq r$ and $p_{r+1} \in E_\eta$.

According to [19, Section 3.2], the flag valuation $\nu := \nu_{E_\bullet}$, defined by $E_\bullet$, satisfies that, for $f \in R = \mathcal{O}_{\mathbb{F}_\delta,p}$, $\nu_{E_\bullet}(f) = (\upsilon_1(f), \upsilon_2(f))$ with $\upsilon_1(f) = \nu_r(f)$ and $\upsilon_2(f) := \nu_\eta(f) + \sum_{p_i \to p_r} \mathrm{mult}_{p_i}(f)$, where ν_η is the divisorial valuation defined by E_η. Up to equivalence of valuations, the value group of ν is $\mathbb{Z}^2$ and $\nu(\mathfrak{m}_r) = (1, 0)$ and $\nu(\mathfrak{m}_{r+1}) = (0, 1)$.

Definition 3.1. Let ν be an exceptional curve valuation of $\mathbb{F}_\delta$ and D a big divisor on $\mathbb{F}_\delta$. The valuation ν is *minimal with respect to* D whenever its first component ν_r is minimal with respect to D. The valuation ν is called *special* (respectively, *non-special*) when its first component ν_r is a special (respectively, non-special) divisorial valuation of $\mathbb{F}_\delta$. Analogously, ν is *non-positive at infinity* whenever ν_r is non-positive at infinity.

Newton-Okounkov bodies are non-empty convex and compact objects attached to flags and give very interesting geometric information [24, 20, 5]. The goal of this section is to explicitly compute the Newton-Okounkov bodies $\Delta_{\nu_{E_\bullet}}(D^*)$, where $E_\bullet$ is a flag as in (3.1) corresponding to a non-positive at infinity exceptional curve valuation $\nu_{E_\bullet}$ and D^* is the pull-back on Z of a big divisor D on $\mathbb{F}_\delta$. We start by defining the Newton-Okounkov body in our case.

Definition 3.2. Let ν be an exceptional curve valuation of $\mathbb{F}_\delta$ and D a big divisor on $\mathbb{F}_\delta$. The *Newton-Okounkov body of D with respect to ν* is defined as

$$\Delta_\nu(D) := \overline{\bigcup_{m\geq 1}\left\{\frac{\nu(f)}{m} \mid f\in H^0(\mathbb{F}_\delta, mD)\setminus\{0\}\right\}},$$

where the upper line means the closed convex hull in $\mathbb{R}^2$.

Notice that, if $E_\bullet$ is a flag as in (3.1) and $\nu=\nu_{E_\bullet}$, then $\Delta_\nu(D)=\Delta_{\nu_{E_\bullet}}(D^*)$. Moreover, the Newton-Okounkov body is a polygon (see [22]) and

$$\mathrm{vol}(D)=\mathrm{vol}_Z(D^*)=2\,\mathrm{vol}_{\mathbb{R}^2}(\Delta_\nu(D)),$$

where $\mathrm{vol}_{\mathbb{R}^2}$ means Euclidean area (see [24]).

Set $g+1$ (respectively, g^*+1) the minimal number of generators of the semigroup of values of the divisorial valuation ν_r (respectively, exceptional curve valuation ν). It holds that $g^*=g$ when p_r and p_{r+1} are satellite points. Otherwise, $g^*=g+1$. Denote by S_ν the semigroup of values of ν, that is, the monoid

$$S_\nu := \{\nu(f)\mid f\in R\setminus\{0\}\}\cup\{0\}\subseteq\mathbb{Z}^2,$$

endowed with the lexicographical ordering. As mentioned, the set S_ν is generated by the set of pairs $\{\overline{\beta}_i(\nu)\}_{i=0}^{g^*}$ (respectively, $\{\overline{\beta}_i(\nu)\}_{i=0}^{g+1}$), where $\overline{\beta}_i(\nu)=(\overline{\beta}_i(\nu_r),\overline{\beta}_i(\nu_\eta))$ (respectively, $\overline{\beta}_i(\nu)=(\overline{\beta}_i(\nu_r),0)$ and $\overline{\beta}_{g+1}(\nu)=(\overline{\beta}_{g+1}(\nu_r),1)$), whenever p_{r+1} is a satellite (respectively, free) point.

Let $\mathfrak{C}(\nu)$ be the convex cone of $\mathbb{R}^2$ spanned by S_ν and $\mathfrak{H}_D(\nu)$ the half-plane $\{(x,y)\in\mathbb{R}^2\mid x\leq\hat{\mu}_D(\nu_r)\}$. Then, the next result follows from Definitions 3.2 and 2.1 and [19, Proposition 3.6].

Proposition 3.3. *The set $\mathfrak{C}(\nu)\cap\mathfrak{H}_D(\nu)$ is a triangle, which contains the Newton-Okounkov body $\Delta_\nu(D)$, and whose vertices are:*

$$(0,0),\quad\left(\hat{\mu}_D(\nu_r),\frac{\hat{\mu}_D(\nu_r)\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)}\right)\quad\textit{and}\quad\left(\hat{\mu}_D(\nu_r),\frac{\hat{\mu}_D(\nu_r)\overline{\beta}_{g^*}(\nu_\eta)}{\overline{\beta}_{g^*}(\nu_r)}\right)$$

whenever $q=p_{r+1}$ is the satellite point $E_r\cap E_\eta$ (with $\eta\neq r$); and

$$(0,0),\quad(\hat{\mu}_D(\nu_r),0)\quad\textit{and}\quad\left(\hat{\mu}_D(\nu_r),\frac{\hat{\mu}_D(\nu_r)}{\overline{\beta}_{g+1}(\nu_r)}\right),$$

otherwise.

Our next result determines the Newton-Okounkov bodies of the minimal exceptional curve valuations of Hirzebruch surfaces.

Theorem 3.4. *Let ν be an exceptional curve valuation of a Hirzebruch surface $\mathbb{F}_\delta$ and D a big divisor on $\mathbb{F}_\delta$. Then, the Newton-Okounkov body $\Delta_\nu(D)$ coincides with the triangle $\mathfrak{C}(\nu)\cap\mathfrak{H}_D(\nu)$ if and only if ν is minimal with respect to D.*

Proof. Proposition 3.3 and [19, Lemma 3.9] prove that the Newton-Okounkov body $\Delta_\nu(D)$ is contained in the triangle $\mathfrak{C}(\nu)\cap\mathfrak{H}_D(\nu)$ whose area is

$$\hat{\mu}_D(\nu_r)^2/2\overline{\beta}_{g+1}(\nu_r).$$

Taking into account that

$$\frac{\hat{\mu}_D(\nu_r)^2}{2\overline{\beta}_{g+1}(\nu_r)} \geq \frac{\mathrm{vol}(D)}{2}$$

by inequality (2.2), it holds that the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_D(\nu)$ will coincide with the Newton-Okounkov body $\Delta_\nu(D)$ when both figures have the same area, which is true only when ν is minimal with respect to D. $\square$

From now on, suppose that ν is an exceptional curve valuation of $\mathbb{F}_\delta$ which is not minimal (or, non-minimal) with respect to a big divisor $D \sim aF + bM$.

When $\delta = 0$, D is also nef. Otherwise ($\delta \neq 0$), D^* may be big but not nef. In this last case, the positive and negative parts of the Zariski decomposition of D^* are

$$P_{D^*} \sim \left(b + \frac{a}{\delta}\right) M^* \text{ and } N_{D^*} = \frac{-a}{\delta}\tilde{M}_0 + \sum_{i=0}^{i_{M_0}} \frac{-a\nu_i(\varphi_{M_0})}{\delta} E_i,$$

where i_{M_0} indicates the last point in C_ν through which the strict transform of M_0 passes. Then we distinguish two cases.

Case 1. p_{r+1} belongs to the support, $\mathrm{supp}(N_{D^*})$, of the negative part N_{D^*} of the Zariski decomposition of the divisor D^*. This fact holds if and only if $g^* = 1, p_1$ is a special point, all the points in $\{p_i\}_{i=1}^{r+1}$ are free, $i_{M_0} = r + 1$ and D is big and not nef.

Case 2. $p_{r+1} \notin \mathrm{supp}(N_{D^*})$. In this case, to compute $\Delta_\nu(D)$ we can assume that D is nef because this assumption does not produce loss of generality. Indeed, if the divisor D is big but not nef, then $b > 0$ and $-b\delta < a < 0$, and, as $p_{r+1} \notin \mathrm{supp}(N_{D^*})$, by [21, Lemma 1.10], it holds that

$$\Delta_\nu(D) = \Delta_\nu(P_D) = \left(b + \frac{a}{\delta}\right) \Delta_\nu(M).$$

Notice that, $\mathrm{vol}(D) = D^2$ and Inequality (2.2) can be written as

$$\hat{\mu}_D(\nu_r) \geq \sqrt{D^2 \overline{\beta}_{g+1}(\nu_r)} \tag{3.2}$$

if $p_{r+1} \notin \mathrm{supp}(N_{D^*})$. Otherwise, we will replace D by P_D.

In the following subsections, we will explicitly get the Newton-Okounkov bodies of big divisors D with respect to non-positive at infinity valuations ν. We start with special valuations, where the case $p_{r+1} \in \mathrm{supp}(N_{D^*})$ might happen.

3.1. **Newton-Okounkov bodies with respect to non-positive at infinity special valuations.** Along this section $D \sim aF + bM$ is a big divisor on $\mathbb{F}_\delta$ and ν a non-positive at infinity special exceptional curve valuation of $\mathbb{F}_\delta$ whose first component is ν_r. Recall that ν is not minimal with respect to D.

The symbol $\theta_1^r(D)$ stands for $a\nu_r(\varphi_{F_1}) - b\nu_r(\varphi_{M_0})$, where F_1 is the fiber which passes through p and M_0 the special section. When $\theta_1^r(D) = 0$, it holds that either $a = b\nu_r(\varphi_{M_0})/\nu_r(\varphi_{F_1})$, or $\nu_r(\varphi_{M_0}) = 0$ and $a = 0$. Notice that, in the second case, some objects that we will introduce are not defined and we will avoid using them. Moreover, if $p_{r+1} \in \mathrm{supp}(N_{D^*})$, then $\theta_1^r(D)$ is always negative.

We start by stating two lemmas which allow us to compute the Zariski decomposition of some key divisors.

Lemma 3.5. *Let ν_r be a non-positive at infinity special divisorial valuation of $\mathbb{F}_\delta$ and D a big and nef divisor on $\mathbb{F}_\delta$. Let also $\theta_1^r(D)$ be as in the above paragraphs. Then, the divisor*

$$D_1 = D^* - \frac{b}{\nu_r(\varphi_{F_1})}\sum_{i=1}^r \nu_r(\mathfrak{m}_i)E_i^* \left(\textit{respectively, } D_2 = D^* - \frac{a}{\nu_r(\varphi_{M_0})}\sum_{i=1}^r \nu_r(\mathfrak{m}_i)E_i^*\right)$$

is nef when $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$).

