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ON THE ALGEBRAS $VN(H)$ AND $VN(H)^*$ OF AN ULTRASPHERICAL HYPERGROUP H

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Abstract

Let H be an ultraspherical hypergroup and let $A(H)$ be the Fourier algebra associated with H. In this paper, we study the dual and the double dual of $A(H)$. We prove among other things that the subspace of all uniformly continuous functionals on $A(H)$ forms a C^*-algebra. We also prove that the double dual $A(H)^{**}$ is neither commutative nor semisimple with respect to the Arens product, unless the underlying hypergroup H is finite. Finally, we study the unit elements of $A(H)^{**}$.

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1. Introduction

Let G be a locally compact group and let $A(G)$ be the Fourier algebra associated with G. In 1973, Dunkl and Ramirez [9] introduced the notion of weakly almost periodic functionals on $A(G)$ extending the classical notion of weakly almost periodic functions on locally compact abelian groups. In 1974, the space of uniformly continuous functionals was introduced by Granirer [13], extending yet another classical notion of uniformly continuous functions. These spaces and their duals were well studied by many authors.

Let H be an ultraspherical hypergroup associated with a locally compact group G and a spherical projector π. Let $A(H)$ denote the Fourier algebra corresponding to the ultraspherical hypergroup H. Also, let $VN(H)$ denote the hypergroup von Neumann algebra of H. The algebra $A(H)$, although introduced a decade ago, has not received much attention. For some of the recent contributions to $A(H)$, see for example, [7, 10, 11, 23]. This paper is a continuation of the series of results obtained by the first and the second authors. The modest aim of this note is to contribute to the study of the dual and the bidual of $A(H)$.

The notion of weakly almost periodic functionals and uniformly continuous functionals on $A(H)$ has been studied in a recent paper [11]. In Section 3 of this paper, we show that the space of all uniformly continuous functionals forms a C*-algebra. This is Theorem 3.4. For the case of locally compact groups, the main ingredient in the proof of this is the fact that singletons are sets of spectral synthesis. In 2014, Degenfeld-Schonburg *et al.* [7] produced examples of ultraspherical hypergroups for which singletons need not be sets of spectral synthesis for the corresponding Fourier algebra. Our idea is to show that the space of uniformly continuous functionals is the image of a C*-algebra under a C*-homomorphism.

In Section 4, we study the dual of $VN(H)$. We characterize the existence of left and right identities for the dual of $VN(H)$. We also characterize when the algebra $VN(H)^*$ is semisimple. Then we take up the dual of the space of uniformly continuous functionals.

One of the classical results of Richard F. Arens says that if G is an infinite locally compact group, then the group algebra $L^1(G)$ cannot be Arens regular. This result has been generalized to the Fourier algebra of a locally compact group by Lau [17] under the assumption that the group G is amenable. In Theorem 4.3, we prove that the Fourier algebra of an ultraspherical hypergroup is Arens regular if and only if H is finite under the assumption that $A(H)$ has a bounded approximate identity.

Finally, in Section 5, we study unit elements in $VN(H)^*$. Our main aim in this section is to characterize when an element of $A(H)^{**}$ is also an element of $A(H)$ in terms of the unit elements.

2. Preliminaries

2.1. Hypergroups. We first recall the definition of a hypergroup. In [15], Jewett calls a hypergroup a convo.

DEFINITION 2.1. A nonempty locally compact Hausdorff space H is said to be a hypergroup if there exists a binary operation $*$ on $M(H)$, the space of all complex-valued bounded Radon measures on H, satisfying the following conditions.

(i) $(M(H), *)$ is an algebra.

(ii) For every $x, y \in H$, $\delta_x * \delta_y$ is a probability measure and the mapping $(x, y) \mapsto \delta_x * \delta_y$ is continuous from $H \times H$ to $M^1(H)$.

(iii) There exists a unique element $e \in H$ such that for all $x \in H$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$.

(iv) There exists a unique homeomorphism $x \mapsto x^-$ of H such that the following hold.

  (a) $(x^-)^- = x$ for all $x \in H$.

  (b) If $\widehat{\mu}$ is defined by $\int_H f(x)\, d\widehat{\mu}(x) = \int_H f(x^-)\, d\mu(x)$ for all $f \in C_c(H)$, then $\widehat{\delta_x * \delta_y} = \delta_{y^-} * \delta_{x^-}$ for all $x, y \in H$.

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(c) For every $x, y \in H$, $\mathrm{supp}(\delta_x * \delta_y)$ is compact. Further, the mapping $(x, y) \mapsto \mathrm{supp}(\delta_x * \delta_y)$ is continuous from $H \times H$ to $\mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the space of all nonempty compact subsets of H equipped with the 'Michael Topology'. See [21].

The following lemma is well known in the context of locally compact groups. As we cannot find any reference, we are providing a proof here.

LEMMA 2.2. *Let H be a hypergroup. If every σ-compact open subhypergroup of H is compact, then H is compact.*

PROOF. The proof is by contradiction. Suppose that H is noncompact. Let H_0 be the subhypergroup of H generated by a compact symmetric neighborhood of e. Then H_0 is open and σ-compact and so under our hypothesis, H_0 is compact. Choose x_1 in H that is not in H_0. Then the union of H_0 and $\{x_1, (x_1)^-\}$ is again a compact symmetric neighborhood of e. Let H_1 be the subhypergroup that it generates. Again, H_1 is open and σ-compact and hence compact. Choose x_2 in H that is not in H_1 and let H_2 be the subghypergroup generated by the union of H_1 and $\{x_2, (x_2)^-\}$. As said earlier, H_2 is compact. We can keep doing this infinitely, thereby obtaining a sequence x_n of elements from H and an increasing sequence H_n of open, compact subgroups with x_n in H_n but not in H_{n-1}. Let H' be the union of all the H_n. Then, once again, H' is an open σ-compact subhypergroup of H and hence is compact. So the sequence $\{x_n\}$ must have a cluster point y in H'. But if y is in H, it is in H_k for some k, so H_k is a neighborhood of y that does not contain x_n for any $n > k$, which is a contradiction. □

We now define the notion of a spherical projector on a locally compact group G [22, Definition 2.1].

DEFINITION 2.3. A map $\pi : C_c(G) \to C_c(G)$ is called a spherical projector if it is linear and satisfies the following for all $f, g \in C_c(G)$.

(i) (a) $\pi^2 = \pi$ and π is positivity preserving;
(b) $\pi(\pi(f)g) = \pi(f)\pi(g)$;
(c) $\langle \pi(f), g \rangle = \langle f, \pi(g) \rangle$;
(d) $\int_G \pi(f)(x)\, dx = \int_G f(x)\, d$.

(ii) $\pi(\pi(f) * \pi(g)) = \pi(f) * \pi(g)$.

(iii) Let $\pi^* : M(G) \to M(G)$ denote the transpose of π and let $O_x = \mathrm{supp}(\pi^*(\delta_x))$, $x \in G$. Then for all $x, y \in G$:

(a) either $O_x \cap O_y = \emptyset$ or $O_x = O_y$;
(b) if $x \in O_y$ then $x^{-1} \in O_{y^{-1}}$;
(c) if $O_{xy} = O_e$ then $O_y = O_{x^{-1}}$;
(d) the map $x \mapsto O_x$ from G to $\mathcal{K}(G)$ is continuous.

Note that π extends to a norm decreasing linear map on various function spaces, including $L^p(G)$, $1 \le p \le \infty$. A function f is called π-radial if $\pi(f) = f$ and similarly, a measure μ is called π-radial if $\pi^*(\mu) = \mu$.

