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Extending Camina pairs

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ABSTRACT

Let G be a finite group and N a nontrivial proper normal subgroup of G. A.R. Camina introduced the class of finite groups G, which extends Frobenius groups, satisfying that for all $g \in G - N$ and $n \in N$, gn is conjugate to g. He proved that under these assumptions one of three possibilities occurs: G is a Frobenius group with kernel N; or N is a p-group; or G/N is a p-group. In this paper we extend this class of groups by investigating the structure of those finite groups G having a nontrivial proper normal subgroup N such that gn is conjugate to either g or g^{-1} for all $g \in G - N$ and all $n \in N$.

1. Introduction

Let G be a finite group and N a nontrivial proper normal subgroup of G. In 1978, A.R. Camina introduced the class of finite groups G, which extends Frobenius groups,

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satisfying that for all $g \in G - N$ and $n \in N$, gn is conjugate to g. This condition is equivalent to assert that every nontrivial coset of N is contained in a conjugacy class of G. Furthermore, a character-theoretic equivalent condition was also given (see Theorem 1 of [3]), though it is not to be used in this paper. The pair (G, N) was subsequently called a *Camina pair*. Camina obtained that if (G, N) is a Camina pair, then one of (non-excluding) three possibilities occurs: G is a Frobenius group with kernel N; or N is a p-group; or G/N is a p-group. Later on, several authors have made significant progress at classifying these groups. For instance, Chillag and MacDonald [4] made an effort in investigating such pairs when G is neither a p-group nor a Frobenius group. Also, Isaacs [10] demonstrated that if G/N is a p-group, (G, N) being a Camina pair, then G possesses normal p-complement. Further progress on this topic can be found in [1] and [5] as well as a survey in [11].

More recently, instead of considering all cosets of N, Navarro and Guralnick [6] independently proved, by appealing to the Classification of Finite Simple Groups, that if just one coset, xN with $x \in G$, lies in a conjugacy class of G, then N must be soluble. In addition, if x is a p-element, then N must be p-nilpotent, which improves above Isaacs's result. On the other hand, the second named author has recently proved that when a single coset of N is contained in the union of a conjugacy class of G and its inverse class, then the same conclusion as Navarro and Guralnick's holds (Theorem A of [2]).

Inspired by the above results, in this paper we generalize Camina's condition by exploring the class of finite groups G having a nontrivial proper normal subgroup N such that gn is conjugate to either g or g^{-1} for all $g \in G - N$ and $n \in N$. Observe that any group satisfying Camina's hypotheses trivially satisfies ours, so the class of groups we are dealing with must include groups satisfying Camina's aforementioned properties. Our study is made depending on whether N is nilpotent or not, and each case yields to the following two theorems.

Theorem A. *Suppose that G is a finite group and N is a nontrivial proper normal subgroup of G such that gn is conjugate to g or g^{-1} for all $g \in G-N$ and all $n \in N$. Assume that N is non-nilpotent. Then G/N is a p-group, and G is p-nilpotent and soluble.*

We recall that a group G is said to be quasi-Frobenius when $G/\mathbf{Z}(G)$ is a Frobenius group. The inverse image in G of the kernel and complement of $G/\mathbf{Z}(G)$ are then called the kernel and complement of G, respectively.

Theorem B. *Suppose that G is a finite group and N is a nontrivial proper normal subgroup of G such that gn is conjugate to g or g^{-1} for all $g \in G - N$ and all $n \in N$. Assume that N is nilpotent. Then G satisfies one of the following conditions:*

(1) G is a Frobenius group with kernel N;
(2) N has prime power order;

(3) $|G/N| = 2$, $C_2 \cong \mathbf{Z}(G) \subseteq N$ *and* G *is a quasi-Frobenius group with complement cyclic of order* 4 *and abelian kernel* N. *In particular,* $G/\mathbf{Z}(G)$ *is a generalized dihedral group.*

These results seem to conclude that our class of groups is quite similar to those groups satisfying Camina's condition. However, a new group structure arises, concretely that given in case (3) of Theorem B. This cannot be excluded since, in fact, it is easy to see that every group described in (3) satisfies our hypothesis too.

2. Preliminaries

In this section, and also throughout the paper, we consider a group G satisfying the hypothesis pointed out in the introduction, which can be reformulated as follows.

Hypothesis. Let G be a finite group and $1 \neq N$ a proper normal subgroup of G such that for every $x \notin N$, the coset xN is contained in $K \cup K^{-1}$ for some conjugacy class K of G. Notice that we can assume without loss that K is the conjugacy class x^G of x in G.

We will prove a set of lemmas that help us to get the proofs of our main theorems. This is done by following the same ideas as those in Section 2 of [3]. Some of the proofs being identical, we omit them and refer the reader to [3].

Lemma 2.1. *Assume that* G *satisfies the hypothesis and let* $x \in G - N$.

(i) If x *is real, then* $|\mathbf{C}_G(x)| = |\mathbf{C}_{G/N}(xN)|$.
(ii) If x *is non-real, then either* $|\mathbf{C}_G(x)| = |\mathbf{C}_{G/N}(xN)|$ *or* $|\mathbf{C}_G(x)| = 2|\mathbf{C}_{G/N}(xN)|$.