Proof. We are going to prove that D_1 is nef. A proof for D_2 runs similarly. As b is a positive integer, one can deduce that

$$\begin{aligned} D_1 &= D^* - \frac{b}{\nu_r(\varphi_{F_1})}\sum_{i=1}^r \nu_r(\mathfrak{m}_i)E_i^* \\ &\sim \frac{b}{\nu_r(\varphi_{F_1})}\left(\frac{a\nu_r(\varphi_{F_1})}{b}F^* + \nu_r(\varphi_{F_1})M^* - \sum_{i=1}^r \nu(\mathfrak{m}_i)E_i^*\right) \\ &= \frac{b}{\nu_r(\varphi_{F_1})}\left(\frac{\theta_1^r(D)}{b}F^* + \Lambda_r\right), \end{aligned}$$

where $\Lambda_r = \nu_r(\varphi_{M_0})F^* + \nu_r(\varphi_{F_1})M^* - \sum_{i=1}^r \nu_r(\mathfrak{m}_i)E_i^*$. The divisors F^* and Λ_r are nef by [17, Theorem 3.6] and then D_1 is also a nef divisor since $\theta_1^r(D)$ is non-negative. □

Lemma 3.6. *Let ν_r be a non-positive at infinity special divisorial valuation of $\mathbb{F}_\delta$ and Z the surface that it defines. Consider a big and nef divisor $D \sim aF + bM$ and recall that $\theta_1^r(D) = a\nu_r(\varphi_{F_1}) - b\nu_r(\varphi_{M_0})$. Then, the following four rational numbers:*

$$t_1 = \frac{b}{\nu_r(\varphi_{F_1})}\overline{\beta}_{g+1}(\nu_r), \quad t_2 = \frac{b}{\nu_r(\varphi_{F_1})}\overline{\beta}_{g+1}(\nu_r) + \theta_1^r(D),$$

$$t_3 = \frac{a}{\nu_r(\varphi_{M_0})}\overline{\beta}_{g+1}(\nu_r) \text{ and } t_4 = \frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r) - \theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}$$

satisfy that t_1 and t_2 (respectively, t_3 and t_4) belong to the set

$$T_{D,\nu_r} := \{t \in \mathbb{Q} \mid 0 \leq t \leq \hat{\mu}_D(\nu_r)\}$$

when $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$). In addition, if $p_{r+1} \in \operatorname{supp}(N_{D^*})$, then $-a\nu_r(\varphi_{M_0})/\delta < t_4 \leq \hat{\mu}_D(\nu_r)$.

Proof. We show that $t_1, t_2 \leq \hat{\mu}_D(\nu_r)$. A proof for the other cases runs similarly.

Let us prove that $t_1 \leq \hat{\mu}_D(\nu_r)$ when $\theta_1^r(D) \geq 0$. By Lemma 3.5, it holds that the divisor $D_1 = D^* - \frac{b}{\nu_r(\varphi_{F_1})}\sum_{i=1}^r \nu_r(\mathfrak{m}_i)E_i^*$ is nef and then, for any curve $C \in |mD|$, m being a positive integer, one has that

$$m(2ab + b^2\delta) - \frac{b}{\nu_r(\varphi_{F_1})}\nu_r(\varphi_C) = D_1 \cdot \tilde{C} \geq 0,$$

where $\tilde{C}$ is the strict transform of C under the birational map defined by ν_r. This shows that

$$2ab + b^2\delta \geq \frac{b}{\nu_r(\varphi_{F_1})}\hat{\mu}_D(\nu_r),$$

which, together with Inequality (3.2), allow us to deduce the inequalities

$$\hat{\mu}_D(\nu_r) \geq \frac{D^2\overline{\beta}_{g+1}(\nu_r)}{\hat{\mu}_D(\nu_r)} = \frac{(2ab+b^2\delta)\overline{\beta}_{g+1}(\nu_r)}{\hat{\mu}_D(\nu_r)} \geq \frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})}.$$

This proves our statement.

To finish the proof, we are going to see that $t_2 \leq \hat{\mu}_D(\nu_r)$ when $\theta_1^r(D) \geq 0$. By Theorem 2.6, it suffices to prove the inequality

$$b(2\nu_r(\varphi_{M_0})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2) \geq b\overline{\beta}_{g+1}(\nu_r),$$

which holds by [17, Theorem 3.6] after noticing that b is a positive integer. $\square$

Remark 3.7. Theorem 2.6 and Corollary 2.7 prove that

$$\hat{\mu}_D(\nu_r) = b\overline{\beta}_{g+1}(\nu_r)/\nu_r(\varphi_{F_1}) = t_1 = t_2\,(= t_3 = t_4,\ \text{when } \nu_r(\varphi_{M_0}) \neq 0),$$

whenever the valuation ν_r is minimal with respect to D.

Otherwise, Lemma 3.6 provides two values, t_1 and t_2 (respectively, t_3 and t_4) when $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$) and D is big and nef. If $\theta_1^r(D) = 0$, then $\hat{\mu}_D(\nu_r) > t_1 = t_2\,(= t_3 = t_4$, when $\nu_r(\varphi_{M_0}) \neq 0)$, and when $\delta > 0, a = 0$ and $\theta_1^r(D) < 0$, then $t_3 = 0$. Moreover, if $2\nu_r(\varphi_{M_0})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2 = \overline{\beta}_{g+1}(\nu_r)$ holds, we obtain that $t_2 = \hat{\mu}_D(\nu_r)$ (respectively, $t_4 = \hat{\mu}_D(\nu_r)$) whenever $\theta_1^r(D) > 0$ (respectively, $\theta_1^r(D) < 0$). Finally, if $p_{r+1} \in \operatorname{supp}(N_{D^*})$, the value t_4 defined in Lemma 3.6 satisfies $t_4 = \hat{\mu}_D(\nu_r)$ when $2\nu_r(\varphi_{M_0})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2 = \overline{\beta}_{g+1}(\nu_r)$.

Lemma 3.8. *Let ν_r be a special divisorial valuation of $\mathbb{F}_\delta$ and D a big and nef divisor on $\mathbb{F}_\delta$. Suppose also that ν_r is non-minimal with respect to D. Let ν_i be the divisorial valuation defined by the exceptional divisor E_i, $1 \leq i \leq r-1$. Then, the intersection matrices determined by the families of divisors $\{\tilde{F}_1, E_1, \ldots, E_{r-1}\}$ and $\{\tilde{M}_0, E_1, \ldots, E_{r-1}\}$ are negative definite. In addition, when $p_{r+1} \in \operatorname{supp}(N_{D^*})$, $\{\tilde{M}_0, E_1, \ldots, E_{r-1}\}$ also determines a negative definite intersection matrix. Notice that in this last case, D is big and not nef.*

Proof. Consider the divisor D_1 defined in Lemma 3.5. We showed that it is nef, let us see that it is also big. Indeed,

$$D_1^2 = D^2 - \frac{b^2\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})^2} \geq D^2 - \frac{b\hat{\mu}_D(\nu_r)}{\nu_r(\varphi_{F_1})}\left(\frac{D^2\overline{\beta}_{g+1}(\nu_r)}{\hat{\mu}_D(\nu_r)^2}\right) > D^2 - \frac{b\hat{\mu}_D(\nu_r)}{\nu_r(\varphi_{F_1})} \geq 0,$$

where the second inequality holds since ν_r is non-minimal with respect to D and the last one by the proof of Lemma 3.6. So, D_1 is a big divisor by [23, Theorem 2.2.16]. Finally, the facts that $D_1 \cdot \tilde{F}_1 = 0$ and $D_1 \cdot E_i = 0$ for $1 \leq i \leq r-1$ prove our statement for $\{\tilde{F}_1, E_1, \ldots, E_{r-1}\}$ by Lemma 4.3 of [3]. The remaining cases can be proved analogously either with the divisor D_2 in Lemma 3.5 or with the nef and big divisor $(b + a/\delta)M^*$. $\square$

Our next result gives the positive part and the negative part of the Zariski decomposition of certain divisors which will be useful.

Proposition 3.9. *Let ν_r be a non-positive at infinity special divisorial valuation of $\mathbb{F}_\delta$, Z the surface defined by ν_r and ν_i the divisorial valuation defined by the exceptional divisor E_i, $1 \leq i \leq r-1$. Set $D \sim aF + bM$ a big and nef divisor on $\mathbb{F}_\delta$ and suppose that ν_r is non-minimal with respect to D. As above, write*

$\theta_1^r(D) = a\nu_r(\varphi_{F_1}) - b\nu_r(\varphi_{M_0})$ *and* $\Lambda_r = \nu_r(\varphi_{M_0})F^* + \nu_r(\varphi_{F_1})M^* - \sum_{i=1}^r \nu_r(\mathfrak{m}_i)E_i^*$. *Consider the divisors* D_1 *and* D_2 *in Lemma 3.5 and the rational numbers* t_1, t_2, t_3 *and* t_4 *given in Lemma 3.6. Then,*

(a) *Assuming* $\theta_1^r(D) \geq 0$, *the positive and negative parts of the Zariski decomposition of the divisors* $D_{t_1} := D^* - t_1E_r$, *and* $D_{t_2} := D^* - t_2E_r$ *are:*

$$P_{D_{t_1}} \sim D_1 \quad \textit{and} \quad N_{D_{t_1}} = \frac{b}{\nu_r(\varphi_{F_1})}\sum_{i=1}^{r-1}\nu_r(\varphi_i)E_i,$$
$$\textit{and } P_{D_{t_2}} \sim \frac{b}{\nu_r(\varphi_{F_1})}\Lambda_r \textit{ and}$$
$$N_{D_{t_2}} = \frac{\theta_1^r(D)}{\nu_r(\varphi_{F_1})}\tilde{F}_1 + \sum_{i=1}^{r-1}\frac{b\nu_r(\varphi_i) + \theta_1^r(D)\nu_i(\varphi_{F_1})}{\nu_r(\varphi_{F_1})}E_i.$$

(b) *When* $\theta_1^r(D) < 0$, *the positive and negative parts of the Zariski decomposition of* $D_{t_3} := D^* - t_3E_r$, *and* $D_{t_4} := D^* - t_4E_r$ *are:*

$$P_{D_{t_3}} \sim D_2 \quad \textit{and} \quad N_{D_{t_3}} = \frac{a}{\nu_r(\varphi_{M_0})}\sum_{i=1}^{r-1}\nu_r(\varphi_i)E_i,$$
$$\textit{and } P_{D_{t_4}} \sim \frac{a+b\delta}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}\Lambda_r \textit{ and}$$
$$N_{D_{t_4}} = \left(\frac{-\theta_1^r(D)}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}\right)\tilde{M}_0$$
$$+\sum_{i=1}^{r-1}\frac{(a+b\delta)\nu_r(\varphi_i) - \theta_1^r(D)\nu_i(\varphi_{M_0})}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}E_i.$$

Moreover, if $p_{r+1} \in \operatorname{supp}(N_{D^*})$, *then the positive and negative parts of* D_{t_4} *are the divisors* $P_{D_{t_4}}$ *and* $N_{D_{t_4}}$ *described before.*

Proof. We only prove Statement (a) since a similar proof can be given for the remaining cases. Let us start with the decomposition of D_{t_1}. It is clear that $P_{D_{t_1}} + N_{D_{t_1}} \sim D_{t_1}$. Also $P_{D_{t_1}}$ is nef, by Lemma 3.5, and orthogonal to each component of $N_{D_{t_1}}$, by the proximity equalities. This concludes the proof after taking into account that the components of $N_{D_{t_1}}$ determine an intersection matrix which is negative definite.