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Let $H = \{\dot{x} = O_x : x \in G\}$, equipped with the natural quotient topology under the quotient map $p : G \to H$. We identify $M(H)$ with the space of all π^*-radial measures in $M(G)$, with the product defined by $\delta_{\dot{x}} * \delta_{\dot{y}} = \pi^*(\pi^*(\delta_x * \delta_y))$ for all $x, y \in G$. With this structure, H becomes a locally compact hypergroup, called a spherical hypergroup [22, Theorem 2.12]. A spherical hypergroup is further called an ultraspherical hypergroup if the modular function on G is π-radial.

The most common example of an ultraspherical hypergroup is the double coset hypergroup. Let G be a locally compact group containing a compact subgroup K. Define $\pi : C_c(G) \to C_c(G)$ as

$$\pi(f)(x) = \int_K \int_K f(k'xk)\, dk\, dk'.$$

Then π defines a spherical projector and the resulting hypergroup is an ultraspherical hypergroup. Another interesting class of examples for ultraspherical hypergroups in the context of Lie groups is due to Damek and Ricci [6]. The spherical projector was called an average projector in [6].

A *left Haar measure* on a hypergroup H is a nonzero regular Borel measure m such that $p_x * m = m$ for all $x \in H$. In this note, by a Haar measure, we mean a left Haar measure. It is well known that commutative (or compact or discrete) hypergroups admit a Haar measure. In fact, a Haar measure on a hypergroup (if it exists) is unique up to a scalar multiple [15]. It remains an open question whether every locally compact hypergroup admits a Haar measure. It is shown in [22] that a Haar measure exists on an ultraspherical hypergroup.

2.2. Fourier algebra. Let G be a locally compact group with a fixed Haar measure denoted by dx. Let $A(G)$ and $B(G)$ denote the Fourier and Fourier–Stieltjes algebras, respectively, introduced by Eymard [12] in his seminal paper. Let H be an ultraspherical hypergroup associated to a locally compact group G and a spherical projector π. The map π extends to be a norm decreasing contraction on $A(G)$.

The Fourier algebra of an ultraspherical hypergroup is defined as the range of π inside $A(G)$. The algebra $A(H)$ is commutative, semisimple, regular and Tauberian, and the character space $\Delta(A(H))$ of $A(H)$ can be canonically identified with H. The Fourier–Stieltjes algebra of H is defined as the algebra of all π-radial functions in $B(G)$. For more details on these algebras, see [22].

Let λ denote the left regular representation of H on $L^2(H)$ given by

$$\lambda(\dot{x})(f)(\dot{y}) = f(\dot{x}^- * \dot{y}) \quad (\dot{x}, \dot{y} \in H, f \in L^2(H)).$$

This can be extended to $L^1(H)$ by $\lambda(f)(g) = f * g$ for all $f \in L^1(H)$ and $g \in L^2(H)$. Let $C^*_\lambda(H)$ denote the completion of $\lambda(L^1(H))$ in $B(L^2(H))$ which is called the reduced C^*-algebra of H. The von Neumann algebra generated by $\{\lambda(\dot{x}) : \dot{x} \in H\}$ is called the von Neumann algebra of H, and is denoted by $VN(H)$. Note that $VN(H)$ is isometrically isomorphic to the dual of $A(H)$. Moreover, $A(H)$ can be considered as an ideal of

$B_\lambda(H)$, where $B_\lambda(H)$ is the dual of $C^*_\lambda(H)$. We denote by $\langle\cdot,\cdot\rangle_0$ the duality between $C^*_\lambda(H)$ and $B_\lambda(H)$.

Let $q : C^*_\lambda(H) \longrightarrow C^*_\lambda(H)^{**}$ be the canonical embedding and $i : C^*_\lambda(H) \longrightarrow VN(H)$ be the inclusion map. Then, by [8, Proposition 12.1.5], there exists an ultraweakly continuous $*$-homomorphism $\tilde{i}$ from $C^*_\lambda(H)^{**}$ onto $VN(H)$ such that

$$\tilde{i}q(T) = i(T) = T \quad (T \in C^*_\lambda(H)).$$

The following lemma is a simple and useful observation, which is a consequence of [8, 12.1.5] and [8, 12.1.3(ii)].

LEMMA 2.4. *If $T \in C^*_\lambda(H)$ and $u \in A(H)$, then $\langle T, u\rangle_0 = \langle u, \tilde{i}(T)\rangle$.*

Let K be a subhypergroup of an ultraspherical hypergroup H and let $L = p^{-1}(K)$. Then L is a closed subgroup of G and the pair $(L, \pi|_L)$ defines the ultraspherical hypergroup K. It follows from [22, Theorem 3.10] that $A(H)$ and $A(K)$ can be identified with $A_\pi(G)$ and $A_{\pi|_L}(L)$ as Banach spaces, respectively.

2.3. Arens product. Let $\mathcal{A}$ be a commutative Banach algebra. For any $a \in \mathcal{A}$, let $\rho_a : \mathcal{A} \longrightarrow \mathcal{A}$ be the mapping defined by $\rho_a(b) = ba$ for $b \in \mathcal{A}$. An element $a \in \mathcal{A}$ is called weakly completely continuous if ρ_a is a weakly compact operator on $\mathcal{A}$ and $\mathcal{A}$ is called weakly completely continuous if every $a \in \mathcal{A}$ is weakly completely continuous.

The Arens products on $\mathcal{A}^{**}$ are defined by the following three steps. For $u, v \in \mathcal{A}$, $T \in \mathcal{A}^*$ and $m, n \in \mathcal{A}^{**}$, we define $T \cdot u, u \cdot T, m \cdot T, T \cdot m \in \mathcal{A}^*$ and $m\Box n, m\Diamond n \in \mathcal{A}^{**}$ as follows:

$$\langle T\cdot u, v\rangle = \langle T, uv\rangle, \quad \langle u\cdot T, v\rangle = \langle T, uv\rangle$$
$$\langle m\cdot T, u\rangle = \langle m, T\cdot u\rangle, \quad \langle T\cdot m, u\rangle = \langle m, u\cdot T\rangle$$
$$\langle mn, T\rangle = \langle m\ , n\cdot T\rangle, \quad \langle m\Diamond n, T\rangle = \langle n, T\cdot m\rangle.$$

$\mathcal{A}$ is said to be Arens regular if $\Box$ and $\Diamond$ coincide on $\mathcal{A}^{**}$.

Let X be a closed topologically invariant subspace of $VN(H)$ containing $\lambda(\dot{e})$. A linear functional $m \in X^*$ is called a topologically invariant mean on X if

$$\|m\| = \langle m, \lambda(\dot{e})\rangle = 1 \quad \text{and} \quad \langle m, u\cdot T\rangle = u(\dot{e})\langle m, T\rangle$$

for every $T \in X$, $u \in A(H)$. We denote by $TIM(\widehat{H})$ the set of all topologically invariant means on $VN(H)$. It is shown in [23] that $VN(H)$ always admits a topologically invariant mean.

Throughout this paper, G will denote a locally compact group, π a spherical projector and H an ultraspherical hypergroup associated with G and π.

3. Uniformly continuous functionals

In this section, we study the space of all uniformly continuous functionals on $A(H)$. We first show that this space is a C^*-algebra. The results of this section, corresponding to locally compact groups, can be found in [14]. We begin with a simple lemma. This is motivated by [22, Theorem 3.9].