Proof. Since xN is contained in $K \cup K^{-1}$, where $K = x^G$, then $(xN)^G \subseteq K \cup K^{-1}$. Also, $(xN)^G$ is a normal subset of G, so it is a union of conjugacy classes of G. Hence, either $(xN)^G = K$ or $(xN)^G = K \cup K^{-1}$. Now, $(xN)^G$ considered as a subset of G is union of distinct conjugates of xN in G/N, say

$$(xN)^G = xN \cup x^{g_2}N \cup \ldots \cup \ldots x^{g_t}N$$

where $g_i \in G$ and $t = |G/N : \mathbf{C}_{G/N}(xN)|$. Therefore, by counting elements in the above equality we get either $|K| = |(xN)^G| = |N|t$ or $|K \cup K^{-1}| = |(xN)^G| = |N|t$. Now, if x is real, that is, if $K = K^{-1}$, then

$$|G/N : \mathbf{C}_{G/N}(xN)||N| = |K| = |G : \mathbf{C}_G(x)|$$

and this gives $|\mathbf{C}_G(x)| = |\mathbf{C}_{G/N}(xN)|$, so (i) is proved. If x is non-real, then we have two possibilities: either $(xN)^G = K$ or $(xN)^G = K \cup K^{-1}$. Similarly, these equalities yield to both cases of (ii), respectively. □

Lemma 2.2. *Assume that G satisfies the hypothesis. If $x \in G - N$ has order m and $y \in \mathbf{C}_N(x)$, then the order of y divides m.*

Proof. As $xy \in xN \subseteq x^G \cup (x^{-1})^G$, then xy has the same order as x, so $1 = (xy)^m = x^m y^m = y^m$, and the lemma is proved. □

Lemma 2.3. *Assume that G satisfies the hypothesis. Let $x \in G - N$ such that xN has order m in G/N. If $y \in \mathbf{C}_N(x)$ has prime order p, then p divides m.*

Proof. Identical to the proof of Lemma 3 of [5]. □

Lemma 2.4. *Assume that G satisfies the hypothesis. If $x \in G - N$ has prime order, then N is nilpotent.*

Proof. Let us consider the subgroup $\langle x \rangle N$. Every element in $\langle x \rangle N - N$ belongs to some coset $x^i N$, with $x^i \notin N$. Since $x^i N \subseteq (x^i)^G \cup (x^{-i})^G$, it follows that every element in $\langle x \rangle N - N$ has prime order, the same as x, and then by Hughes-Thompson-Kegel Theorem (V.8 of [9]), N is nilpotent. □

Theorem 2.5. *Assume that G satisfies the hypothesis and that N and G/N have coprime orders. Then G is a Frobenius group with kernel N.*

Proof. By the Schur-Zassenhauss theorem it is known that N has a complement in G, say H, and by using Lemma 2.3, we get that if $1 \neq h \in H$, then $\mathbf{C}_N(h) = 1$. This condition characterizes Frobenius groups. □

Independently of the above results, we know that if G satisfies the hypothesis and G/N has p-elements for some prime p, then N has normal p-complement by Theorem A(c) of [2]. As a consequence, for every group G satisfying the hypothesis, N has a normal $\pi(G/N)$-complement.

Lemma 2.6. *Assume that G satisfies the hypothesis. If xN has order pq in G/N, with p and q distinct primes, then N is nilpotent.*

Proof. Identical to the proof of Lemma 5 of [5]. □

Lemma 2.7. *Assume that G satisfies the hypothesis and let P be a Sylow p-subgroup of G for some odd prime p. Suppose that there is $x \in P$ such that $x \notin N$ and $\mathbf{C}_P(x)(P \cap N) = P$. Then $P \cap N = 1$.*

Proof. As $\mathbf{C}_P(x)(P \cap N) = P$, then by taking cardinalities,

$$\frac{|\mathbf{C}_P(x)||P \cap N)|}{|\mathbf{C}_{P \cap N}(x)|} = |P|,$$

or equivalently

$$|\mathbf{C}_P(x)| = |\mathbf{C}_{P\cap N}(x)||P/(P\cap N)|. \quad \text{(I)}$$

On the other hand, by applying Lemma 2.1, we have

$$|\mathbf{C}_P(x)| \leq |\mathbf{C}_G(x)|_p = |\mathbf{C}_{G/N}(xN)|_p \leq |P/P\cap N|,$$

which, joint with Eq. (I), gives $|\mathbf{C}_P(x)| = |P/P\cap N|$ and $\mathbf{C}_{P\cap N}(x) = 1$. This is a contradiction unless $P\cap N = 1$, because if $P\cap N \neq 1$, then $\mathbf{Z}(P)\cap N \neq 1$, which obviously centralizes x. □

Lemma 2.8. *Assume that G satisfies the hypothesis. Let p be an odd prime dividing both $|N|$ and $|G/N|$ and let P be a Sylow p-subgroup of G. Then $P/P\cap N$ contains an elementary abelian subgroup of order p^2.*

Proof. Every odd order p-group having no elementary abelian subgroups of order p^2 is necessarily cyclic. But then, if $x(P\cap N)$ is a generator of $P/P\cap N$, we have

$$\frac{P}{P\cap N} = \frac{\langle x\rangle(P\cap N)}{P\cap N}.$$

This clearly implies that $\mathbf{C}_P(x)(P\cap N) = P$, against Lemma 2.7. □

Remark 2.9. Lemmas 2.7 and 2.8 do not hold in general for $p = 2$. The next example shows it. Let G be the semidirect product of C_3^2 and C_4 acting faithfully, that is,

$$G = \langle a, b, x \mid a^3 = b^3 = x^4 = 1, ab = ba, xax^{-1} = a^{-1}b, xbx^{-1} = ab\rangle.$$