Finally, we prove the claim for the divisor D_{t_2}. By [17, Proposition 3.3 and Theorem 3.6], $P_{D_{t_2}}$ is nef and orthogonal to each component of $N_{D_{t_2}}$. As well, it follows from Lemma 3.8 that the intersection matrix determined by the components of $N_{D_{t_2}}$ is negative definite. To conclude, adding the following two expressions:

$$D - \frac{b}{\nu_r(\varphi_{F_1})}\overline{\beta}_{g+1}(\nu_r)E_r \sim \frac{b}{\nu_r(\varphi_{F_1})}\Lambda_r + \frac{\theta_1^r(D)}{\nu_r(\varphi_{F_1})}F^* + \frac{b}{\nu_r(\varphi_{F_1})}\sum_{i=1}^{r-1}\nu_r(\varphi_i)E_i$$

and

$$-\theta_1^r(D)E_r = \frac{\theta_1^r(D)}{\nu_r(\varphi_{F_1})}\left(\sum_{i=1}^{r-1}\nu_i(\varphi_{F_1})E_i - \sum_{i=1}^{i_{F_1}}E_i^*\right),$$

and taking into account that $\tilde{F}_1 \sim F^* - \sum_{i=1}^{i_{F_1}} E_i^*$, where i_{F_1} indicates the last point in the configuration of infinitely near points $\mathcal{C}_{\nu_r}$ of the valuation ν_r through which the strict transform of F_1 goes, we get $D_{t_2} \sim P_{D_{t_2}} + N_{D_{t_2}}$. This completes the proof. $\square$

Next, we are going to state the main results in this subsection. Recall that ν is a special exceptional curve valuation whose first component is ν_r. Moreover, ν is non-positive at infinity and non-minimal with respect to a big divisor $D \sim aF + bM$.

Our results determine the coordinates of the vertices of the Newton-Okounkov bodies $\Delta_\nu(D)$. We divide our study in three cases.

Case A: Either $g^* > 1$, or $g^* = 1$, $\nu(\varphi_{F_1}) \neq \overline{\beta}_1(\nu)$ and $\nu(\varphi_{M_0}) \neq \overline{\beta}_1(\nu)$.

Case B: The value g^* equals 1 and $\nu(\varphi_{F_1}) = \overline{\beta}_1(\nu)$.

Case C: The value g^* equals 1 and $\nu(\varphi_{M_0}) = \overline{\beta}_1(\nu)$.

We start with Case A. Here we can assume that D is also nef (see the paragraph below Theorem 3.4). According with [24, Theorem 6.4], by Remark 2.3, the Newton-Okounkov body $\Delta_\nu(D)$ coincides with the set

$$\{(t,y) \in \mathbb{R}^2 \mid 0 \leq t \leq \hat{\mu}_D(\nu_r) \text{ and } \alpha(t) \leq y \leq \beta(t)\},$$

where, for all $t \in [0, \hat{\mu}_D(\nu_r)]$, $\alpha(t) := \mathrm{ord}_{p_{r+1}}(N_{D_t}|_{E_r})$ and $\beta(t) := \alpha(t) + P_{D_t} \cdot E_r$; here P_{D_t} and N_{D_t} are, respectively, the positive and negative parts of the divisor $D_t = D^* - tE_r$. As a consequence, by Proposition 3.9, the points

$$Q_1 = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})}, \frac{b\nu_r(\varphi_\eta)}{\nu_r(\varphi_{F_1})}\right) \left(\text{respectively, } Q_1 = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})}, 0\right)\right),$$

$$Q_2 = Q_1 + \left(0, \frac{b}{\nu_r(\varphi_{F_1})}\right),$$

$$Q_3 = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})} + \theta_1^r(D), \frac{b\nu_r(\varphi_\eta) + \theta_1^r(D)\nu_\eta(\varphi_{F_1})}{\nu_r(\varphi_{F_1})}\right)$$

$$\left(\text{respectively, } Q_3 = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})} + \theta_1^r(D), 0\right)\right) \text{ and } Q_4 = Q_3 + \left(0, \frac{b}{\nu_r(\varphi_{F_1})}\right)$$

are in $\Delta_\nu(D)$ whenever $\theta_1^r(D) \geq 0$ and the point p_{r+1} is satellite (respectively, free). When $\theta_1^r(D) < 0$ and the point p_{r+1} is satellite (respectively, free), the points are

$$Q_5 = \left(\frac{a\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_0})}, \frac{a\nu_r(\varphi_\eta)}{\nu_r(\varphi_{M_0})}\right) \left(\text{respectively, } Q_5 = \left(\frac{a\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_0})}, 0\right)\right),$$

$$Q_6 = Q_5 + \left(0, \frac{a}{\nu_r(\varphi_{M_0})}\right),$$

$$Q_7 = \left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r) - \theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}, \frac{(a+b\delta)\nu_r(\varphi_\eta) - \theta_1^r(D)\nu_\eta(\varphi_{M_0})}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}\right)$$

$$\left(\text{respectively, } Q_7 = \left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r) - \theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}, 0\right)\right)$$
$$\text{and } Q_8 = Q_7 + \left(0, \frac{a+b\delta}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}\right).$$

Notice that, as mentioned at the beginning of Section 3, p_{r+1} is the intersection point $E_\eta \cap E_r$ when it is satellite. Moreover, Q_5 and Q_6 may not be well-defined when $\theta_1^r(D) = 0$, in fact this happens if $a = 0 = \nu_r(\varphi_{M_0})$.

By definition, it also holds that the point $Q_9 = (\hat{\mu}_D(\nu_r), \hat{\mu}_D(\nu_\eta))$ (respectively, $Q_9 = (\hat{\mu}_D(\nu_r), 0))$ when p_{r+1} is satellite (respectively, free) belongs to $\Delta_\nu(D)$. By Theorem 2.6, we are able to compute explicitly this point. Now, we state our first main result where we use the symbol $\preccurlyeq$ defined in Section 2.1.

Theorem 3.10. *Let ν be a valuation in Case A. With the notation as in the previous two paragraphs, the Newton-Okounkov body $\Delta_\nu(D)$ of D with respect to ν is a quadrilateral if and only if $a \neq 0$ and $\theta_1^r(D) \neq 0$. Otherwise, it is a triangle.*

The vertices of the quadrilateral are:

- (a) *$(0,0), Q_1, Q_3$ (respectively, Q_5, Q_7) and Q_9 when $\theta_1^r(D) > 0$ (respectively, $\theta_1^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r$ and $r \not\preccurlyeq \eta$.*
- (b) *$(0,0), Q_2, Q_4$ (respectively, Q_6, Q_8) and Q_9 when $\theta_1^r(D) > 0$ (respectively, $\theta_1^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r$ and $r \preccurlyeq \eta$.*
- (c) *$(0,0), Q_2, Q_4$ (respectively, Q_6, Q_8) and Q_9 when $\theta_1^r(D) > 0$ (respectively, $\theta_1^r(D) < 0$) and p_{r+1} is a free point.*

When $\delta > 0, a = 0$ and $\theta_1^r(D) < 0$, $Q_5 = Q_6 = (0,0)$ and the vertices of the triangle $\Delta_\nu(D)$ are as described in items (a), (b) and (c).

Finally, replacing $\theta_1^r(D) > 0$ (or $\theta_1^r(D) < 0$) with $\theta_1^r(D) = 0$ in items (a), (b) and (c), we obtain the vertices of the triangle $\Delta_\nu(D)$. This is because $Q_1 = Q_3\,(= Q_5 = Q_7$, when $\nu(\varphi_{M_0}) \neq 0)$ in Case (a) and $Q_2 = Q_4\,(= Q_6 = Q_8$, when $\nu(\varphi_{M_0}) \neq 0)$ otherwise.

Proof. First we show that $D^2/2$ is the area of the convex sets Δ and Δ' defined, respectively, by the sets of points $\{(0,0), Q_1, Q_2, Q_3, Q_4, Q_9\}$ and $\{(0,0), Q_5, Q_6, Q_7, Q_8, Q_9\}$.

Let us start with Δ. The area of the triangle $(0,0), Q_1$ and Q_2 (respectively, Q_3, Q_4 and Q_9) is

$$\frac{b^2\overline{\beta}_{g+1}(\nu_r)}{2\nu_r(\varphi_{F_1})^2}\left(\text{respectively, } \frac{b}{2\nu_r(\varphi_{F_1})}\Big(\hat{\mu}_D(\nu_r) - \big(\frac{b}{\nu_r(\varphi_{F_1})}\overline{\beta}_{g+1}(\nu_r) + \theta_1^r(D)\big)\Big)\right).$$

The area of the parallelogram Q_1, Q_2, Q_3 and Q_4 is $\frac{b}{\nu_r(\varphi_{F_1})}\theta_1^r(D)$. Thus, the area of Δ will be the sum of the above areas, which is

$$\frac{2ab + b^2\delta}{2} = \frac{D^2}{2}.$$

With respect to Δ', we have to add the area of the triangles with vertices $(0,0), Q_5$ and Q_6, and Q_7, Q_8 and Q_9, to the area of a trapezium whose vertices are Q_5, Q_6, Q_7 and Q_8. The areas of the triangles are equal to $\frac{a^2}{2\nu_r(\varphi_{M_0})^2}\overline{\beta}_{g+1}(\nu_r)$

and

$$\frac{a+b\delta}{2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))}\left(\hat{\mu}_D(\nu_r)-\left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r)-\theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1})}\right)\right).$$

The length of the parallel sides of the trapezium and the distance between them are

$$\frac{a}{\nu_r(\varphi_{M_0})},\quad \frac{a+b\delta}{\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1})}\text{ and}$$
$$\frac{-\theta_1^r(D)(\delta\overline{\beta}_{g+1}(\nu_r)+\nu_r(\varphi_{M_0})^2)}{\nu_r(\varphi_{M_0})(\nu_r(\varphi_{M_0})-\delta\nu_r(\varphi_{F_1}))},$$

and the area is

$$\frac{-\theta_1^r(D)\left((2a+b\delta)\nu_r(\varphi_{M_0})+a\delta\nu_r(\varphi_{F_1})\right)(\delta\overline{\beta}_{g+1}(\nu_r)+\nu_r(\varphi_{M_0})^2)}{2\nu_r(\varphi_{M_0})^2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))^2}.$$

When adding, the coefficients of $\overline{\beta}_{g+1}(\nu_r)$ are cancelled. Therefore, we only have to add the following fractions

$$\frac{(a+b\delta)\hat{\mu}_D(\nu_r)}{2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))},\ \frac{\theta_1^r(D)(a+b\delta)\nu_r(\varphi_{M_0})}{2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))^2}\text{ and}$$
$$\frac{-\theta_1^r(D)\nu_r(\varphi_{M_0})^2((2a+b\delta b)\nu_r(\varphi_{M_0})+a\delta\nu_r(\varphi_{F_1}))}{2\nu_r(\varphi_{M_0})^2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))^2},$$

giving rise to the desired value $D^2/2$.