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LEMMA 3.1. *The mapping π^* is a weak*–weak*-continuous isomorphism from $VN(H)$ onto $VN_\pi(G)$.*

PROOF. Let $\psi : VN_\pi(G) \to VN(H)$ be defined as $\psi(T) = T|_{L^2(H)}$. It is shown by Muruganandam [22] that ψ is a *-isomorphism and therefore ψ^{-1} is also a *-isomorphism. Thus, to prove this lemma, it would be enough to show that $\pi^* = \psi^{-1}$. Since $\{\lambda_H(f) : f \in C_c(H)\}$ is weak*-dense in $VN(H)$, it would suffice to show that $\pi^*(\lambda_H(f)) = \psi^{-1}(\lambda_H(f))$ for all $f \in C_c(H)$.

Let $f \in C_c(H)$. Then there exists a unique radial element $\widetilde{f} \in C_c(G)$ such that $\pi(\widetilde{f}) = f$. Further, $\psi(\lambda_G(\widetilde{f})) = \lambda_H(f)$. Hence, for $u \in A(G)$, we have

$$\begin{aligned}\langle u, \pi^*(\lambda_H(f))\rangle &= \langle \pi(u), \lambda_H(f)\rangle = \int_H \pi(u)(\dot{x}) f(\dot{x})\, d\dot{x}\\ &= \int_G \pi(u)(x)\widetilde{f}(x)\, dx = \int_G u(x)\pi(\widetilde{f})(x)\, dx\\ &= \int_G u(x)\widetilde{f}(x)\, dx = \langle u, \lambda_G(\widetilde{f})\rangle = \langle u, \psi^{-1}(\lambda_H)(f)\rangle.\end{aligned}$$

Hence, the lemma is proved. □

DEFINITION 3.2. Let $T \in VN(H)$. Then the support of T, denoted supp(T), is the closed set consisting of all elements $x \in H$ such that for every neighbourhood U_x of x, there exists $v \in A(H)$ with supp$(v) \subseteq U_x$ and $\langle T, v\rangle \neq 0$.

Let $UCB(\widehat{H})$ denote the closed linear span of $\{u \cdot T : u \in A(H), T \in VN(H)\}$. The elements in $UCB(\widehat{H})$ are called uniformly continuous functionals on $A(H)$. Let

$$UC_c(\widehat{H}) = \{T \in VN(H) : \text{supp}(T) \text{ is compact}\}.$$

REMARK 3.3. It can be shown as in the group case that if $T \in UC_c(\widehat{H})$, then there exists $u \in A(H)$ such that supp(u) is compact and $u \cdot T = T$. Thus, it follows that $UC_c(\widehat{H})$ is dense in $UCB(\widehat{H})$.

Here is our first main result. This is an analogue of [14, Proposition 2]. In this proof, ι will denote the canonical inclusion of $A(H)$ inside $A(G)$.

THEOREM 3.4. *The space $UCB(\widehat{H})$ is a C^*-subalgebra of $VN(H)$.*

PROOF. Our first claim is $\pi^*(UC_c(\widehat{H})) = UC_c(\widehat{G}) \cap VN_\pi(G)$.

Let $T \in UC_c(\widehat{H})$. Then, by Remark 3.3, there exists $u \in A(H) \cap C_c(H)$ such that $u \cdot T = T$. Let $\widetilde{u} \in A(G) \cap C_c(G)$ with $\pi(\widetilde{u}) = u$. Then, for $w \in A(G)$, we have

$$\begin{aligned}\langle w, \pi^*(T)\rangle &= \langle w, \pi^*(u \cdot T)\rangle = \langle \pi(w), u \cdot T\rangle\\ &= \langle \pi(w) \cdot u, T\rangle = \langle \pi(w) \cdot \pi(\widetilde{u}), T\rangle\\ &= \langle \pi(w \cdot \pi(\widetilde{u})), T\rangle = \langle w \cdot \iota(\pi(\widetilde{u})), \pi^*(T)\rangle\\ &= \langle w, \iota(\pi(\widetilde{u})) \cdot \pi^*(T)\rangle.\end{aligned}$$

Therefore, $\pi^*(T) = \iota(\pi(\widetilde{u})) \cdot \pi^*(T) \in UC_c(\widehat{G}) \cap VN_\pi(G)$.

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We now prove the other inclusion. Let $\widetilde{T} \in UC_c(\widehat{G}) \cap VN_\pi(G)$. By Lemma 3.1, there is a unique $T \in VN(H)$ such that $\pi^*(T) = \widetilde{T}$. Choose $v \in A_\pi(G)$ such that $\mathrm{supp}(v)$ is compact and $v \equiv 1$ on some open set containing $\mathrm{supp}(\widetilde{T})$. It is clear that $v.\widetilde{T} = \widetilde{T}$ and $\pi(v).T \in UC_c(\widehat{H})$. Now, for any $w \in A(G)$, we have

$$\begin{aligned}\langle w, \pi^*(\pi(v)\cdot T)\rangle &= \langle \pi(w), \pi(v)\cdot T\rangle = \langle \pi(v)\cdot\pi(w), T\rangle \\ &= \langle \pi(\pi(v)\cdot w), T\rangle = \langle \pi(v)\cdot w, \pi^*(T)\rangle \\ &= \langle w, \pi(v)\cdot\pi^*(T)\rangle = \langle w, v\cdot\widetilde{T}\rangle = \langle w, \widetilde{T}\rangle.\end{aligned}$$

Therefore, $\pi^*(\pi(v)\cdot T) = \widetilde{T}$. Since $\pi(v)\cdot T \in UC_c(\widehat{H})$, the converse follows.

Our second claim is $\pi^*(UCB(\widehat{H})) = UCB(\widehat{G}) \cap VN_\pi(G)$.

Let ψ be the map defined in Lemma 3.1. As π^* is a $*$-isomorphism between $VN(H)$ and $VN_\pi(G)$, it follows that π^* is a closed map. Hence, by the first claim, $\pi^*(UCB(\widehat{H}))$ is a closed subspace of $VN_\pi(G)$ containing $UC_c(\widehat{G}) \cap VN_\pi(G)$. Since $UC_c(\widehat{G}) \cap VN_\pi(G)$ is dense in $UCB(\widehat{G}) \cap VN_\pi(G)$, it follows that $\pi^*(UCB(\widehat{H})) = UCB(\widehat{G}) \cap VN_\pi(G)$.

Thus, we have $\psi(UCB(\widehat{G}) \cap VN_\pi(G)) = UCB(\widehat{H})$. Since $UCB(\widehat{G})$ and $VN_\pi(G)$ are C*-algebras and also as ψ is a *-isomorphism, it follows that $UCB(\widehat{H})$ is a C*-algebra. □

REMARK 3.5. The above theorem of course extends [14, Proposition 2]. However, our proof makes use of the same method.

For an ultraspherical hypergroup H, let

$$C(\widehat{H}) = \{T \in VN(H) : TC^*_\lambda(H) \cup C^*_\lambda(H)T \subseteq C^*_\lambda(H)\}.$$

Our next result says that the space $C(\widehat{H})$ is a C^*-algebra. For the case of locally compact groups, see [14]. As the proof of Proposition 3.6 follows the same lines as in [14], we omit the proof.