Take $N = \langle a, b, x^2\rangle \cong C_3 \rtimes S_3$ and $K = x^G$. It is easy to check that $xN = K \cup K^{-1}$ and moreover, this is the only nontrivial coset of N in G, so G satisfies the hypothesis. Now, if you take $P = \langle x\rangle$ of order 4, then $P/(P\cap N)$ is cyclic of order 2 and thus Lemma 2.8 fails. Notice that Lemma 2.7 does not hold either in this example, since $1 \neq P\cap N = \langle x^2\rangle$. Nevertheless, by taking into account Lemma 2.1 it can be easily seen that Lemma 2.7 does hold for $p = 2$ whenever x is real.

3. Proof of Theorem A

In this section we assume that N is non-nilpotent. For proving Theorem A, we obtain first the solubility of G/N by employing the classification of those finite simple groups all of whose elements have prime power order [7]. This classification relies on the Classification of Finite Simple Groups.

Theorem 3.1. *Assume that G satisfies the hypothesis and that N is not nilpotent. Then G/N is soluble.*

Proof. Since N is non-nilpotent, we know by Lemma 2.6 that G/N is a CP-group, that is, every element of G/N has prime power order. Also, by applying Lemma 2.4, we may assume that every prime dividing $|G/N|$ divides $|N|$. Suppose that G/N is non-soluble and seek a contradiction. Then, by Proposition 2 of [7] (which gives the structure of non-soluble CP-groups), there exist normal subgroups $B/N \subseteq C/N \subseteq G/N$ such that B/N is a 2-group, $C/B \cong (C/N)/(B/N)$ is non-abelian simple and $G/C \cong (G/N)/(C/N)$ is a p-group (cyclic or generalized quaternion). Furthermore, Proposition 2 of [7] also claims that if $B/N \neq 1$, then $p = 2$. Moreover, as C/B is a non-abelian simple CP-group, we also know by Proposition 3 of [7] that C/B is isomorphic to one of the following groups: $\mathrm{L}_2(4), \mathrm{L}_2(7), \mathrm{L}_2(8), \mathrm{L}_2(9), \mathrm{L}_2(17), \mathrm{L}_3(4), \mathrm{Sz}(8)$ or $\mathrm{Sz}(32)$.

Next we show that G/N must have a cyclic Sylow p-subgroup for some odd prime p. In this case, since p divides both $|N|$ and $|G/N|$, then Lemma 2.8 will provide a contradiction, as required. Observe that each of the above listed groups except $\mathrm{L}_2(9)$ possesses cyclic Sylow subgroups for two distinct odd primes. These primes are: 3 and 5 for $\mathrm{L}_2(5)$; 3 and 7 for $\mathrm{L}_2(7)$; 3 and 7 for $\mathrm{L}_2(8)$; 3 and 17 for $\mathrm{L}_2(17)$; 5 and 7 for $\mathrm{L}_3(4)$; 5, 7 and 13 for $\mathrm{Sz}(8)$; and 31 and 41 for $\mathrm{Sz}(32)$. Therefore, in all these cases, by the given structure of G/N, we get that G/N has cyclic Sylow subgroups for at least one odd prime, as wanted. Suppose now that $C/B \cong \mathrm{L}_2(9)$, which has cyclic Sylow subgroups only for the prime 5. If $B/N \neq 1$, then G/C is a 2-group (possibly trivial), and thus G/N also has cyclic Sylow 5-subgroups, so we are finished. On the contrary, if $B/N = 1$, as $C/N \cong \mathrm{L}_2(9)$ is the only minimal normal subgroup of G, then G/N is immersed in $\mathrm{Aut}(\mathrm{L}_2(9))$. However, $\mathrm{Out}(\mathrm{L}_2(9)) \cong C_2 \times C_2$, and this implies that G/N has cyclic Sylow 5-subgroups too, so the solubility of G/N is proved. □

Theorem 3.2. *Assume that G satisfies the hypothesis and that N is not nilpotent. Then G/N is a p-group.*

Proof. We prove the theorem in several steps as follows:

Step 1. Either G/N has prime power order or G/N is a $\{2, p\}$-group for some odd prime p. In the second case, if $M/N = \mathbf{O}_p(G/N)$ then G/N is a Frobenius group with kernel M/N, and thus, G/M is a cyclic 2-group or generalized quaternion.

Proof. By Theorem 3.1, we know that G/N is soluble and we will assume that it does not have prime power order. Then we can choose $M/N = \mathbf{O}_p(G/N) \neq 1$ for some prime p. Let q be a prime divisor of $|G/N|$ distinct from p and let Q be a Sylow q-subgroup of G. Since G/N is a CP-group, Lemma 2.6 implies that $QN/N \cong Q/Q \cap N$ acts fixed-point-freely on M/N, so $Q/Q \cap N$ is cyclic or generalized quaternion. If q is odd, then $Q/Q \cap N$ is cyclic and since q also divides $|N|$ by Lemma 2.4, then Lemma 2.8 provides a contradiction. Henceforth, $q = 2$ and consequently, p is odd. On the other hand, Theorem 1 of [8] (which gives the structure of soluble CP-groups) establishes that G/M is either cyclic of order a power of a prime other than p, or generalized quaternion, or a $\{q, p\}$-group with q of the form $kp^a + 1$ with $a \geq 1$. Since $q = 2$, the latter possibility is not possible, so we conclude that G/M is a cyclic 2-group or generalized quaternion and the step is proved.

