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Endograph Metric and a Version of the Arzelà–Ascoli Theorem for Fuzzy Sets

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Abstract: In this paper, we provide several Arzelà–Ascoli-type results on the space of all continuous functions from a Tychonoff space X into the fuzzy sets of $\mathbb{R}^n$, $(F_{USCB}(\mathbb{R}^n), H_{end})$, which are upper semi-continuous and have bounded support endowed with the endograph metric. Namely, we obtain positive results when X is considered to be a k_r-space and $C(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ is endowed with the compact open topology, as well as when we assume that X is pseudocompact and $C(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ is equipped with the uniform topology.

Keywords: Arzelà–Ascoli theorem; compactness; fuzzy sets; endograph metric

MSC: 03E72; 54A40; 46S40

1. Introduction

Since L.A. Zadeh introduced the notion of a fuzzy set in 1965 [1,2], its applications have covered a wide spectrum of fields of mathematics: from fuzzy logic [3,4] to fuzzy differential equations [5,6] through fuzzy control theory [7,8], fuzzy optimization theory [9–12], fuzzy analysis [13,14] or dynamical systems [15,16].

In the framework of classical analysis, Arzelà–Ascoli *-type* theorems are a powerful tool in the applications of function spaces. Thus, these kinds of theorems have potential applications of interest in the fuzzy context. In essence, given a function space $\mathbf{F}(X, Y)$ endowed with a topology τ, the aim is to characterize compact subsets of $\mathbf{F}(X, Y)$. For example, the classical Arzelà–Ascoli theorem states that a subset K of the space of all real-valued continuous functions on the unit interval equipped with the unifom topology is compact if and only if K is closed, bounded and equicontinuous. Actually, given a uniform space Y, an Arzelà–Ascoli-type theorem characterizes compactness in the function space $C(X, Y)$ by means of equicontinuity plus *natural conditions*. In fuzzy analysis, examples of this situation are the Arzelà–Ascoli-type theorems presented in [17]: the authors characterize compact subsets of $C_{\tau_\alpha}(X, (\mathbb{E}^1, d_\infty))$, the space of all continuous functions from a Tychonoff space X into the space of real (compact) fuzzy numbers (endowed with the supremum distance) where τ_α is the topology of the uniform convergence on the members of a cover α. The characterization is obtained in terms of equicontinuity *plus fuzzy conditions*.

Following this pattern, our goal is to obtain a fuzzy Arzelà–Ascoli theorem on space $C(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ of all continuous functions from Tychonoff space X into the fuzzy sets u of $\mathbb{R}^n$, which are upper semi-continuous, and the support of u is a bounded set in $\mathbb{R}^n$ endowed with the endograph metric. The function space $C(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ will be equipped with the topology of uniform convergence on the members of some covers of X. Taking advantage of the fact that $\mathbb{E}^n$, the set of all fuzzy numbers, endowed with the endograph metric H_{end}, is a closed subset of $(F_{USCB}(\mathbb{R}^n), H_{end})$, we establish the corresponding ones for $C(X, (\mathbb{E}^n, H_{end}))$.

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2. Preliminaries

We deal with fuzzy sets on $\mathbb{R}^n$, i.e., functions from $\mathbb{R}^n$ into the unit interval $[0,1]$. As usual, the symbol $F(\mathbb{R}^n)$ stands for the family of all fuzzy sets on $\mathbb{R}^n$.

For each $u \in F(\mathbb{R}^n)$, let $[u]_\alpha$ denote the α-cut of u, i.e.,

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R}^n \mid u(x) \geq \alpha\}, & \alpha \in (0,1], \\ \mathrm{cl}_{\mathbb{R}^n}\{x \in \mathbb{R}^n \mid u(x) > 0\}, & \alpha = 0. \end{cases}$$

where, as usual, the symbol $\mathrm{cl}_{\mathbb{R}^n}(A)$ stands for the closure of a subset A of $\mathbb{R}^n$. Notice that $[u]_0$ is the support of u. We will now consider the subset $F_{USCB}(\mathbb{R}^n)$ of $F(\mathbb{R}^n)$ of all fuzzy sets u, which are upper semi-continuous, and $[u]_0$ is a bounded set in $\mathbb{R}^n$ equipped with the endograph metric H_{end}. It is worth noting that, if $u \in F_{USCB}(\mathbb{R}^n)$, then $[u]_0$ is a closed bounded subset of $\mathbb{R}^n$. Consequently, it is a compact set. This implies that for all $\alpha \in [0,1]$, $[u]_\alpha$ is compact as well: indeed, since u is upper semi-continuous, $[u]_\alpha$ is closed; the result now follows from the fact that $[u]_\alpha \subseteq [u]_0$.

In order to introduce the endograph metric, we first need to define the *Hausdorff metric* H on the set $K(\mathbb{R}^n)$ of all nonempty compact subsets of $\mathbb{R}^n$. To do this, given A, $B \in K(\mathbb{R}^n)$, consider

$$D(A,B) = \sup_{a \in A} d(a,B) = \sup_{a\in A} \inf_{b \in B} d(a,b),$$

where d stands for the Euclidean metric on $\mathbb{R}^n$. Now, the Hausdorff metric H is defined as

$$H(K_1, K_2) = \max\{D(K_1,K_2), D(K_2,K_1)\}$$

for all $K_1, K_2 \in K(\mathbb{R}^n)$.