PROPOSITION 3.6. *Let H be an ultraspherical hypergroup. Then $C(\widehat{H})$ is a C^*-subalgebra of $VN(H)$ containing $UCB(\widehat{H})$.*

Before we proceed to the next result, here is some notation. Let $\mathcal{A}$ be a C^*-algebra. We let $w^\sim$ denote the topology on $\mathcal{A}^*$ such that $w^\sim - \lim_\alpha f_\alpha = f$ in $\mathcal{A}^*$ means:

(i) $\langle f_\alpha, a\rangle \longrightarrow \langle f, a\rangle$ for each $a \in \mathcal{A}$;
(ii) $\|f_\alpha\| \longrightarrow \|f\|$.

We write

$$\begin{aligned} M_\ell(\mathcal{A}) &= \{m \in \mathcal{A}^{**} : m\mathcal{A} \subseteq \mathcal{A}\},\\ M_r(\mathcal{A}) &= \{m \in \mathcal{A}^{**} : \mathcal{A}m \subseteq \mathcal{A}\},\\ M_{\ell+r}(\mathcal{A}) &= M_\ell(\mathcal{A}) + M_r(\mathcal{A}).\end{aligned}$$

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Our next result gives a characterization of when the algebras $C(\widehat{H})$ and $VN(H)$ are equal under the assumption that the underlying group is amenable. The proof of this is an adaptation of the proof given in [14] for the case of locally compact groups.

THEOREM 3.7. *Let H be an ultraspherical hypergroup on an amenable locally compact group G. Then $C(\widehat{H}) = VN(H)$ if and only if H is compact.*

PROOF. If H is compact, then $1 \in A(H)$ and therefore $VN(H) = 1 \cdot VN(H) \subseteq UCB(\widehat{H})$. Hence, by Proposition 3.6, $C(\widehat{H}) = VN(H)$.

Suppose that $C(\widehat{H}) = VN(H)$. To show that H is compact, by Lemma 2.2, it is enough to show that each σ-compact open subhypergroup K of H is compact.

Let K be a σ-compact open subhypergroup of H. It follows from [22, Definition 2.1] and [15, Lemma 13.1C] that the set $L = p^{-1}(K)$ is a noncompact σ-compact subgroup of G. Since G is amenable, it follows that L is amenable and hence by [7, Lemma 3.7], $A(K)$ possesses a bounded approximate identity $(e_n)_{n\in\mathbb{N}} \subseteq P(K) \cap C_c(K)$ bounded by 1. Now, by [23, Lemma 5.1], we can assume that $(e_n)_n \subseteq A(H)$. Note that since G is amenable, by [10, Theorem 3.4], $\mathbf{1}_K \in B_\lambda(H)$. It is easy to see that $e_n \longrightarrow \mathbf{1}_K$ uniformly on compacta on H and since $(e_n)_n$ is bounded, we conclude that $e_n \longrightarrow \mathbf{1}_K$ in $\sigma(B_\lambda(H), C^*_\lambda(H))$. Note that

$$\|e_n\| \longrightarrow 1 = \mathbf{1}_K(\dot{e}) = \|\mathbf{1}_K\|$$

and so $e_n \longrightarrow \mathbf{1}_K$ in the $w^\sim$-topology on $B_\lambda(H) = C^*_\lambda(H)^*$. Hence, it follows from [1, Proposition 2] that

$$\langle m, e_n\rangle \longrightarrow \langle m, \mathbf{1}_K\rangle, \quad (m \in M_{\ell+r}(C^*_\lambda(H))).$$

Now, using the fact that $C(\widehat{H}) = VN(H)$, it can be shown as in [14] that $\tilde{i}(M_{\ell+r}(C^*_\lambda(H))) = VN(H)$. As a consequence, it follows from Lemma 2.4 that the sequence $\{e_n\}$ is a weak Cauchy sequence in $A(H)$. Since $A(H)$ is the predual of the von Neumann algebra $VN(H)$, it is weakly sequentially complete and hence there exists $u \in A(H)$ such that $e_n \to u$ in the weak topology. As $e_n \to \mathbf{1}_K$ uniformly on compacta, it follows that $\mathbf{1}_K \in A(H)$. Thus, K is compact. □

4. The dual of $VN(H)$ and $UCB(\widehat{H})$

In this section, we study the dual of $VN(H)$. Our first two results generalize [17, Proposition 3.2].

PROPOSITION 4.1. *Let H be an ultraspherical hypergroup on a locally compact group G. Then $VN(H)^*$ has a right identity if and only if G is amenable.*

PROOF. Note that, by [7, Lemma 3.7], amenability of G implies the existence of a bounded approximate identity in $A(H)$, which is further equivalent to the existence of a right identity in $VN(H)^*$ by [3, Proposition III.28.7, page 146].

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For the converse, let E be a right identity in $VN(H)^*$. Then, by Goldstine's theorem, there is a net $(e_\alpha)_{\alpha\in I}$ in $A(H)$ with $\|e_\alpha\| \leq \|E\|$ such that

$$\text{weak}^*\text{-}\lim_{\alpha} e_\alpha = E.$$

Thus, for every $a \in A(H), T \in VN(H)$, we have

$$\langle a, T\rangle = \langle E\Box a, T\rangle = \lim_{\alpha}\langle e_\alpha a, T\rangle.$$

Hence, $(e_\alpha)_{\alpha\in I}$ is a bounded weak approximate identity for $A(H)$. Thus, by [3, Proposition I.11.4, page 58], $A(H)$ has a bounded approximate identity. Therefore, G is amenable, by [2, Theorem 4.4]. □

Proposition 4.2. *Let H be an ultraspherical hypergroup on an amenable locally compact group G. Then $VN(H)^*$ has a left identity if and only if H is compact.*

Proof. If H is compact, then $1 \in A(H)$. Therefore, $A(H)$ has an identity. Now, since $A(H)$ is in the centre of $VN(H)^*$ and Arens multiplication ($\Box$) is weak* continuous from the left, it follows that the identity of $A(H)$ is also the identity for $VN(H)^*$. Conversely, if H is not compact, then $UCB(\widehat{H})$ is a closed proper subspace of $VN(H)$, by Proposition 3.6 and Theorem 3.7. Hence, by the Hahn–Banach theorem, there exists a nonzero $m \in VN(H)^*$ such that $m = 0$ on $UCB(\widehat{H})$. So, $m\Box T = 0$ for each $T \in VN(H)$. It follows that $n\Box m = 0$ for each $n \in VN(H)^*$. Therefore, $VN(H)^*$ cannot have a left identity. □

Our next result characterizes Arens regularity of $A(H)$ when the underlying group is amenable. For the corresponding on locally compact groups, see [17, Proposition 3.3]. We denote by $W(\widehat{H})$ the set of all T in $VN(H)$ such that the map $u \longrightarrow u \cdot T$ from $A(H)$ into $VN(H)$ is weakly compact.

Theorem 4.3. *Let H be an ultraspherical hypergroup for which $A(H)$ has a bounded approximate identity. Then $A(H)$ is Arens regular if and only if H is finite.*

Proof. Let $A(H)$ be Arens regular. Then, it follows from [5, Theorem 3.14] that $W(\widehat{H}) = VN(H)$ and hence $VN(H)$ has a unique topologically invariant mean. Therefore, H is discrete by [23, Theorem 1.7]. Now, since $A(H)$ is the predual of the von Neumann algebra $VN(H)$, it is weakly sequentially complete. Finally, [4, Theorem 2.9.39] implies that $A(H)$ is unital, and thus H must be finite.