The rest of the proof consists in proving that the second possibility of Step 1 is not possible. In the sequel, we fix Q to be a Sylow 2-subgroup of G, so QN/N is a Sylow 2-subgroup of G/N.

Step 2. Suppose that G/M is a generalized quaternion group or a cyclic group of order 2^a, for some integer a. Let $x(Q \cap N) \in Q/(Q \cap N)$ be an element of maximal order 2^b, where $b \in \{a-1, a\}$ depending on the structure of G/M. Then $\mathbf{C}_G(x) = \langle x \rangle$, $|x| = 2^{b+1}$ and $\mathbf{Z}(Q) \cap N = \langle x^{2^b} \rangle$.

Proof. By Step 1, we know that G/N is Frobenius. Let $z = x^{2^{b-1}}$ and observe that $z^2 \in N$. By applying Lemma 2.4, we have $z^2 \neq 1$, and this implies that $|x| \geq 2^{b+1}$. Now, taking into account Lemma 2.1, we have two possibilities. First

$$2^{b+1} \leq |\langle x \rangle| \leq |\mathbf{C}_G(x)| = |\mathbf{C}_{G/N}(xN)| = 2^b,$$

where the last equality follows by the Frobenius structure of G/N. This is trivially a contradiction. The second possibility is

$$2^{b+1} \leq |\langle x \rangle| \leq |\mathbf{C}_G(x)| = 2|\mathbf{C}_{G/N}(xN)| = 2^{b+1},$$

which forces $\mathbf{C}_G(x) = \langle x \rangle$ and $|x| = 2^{b+1}$. By noting that $\mathbf{Z}(Q) \cap N \subseteq \mathbf{C}_N(x) = \langle x \rangle \cap N$, we easily deduce that $\mathbf{Z}(Q) \cap N = \langle x \rangle \cap N = \langle x^{2^b} \rangle$, as claimed.

Step 3. G/M cannot be a generalized quaternion group of order greater that 8 or a cyclic 2-group of order greater than 2.

Proof. Suppose that G/M is either generalized quaternion of order at least 16 or a cyclic 2-group of order at least 4. With the same notation used in Step 2, we have $|z| = 4$ and $z^2 \in N$. Then the hypotheses assert that xz^2 is conjugated to x or to x^{-1}, so there exists some $g \in G$ such that $xz^2 = x^g$ or $xz^2 = (x^{-1})^g$. In both cases $g \in \mathrm{Aut}(\langle x \rangle)$ (which is a 2-group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{b-1}}$).

Assume first that $x^g = xz^2$. Then $(x^2)^g = (xz^2)^2 = x^2$, that is, g fixes x^2. As we have $x^2 \neq z$ when G/M is generalized quaternion of order at least 16, and we have $x^2 \notin N$ when G/M is cyclic of order at least 4, we can reason as in Step 2 by employing Lemma 2.1 to obtain $|\mathbf{C}_G(x^2)| = 2|\mathbf{C}_{G/N}(x^2N)| = 2^{b+1}$. The latter equality follows by applying that G/N is Frobenius and that the centralizer of x^2N in QN/N coincides with the centralizer of xN. By Step 2, it follows that $\mathbf{C}_G(x^2) = \mathbf{C}_G(x)$, in particular g centralizes x, which is a contradiction.

Suppose now that $(x^{-1})^g = xz^2$. Then $(x^{-2})^g = (xz^2)^2 = x^2$. This means that x^2 is a real element, so again by Lemma 2.1, we easily obtain $|\mathbf{C}_G(x^2)| = |\mathbf{C}_{G/N}(x^2N)| = 2^b$.

This provides a contradiction by order considerations, since obviously we always have $\mathbf{C}_G(x) \subseteq \mathbf{C}_G(x^2)$.

Step 4. G/M is not isomorphic to Q_8.

Proof. Suppose that $G/M \cong Q/(Q \cap N) \cong Q_8$. By Step 2, if $x(Q \cap N)$ is an element of order 4, then $|x| = 8$, $|\mathbf{C}_G(x)| = |\mathbf{C}_Q(x)| = 8$ and $T := \mathbf{Z}(Q) \cap N = \langle x^4 \rangle$. Next, we distinguish two cases depending on whether $xx^4 \in xN$ is conjugate to x or x^{-1}, and show that each one leads to a contradiction.

Case 1. There exists $g \in G$ such that $(x^{-1})^g = xx^4 = x^5$. Then $(x^{-2})^g = x^2$, and hence, x^2 is real. By Lemma 2.1, we have $|\mathbf{C}_G(x^2)| = |\mathbf{C}_{G/N}(x^2N)| = 8$ and so $\mathbf{C}_G(x^2) = \langle x \rangle$. We claim that $Q \cap N = T$. Suppose not and take $1 \neq tT \in \mathbf{Z}(Q/T) \cap (N \cap Q)/T$. Hence $[x, t] = x^4$, or equivalently $x^t = x^5$. This is a contradiction because x^5 is conjugate to x^{-1} and x is not real. Therefore, $\mathbf{Z}(Q/T) \cap (N \cap Q)/T = 1$, and this certainly implies that $N \cap Q = T$, as claimed. As a consequence, Q must have order 16. Noting that $g^2 \in \mathbf{C}_G(x^2) = \langle x \rangle$, it follows that $\langle x, g \rangle$ has order 16 too. Hence $Q \cong \langle x, g \rangle$ and satisfies the relations $x^8 = 1$ and $x^g = x^3$. According to the classification of groups of order 16, we have $Q \cong SD_{16}$ (the semi-dihedral group of order 16). This is not possible, however, because this group has no quotients isomorphic to Q_8 (indeed $Q/(Q \cap N)$ is isomorphic to the dihedral group D_8).