Let us show that the defining points of Δ and Δ' that do not appear in the items (a), (b) and (c) belong to the line L which goes through $(0,0)$ and Q_9. It is clear that $(0,0), Q_1, Q_3$ (respectively, Q_5 and Q_7) and Q_9 are in L when $\theta_1^r(D)\geq 0$ (respectively, $\theta_1^r(D)<0$) and p_{r+1} is a free point. This corresponds to Item (c).

Now we suppose that p_{r+1} is satellite and $r\preccurlyeq\eta$. Then, p_r is also a satellite point, $g^*=g$ and, by [19, Proposition 2.5], one obtains that

$$\nu_r(\varphi_\eta)=e_{g-1}(\nu_\eta)\overline{\beta}_g(\nu_r)=e_{g-1}(\nu_r)\overline{\beta}_g(\nu_r)\frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)}=\overline{\beta}_{g+1}(\nu_r)\frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)},$$

where $e_{g-1}(\nu_i)=\gcd(\overline{\beta}_0(\nu_i),\overline{\beta}_1(\nu_i),\ldots,\overline{\beta}_{g-1}(\nu_i))$, for $i=r$ or η. Moreover, by the proof of Lemma 3.9 in [19], it holds that $e_{g-1}(\nu_\eta)\overline{\beta}_g(\nu_r)-e_{g-1}(\nu_r)\overline{\beta}_g(\nu_\eta)=-1$ and then

$$\nu_r(\varphi_\eta)+1=e_{g-1}(\nu_r)\overline{\beta}_g(\nu_\eta)=\overline{\beta}_{g+1}(\nu_r)\frac{\overline{\beta}_g(\nu_\eta)}{\overline{\beta}_g(\nu_r)}.$$

Also, we have that

$$\nu_\eta(\varphi_{F_1})=\frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)}\nu_r(\varphi_{F_1})\text{ and }\nu_\eta(\varphi_{M_0})=\frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)}\nu_r(\varphi_{M_0}).$$

As a result, it is easy to check that the points $(0,0)$ and Q_1, Q_3 (respectively, Q_5, Q_7) and Q_9 are in the line $L\equiv\overline{\beta}_0(\nu_r)y=\overline{\beta}_0(\nu_\eta)x$ when $\theta_1^r(D)\geq 0$ (respectively, $\theta_1^r(D)<0$), which corresponds to Item (b).

A similar reasoning can be applied to the case when p_{r+1} is satellite and $r\not\preccurlyeq\eta$. Notice that, in this case, Q_2, Q_4 (respectively, Q_6, Q_8) and Q_9 are in the line L when $\theta_1^r(D)\geq 0$ (respectively, $\theta_1^r(D)<0$).

As a consequence of our reasoning $\Delta_\nu(D)$ is a quadrilateral or a triangle. To conclude the proof, we show that $\Delta_\nu(D)$ is a triangle if and only if the conditions in the last two paragraphs of the statement hold. Otherwise, $\Delta_\nu(D)$ will be a quadrilateral.

Assume, for instance, that p_{r+1} is a satellite point and $r \preccurlyeq \eta$. Suppose also that $\theta_1^r(D) \geq 0$. In this case, $\Delta_\nu(D)$ is a triangle if and only if one of the following conditions is satisfied: Q_4 belongs to the line with equation $\overline{\beta}_g(\nu_\eta)x = \overline{\beta}_g(\nu_r)y$, or Q_4 belongs to the line which goes through Q_2 and Q_9. These two conditions happen if and only if $\theta_1^r(D) = 0$, which proves our statement. Now assume that $\theta_1^r(D) < 0$. Here, $\Delta_\nu(D)$ is a triangle if and only if one of the next conditions holds: Q_8 belongs to the line with equation $\overline{\beta}_g(\nu_\eta)x = \overline{\beta}_g(\nu_r)y$; Q_8 belongs to the line which goes through Q_6 and Q_9; or $Q_6 = (0,0) = Q_5$. The first and second conditions are true if and only if $\theta_1^r(D) = 0$, which is a contradiction because we have supposed that $\theta_1^r(D)$ is negative. The third one happens if and only if $\delta > 0$ and $a = 0$. This completes the proof after noticing that the remaining cases can be proved analogously. $\square$

An example that corresponds to Statement (a) of the above theorem is the next one.

Example 3.11. Let p be a special point of the Hirzebruch surface $\mathbb{F}_2$ and ν_r a special divisorial valuation centered at $\mathcal{O}_{\mathbb{F}_2,p}$ whose sequence of maximal contact values is $\{\overline{\beta}_i(\nu_r)\}_{i=0}^3 = \{20, 28, 153, 612\}$. Let $\mathcal{C}_{\nu_r} = \{p_i\}_{i=1}^{12}$ (with $p = p_1$) its configuration of infinitely near points and set F_1 the fiber which passes through p. Suppose that the strict transform of M_0 passes through p_2. Then, $\nu_r(\varphi_{F_1}) = 20$, $\nu_r(\varphi_{M_0}) = 28$ and $2\nu_r(\varphi_{F_1})\nu_r(\varphi_{M_0}) + \nu_r(\varphi_{F_1})^2\delta = 1920 > 612 = \overline{\beta}_{g+1}(\nu_r)$. So, ν_r is non-positive at infinity by [17, Theorem 3.6].

Let $\nu = \nu_{E_\bullet}$ be the valuation defined by the flag $E_\bullet = \{Z = Z_{12} \supset E_{12} \supset \{p_{13}\}\}$, where p_{13} is the intersection point $E_8 \cap E_{12}$. According to Theorem 3.10, $\Delta_\nu(F + 2M)$ is a quadrilateral with vertices

$$(0,0), Q_5 = \left(\frac{612}{28}, \frac{152}{28}\right), Q_7 = \left(\frac{4068}{68}, \frac{1012}{68}\right) \text{ and } Q_9 = (156, 39),$$

since ν_r is non-minimal with respect to $F + 2M$ by Corollary 2.8, $\theta_1^r(F + 2M) < 0$ and $12 \not\preccurlyeq 8$. Figure 1 shows the Newton-Okounkov body $\Delta_\nu(F + 2M)$ (in dark) and the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_{F+2M}(\nu)$ given in Proposition 3.3.

Now we are going to determine the Newton-Okounkov $\Delta_\nu(D)$ for valuations ν as in Case B introduced before Theorem 3.10. That is, we assume that $g^* = 1$ and $\nu(\varphi_{F_1}) = \overline{\beta}_1(\nu)$. Note that, in this case, it could happen that $\nu(\varphi_{M_0}) = (0,0)$ and then $\theta_1^r(D) = a\nu_r(\varphi_{F_1}) \geq 0$. In addition, we can assume that D is a big and nef divisor because when D is big and not nef, $\Delta_\nu(D)$ can be computed as we explained in the paragraph under Theorem 3.4. Following [24, Theorem 6.4] and Proposition 3.9, if p_{r+1} is the satellite point $E_r \cap E_\eta$ and $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), then the points Q_1, Q_2, Q_3, Q_4 (respectively, Q_5, Q_6, Q_7, Q_8) and Q_9, described before Theorem 3.10 for the satellite case, belong to $\Delta_\nu(D)$. Otherwise

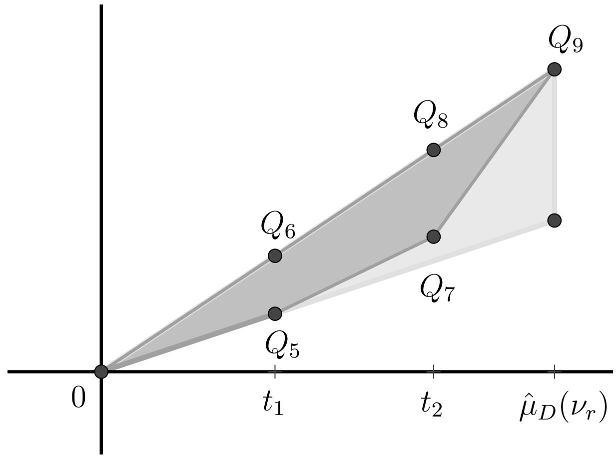

FIGURE 1. $\Delta_\nu(F+2M)$ and $\mathfrak{C}(\nu)\cap\mathfrak{H}_{F+2M}(\nu)$ in Example 3.11.

(p_{r+1} is free), the points

$$\begin{aligned} Q_1 &= \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})}, 0\right), Q_2 = Q_1 + \left(0, \frac{b}{\nu_r(\varphi_{F_1})}\right), \\ Q_3 &= \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})} + \theta_1^r(D), \frac{\theta_1^r(D)}{\nu_r(\varphi_{F_1})}\right) \text{ and } Q_4 = Q_3 + \left(0, \frac{b}{\nu_r(\varphi_{F_1})}\right) \end{aligned} \tag{3.3}$$

(respectively, Q_5, Q_6, Q_7, Q_8 given before Theorem 3.10 for the free case) and

$$Q_9 = (\hat{\mu}_D(\nu_r), a + b\delta)$$

are in $\Delta_\nu(D)$ if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$).

Theorem 3.12. *Let ν be a valuation in Case B. With the notation as in the previous paragraph, the Newton-Okounkov body $\Delta_\nu(D)$ of D with respect to ν is a quadrilateral if and only if $a \neq 0$. Otherwise, it is a triangle.*

(a) *When $\nu(\varphi_{M_0}) = (0,0)$, the vertices of the quadrilateral are:*
  (a.1) *$(0,0), Q_2, Q_4$ (respectively, Q_1, Q_3) and Q_9 if p_{r+1} is the satellite point $E_\eta \cap E_r$ and $r \not\preccurlyeq \eta$, (respectively, $r \preccurlyeq \eta$).*
  (a.2) *$(0,0), Q_1, Q_3$ and Q_9 whenever p_{r+1} is a free point.*
  In addition, if $\delta > 0$ and $a = 0$, then the vertices of the triangle $\Delta_\nu(D)$ are the above ones, where $Q_1 = Q_3$ and $Q_2 = Q_4$.