Conversely, if H is finite, then $A(H)$ is finite dimensional. In particular, $A(H)$ is reflexive. It then follows that $VN(H)^*$ is a commutative Banach algebra, which is equivalent to saying that $A(H)$ is Arens regular. □

In the next result, we characterize semisimplicity of the algebra $VN(H)^*$ assuming that the locally compact group G is amenable. See [17, Theorem 3.4] for the case of locally compact groups.

Proposition 4.4. *The Banach algebra $VN(H)^*$ is semisimple if and only if H is finite.*

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PROOF. Suppose that $VN(H)^*$ is semisimple. Let

$$\mathcal{I} = \{m \in VN(H)^* : \langle m, \lambda(\dot{e})\rangle = 0 \text{ and } m\Box v = v(\dot{e})m \text{ for each } v \in A(H)\}.$$

We first claim that $\mathcal{I}$ is an ideal in $VN(H)^*$. Indeed, let $m \in \mathcal{I}$ and $n \in VN(H)^*$. Then,

$$\langle n\Box m, \lambda(\dot{e})\rangle = \langle n, m\Box\lambda(\dot{e})\rangle = \langle n, \langle m, \lambda(\dot{e})\rangle\lambda(\dot{e})\rangle = \langle n, \lambda(\dot{e})\rangle\langle m, \lambda(\dot{e})\rangle = 0.$$

Similarly, $\langle m\Box n, \lambda(\dot{e})\rangle = 0$. Now, let $v \in A(H)$. Then,

$$(n\Box m)\Box v = n\Box(m\Box v) = v(\dot{e})(n\Box m).$$

Using the fact that $A(H)$ is in the centre of $VN(H)^*$, one can show that

$$(m\Box n)\Box v = v(\dot{e})(m\Box n).$$

Thus, $\mathcal{I}$ is an ideal in $VN(H)^*$.

Let $m, n \in \mathcal{I}$. Then, for each $T \in VN(H)$, we have

$$\langle m\Box n, T\rangle = \langle m, n \cdot T\rangle = \langle m, \lambda(\dot{e})\rangle\langle n, T\rangle = 0,$$

that is, $m\Box n = 0$. In particular, $\mathcal{I}$ is nil and hence, by [4, Proposition 1.5.6], $\mathcal{I} \subseteq Rad(VN(H)^*)$. If m_1 and m_2 are any two distinct elements of $TIM(\widehat{H})$, then it is clear that $m_1 - m_2 \in \mathcal{I}$. In particular, $\mathcal{I} \neq \emptyset$, which forces us to conclude that $VN(H)^*$ is not semisimple, which is a contradiction. Therefore, $TIM(\widehat{H})$ is a singleton and hence, by [24, Theorem 1.7], H is discrete.

We finally claim that H is finite. We show this by contradiction. Suppose that H is not finite. Let $\mathcal{I}$ denote the annihilator of $UCB(\widehat{H})$ in $VN(H)^*$. Then, as earlier, one can show that $\mathcal{I}$ is a nonzero ideal contained in $Rad(VN(H)^*)$, which will again force us to conclude that $VN(H)^*$ is not semisimple.

For the converse, if H is finite, then $A(H)$ is reflexive and hence $VN(H)^*$ is semisimple. □

For a Banach algebra $\mathcal{A}$, we denote by $\mathrm{Rad}(\mathcal{A})$ the radical of $\mathcal{A}$. Let

$$C^*_\lambda(H)^\perp = \{m \in VN(H)^* : \langle m, T\rangle = 0 \text{ for each } T \in C^*_\lambda(H)\}.$$

PROPOSITION 4.5. *The space $C^*_\lambda(H)^\perp$ is a weak*-closed ideal in $VN(H)^*$ containing $Rad(VN(H)^*)$. Furthermore, the Banach algebra $VN(H)^*/C^*_\lambda(H)^\perp$ is isometrically isomorphic to $B_\lambda(H)$.*

PROOF. Let $n \in VN(H)^*$ and let $T \in C^*_\lambda(H)$. Then, $n\Box T \in C^*_\lambda(H)$ by [10, Proposition 4.6]. Hence, for each $m \in C^*_\lambda(H)^\perp$ and $T \in C^*_\lambda(H)$, we have

$$\langle m\Box n, T\rangle = \langle m, n\Box T\rangle = 0.$$

This implies that $C^*_\lambda(H)^\perp$ is a right ideal. Since $C^*_\lambda(H)^\perp$ is weak*-closed by [19, Proposition 2.6.6], a simple weak* approximation argument gives that $C^*_\lambda(H)^\perp$ is an ideal.

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Let $P : VN(H)^*/C^*_\lambda(H)^\perp \longrightarrow B_\lambda(H)$ be defined by

$$P(m + C^*_\lambda(H)^\perp)(T) = \langle \psi, T\rangle \quad (T \in C^*_\lambda(H)),$$

where $\psi \in B_\lambda(H)$ is the restriction of m on $C^*_\lambda(H)$. Then, since $B_\lambda(H) = C^*_\lambda(H)^*$, the map P is an isometric isomorphism by [19, Theorem 1.10.16]. Since the Arens product on $B_\lambda(H)$ agrees with the pointwise multiplication [10, Proposition 4.12], it follows that P is an algebra isomorphism.

Finally, as P is an epimorphism of the Banach algebra $VN(H)^*/C^*_\lambda(H)^\perp$ onto the semisimple Banach algebra $B_\lambda(H)$, it follows from [3, Proposition III. 25.10, page 131] that $\mathrm{Rad}(VN(H)^*) \subseteq C^*_\lambda(H)^\perp$. □

PROPOSITION 4.6. *The spaces $C^*_\lambda(H)^\perp$ and $Rad(VN(H)^*)$ are equal if and only if the ultraspherical hypergroup H is discrete.*

PROOF. If H is discrete, then $C^*_\lambda(H) = UCB(\widehat{H})$ by [10, Proposition 4.4]. Now let $m, n \in C^*_\lambda(H)^\perp$ and let (v_α) be a net in $A(H)$ such that weak*-$\lim_\alpha v_\alpha = m$. Then we have

$$\langle m\square n, T\rangle = \lim_\alpha \langle v_\alpha \square n, T\rangle = \lim_\alpha \langle n, v_\alpha \cdot T\rangle = 0 \quad (T \in VN(H)),$$

and hence $(C^*_\lambda(H)^\perp)^2 = \{0\}$, which implies that $C^*_\lambda(H)^\perp \subseteq \mathrm{Rad}(VN(H)^*)$. It then follows from Proposition 4.5 that $C^*_\lambda(H)^\perp = \mathrm{Rad}(VN(H)^*)$.

For the converse, suppose that H is not discrete. Then $\lambda(\dot{e}) \notin C^*_\lambda(H)$ by [10, Corollary 4.8]. Now choose $m \in VN(H)^*$ such that $\langle m, \lambda(\dot{e})\rangle \neq 0$ and $\langle m, T\rangle = 0$ for each $T \in C^*_\lambda(H)$. If $C^*_\lambda(H)^\perp = \mathrm{Rad}(VN(H)^*)$, then $0 \neq m \in \mathrm{Rad}(VN(H)^*)$, which is impossible since $\mathrm{Rad}(VN(H)^*) \subseteq \ker\lambda(\dot{e})$. □

We finish this section with an analogue of Proposition 4.4 for $UCB(\widehat{H})^*$. For the corresponding result for the case of locally compact groups, see [18, Theorem 5.6].

PROPOSITION 4.7. *The Banach algebra $UCB(\widehat{H})^*$ is semisimple if and only if H is discrete.*

PROOF. If H is discrete, then $UCB(\widehat{H}) = C^*_\lambda(H)$ by [10, Proposition 4.6] and hence $UCB(\widehat{H})^* = B_\lambda(H)$ which is semisimple.