Case 2. There exists $g \in G$ such that $x^g = xx^4 = x^5$, and hence g fixes x^2. In this case, by applying Lemma 2.1 and the fact that $x^2(Q \cap N) \in \mathbf{Z}(Q/(Q \cap N))$, we have $16 \leq |\langle x, g \rangle| \leq |\mathbf{C}_G(x^2)| \leq 2|\mathbf{C}_{G/N}(x^2N)| = 16$, and so $\mathbf{C}_G(x^2) = \langle g, x \rangle$ has order 16. By using the classification of groups of order 16 having a cyclic subgroup of order 8, we obtain $\mathbf{C}_G(x^2) \cong M_{16}$. Thus, there is no loss if we assume $|g| = 2$ and so $g \in N$, as $G - N$ does not contain elements of order 2 as a result of Lemma 2.4. Moreover, there is no loss if we assume $g \in Q$.

Therefore, in this case we have got the following properties: $x(Q \cap N)$ is an element of order 4 in the quaternion group $Q/(Q \cap N)$, with $|x| = 8$, and there exists $g \in Q \cap N$ of order 2 such that $x^g = x^5$ and $\mathbf{C}_G(x^2) = \langle g, x \rangle \cong M_{16}$. The rest of this case involves in proving that from exactly these properties we derive a contradiction. We want to remark that we do not utilize the fact that N is not nilpotent in this part of the proof.

Assume first that $\mathbf{Z}(Q/T) \cap (N \cap Q)/T = 1$, so $N \cap Q = T$, and as a result, Q has order 16. By the above paragraph, this forces that $Q \cong M_{16}$, but this group has no factors isomorphic to Q_8 (indeed $Q/(Q \cap N) \cong C_4 \times C_2$), so we have a contradiction. Therefore, we can take $1 \neq tT \in \mathbf{Z}(Q/T) \cap (N \cap Q)/T$. Then $[x, t] = x^4$, or equivalently, $x^t = x^5$. From this, we deduce that $t \in L := \mathbf{C}_N(x^2) = \langle x^4, g \rangle \cong C_2 \times C_2$. This implies that $\mathbf{Z}(Q/T) \cap (N \cap Q)/T = L/T$. In particular, $L \trianglelefteq Q$ and we may consider Q/L. Assume now that $1 \neq hL \in \mathbf{Z}(Q/L) \cap (N \cap Q)/L$. Then $[x, h] \in \{x^4, g, x^4g\}$. If the first possibility occurs, then $x^h = x^5$, and as a consequence $h \in \mathbf{C}_G(x^2)$. But we also have $h \in N$, so $h \in L$, a contradiction. If the second or third possibilities occur, that is, if $x^h = xg$ or $x^h = x^5g$, then both equalities lead to $(x^2)^h = x^{-2}$, that is, x^2 is real. However, this is not possible by Lemma 2.1 since $|\mathbf{C}_G(x^2)| \neq |\mathbf{C}_{G/N}(x^2N)|$. This contradiction shows that $Q \cap N = L$, which implies that Q has order 32.

On the other hand, we write $Q_8 \cong Q/(N \cap Q) = \langle \bar{x}, \bar{y} \rangle$ where $\bar{x}, \bar{y}$ have order 4, and $\bar{x}^2 = \bar{y}^2$ and $\bar{x}^{\bar{y}} = \bar{x}^{-1}$. As we have proved above that $N \cap Q = \langle x^4, g \rangle$, the equality $\bar{x}^{\bar{y}} = \bar{x}^{-1}$ gives $xx^y \in \{1, x^4, g, x^4g\}$. Next we prove that each of these possibilities provides a contradiction. Of course $xx^y \neq 1$, otherwise x would be real and this is not possible. If $xx^y = x^4$, then $x^y = x^3$, and this gives $x^{yg} = (x^3)^g = x^{-1}$, again a contradiction. Suppose finally that $xx^y = g$ or $xx^y = x^4g$. In the first case

$$(x^2)^y = (x^y)^2 = (x^{-1}g)^2 = x^{-1}(x^{-1})^g = x^{-1}x^3 = x^2,$$

and in the second case

$$(x^2)^y = (x^y)^2 = (x^3g)^2 = x^3(x^3)^g = x^3x^7 = x^2,$$

that is, in both cases $y \in \mathbf{C}_G(x^2)$. Now, we have proved at the beginning of this case that this subgroup coincides with $\langle x, g \rangle$ and has order 16. However, Q is certainly generated by x, y and $Q \cap N$, so $Q = \langle x, y, g \rangle$. Thus, since $y \in \langle x, g \rangle$, then $Q = \langle x, g \rangle$, contradicting the fact that Q has order 32.

Step 5. G/M is not cyclic of order 2. Consequently, G/N is a p-group.