(b) *When $\nu(\varphi_{M_0}) \neq (0,0)$, the vertices of the quadrilateral are:*
  (b.1) *$(0,0), Q_2, Q_3$ (respectively, Q_5, Q_8) and Q_9 if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r$ and $r \not\preccurlyeq \eta$.*
  (b.2) *$(0,0), Q_1, Q_4$ (respectively, Q_6, Q_7) and Q_9 if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r$ and $r \preccurlyeq \eta$.*
  (b.3) *$(0,0), Q_1, Q_4$ (respectively, Q_6, Q_7) and Q_9 if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$) and p_{r+1} is a free point.*
  Moreover, if $\delta > 0$ and $a = 0$, the vertices of the triangle $\Delta_\nu(D)$ are the above ones where $Q_5 = (0,0) = Q_6$.

Proof. Consider the convex sets defined by the points $\{(0,0), Q_1, Q_2, Q_3, Q_4, Q_9\}$ and $\{(0,0), Q_5, Q_6, Q_7, Q_8, Q_9\}$. Reasoning as in the proof of Theorem 3.10, we deduce that the area of both sets is $D^2/2$.

To prove items (a.1) and (a.2), it suffices to check that the points defining $\Delta_\nu(D)$ that do not appear in the statement belong to the line L that passes through $(0,0)$ and Q_9. L is defined by the equation $L \equiv \overline{\beta}_{g^*}(\nu_r)y = \overline{\beta}_{g^*}(\nu_\eta)x$ (respectively, $L \equiv y = x/\overline{\beta}_{g+1}(\nu_r)$) when p_{r+1} is a satellite point (respectively, p_{r+1} is a free point). This is because

$$\nu_\eta(\varphi_{F_1}) = \nu_r(\varphi_{F_1})\frac{\overline{\beta}_{g^*}(\nu_\eta)}{\overline{\beta}_{g^*}(\nu_r)} \quad \left(\text{respectively, } \nu_r(\varphi_{F_1}) = \overline{\beta}_{g+1}(\nu_r)\right) \tag{3.4}$$

if p_{r+1} is a satellite point (respectively, p_{r+1} is a free point). To verify the point alignment we can use [19, Proposition 2.5 and Lemma 3.9], which proves that

$$\nu_r(\varphi_\eta) = \overline{\beta}_{g+1}(\nu_r)\frac{\overline{\beta}_{g^*}(\nu_\eta)}{\overline{\beta}_{g^*}(\nu_r)} \quad \left(\text{respectively, } \nu_r(\varphi_\eta)+1 = \overline{\beta}_{g+1}(\nu_r)\frac{\overline{\beta}_{g^*}(\nu_\eta)}{\overline{\beta}_{g^*}(\nu_r)}\right) \tag{3.5}$$

if $r \not\preccurlyeq \eta$ (respectively, $r \preccurlyeq \eta$).

To conclude the proof, we only show Item (b.1) since the remaining items (b.2) and (b.3) run similarly. First, we suppose that $\theta_1^r(D) \geq 0$. By (3.4) and (3.5), the points $(0,0), Q_1$ and Q_3 belong to the line with equation $\overline{\beta}_{g^*}(\nu_r)y = \overline{\beta}_{g^*}(\nu_\eta)x$. Moreover, it is easily seen that Q_4 belongs to the line which goes through Q_2 and Q_9. Finally, the point Q_9 does not belong neither to the line with equation $\overline{\beta}_0(\nu_r)y = \overline{\beta}_0(\nu_\eta)x$ nor to that with equation $\overline{\beta}_{g^*}(\nu_r)y = \overline{\beta}_{g^*}(\nu_\eta)x$, which finishes the proof in this case where $\theta_1^r(D) \geq 0$.

It only remains to assume that $\theta_1^r(D) < 0$, then

$$\nu_\eta(\varphi_{M_0}) = \nu_r(\varphi_{M_0})\frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)} \text{ and } \nu_r(\varphi_\eta)+1 = \overline{\beta}_{g+1}(\nu_r)\frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)}.$$

Therefore, $(0,0), Q_6$ and Q_8 belong to the line with equation $\overline{\beta}_0(\nu_r)y = \overline{\beta}_0(\nu_\eta)x$. Moreover Q_7 is in the line which goes through Q_5 and Q_9, which completes the proof. □

We finish this section by describing the Newton-Okounkov bodies $\Delta_\nu(D)$ in Case C introduced before Theorem 3.10. Then, suppose that $g^* = 1$ and $\nu(\varphi_{M_0}) = \overline{\beta}_1(\nu)$. Here, we can assume that D is a big and nef divisor except for the case when all the points $\{p_i\}_{i=1}^{r+1}$ are free. In this last situation, $p_{r+1} \in \operatorname{supp}(N_{D^*})$ if and only if D is big and not nef.

First assume that D is big and nef. According to [24, Theorem 6.4] and Proposition 3.9, when p_{r+1} is the satellite point $E_r \cap E_\eta$ and $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), the points Q_1, Q_2, Q_3, Q_4 (respectively, Q_5, Q_6, Q_7, Q_8) and Q_9, described before Theorem 3.10 for the satellite case, are in $\Delta_\nu(D)$.

When p_{r+1} is free, the points

$$\begin{gathered} Q_5 = \left(\frac{a\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_0})}, 0\right), Q_6 = Q_5 + \left(0, \frac{a}{\nu_r(\varphi_{M_0})}\right), \\ Q_7 = \left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r) - \theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}, \frac{-\theta_1^r(D)}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}\right), \\ Q_8 = Q_7 + \left(0, \frac{a+b\delta}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}\right) \end{gathered} \tag{3.6}$$

(respectively, Q_1, Q_2, Q_3, Q_4 provided before Theorem 3.10 for the free case) and $Q_9 = (\hat{\mu}_D(\nu_r), b)$ belong to $\Delta_\nu(D)$ if $\theta_1^r(D) < 0$ (respectively, $\theta_1^r(D) \geq 0$).

Finally, assume that D is big and not nef and all the points in $\{p_i\}_{i=1}^{r+1}$ are free. Recall that these assumptions are equivalent to the fact that $p_{r+1} \in \operatorname{supp}(N_{D^*})$ (see the paragraph after Theorem 3.4). Then, the points

$$P_1 = \left(\frac{-a\nu_r(\varphi_{M_0})}{\delta}, \frac{-a}{\delta}\right),$$

$$P_2 = \left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r) - \theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}, \frac{-\theta_1^r(D)}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}\right),$$

$$P_3 = P_2 + \left(0, \frac{a+b\delta}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}\right) \text{ and } P_4 = (\hat{\mu}_D(\nu_r), b)$$

are in $\Delta_\nu(D)$.

Next, we state our result for Case C, where, as mentioned, D is big and nef except when $p_{r+1} \in \operatorname{supp}(N_{D^*})$. We recall that the Newton-Okounkov bodies $\Delta_\nu(D)$ for the remaining cases where D is big but not nef can be reduced to the big and nef situation (see the paragraph below Theorem 3.4).

Theorem 3.13. *Let ν be a valuation in Case C. Under the above assumptions and notation, the Newton-Okounkov body $\Delta_\nu(D)$ of D with respect to ν is a quadrilateral if and only if $a \neq 0$ and D is nef. Otherwise, it is a triangle.*

(a) *When D is a big and nef divisor, the vertices of the quadrilateral are:*
   (a.1) *$(0,0), Q_1, Q_4$ (respectively, Q_6, Q_7) and Q_9 if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r$ and $r \not\preccurlyeq \eta$.*
   (a.2) *$(0,0), Q_2, Q_3$ (respectively, Q_5, Q_8) and Q_9 if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r$ and $r \preccurlyeq \eta$.*
   (a.3) *$(0,0), Q_2, Q_3$ (respectively, Q_5, Q_8) and Q_9 if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$) and p_{r+1} is a free point.*

   Moreover, if $\delta > 0$ and $a = 0$, the vertices of the triangle $\Delta_\nu(D)$ are the above ones where $Q_5 = (0,0) = Q_6$.

(b) *If D is big but not nef and all the points in $\{p_i\}_{i=1}^{r+1}$ are free, the vertices of the triangle $\Delta_\nu(D)$ are P_1, P_3 and P_4.*

Proof. Item (a) follows as in Theorem 3.12 (b) after considering that

$$\nu_\eta(\varphi_{M_0})\overline{\beta}_{g^*}(\nu_r) = \nu_r(\varphi_{M_0})\overline{\beta}_{g^*}(\nu_\eta) \text{ and } \nu_\eta(\varphi_{F_1})\overline{\beta}_0(\nu_r) = \nu_r(\varphi_{F_1})\overline{\beta}_0(\nu_\eta).$$

Now we are going to prove Item (b). For a start, the area of the convex set Δ generated by P_1, P_2, P_3 and P_4 is $P_{D^*}^2/2$. Indeed, the area of the triangle generated by P_1, P_2 and P_3 (respectively, P_2, P_3 and P_4) is

$$\frac{(a+b\delta)\left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r) - \theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1})} - \frac{-a\nu_r(\varphi_{M_0})}{\delta}\right)}{2(\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1}))}$$

$$\left(\text{respectively, } \frac{(a+b\delta)\left(\hat{\mu}_D(\nu_r) - \frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r) - \theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1})}\right)}{2(\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1}))}\right).$$

Therefore, the area of Δ is the sum of the two above areas, which is

$$\frac{(a+b\delta)\left(\hat{\mu}_D(\nu_r)-\frac{-a\nu_r(\varphi_{M_0})}{\delta}\right)}{2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))}=\frac{(a+b\delta)\left(b+\frac{a}{\delta}\right)(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))}{2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))}$$
$$=\frac{\left(\left(b+\frac{a}{\delta}\right)M^*\right)^2}{2}=\frac{P_{D^*}^2}{2}.$$

Finally, P_2 belongs to the line which goes through P_1 and P_4, and P_4 is not in the line with equation $\overline{\beta}_{g+1}(\nu_r)y=x$, which completes the proof of Item (b). □

Remark 3.14. The forthcoming Table 1 (Subsection 3.3) summarizes Theorems 3.10, 3.12 and 3.13. Moreover, the particular cases $\delta=1$, $a=0$ and $\theta_1^r(D)<0$ in those theorems provide the Newton-Okounkov bodies described in [19, Corollary 5.2]. This holds because $\mathbb{F}_1$ is the blowup of the projective plane $\mathbb{P}^2$ at a point, and the special section, in this case, is the exceptional divisor.