For the converse, let

$$\mathcal{I} = \{m \in UCB(\widehat{H})^* : \langle m, \lambda(\dot{e})\rangle = 0 \text{ and } m\square v = v(\dot{e})m \text{ for each } v \in A(H)\}.$$

By repeating the arguments as in the proof of Proposition 4.4, one can deduce that $\mathcal{I}$ is a closed ideal of $UCB(\widehat{H})^*$ and is contained in $\mathrm{Rad}(UCB(\widehat{H})^*)$. However then, $\mathcal{I} = 0$ by semisimplicity of $UCB(\widehat{H})^*$. By [10, Lemma 4.5], the set of topologically invariant means on $UCB(\widehat{H})$ is nonempty. Further, as the difference of any two topologically invariant means on $UCB(\widehat{H})$ lies in $\mathcal{I}$, we conclude that $UCB(\widehat{H})$ has a unique topologically invariant mean. Hence, by [10, Proposition 4.6], H is discrete. □

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5. Unit elements

In this section, we study the unit elements in $VN(H)^*$ in the spirit of [20]. The proofs given are modifications of the proofs given in [20]. Our main aim in this section is to characterize when an element of the double dual of $A(H)$ belongs to $A(H)$.

Let H be an ultraspherical hypergroup on a locally compact group G. Let K be an open and closed subhypergroup of H. Then, by [23, Lemma 5.1], the restriction map $u \longrightarrow u|_K$ is a norm decreasing algebra homomorphism from $A(H)$ onto $A(K)$, denoted by P_K.

Let H be an ultraspherical hypergroup on an amenable locally compact group G, let $\mathcal{E} = \mathcal{E}(A(H))$ denote the set of all right identities of the Banach algebra $A(H)^{**}$ that are the cluster points of all approximate identities in $A(H)$ bounded by 1.

REMARK 5.1. If $\mathcal{X}$ is a Banach space and if P is a bounded projection, then P^* is a projection and $P^*(\mathcal{X}^*)$ is identified by $P(\mathcal{X})^*$.

DEFINITION 5.2. Let $\mathcal{B}$ be a closed subalgebra of a Banach algebra $\mathcal{A}$ and let $P : \mathcal{A} \longrightarrow \mathcal{B}$ be a bounded projection. Let E be a right unit of $\mathcal{B}^{**}$ in $\mathcal{A}^{**}$. A right unit $\widetilde{E}$ of $\mathcal{A}^{**}$ is called an extension of E if $\langle \widetilde{E}, P^*(f)\rangle = \langle E, P^*(f)\rangle$ holds for all $f \in \mathcal{A}^*$.

LEMMA 5.3. *Let H be an ultraspherical hypergroup on an amenable locally compact group G. Let K be an open subhypergroup of H. Then every right unit of $A(K)^{**}$ in $A(H)^{**}$ can be extended to a right unit of $A(H)^{**}$. In particular, $A(H)^{**}$ has a unique right unit if and only if H is compact.*

PROOF. Since G is amenable, $A(H)$ has a bounded approximate identity, by [7, Lemma 3.7]. Since $A(K)$ is a closed ideal of $A(H)$ and $P_K : A(H) \longrightarrow A(K)$ is a bounded projection which is also a multiplier, it now follows from [20, Theorem 2.3] that every right unit of $A(K)^{**}$ in $A(H)^{**}$ can be extended to a right unit of $A(H)^{**}$.

Let E be a right identity of $A(H)^{**}$ and (e_α) be a bounded approximate identity of $A(H)$ associated with E. If H is compact, then $1 \in A(H)$ and $\lim_\alpha \|e_\alpha - 1\| = 0$. Then, for each $T \in A(H)^*$ and each $m \in A(H)^{**}$, we have

$$\begin{aligned}\langle E\Box m, T\rangle &= \lim_\alpha \langle e_\alpha \Box m, T\rangle \\ &= \lim_\alpha \langle e_\alpha, m\cdot T\rangle \\ &= \lim_\alpha \langle m\cdot T, 1\rangle \\ &= \langle m, T\rangle.\end{aligned}$$

Thus, E is also a left unit for $(A(H)^{**}, \Box)$, and hence E is unique.

If H is not compact, then there exists an open σ-compact and noncompact subhypergroup K of H, by Lemma 2.2. It follows from the definition of the spherical hypergroup and by [15, Lemma 13.1C] that the set $L = p^{-1}(K)$ is a σ-compact and noncompact subgroup of G. Hence, $A(L)$ has a sequential bounded approximate identity, which implies that $A(K)$ must have a sequential bounded approximate identity $\{e_n\}$, by [7, Lemma 3.7]. Towards a contradiction, suppose that $\{e_n\}$ has a unique weak*

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cluster point denoted by E. Then $\{e_n\}$ must be a weakly Cauchy sequence, and since $A(K)$ is weakly sequentially complete, we have weak-$\lim_n e_n = E$. Therefore, $A(K)$ is unital, which is impossible since K is not compact. Thus, $A(K)^{**}$ at least has two distinct right identities. Now, let E_1 and E_2 be distinct right units of $A(K)^{**}$ in $A(H)^{**}$. Then, by [20, Theorem 2.3], E_1 and E_2 can be extended to distinct right units for $A(H)^{**}$, which is a contradiction. □

LEMMA 5.4. *Let H be an ultraspherical hypergroup and let $m \in A(H)^{**}$. Suppose that for every open σ-compact subhypergroup K of H, $P_K^{**}(m) \in A(K)$. Then, for each $n \in \mathbb{N}$, there exists a compact subset K_n of H such that $|\langle m, T\rangle| < (1/n)$ for $T \in UCB(\widehat{H})$ with $\|T\| \leq 1$ and supp$(T) \subseteq H \setminus K_n$.*

PROOF. Suppose to the contrary that the statement is false. By the same argument as in the proof of [20, Lemma 3.1], we can construct a sequence $\{T_n\}$ in $UCB(\widehat{H})$ and a sequence of symmetric, relatively compact neighbourhoods $\{U_n\}$ of $\dot{e}$ in H such that $U_n^2 \subseteq U_{n+1}$ and:

(1) $\operatorname{supp}(T_n) \subseteq U_n$ and $\operatorname{supp}(T_{n+1}) \subseteq H \setminus \overline{U_n}$;
(2) $\|T_n\| \leq 1$ and $|\langle m, T_n\rangle| \geq \epsilon$.

Let $K = \bigcup_n U_n$. Since $U_n^2 \subseteq U_{n+1}$ for each n, it follows that K is an open σ-compact subhypergroup of H. Therefore, by hypothesis, $P_K^{**}(m) \in A(K)$. Now, by the density of the subspace $A(H) \cap C_c(H)$ in $A(H)$, there exists an element $v \in A(H) \cap C_c(H)$ such that $\|P_K^{**}(m) - v\|_{A(H)} < \epsilon/2$ and $\operatorname{supp}(v) \subseteq K$. Let V be a relatively compact neighbourhood of $\operatorname{supp}(v)$ in K. Then, for any $T \in VN(H)$ with $\|T\| \leq 1$ and $\operatorname{supp}(T) \subseteq H \setminus \overline{V}$, we have

$$|\langle P_K^{**}(m), T\rangle| \leq |\langle P_K^{**}(m) - v, T\rangle| + |\langle v, T\rangle| \leq \|P_K^{**}(m) - v\|_{A(H)} < \frac{\epsilon}{2}.$$

Since $\overline{V} \subseteq K$ is compact and $\{U_n\}$ is an increasing sequence of open sets, there is an $n \in \mathbb{N}$ such that $\overline{V} \subseteq U_n$. Hence, the inequality $|\langle P_K^{**}(m), T_{n+1}\rangle| \leq \epsilon/2$ follows from the fact that $\operatorname{supp}(T_{n+1}) \subseteq H \setminus \overline{U_n} \subseteq H \setminus \overline{V}$.