Proof. Assume that $Q/(Q \cap N)$ is cyclic of order 2 and let $x(Q \cap N) \in Q/(Q \cap N)$ be a generator. By Step 2, we have $\mathbf{C}_G(x) = \langle x \rangle$ and $|x| = 4$. Now, by using Lemma 2.1, we know that x cannot be real in G, that is, there is no $g \in G$, such that $x^g = x^{-1} = x^3$. But this implies that $\mathbf{N}_G(\langle x \rangle) = \mathbf{C}_G(\langle x \rangle) = \langle x \rangle$ (because any $g \in \mathbf{N}_G(\langle x \rangle)$ must centralize x). In particular, $\mathbf{N}_Q(\langle x \rangle) = \mathbf{C}_Q(\langle x \rangle) = \langle x \rangle$. As normalizers grow in any prime power order group we conclude that $Q = \langle x \rangle$.

On the other hand, by the observation made after Theorem 2.5, we know that N has normal 2-complement, say K. Hence K is normal in G and $\overline{G} = G/K$ is a $\{2, p\}$-group. Since we have proved that $\overline{G}$ has cyclic Sylow 2-subgroups of order 4, then $\overline{G}$ has a normal Sylow p-subgroup, say $\overline{P}$. Also, since $\langle \overline{x^2} \rangle = \overline{N} \trianglelefteq \overline{G}$, then $\overline{x^2} \in \mathbf{C}_{\overline{G}}(\overline{P})$. Hence, for every $y \in P - (N \cap P)$, since $P \cap N = P \cap K$ we have $1 \neq \overline{y} \in \overline{P}$, and $\overline{yx^2}$ has order $2|\overline{y}|$. As a consequence, $|yx^2|$ is divisible by $2|y|$. However, the fact that $yx^2 \in yN \subseteq y^G \cup (y^{-1})^G$ ensures that yx^2 and y have the same order, a contradiction. This contradiction also allows us to conclude that case 2 of Step 1 cannot happen. Hence G/N is a p-group and the proof is finished. □

Proof of Theorem A. Theorem 3.2 proves that G/N is a p-group for some prime p. The solubility of N and the fact that N is p-nilpotent follow straightforwardly by Theorem A(a) and (c) of [2], so G is soluble and p-nilpotent as well.

Example. The example given in Remark 2.9 illustrates a group satisfying our hypothesis with N non-nilpotent and $|G/N| = 2$.

4. Proof of Theorem B

In this section, we assume that N is nilpotent and by tracking the following theorem and lemmas, we prove Theorem B.

Theorem 4.1. *Assume that G satisfies the hypothesis and that N is nilpotent. Then one of the following occurs:*

(a) G is a Frobenius group with kernel N;
(b) N has prime power order;
(c) $(|G/N|, |N|) = 2^a$ for some $a \geq 1$. In addition, if N is not a 2-group, then every Sylow subgroup of G/N acts fixed-point-freely on the 2-complement of N.

Proof. Assume that G is not a Frobenius group and N has not prime power order and we will prove (c). Then, by Theorem 2.5, we have $(|N|, |G/N|) \neq 1$, so we take a prime p dividing both $|N|$ and $|G/N|$, and P a Sylow p-subgroup of G. Let $r \neq p$ be a prime dividing $|N|$, and R a Sylow r-subgroup of N. As N is nilpotent, then $P \subseteq \mathbf{N}_G(R)$ and $P \cap N \subseteq \mathbf{C}_G(R)$. Hence $P/(P \cap N)$ acts fixed-point-freely on R by using Lemma 2.3. In particular, $P/(P \cap N)$ has no elementary abelian subgroups of order p^2. But if p is odd, since $P \cap N \neq 1$ and $P/(P \cap N) \neq 1$, this contradicts Lemma 2.7. Therefore $p = 2$, and the first part of (c) is proved.

Assume further that N is not a 2-group. We can use the same reasonings to deduce that if q is any prime dividing $|G/N|$ and K denotes the (normal and nontrivial) 2-complement of N, then every Sylow q-subgroup of G/N acts fixed-point-freely on K. Hence, the lemma is proved. □

Lemma 4.2. *Assume that G satisfies the hypothesis, N is nilpotent and Case (c) of Theorem 4.1, including that N is not a 2-group, occurs. Let Q be a Sylow 2-subgroup of G and $x(Q \cap N)$ an element of $Q/(Q \cap N)$ of maximal order. Then the following assertions occur:*

(i) $|\mathbf{C}_G(x)| = 2|\mathbf{C}_{G/N}(xN)|$.
(ii) $\mathbf{C}_G(x)N/N = \mathbf{C}_{G/N}(xN)$.
(iii) Either $|G/N|_2 = 2$ or the Sylow 2-subgroups of G/N are quaternion.

Proof. As G/N acts Frobeniusly on the (nontrivial) 2-complement of N, then every Sylow 2-subgroup of G/N is either cyclic or generalized quaternion. Let $x(N \cap Q) \in Q/(N \cap Q)$ be an element of maximal order 2^a. Attending to Lemma 2.1, suppose first that $|\mathbf{C}_G(x)| = |\mathbf{C}_{G/N}(xN)|$. Note that $2^a \leq |\langle x \rangle| \leq |\mathbf{C}_G(x)|_2 = |\mathbf{C}_{G/N}(xN)|_2 = 2^a$, which means that $x^{2^a} = 1$ and so $\langle x \rangle \cap N = 1$. However, $1 \neq \mathbf{Z}(Q) \cap N \subseteq \mathbf{C}_G(x)$, and this leads to a contradiction. Hence, $|\mathbf{C}_G(x)| = 2|\mathbf{C}_{G/N}(xN)|$, which shows the validity of (i).