3.2. **Newton-Okounkov bodies with respect to non-positive at infinity non-special valuations.** In this last subsection, we complete Subsection 3.1 by considering non-special valuations. Denote by ν a non-positive at infinity non-special exceptional curve valuation whose first component is ν_r. Also, assume that $D\sim aF+bM$ is a big and nef divisor on $\mathbb{F}_\delta$ (since p_1 is a general point by Definition 2.4). We will use the notation $\theta_2^r(D)$ for the value $a\nu_r(\varphi_{F_1})-b\big(\nu_r(\varphi_{M_1})-\delta\nu_r(\varphi_{F_1})\big)$, where F_1 and M_1 are as defined below Definition 2.4. The following results translate to the non-special case what happens in Subsection 3.1 for the special one.

Lemma 3.15. *Let ν_r be a non-positive at infinity non-special divisorial valuation of $\mathbb{F}_\delta$. Set D and $\theta_2^r(D)$ as above. Then, the divisor*

$$D_3=D^*-\frac{b}{\nu_r(\varphi_{F_1})}\sum_{i=1}^r\nu_r(\mathfrak{m}_i)E_i^*\left(\textit{respectively, } D_4=D^*-\frac{a+b\delta}{\nu_r(\varphi_{M_1})}\sum_{i=1}^r\nu_r(\mathfrak{m}_i)E_i^*\right)$$

is nef when $\theta_2^r(D)\geq 0$ (respectively, $\theta_2^r(D)<0$).

Proof. We are going to show that D_4 is nef when $\theta_2^r(D)<0$. The fact that the divisor D_3 is nef follows from a similar reasoning as that used in Lemma 3.5.

Write

$$\Delta_r:=(\nu_r(\varphi_{M_1})-\delta\nu_r(\varphi_{F_1}))F^*+\nu_r(\varphi_{F_1})M^*-\sum_{i=1}^r\nu_r(\mathfrak{m}_i)E_i^*\text{ and}$$

$$\Gamma_r:=\nu_r(\varphi_{M_1})M^*-\delta\sum_{i=1}^r\nu_r(\mathfrak{m}_i)E_i^*.$$

Both divisors are nef by [17, Theorem 4.8] and this concludes the proof since

$$D_4\sim\frac{a}{\nu_r(\varphi_{M_1})-\delta\nu_r(\varphi_{F_1})}\Delta_r+\frac{-\theta_2^r(D)}{\nu_r(\varphi_{M_1})-\delta\nu_r(\varphi_{F_1})}\Gamma_r$$

and $-\theta_2^r(D)>0$. □

The following result can be proved reasoning as in the proof of Lemma 3.6. Notice that we are considering a non-special divisorial valuation whose non-positivity at infinity can be checked with the inequality below Definition 2.5. Recall that we are considering a big and nef divisor $D \sim aF + bM$ on $\mathbb{F}_\delta$. We will also use the value $\theta_2^r(D)$.

Lemma 3.16. *Let ν_r be a non-positive at infinity non-special divisorial valuation of $\mathbb{F}_\delta$. Then, the rational numbers*

$$t_5 = \frac{b}{\nu_r(\varphi_{F_1})}\overline{\beta}_{g+1}(\nu_r) \text{ and } t_6 = \frac{b}{\nu_r(\varphi_{F_1})}\overline{\beta}_{g+1}(\nu_r) + \theta_2^r(D)$$

$$\left(\text{respectively, } t_7 = \frac{a+b\delta}{\nu_r(\varphi_{M_1})}\overline{\beta}_{g+1}(\nu_r) \text{ and } t_8 = \frac{a\overline{\beta}_{g+1}(\nu_r) - \nu_r(\varphi_{M_1})\theta_2^r(D)}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}\right)$$

belong to the set $T_{D,\nu_r} := \{t \in \mathbb{Q} \mid 0 \leq t \leq \hat{\mu}_D(\nu_r)\}$ when $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D) < 0$).

Remark 3.17. As in the special divisorial valuations case, if ν_r is minimal with respect to D, by Theorem 2.6 and Corollary 2.7, one gets

$$\hat{\mu}_D(\nu_r) = b\overline{\beta}_{g+1}(\nu_r)/\nu_r(\varphi_{F_1}) = t_5 = t_6 = t_7 = t_8.$$

Otherwise, Lemma 3.16 provides two values, t_5 and t_6 (respectively, t_7 and t_8) when $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D) < 0$). When $\theta_2^r(D) = 0$, one has that $\hat{\mu}_D(\nu_r) > t_5 = t_6 = t_7 = t_8$, and if $a = 0$ and $\theta_2^r(D) < 0$, then $t_8 = \hat{\mu}_D(\nu_r)$. Moreover, if the equality $2\nu_r(\varphi_{M_1})\nu_r(\varphi_{F_1}) - \delta\nu_r(\varphi_{F_1})^2 = \overline{\beta}_{g+1}(\nu_r)$ holds, we deduce that $t_6 = \hat{\mu}_D(\nu_r)$ (respectively, $t_8 = \hat{\mu}_D(\nu_r)$) whenever $\theta_2^r(D) > 0$ (respectively, $\theta_2^r(D) < 0$).

Reasoning as in Lemma 3.8, one proves that the divisors D_3 and D_4 in Lemma 3.15 are big. Moreover, $D_3 \cdot \tilde{F}_1 = 0$, $D_4 \cdot \tilde{M}_1 = 0$, and $D_3 \cdot E_i = 0$ and $D_4 \cdot E_i = 0$, for $1 \leq i \leq r-1$. As a consequence, one gets the following result.

Lemma 3.18. *Let ν_r be a divisorial valuation and D a divisor as in Lemma 3.15. Assume also that ν_r is non-minimal with respect to D. The intersection matrix determined by the set of divisors $\{\tilde{F}_1, E_1, \ldots, E_{r-1}\}$ (respectively, $\{\tilde{M}_1, E_1, \ldots, E_{r-1}\}$) is negative definite.*

Our upcoming proposition considers a valuation ν_r and a divisor D as stated before Lemma 3.15 and determines the Zariski decomposition of the divisors $D^* - t_iE_r$, $5 \leq i \leq 8$, where t_i are the rational numbers defined in Lemma 3.16. We will use the above defined value $\theta_2^r(D)$ and the divisors D_3, D_4 and $\Delta_r = (\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))F^* + \nu_r(\varphi_{F_1})M^* - \sum_{i=1}^r \nu_r(\mathfrak{m}_i)E_i^*$ given in Lemma 3.15 and its proof.

Proposition 3.19. *The following statements hold.*

(a) *The positive and negative parts of the Zariski decomposition of the divisor $D_{t_5} = D^* - t_5 E_r$ (respectively, $D_{t_6} = D^* - t_6 E_r$) are*

$$P_{D_{t_5}} \sim D_3 \quad \textit{and} \quad N_{D_{t_5}} = \frac{b}{\nu_r(\varphi_{F_1})} \sum_{i=1}^{r-1} \nu_r(\varphi_i) E_i$$

$$\left(\textit{respectively, } P_{D_{t_6}} \sim \frac{b}{\nu_r(\varphi_{F_1})} \Delta_r \textit{ and} \right.$$

$$\left. N_{D_{t_6}} = \frac{\theta_2^r(D)}{\nu_r(\varphi_{F_1})} \tilde{F} + \sum_{i=1}^{r-1} \frac{b\nu_r(\varphi_i) + \theta_2^r(D)\nu_i(\varphi_{F_1})}{\nu_r(\varphi_{F_1})} E_i \right),$$

when $\theta_2^r(D) \geq 0$.

(b) *The positive and negative parts of the Zariski decomposition of $D_{t_7} = D^* - t_7 E_r$ (respectively, $D_{t_8} = D^* - t_8 E_r$) are*

$$P_{D_{t_7}} \sim D_4 \quad \textit{and} \quad N_{D_{t_7}} = \frac{a + b\delta}{\nu_r(\varphi_{M_1})} \sum_{i=1}^{r-1} \nu_r(\varphi_i) E_i$$

$$\left(\textit{respectively, } P_{D_{t_8}} \sim \frac{a}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})} \Delta_r \textit{ and} \right.$$

$$\left. N_{D_{t_8}} = \frac{-\theta_2^r(D)}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})} \tilde{M}_1 + \sum_{i=1}^{r-1} \frac{a\nu_r(\varphi_i) - \theta_2^r(D)\nu_i(\varphi_{M_1})}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})} E_i \right),$$

when $\theta_2^r(D) < 0$.

Proof. We are going to prove Statement (b). A proof for (a) runs similarly. On the one hand, the components of the divisor $N_{D_{t_7}}$ determine a negative definite intersection matrix. On the other hand, the divisor $P_{D_{t_7}}$ is nef by Lemma 3.15 and orthogonal to each component of $N_{D_{t_7}}$ by the proximity equalities. So, $P_{D_{t_7}} + N_{D_{t_7}}$ gives the Zariski decomposition of D_{t_7}.

Let us show the claim for D_{t_8}. By Lemma 3.18, the components of $N_{D_{t_8}}$ determine a negative definite intersection matrix and, by [17, Proposition 4.1 and Theorem 4.8], the divisor $P_{D_{t_8}}$ is nef and orthogonal to each component of $N_{D_{t_8}}$. Finally, we are going to see that $P_{D_{t_8}} + N_{D_{t_8}} \sim D_{t_8}$, which completes the proof. Indeed, let $p_{i_{M_1}}$ be the last point in the configuration of infinitely near points $\mathcal{C}_{\nu_r}$ of the valuation ν_r through which the strict transform of M_1 goes. Since $\tilde{M}_1 \sim M^* - \sum_{i=1}^{i_{M_1}} E_i^*$, it holds that

$$\frac{a(\Delta_r + \sum_{i=1}^{r-1} \nu_r(\varphi_i) E_i) + \theta_2^r(D) M^*}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})} \sim D - \frac{a\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})} E_r.$$

In addition,

$$\frac{-\theta_2^r(D)}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{M_1})} \left(\sum_{i=1}^{r-1} \nu_i(\varphi_{M_1}) E_i - \sum_{i=1}^{i_{M_1}} E_i^* \right) = \frac{-\theta_2^r(D)\nu_r(\varphi_{M_1})}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})} E_r,$$

and the result follows after adding both expressions. □

We conclude our paper by determining the vertices of the Newton-Okounkov bodies $\Delta_\nu(D)$, where D and ν are as in the paragraph before Lemma 3.15. Recall that ν_r is the first component of ν. We again divide our description of $\Delta_\nu(D)$ in two cases:

Case D: Either $g^* > 1$ or $g^* = 1$ and $\nu(\varphi_{M_1}) \neq \overline{\beta}_1(\nu)$.

Case E: The value g^* equals 1 and $\nu(\varphi_{M_1}) = \overline{\beta}_1(\nu)$.