Now, we show that $P_K^*(T_{n+1}) = T_{n+1}$. Let $u \in A(H) \cap C_c(H)$ be such that $\operatorname{supp}(u) \cap K^c \neq \varnothing$. Let $v \in A(H) \cap C_c(H)$ be such that $v(\dot{x}) = 1$ for each $\dot{x} \in \operatorname{supp}(u) \cap K^c$ and $\operatorname{supp}(v) \subseteq K^c$. Then,

$$\begin{aligned}\langle T_{n+1}, u\rangle &= \langle T_{n+1}, u|_K\rangle + \langle T_{n+1}, uv\rangle \\ &= \langle T_{n+1}, u|_K\rangle + \langle v \cdot T_{n+1}, u\rangle \\ &= \langle T_{n+1}, u|_K\rangle \\ &= \langle T_{n+1}, P_K(u)\rangle \\ &= \langle P_K^*(T_{n+1}), u\rangle,\end{aligned}$$

and since $\operatorname{supp}(T_{n+1}) \cap \operatorname{supp}(v) = \varnothing$, we have $v \cdot T_{n+1} = 0$, so the third equality holds. Hence, a simple approximation argument gives that $\langle T_{n+1}, u\rangle = \langle P_K^*(T_{n+1}), u\rangle$ for all

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$u \in A(H)$. Therefore,

$$|\langle P_K^{**}(m), T_{n+1}\rangle| = |\langle m, P_K^*(T_{n+1})\rangle| = |\langle m, T_{n+1}\rangle| \geq \epsilon,$$

which is a contradiction. □

PROPOSITION 5.5. *Let $m \in A(H)^{**}$. Suppose that for every open σ-compact subhypergroup K of H, $P_K^{**}(m)$ is in $A(K)$. Then the restriction of m to $UCB(\widehat{H})$ is in $A(H)$.*

PROOF. By Lemma 5.4, for each $n \in \mathbb{N}$, there exists a compact subset K_n of H such that $|\langle m, T\rangle| < (1/n)$ for $T \in UCB(\widehat{H})$ with $\|T\| \leq 1$ and $\text{supp}(T) \subseteq H \setminus K_n$. Let K be an open σ-compact subhypergroup of H containing all K_n. Then for each $n \in \mathbb{N}$, $H \setminus K \subseteq H \setminus K_n$. Hence, for each $T \in UCB(\widehat{H})$ with $\text{supp}(T) \subseteq H \setminus K$, we have $\langle m, T\rangle = 0$. Let $T \in UCB(\widehat{H})$, $\dot{x} \in K$, and let $U_{\dot{x}} \subseteq K$ be a neighbourhood of $\dot{x}$. Then for each $u \in A(H)$ with $\text{supp}(u) \subseteq U_{\dot{x}}$, we have $P_K(u) = u$. Consequently,

$$\langle T - P_K^*(T), u\rangle = \langle T, u\rangle - \langle P_K^*(T), u\rangle = 0,$$

which implies that $\text{supp}(T - P_K^*(T)) \subseteq H \setminus K$. Therefore, for each $T \in UCB(\widehat{H})$, we have

$$\langle m, T\rangle = \langle m, T - P_K^*(T)\rangle + \langle m, P_K^*(T)\rangle = \langle m, P_K^*(T)\rangle = \langle P_K^{**}(m), T\rangle.$$

Since $P_K^{**}(m)$ is in $A(K) \subseteq A(H)$, the restriction of m to $UCB(\widehat{H})$ is in $A(H)$. □

With the preceding proposition at hand, we can now deduce the main result of this section.

THEOREM 5.6. *Let H be an ultraspherical hypergroup on an amenable locally compact group G. Then for an element $m \in A(H)^{**}$, $m \in A(H)$ if and only if $A(H)\square m \subseteq A(H)$ and $E\square m = m$ for all E in $\mathcal{E}$.*

PROOF. If $m \in A(H)$, then $A(H)\square m \subseteq A(H)$ and for any E in $\mathcal{E}$, $E\square m = m$.

We now prove the converse. Suppose that $m \in A(H)^{**}$ is such that m satisfies the assumptions. If H is compact, then $\mathbf{1}_H \in A(H)$, and hence $m = \mathbf{1}_H\square m \in A(H)$. Let H be noncompact and let K be a σ-compact, open subhypergroup of H. Let $\{K_i\}$ be an increasing sequence of compact subsets of K such that $K = \bigcup_i K_i$. Let $L = p^{-1}(K)$. Then L is a σ-compact, open and closed subgroup of G and $L = \bigcup_i p^{-1}(K_i)$.

It follows from the amenability of L and [16, Corollary 2.7.3] that there exists a sequence $\{u_i\}$ in $A(L)$ such that for each i, $u_i = 1$ on $p^{-1}(K_i)$ and $\|u_i\|_{A(L)} \leq 1 + 1/i$. Therefore, $u_i \longrightarrow \mathbf{1}_L$ in the $\sigma(B(L), C^*(L))$ topology. Hence, it follows from [16, Theorem 3.7.7] that $\|u_i v - v\|_{A(L)} \longrightarrow 0$ for all $v \in A(L)$. Therefore, by identifying $A(K)$ with $A_{\pi|L}(L)$, the sequence $\{\pi(u_i)\}$ is in $A(K)$. Now, if $\dot{x} \in K_i$, then $O_x \subseteq p^{-1}(K_i)$. So $u_i = 1$ on O_x and since $\pi(\delta_x)$ is a probability measure, we have

$$\pi(u_i)(\dot{x}) = \langle u_i, \pi(\delta_x)\rangle = \int_{O_x} u_i(z)\, d\pi(\delta_x)(z) = 1.$$

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Further, for each $v \in A(K)$,

$$\begin{aligned} \|\pi(u_i)v - v\|_{A(K)} &= \|\pi(u_i)\pi(v) - \pi(v)\|_{A(K)} \\ &= \|\pi(u_i v - v)\|_{A(K)} \\ &\leq \|u_i v - v\|_{A(L)} \longrightarrow 0. \end{aligned}$$

By hypothesis, $\pi(u_i)\Box m \in A(H)$ for all i. Then, by using Lemma 5.3, we can show as in [20, Theorem 3.2] that $\{\pi(u_i)\Box m\}$ is a weakly Cauchy sequence. Since $A(H)$ is the predual of the von Neumann algebra $VN(H)$, it is weakly sequentially complete, and thus $\{\pi(u_i)\Box m\}$ converges weakly to a point in $A(H)$. Let $f \in L^1(H)$ with compact support. Using the fact that each $\pi(u_i) \in A(K)$, we have $\langle \pi(u_i)\Box m, \lambda(f)\rangle = \langle \pi(u_i)\Box m, \lambda(\mathbf{1}_K f)\rangle$. Now, for each $u \in A(H)$,

$$\langle P_K^*(\lambda(((\mathbf{1}_H - \mathbf{1}_K)f))), u\rangle = \langle \lambda((\mathbf{1}_H - \mathbf{1}_K)f), P_K(u)\rangle = 0.$$