Set $C/N := \mathbf{C}_{G/N}(xN)$. Observe that $|\mathbf{C}_G(x)N/N| = |\mathbf{C}_G(x) : \mathbf{C}_G(x) \cap N|$ and that x cannot fix any $2'$-element of N by hypothesis, hence $\mathbf{C}_G(x) \cap N$ is a 2-group. Thus, $|\mathbf{C}_G(x)N/N|_{2'} = |\mathbf{C}_G(x)|_{2'} = |C/N|_{2'}$. On the other hand, it is clear that $L/N = \langle x \rangle N/N$ is a Sylow 2-subgroup of C/N and is contained in $\mathbf{C}_G(x)N/N$. Therefore, $C/N \subseteq \mathbf{C}_G(x)N/N$, which implies that $C = \mathbf{C}_G(x)N$, so (ii) is proved.

To prove (iii), suppose that $Q/(Q \cap N)$ is neither cyclic of order 2 nor quaternion. Then, as $\mathbf{C}_G(x) \subseteq \mathbf{C}_G(x^2)$, according to Lemma 2.1 and the structure of the Sylow 2-subgroup of G/N given by the hypotheses, we have $|\mathbf{C}_G(x)|_2 = 2|\mathbf{C}_{G/N}(xN)|_2 = 2|\mathbf{C}_{G/N}(x^2N)|_2 = |\mathbf{C}_G(x^2)|_2$. Now, as $2^a = |\langle x \rangle N/N| \leq |\mathbf{C}_Q(x)N/N| \leq |\mathbf{C}_{G/N}(xN)|_2 = 2^a$, we conclude that $\mathbf{C}_Q(x) \cap N = \mathbf{Z}(Q) \cap N$ has order 2. The same discussion yields that $\mathbf{C}_Q(x^2) \cap N = \mathbf{Z}(Q) \cap N$.

Take $y \in \mathbf{Z}(Q) \cap N$ to be the element of order 2. We distinguish two cases, depending on the fact that yx is conjugate to either x or x^{-1}. Suppose first that $x^g = yx$ for some $g \in G$. Then, as $[x, g] \in N$, by (ii) we can write $g = cn$, where $c \in \mathbf{C}_G(x)$ and $n \in N$. Thus, we may assume without loss that $g \in N$. On the other hand, notice that $g \in \mathbf{C}_G(x^2)$. Since x^2 cannot fix any $2'$-element of N, we deduce that g is a 2-element, so g belongs to $\mathbf{Z}(Q) \cap N \subseteq \mathbf{C}_G(x)$, which is a contradiction. Finally, if $(x^{-1})^g = yx$ for some $g \in G$, it follows that x^2 is real, again a contradiction. The proof of (iii) is complete. $\square$

Lemma 4.3. *Assume that G satisfies the hypothesis, N is nilpotent and Case (c) of Theorem 4.1, including that N is not a 2-group, occurs. Then the Sylow 2-subgroups of G/N are not quaternion.*

Proof. Assume on the contrary that the Sylow 2-subgroups of G/N are isomorphic to Q_8 and take $x(Q \cap N)$ an element of order 4, where Q is a Sylow 2-subgroup of G. According to Lemma 2.1 we have either $|\mathbf{C}_G(x^2)|_2 = |\mathbf{C}_{G/N}(x^2N)|_2 = 8$ or $|\mathbf{C}_G(x^2)|_2 = 2|\mathbf{C}_{G/N}(x^2N)|_2 = 16$. We study both cases separately.

Case 1. Suppose that the former case occurs. By applying Lemma 4.2(i), it follows that $\mathbf{C}_G(x^2) = \mathbf{C}_G(x)$ and so $\mathbf{C}_Q(x^2) = \mathbf{C}_Q(x)$. On the other hand, $4 = |\mathbf{C}_Q(x)N/N| = |\mathbf{C}_Q(x)/\mathbf{C}_{Q\cap N}(x)|$. Thus, $4 = |\langle x \rangle N/N| \leq |\mathbf{C}_Q(x)N/N| \leq |\mathbf{C}_{G/N}(xN)|_2 = 4$, which means that $\mathbf{Z}(Q) \cap N = \mathbf{C}_Q(x) \cap N = \mathbf{C}_Q(x^2) \cap N$ is a group of order 2. Set $T = \mathbf{Z}(Q) \cap N = \langle y \rangle$. By hypothesis, we know that there exists $g \in G$, such that either $x^g = yx$ or $(x^{-1})^g = xy$. In the following we exclude both cases.

Suppose first that $x^g = yx$ for some $g \in G$. Since $[x, g] \in N$, then by Lemma 4.2(ii) we can write $g = cn$, where $c \in \mathbf{C}_G(x)$ and $n \in N$. Thus, we may assume $g \in N$. On the other hand, $g \in \mathbf{C}_G(x^2)$. Since x^2 does not fix any $2'$-element of N by Theorem 4.1(c), we deduce that g is a 2-element. As N is nilpotent, then g belongs to the only Sylow 2-subgroup of N, that is, $g \in Q \cap N$. Then g must belong to $T = \mathbf{Z}(Q) \cap N \subseteq \mathbf{C}_G(x)$, a contradiction.