Let us start with the case D. Arguing as before Theorem 3.10, the points

$$Q_{10} = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})}, \frac{b\nu_r(\varphi_\eta)}{\nu_r(\varphi_{F_1})}\right) \left(\text{respectively, } Q_{10} = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})}, 0\right)\right),$$
$$Q_{11} = Q_{10} + \left(0, \frac{b}{\nu_r(\varphi_{F_1})}\right),$$
$$Q_{12} = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})} + \theta_2^r(D), \frac{b\nu_r(\varphi_\eta) + \theta_2^r(D)\nu_\eta(\varphi_{F_1})}{\nu_r(\varphi_{F_1})}\right)$$
$$\left(\text{respectively } Q_{12} = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})} + \theta_2^r(D), 0\right)\right) \text{ and } Q_{13} = Q_{12} + \left(0, \frac{b}{\nu_r(\varphi_{F_1})}\right) \tag{3.7}$$

belong to $\Delta_\nu(D)$ when $\theta_2^r(D) \geq 0$ and the point p_{r+1} is satellite (respectively, free). When $\theta_2^r(D) < 0$ and the point p_{r+1} is satellite (respectively, free)), the points in $\Delta_\nu(D)$ are:

$$Q_{14} = \left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_1})}, \frac{(a+b\delta)\nu_r(\varphi_\eta)}{\nu_r(\varphi_{M_1})}\right)$$
$$\left(\text{respectively, } Q_{14} = \left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_1})}, 0\right)\right), Q_{15} = Q_{14} + \left(0, \frac{a+b\delta}{\nu_r(\varphi_{M_1})}\right),$$
$$Q_{16} = \left(\frac{a\overline{\beta}_{g+1}(\nu_r) - \theta_2^r(D)\nu_r(\varphi_{M_1})}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}, \frac{a\nu_r(\varphi_\eta) - \theta_2^r(D)\nu_\eta(\varphi_{M_1})}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}\right)$$
$$\left(\text{respectively, } Q_{16} = \left(\frac{a\overline{\beta}_{g+1}(\nu_r) - \theta_2^r(D)\nu_r(\varphi_{M_1})}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}, 0\right)\right)$$
$$\text{and } Q_{17} = Q_{16} + \left(0, \frac{a}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}\right). \tag{3.8}$$

Also, when p_{r+1} is satellite (respectively, free), the point $Q_{18} = (\hat{\mu}_D(\nu_r), \hat{\mu}_D(\nu_\eta))$ (respectively, $Q_{18} = (\hat{\mu}_D(\nu_r), 0)$) belongs to $\Delta_\nu(D)$ by Theorem 2.6.

Theorem 3.20. *Let ν be a valuation in Case D. With the notation as in the previous paragraphs, the Newton-Okounkov body $\Delta_\nu(D)$ of D with respect to ν is a quadrilateral if and only if $a \neq 0$ and $\theta_2^r(D) \neq 0$. Otherwise, it is a triangle.*

The vertices of the quadrilateral are:

(a) *$(0,0), Q_{10}, Q_{12}$ (respectively, Q_{14}, Q_{16}) and Q_{18} when $\theta_2^r(D) > 0$ (respectively, $\theta_2^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r$ and $r \not\preccurlyeq \eta$.*

(b) *$(0,0), Q_{11}, Q_{13}$ (respectively, Q_{15}, Q_{17}) and Q_{18} when $\theta_2^r(D) > 0$ (respectively, $\theta_2^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r$ and $r \preccurlyeq \eta$.*

(c) *$(0,0), Q_{11}, Q_{13}$ (respectively, Q_{15}, Q_{17}) and Q_{18} when $\theta_2^r(D) > 0$ (respectively, $\theta_2^r(D) < 0$) and p_{r+1} is a free point.*

When $a = 0$ and $\theta_2^r(D) < 0$, $Q_{16} = Q_{18} = Q_{17}$ and the vertices of the triangle $\Delta_\nu(D)$ are as described in items (a), (b) and (c).

Finally, replacing $\theta_2^r(D) > 0$ or $\theta_2^r(D) < 0$ with $\theta_2^r(D) = 0$ in items (a), (b) and (c) we obtain the vertices of the triangle $\Delta_\nu(D)$ because $Q_{10} = Q_{12} = Q_{14} = Q_{16}$ in Case (a) and $Q_{11} = Q_{13} = Q_{15} = Q_{17}$ otherwise.

Proof. We are going to show that $D^2/2$ is the area of the convex set Δ generated by the points $(0,0), Q_{14}, Q_{15}, Q_{16}, Q_{17}$ and Q_{18}. The case concerning the points $(0,0), Q_{10}, Q_{11}, Q_{12}, Q_{13}$ and Q_{18} and the fact of being a quadrilateral or a triangle follow as in the proof of Theorem 3.10.

The area of the triangle with vertices $(0,0), Q_{14}$ and Q_{15} (respectively, Q_{16}, Q_{17} and Q_{18}) is

$$\frac{(a+b\delta)^2}{2\nu_r(\varphi_{M_1})^2}\overline{\beta}_{g+1}(\nu_r)\Bigg(\text{respectively,}$$

$$\frac{a}{2(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))}\left(\hat{\mu}_D(\nu_r) - \left(\frac{a\overline{\beta}_{g+1}(\nu_r) - \theta_2^r(D)\nu_r(\varphi_{M_1})}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}\right)\right)\Bigg).$$

The area of the trapezium given by Q_{14}, Q_{15}, Q_{16} and Q_{17} is

$$\frac{-\theta_2^r(D)\left((a+b\delta)(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})) + a\nu_r(\varphi_{F_1})\right)\left(\nu_r(\varphi_{M_0})^2 - \delta\overline{\beta}_{g+1}(\nu_r)\right)}{2\nu_r(\varphi_{M_1})^2(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))^2}.$$

Adding the above three areas, we notice that the coefficient of $\overline{\beta}_{g+1}(\nu_r)$ vanishes and it suffices to add the following three fractions:

$$\frac{a\hat{\mu}_D(\nu_r)}{2(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))}, \frac{a\theta_2^r(D)\nu_r(\varphi_{M_1})}{2(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))^2} \text{ and}$$
$$\frac{-\theta_2^r(D)\nu_r(\varphi_{M_1})^2((a+b\delta)(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})) + a\nu_r(\varphi_{F_1}))}{2\nu_r(\varphi_{M_1})^2(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))^2}.$$

After computing, one gets $(2ab + \delta b^2)/2$, which concludes the proof. □

Example 3.21. Let p be a general point of the Hirzebruch surface $\mathbb{F}_2$ and ν_r a non-special divisorial valuation centered at $\mathcal{O}_{\mathbb{F}_2,p}$, whose sequence of maximal contact values is $\{\overline{\beta}_i(\nu_r)\}_{i=0}^3 = \{15, 51, 262, 786\}$. Let $\mathcal{C}_{\nu_r} = \{p_i\}_{i=1}^{12}$ (with $p = p_1$) be its configuration of infinitely near points, F_1 the fiber which passes through p, and M_1 the irreducible section linearly equivalent to M that passes through p and whose strict transform passes through p_2 and p_3. Notice that this means that the self-intersection of $\tilde{M}_1$ is negative. Then, $\nu_r(\varphi_{F_1}) = 15$ and $\nu_r(\varphi_{M_1}) = 45$ and so $2\nu_r(\varphi_{F_1})\nu_r(\varphi_{M_1}) - \nu_r(\varphi_{F_1})^2\delta = 900 > 786 = \overline{\beta}_{g+1}(\nu_r)$. As a consequence, ν_r is non-positive at infinity by [17, Theorem 4.8].

Let $\nu = \nu_{E_\bullet}$ be the valuation defined by the flag

$$E_\bullet = \{Z = Z_{12} \supset E_{12} \supset \{p_{13}\}\},$$

where p_{13} is the intersection point $E_9 \cap E_{12}$. By Theorem 3.20, the coordinates of the vertices of the Newton-Okounkov body $\Delta_\nu(2F + 5M)$ are

$$(0,0), Q_{14} = \left(\frac{9432}{45}, \frac{3132}{45}\right), Q_{16} = \left(\frac{3597}{15}, \frac{1197}{15}\right) \text{ and } Q_{18} = (255, 85),$$

since ν_r is non-minimal with respect to $2F+5M$ by Corollary 2.8, $\theta_2^r(D)<0$ and $12=r\not\preccurlyeq\eta=9$.

Finally, assume that ν is in Case E. By [24, Theorem 6.4] and Proposition 3.19, if p_{r+1} is the satellite point $E_r\cap E_\eta$ and $\theta_2^r(D)\geq 0$ (respectively, $\theta_2^r(D)<0$), the points $Q_{10},Q_{11},Q_{12},Q_{13}$ (respectively, $Q_{14},Q_{15},Q_{16},Q_{17}$) and Q_{18} provided before Theorem 3.20 for the satellite case belong to $\Delta_\nu(D)$. When p_{r+1} is a free point and $\theta_2^r(D)<0$ (respectively, $\theta_2^r(D)\geq 0$), the points

$$\begin{gathered}Q_{14}=\left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_1})},0\right),Q_{15}=Q_{14}+\left(0,\frac{a+b\delta}{\nu_r(\varphi_{M_1})}\right),\\ Q_{16}=\left(\frac{a\overline{\beta}_{g+1}(\nu_r)-\theta_2^r(D)\nu_r(\varphi_{M_1})}{\nu_r(\varphi_{M_1})-\delta\nu_r(\varphi_{F_1})},\frac{-\theta_2^r(D)}{\nu_r(\varphi_{M_1})-\delta\nu_r(\varphi_{F_1})}\right),\\ Q_{17}=Q_{16}+\left(0,\frac{a}{\nu_r(\varphi_{M_1})-\delta\nu_r(\varphi_{F_1})}\right)\end{gathered}\tag{3.9}$$

(respectively, $Q_{10},Q_{11},Q_{12},Q_{13}$ given before Theorem 3.20 for the free case) and $Q_{18}=(\hat{\mu}_D(\nu_r)),b)$ are in $\Delta_\nu(D)$.

Theorem 3.22. *Let ν be a valuation in Case E. Under the above assumptions and notation, the Newton-Okounkov body $\Delta_\nu(D)$ of D with respect to ν is a quadrilateral if and only if $a\neq 0$. Otherwise, it is a triangle.*

The vertices of the quadrilateral are:

(a) $(0,0),Q_{10},Q_{13}$ *(respectively, Q_{15},Q_{16}) and Q_{18} if $\theta_1^r(D)\geq 0$ (respectively, $\theta_1^r(D)<0$), p_{r+1} is the satellite point $E_r\cap E_\eta$ and $r\not\preccurlyeq\eta$.*

(b) $(0,0),Q_{11},Q_{12}$ *(respectively, Q_{14},Q_{17}) and Q_{18} if $\theta_1^r(D)\geq 0$ (respectively, $\theta_1^r(D)<0$), p_{r+1} is the satellite point $E_r\cap E_\eta$ and $r\preccurlyeq\eta$.*

(c) $(0,0),Q_{11},Q_{12}$ *(respectively, Q_{14},Q_{17}) and Q_{18} if $\theta_1^r(D)\geq 0$ (respectively, $\theta_1^r(D)<0$) and p_{r+1} is a free point.*

In addition, if $a=0$, then the vertices of the triangle $\Delta_\nu(D)$ are the previous ones where $Q_{16}=Q_{18}=Q_{17}$.