Also,

$$\begin{aligned} \langle P_K^*(\lambda((\mathbf{1}_K f)) \cdot \pi(u_i)), u\rangle &= \langle \lambda(\mathbf{1}_K f), \pi(u_i)P_K(u)\rangle \\ &= \langle \lambda(\mathbf{1}_K f), P_K(\pi(u_i))u\rangle \\ &= \langle \lambda(\mathbf{1}_K f) \cdot P_K(\pi(u_i)), u\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \pi(u_i)\Box m, \lambda(f)\rangle &= \langle \pi(u_i)\Box m, \lambda(\mathbf{1}_K f)\rangle = \langle m, \lambda(\mathbf{1}_K f) \cdot \pi(u_i)\rangle \\ &= \langle m, \lambda(\mathbf{1}_K f) \cdot P_K(\pi(u_i))\rangle = \langle m, P_K^*(\lambda(\mathbf{1}_K f) \cdot \pi(u_i))\rangle \\ &= \langle P_K^{**}(m), \lambda(\mathbf{1}_K f) \cdot \pi(u_i)\rangle \longrightarrow \langle P_K^{**}(m), \lambda(\mathbf{1}_K f)\rangle \\ &= \langle P_K^{**}(m), \lambda(f)\rangle, \end{aligned}$$

since $\operatorname{supp}(\mathbf{1}_K f) \subset K_i$ for some i. Thus, $\pi(u_i)\Box m \xrightarrow{w} P_K^{**}(m)$, which implies that $P_K^{**}(m) \in \overline{A(H)}^w = A(H)$. Since each $\pi(u_i)\Box m \in A(K)$ and $A(K)$ is a closed convex subset of $A(H)$, we have $P_K^{**}(m) \in A(K)$. Let $m|_{UCB(\widehat{H})} = u$. Then, by Proposition 5.5, $u \in A(H)$. Since G is amenable, $A(H)$ has a bounded approximate identity, say $\{e_\alpha\}$. It is easy to see that $e_\alpha\Box m \longrightarrow E\Box m$ in the $\sigma(A(H)^{**}, A(H)^*)$-topology for some $E \in \mathcal{E}$. By assumption $E\Box m = m$. Hence, $e_\alpha\Box m \longrightarrow m$. Since $e_\alpha\Box m = e_\alpha u$ and $e_\alpha u \longrightarrow u$ in norm, thus $u = m$. In fact, for each $T \in VN(H)$, we have

$$\langle e_\alpha\Box m, T\rangle = \langle m, e_\alpha \cdot T\rangle = \langle u, e_\alpha \cdot T\rangle = \langle ue_\alpha, T\rangle,$$

so $e_\alpha\Box m = e_\alpha u$. Therefore, $m \in A(H)$. □

Corollary 5.7. *Let H be a discrete ultraspherical hypergroup on an amenable locally compact group G and $m \in A(H)^{**}$. Then, $m \in A(H)$ if and only if $E\Box m = m$ for each $E \in \mathcal{E}$.*

Proof. If H is discrete, then $A(H)$ is an ideal in $A(H)^{**}$ by [11, Proposition 2.7]. Hence, the result follows from Theorem 5.6. □

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COROLLARY 5.8. *Let H be an ultraspherical hypergroup on an amenable locally compact group G. If $m \in \mathfrak{Z}(A(H)^{**})$ and $A(H)\square m \subseteq A(H)$, then $m \in A(H)$. In particular, if H is discrete, then $\mathfrak{Z}(A(H)^{**}) = A(H)$.*

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References

[1] C. A. Akeman and M. E. Walter, 'Non-abelian Pontriagin duality', *Duke Math. J.* **39** (1972), 451–463.
[2] M. Alaghmandan, 'Remarks on weak amenability of hypergroups', Preprint, 2018, arXiv:1808.03805.
[3] F. F. Bonsall and J. Duncan, *Complete Normed Algebras* (Springer, New York, 1973).
[4] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs, 24 (Clarendon Press, Oxford, 2000).
[5] H. G. Dales and A. T.-M. Lau, 'The second dual of Beurling algebras', *Mem. Amer. Math. Soc.* **177**(836) (2005), vi+191.
[6] E. Damek and F. Ricci, 'Harmonic analysis on solvable extensions of H-type groups', *J. Geom. Anal.* **2** (1992), 213–248.
[7] S. Degenfeld-Schonburg, E. Kaniuth and R. Lasser, 'Spectral synthesis in Fourier algebras of ultraspherical hypergroups', *J. Fourier Anal. Appl.* **20** (2014), 258–281.
[8] J. Dixmier, *C*-Algebras* (North-Holland Publishing Co., Amsterdam–New York–Oxford, 1977).
[9] C. F. Dunkl and D. E. Ramirez, 'Weakly almost periodic functionals on the Fourier algebra', *Trans. Amer. Math. Soc.* **185** (1973), 501–514.
[10] R. Esmailvandi and M. Nemati, 'Multipliers and uniformly continuous functionals over Fourier algebras of ultraspherical hypergroups', *Filomat.* **35**(9) (2021), 3139–3150.
[11] R. Esmailvandi and M. Nemati, 'Compact and weakly compact multipliers over Fourier algebras of ultraspherical hypergroups', *Mediterr. J. Math.* **18** (2021), 33; doi:10.1007/s00009-020-01651-y.
[12] P. Eymard, 'L'algèbre de Fourier d'un groupe localement compact', *Bull. Soc. Math. France* **92** (1964), 181–236.
[13] E. E. Granirer, 'Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group', *Trans. Amer. Math. Soc.* **189** (1974), 371–382.
[14] E. E. Granirer, 'Density theorems for some linear subspaces and some C*-algebras of $VN(G)$', *Symp. Math. Istit. Naz. Alta Mat.* **XXII** (1977), 61–70.
[15] R. I. Jewett, 'Spaces with an abstract convolution of measures', *Adv. Math.* **18** (1975), 1–101.
[16] K. Kaniuth and A. T.-M. Lau, *Fourier and Fourier–Stieljes Algebras on Locally Compact Groups*, Mathematical Surveys and Monographs, 231 (American Mathematical Society, Providence, RI, 2018).
[17] A. T.-M. Lau, 'The second conjugate algebra of the Fourier algebra of a locally compact group', *Trans. Amer. Math. Soc.* **267** (1981), 53–63.
[18] A. T.-M. Lau, R. J. Loy and G. A. Willis, 'Amenability of Banach and C*-algebras on locally compact groups', *Studia Math.* **119** (1996), 161–178.
[19] R. E. Megginson, *An Introduction to Banach Space Theory* (Springer-Verlag, New York, 1998).

[20] T. Miao, 'Unit elements in the double dual of a subalgebra of the Fourier algebra $A(G)$', *Canad. J. Math.* **61** (2009), 382–394.
[21] E. Michael, 'Topologies on spaces of subsets', *Trans. Amer. Math. Soc.* **71** (1951), 152–182.
[22] V. Muruganandam, 'Fourier algebra of a hypergroup. II. Spherical hypergroups', *Math. Nachr.* **11** (2008), 1590–1603.
[23] N. Shravan Kumar, 'Invariant means on a class of von Neumann algebras related to ultraspherical hypergroups', *Studia Math.* **225** (2014), 235–247.
[24] N. Shravan Kumar, 'Invariant means on a class of von Neumann algebras related to ultraspherical hypergroups II', *Canad. Math. Bull.* **60** (2017), 402–410.

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