Therefore, we may assume $(x^{-1})^g = yx$ for some $g \in G$, and it follows that x^2 is real. Now, assume $1 \neq tT \in \mathbf{Z}(Q/T) \cap (N \cap Q)/T$. Then $[t, x] = y$, which implies that $x^t =$

$xy = (x^{-1})^g$. This contradicts the fact that x is non-real. Hence $\mathbf{Z}(Q/T) \cap (N \cap Q)/T = 1$ and so $N \cap Q = T$, which gives that Q has order 16. According to the classification of groups of order 16 that have a factor isomorphic to Q_8, we obtain that Q is isomorphic to either $Q_8 \times C_2$ or $C_4 \rtimes C_4$. But in both cases, all elements of order 2 are central. This would imply that $|\mathbf{C}_Q(x^2)| = 16$, a contradiction.

Case 2. Assume $|\mathbf{C}_G(x^2)|_2 = 2|\mathbf{C}_{G/N}(x^2N)|_2 = 16$. By Lemma 2.1, we have that x^2 is not real, and consequently, $|x^2| > 2$, which means that $|x| > 4$. Since $|\mathbf{C}_Q(x)| = 8$, we get $\mathbf{C}_Q(x) = \langle x \rangle$, $|x| = 8$ and $\mathbf{Z}(Q) \cap N = \langle x^4 \rangle$ is a group of order 2. By hypothesis, there exists $g \in G$, such that $x^g = x^5$ or $(x^{-1})^g = x^5$. The latter case yields that x^2 is real, a contradiction. Therefore, we assume $x^g = x^5$ for some $g \in G$. Then, as $[x, g] \in N$, by applying Lemma 4.2(ii) we can write $g = cn$, where $c \in \mathbf{C}_G(x)$ and $n \in N$, so we assume $g \in N$. On the other hand, $g \in \mathbf{C}_G(x^2)$. Since x^2 does not fix any $2'$-element of N, we deduce g is a 2-element. As N is nilpotent we have $g \in Q \cap N$. Thus, $g \in \mathbf{C}_Q(x^2) \setminus \mathbf{C}_Q(x)$, $\mathbf{C}_Q(x^2) = \langle x, g \rangle$ is a group of order 16 and $\mathbf{C}_{Q \cap N}(x^2) = \langle g, x^4 \rangle$. By the classification of groups of order 16 with a cyclic maximal subgroup we obtain $\mathbf{C}_G(x^2) \cong M_{16}$ and $|g| = 2$.

Therefore, we have got the following properties: $x(Q \cap N)$ is an element of order 4 in the quaternion group $Q/(Q \cap N)$, with $|x| = 8$, and there exists $g \in Q \cap N$ of order 2 such that $x^g = x^5$ and $\mathbf{C}_G(x^2) = \langle g, x \rangle \cong M_{16}$. Observe that these properties are exactly the same as the ones obtained in case 2 of Step 4 in the proof of Theorem A. Thus, by using the same argument we get a contradiction. As noticed before, the fact that N is nilpotent is not employed. □

Lemma 4.4. *Assume that G satisfies the hypothesis, N is nilpotent, $|G/N|_2 = 2$ and case (c) of Theorem 4.1, including that N is not a 2-group, occurs. Then, $|G/N| = 2$, $C_2 \cong \mathbf{Z}(G) \subseteq N$ and G is a quasi-Frobenius group with complement cyclic of order 4 and abelian kernel N. In particular, $G/\mathbf{Z}(G)$ is a generalized dihedral group.*

Proof. Let x and Q be as described in Lemma 4.2. By applying Lemma 4.2(i), we have $|\mathbf{C}_G(x)|_2 = 2|\mathbf{C}_{G/N}(xN)|_2$, which means that x is not real by Lemma 2.1, and in particular $|x| \geq 4$. But on the other hand, as $|\mathbf{C}_G(x)|_2 = 4$, we get $|x| = 4$.

Since x is not real, then for every $g \in \mathbf{N}_G(\langle x \rangle)$, we have $g \in \mathbf{C}_G(\langle x \rangle)$. In particular, $\mathbf{N}_Q(\langle x \rangle) = \mathbf{C}_Q(\langle x \rangle) = \langle x \rangle$, which forces that $\langle x \rangle = Q$. It is well-known then that G has a normal 2-complement, say H. As N is nilpotent, $Q \cap N = \langle x^2 \rangle$ is normal in G, so we trivially have $\langle x^2 \rangle \subseteq \mathbf{Z}(G)$. Furthermore, by hypothesis $x^2 \in N$ cannot centralize any odd order element out of N, so $H \leq N = H\langle x^2 \rangle$ and $|G/N| = 2$. Also, by Theorem 4.1(c) we know that G/N acts Frobeniusly on H, so necessarily $\mathbf{Z}(G) = \langle x^2 \rangle$. Moreover, notice that H is abelian since it is an odd order group acted on by a fixed-point-free automorphism of order 2. We conclude then that G is a quasi-Frobenius group with complement cyclic or order 4 and abelian kernel N. In particular, $G/\mathbf{Z}(G)$ is generalized dihedral. □

Proof of Theorem B. Theorem 4.1 asserts that we have three possibilities for G, the first two of which coincide with the first two possibilities of Theorem B. Suppose that

Theorem 4.1(c) occurs including the hypothesis that N is not a 2-group. Then, by following Lemmas 4.2-4.4, we conclude that G satisfies the assertion (3) of Theorem B, as wanted.

Data availability

No data was used for the research described in the article.

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