Proof. It follows reasoning as in the proof of Theorem 3.20 to compute the area of the convex sets generated by the points given in the statement, and arguing as in Theorem 3.4 (b), after taking into account the equalities

$$\nu_\eta(\varphi_{M_1})\overline{\beta}_{g^*}(\nu_r)=\nu_r(\varphi_{M_1})\overline{\beta}_{g^*}(\nu_\eta)\text{ and }\nu_\eta(\varphi_{F_1})\overline{\beta}_0(\nu_r)=\nu_r(\varphi_{F_1})\overline{\beta}_0(\nu_\eta).$$

□

Table 2 in the next subsection summarizes Theorem 3.20 and Theorem 3.22.

3.3. **Tables.** In this subsection, and for the reader's convenience, we provide two tables summarizing the results of our main theorems on Newton-Okounkov bodies with respect to non-minimal exceptional curve valuations of Hirzebruch surfaces (the minimal case is described in Theorem 3.4). Thus, Table 1 summarizes Theorems 3.10, 3.12 and 3.13 given in Subsection 3.1, while Table 2 summarizes Theorems 3.20 and 3.22, which appear in Subsection 3.2.

Next, we give some additional information to ease the reading of the tables.

Conditions \ Theorems	Theorem 3.10 (Case A)	Theorem 3.12 (Case B)	Theorem 3.13 (Case C)
$\theta_1^r(D) > 0, p_{r+1}$ is the satellite point $E_\eta \cap E_r$ and $r \not\preccurlyeq \eta$	$(0,0), Q_1, Q_3, Q_9$	$(0,0), Q_2, Q_4, Q_9$ $(0,0), Q_2, Q_3, Q_9$	$(0,0), Q_1, Q_4, Q_9$
$\theta_1^r(D) = 0, p_{r+1}$ is the satellite point $E_\eta \cap E_r$ and $r \not\preccurlyeq \eta$	$(0,0), Q_1 = Q_3, Q_9$	$(0,0), Q_2 = Q_4, Q_9$ $(0,0), Q_2, Q_3, Q_9$	$(0,0), Q_1, Q_4, Q_9$
$\theta_1^r(D) < 0, p_{r+1}$ is the satellite point $E_\eta \cap E_r$ and $r \not\preccurlyeq \eta$	$(0,0), Q_5, Q_7, Q_9$	 $(0,0), Q_5, Q_8, Q_9$	$(0,0), Q_6, Q_7, Q_9$
$\theta_1^r(D) > 0, p_{r+1}$ is the satellite point $E_\eta \cap E_r$ and $r \preccurlyeq \eta$	$(0,0), Q_2, Q_4, Q_9$	$(0,0), Q_1, Q_3, Q_9$ $(0,0), Q_1, Q_4, Q_9$	$(0,0), Q_2, Q_3, Q_9$
$\theta_1^r(D) = 0, p_{r+1}$ is the satellite point $E_\eta \cap E_r$ and $r \preccurlyeq \eta$	$(0,0), Q_2 = Q_4, Q_9$	$(0,0), Q_1 = Q_3, Q_9$ $(0,0), Q_1, Q_4, Q_9$	$(0,0), Q_2, Q_3, Q_9$
$\theta_1^r(D) < 0, p_{r+1}$ is the satellite point $E_\eta \cap E_r$ and $r \preccurlyeq \eta$	$(0,0), Q_6, Q_8, Q_9$	 $(0,0), Q_6, Q_7, Q_9$	$(0,0), Q_5, Q_8, Q_9$
$\theta_1^r(D) > 0$ and p_{r+1} is a free point	$(0,0), Q_2, Q_4, Q_9$	$(0,0), Q_1, Q_3, Q_9$ $(0,0), Q_1, Q_4, Q_9$	$(0,0), Q_2, Q_3, Q_9$
$\theta_1^r(D) = 0$ and p_{r+1} is a free point	$(0,0), Q_2 = Q_4, Q_9$	$(0,0), Q_1 = Q_3, Q_9$ $(0,0), Q_1, Q_4, Q_9$	$(0,0), Q_2, Q_3, Q_9$
$\theta_1^r(D) < 0$ and p_{r+1} is a free point	$(0,0), Q_6, Q_8, Q_9$	 $(0,0), Q_6, Q_7, Q_9$	$(0,0), Q_5, Q_8, Q_9$
$\delta > 0, a = 0, \theta_1^r(D) \leq 0$, p_{r+1} is the satellite point $E_\eta \cap E_r$ and $r \not\preccurlyeq \eta$	$(0,0) = Q_5, Q_7, Q_9$	$(0,0), Q_2 = Q_4, Q_9$ $(0,0) = Q_5, Q_8, Q_9$	$(0,0) = Q_6, Q_7, Q_9$
$\delta > 0, a = 0, \theta_1^r(D) \leq 0$, p_{r+1} is the satellite point $E_\eta \cap E_r$ and $r \preccurlyeq \eta$	$(0,0) = Q_6, Q_8, Q_9$	$(0,0), Q_1 = Q_3, Q_9$ $(0,0) = Q_6, Q_7, Q_9$	$(0,0) = Q_5, Q_8, Q_9$
$\delta > 0, a = 0, \theta_1^r(D) \leq 0$, and p_{r+1} is a free point	$(0,0) = Q_6, Q_8, Q_9$	$(0,0), Q_1 = Q_3, Q_9$ $(0,0) = Q_6, Q_7, Q_9$	$(0,0) = Q_5, Q_8, Q_9$

TABLE 1. Vertices of the Newton-Okounkov bodies described in Subsection 3.1.

The tables show the vertices of the Newton-Okounkov bodies $\Delta_\nu(D)$, introduced in Definition 3.2, of big and nef divisors $D = aF + bM$ on surfaces $\mathbb{F}_\delta, \delta \geq 0$ (see Subsection 2.1 for the definition of F and M). These Newton-Okounkov

Conditions \ Theorems	Theorem 3.20 (Case D)	Theorem 3.22 (Case E)
$\theta_2^r(D) > 0, p_{r+1}$ is the satellite point $E_\eta \cap E_r$ and $r \not\preccurlyeq \eta$	$(0,0), Q_{10}, Q_{12}, Q_{18}$	$(0,0), Q_{10}, Q_{13}, Q_{18}$
$\theta_2^r(D) = 0, p_{r+1}$ is the satellite point $E_\eta \cap E_r$ and $r \not\preccurlyeq \eta$	$(0,0), Q_{10} = Q_{12} = Q_{14} = Q_{16}, Q_{18}$	$(0,0), Q_{10}, Q_{13}, Q_{18}$
$\theta_2^r(D) < 0, p_{r+1}$ is the satellite point $E_\eta \cap E_r$ and $r \not\preccurlyeq \eta$	$(0,0), Q_{14}, Q_{16}, Q_{18}$	$(0,0), Q_{15}, Q_{16}, Q_{18}$
$\theta_2^r(D) > 0, p_{r+1}$ is the satellite point $E_\eta \cap E_r$ and $r \preccurlyeq \eta$	$(0,0), Q_{11}, Q_{13}, Q_{18}$	$(0,0), Q_{11}, Q_{12}, Q_{18}$
$\theta_2^r(D) = 0, p_{r+1}$ is the satellite point $E_\eta \cap E_r$ and $r \preccurlyeq \eta$	$(0,0), Q_{11} = Q_{13} = Q_{15} = Q_{17}, Q_{18}$	$(0,0), Q_{11}, Q_{12}, Q_{18}$
$\theta_2^r(D) < 0, p_{r+1}$ is the satellite point $E_\eta \cap E_r$ and $r \preccurlyeq \eta$	$(0,0), Q_{15}, Q_{17}, Q_{18}$	$(0,0), Q_{14}, Q_{17}, Q_{18}$
$\theta_2^r(D) > 0$ and p_{r+1} is a free point	$(0,0), Q_{11}, Q_{13}, Q_{18}$	$(0,0), Q_{11}, Q_{12}, Q_{18}$
$\theta_2^r(D) = 0$ and p_{r+1} is a free point	$(0,0), Q_{11} = Q_{13} = Q_{15} = Q_{17}, Q_{18}$	$(0,0), Q_{11}, Q_{12}, Q_{18}$
$\theta_2^r(D) < 0$ and p_{r+1} is a free point	$(0,0), Q_{15}, Q_{17}, Q_{18}$	$(0,0), Q_{14}, Q_{17}, Q_{18}$
$a = 0, \theta_2^r(D) < 0$, p_{r+1} is the satellite point $E_\eta \cap E_r$ and $r \not\preccurlyeq \eta$	$(0,0), Q_{14}, Q_{16} = Q_{18}$	$(0,0), Q_{15}, Q_{16} = Q_{18}$
$a = 0, \theta_2^r(D) < 0$, p_{r+1} is the satellite point $E_\eta \cap E_r$ and $r \preccurlyeq \eta$	$(0,0), Q_{15}, Q_{17} = Q_{18}$	$(0,0), Q_{14}, Q_{17} = Q_{18}$
$a = 0, \theta_2^r(D) < 0$, and p_{r+1} is a free point	$(0,0), Q_{15}, Q_{17} = Q_{18}$	$(0,0), Q_{14}, Q_{17} = Q_{18}$

TABLE 2. Vertices of the Newton-Okounkov bodies described in Subsection 3.2.

bodies are with respect to non-minimal (Definition 2.2) non-positive at infinity (Definition 2.5) valuations ν of $\mathbb{F}_\delta$. Table 1 considers the cases where ν is special and Table 2 those where ν is non-special.

With respect to the notation in our tables, divisors E_r, E_η defining the divisorial valuations ν_r and ν_η, respectively, and the point p_{r+1} are introduced at the beginning of Section 3. The concepts of satellite and free point and the ordering $\preccurlyeq$ on the set of vertices of the dual graph of a valuation are given in Subsection 2.1.

The definition of the value $\theta_1^r(D)$ (respectively, $\theta_2^r(D)$) can be found at the beginning of Subsection 3.1 (respectively, Subsection 3.2). The conditions to distinguish cases A, B and C (respectively, D and E) are given after the proof of Proposition 3.9 (respectively, Proposition 3.19). The points Q_i appearing in our tables can be found after stating the above mentioned cases.

Finally, the data to understand which are the above cases and the coordinates of the points are in Subsection 2.1 (maximal contact values $\{\overline{\beta}_j\}_{j=0}^{g+1}$ and germs φ_C of curves C), and before Proposition 3.3 for the value g^*.

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