Given $u \in F(\mathbb{R}^n)$, the endograph of u is defined as

$$\mathrm{end}\, u = \{(x,t) \in \mathbb{R}^n \times [0,1] \mid u(x) \geq t\}.$$

and the endograph metric H_{end} is defined by using the Hausdorff metric H:

$$H_{end}(u,v) = H(\mathrm{end}\, u, \mathrm{end}\, v).$$

The endograph metric was introduced by Kloeden in the family of all semi-continuous fuzzy sets by means of the so-called *extended Hausdorff distance* (See [18] for details. See also [19].).

Given a Tychonoff space, X, we will present Arzelà–Ascoli-type theorems on the function space $C_{\tau_\alpha}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ where τ_α is the topology of uniform convergence on members of special covers α of X. It is worth remarking that the space $C_{\tau_\alpha}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ is a Tychonoff space: indeed, the family of all subsets $U(A,\varepsilon)$ of $C(X, (F_{USCB}(\mathbb{R}^n), H_{end})) \times C(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ of the form

$$\{(f,g) \in C(X,(F_{USCB}(\mathbb{R}^n),H_{end})) \times C(X,(F_{USCB}(\mathbb{R}^n),H_{end})) \mid \sup_{a\in A} H_{end}(f(a),g(a)) < \varepsilon\},$$

for all $A \in \alpha$ and all $\varepsilon > 0$, is a subbase for a (Hausdorff) uniformity $\mathcal{U}_\alpha$ on the function space $C(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$, which induces the topology τ_α. The closure of a subset S in the τ_α topology will be denoted by $\mathrm{cl}_{\tau_\alpha} A$.

Recall that space X is a k_r-space if every real-valued continuous function that is continuous on the compact sets is continuous on the whole X. It is a well-known fact that it is possible to replace *real-valued* by *into any Tychonoff space*. In particular, we can replace *real-valued* by *into* $(F_{USCB}(\mathbb{R}^n), H_{end})$. Wpace X is pseudocompact if every real-valued continuous function on X is bounded. Equivalently, any locally finite sequence of pairwise disjoint open sets in X is finite. Our main results establish that our fuzzy version of Arzelà–Ascoli theorem is satisfied when: (1) X is a k_r-space and $C(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ is endowed with the compact open topology, and (2) X is pseudocompact and $C(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ is equipped with the uniform topology. Moreover, the converse of (2) is also valid. The results also apply in the case of fuzzy numbers.

Our terminology and notation are standard. For instance, $\mathbb{N}$ stands for the set of natural numbers and $f|_A$ means the restriction of a function f to a subset A. We denote by $\mathbb{R}^m$ the product space of m copies of the reals. The symbol $C(X,(F_{USCB}(\mathbb{R}^n), H_{end}))$ (respectively, $C(X,(\mathbb{E}^n, H_{end}))$ stands for the set of all continuous functions from X into $(F_{USCB}(\mathbb{R}^n), H_{end})$ (respectively, into $(\mathbb{E}^n, H_{end})$). Notice that we obtain τ_p, the topology of the pointwise convergence, by taking α the cover of X consisting of its points or, equivalently, of all its finite subsets. If $\alpha = \{X\}$, then we obtain the topology, τ_u, of uniform convergence on X. The cover k of all compact subsets of a topological space X induces the so-called compact-open topology on $C(X,(F_{USCB}(\mathbb{R}^n), H_{end}))$ denoted by τ_{co}. It is worth noting that the pointwise convergence topology on the set of all functions from X into $F_{USCB}(\mathbb{R}^n)$ coincides with the product topology on $(F_{USCB}(\mathbb{R}^n))^X$. This is equivalent to considering τ_p on $C(X, F_{USCB}(\mathbb{R}^n), H_{end}))$ when X is endowed with the discrete topology. Throughout the whole paper, *space* means a *Tychonoff* space. For notions that are not explicitly defined here, the reader might consult [20].

3. Results

Given subset $B \subseteq F_{USCB}(\mathbb{R}^n)$ and $\alpha \in (0,1]$, we define

$$B(\alpha) = \bigcup_{u \in B} [u]_\alpha.$$

Our starting point is the following theorem by H. Huang:

Theorem 1 ([21], Theorem 8.6)**.** *Subset $B \subset (F_{USCB}(\mathbb{R}^n), H_{end})$ is relatively compact if and only if the following hold:*

(i) For each $\alpha \in (0,1]$, $B(\alpha)$ is a bounded subset in $\mathbb{R}^n$;
(ii) Given sequence $\{u_n\}_{n=1}^\infty \subset B$, there exists $r \in (0,1]$ and a subsequence, $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty \subset B$, such that $\lim_{k\to\infty}[u_{n_k}]_r = 0$.

A fuzzy set $u \in F_{USCB}(\mathbb{R}^n)$ is called a *fuzzy number* if it is normal; that is, there exists $x \in \mathbb{R}^n$ such that $u(x) = 1$, and u is *fuzzy convex*, that is, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$. The set of all fuzzy numbers is denoted by $\mathbb{E}^n$. It is worth noting that $\mathbb{R}^n$ is a closed subset of $(\mathbb{E}^n, H_{end})$: indeed, for each $x \in \mathbb{R}^n$, it suffices to consider the fuzzy set defined by the characteristic function of $\{x\}$. Moreover, it is well known that $(\mathbb{E}^n, H_{end})$ is a closed subspace of $(F_{USCB}(\mathbb{R}^n), H_{end})$ (see, for example, [21], p. 79). Thus, the following result is a straightforward consequence of Theorem 1.

Theorem 2. *Subset $B \subset (\mathbb{E}^n, H_{end})$ is relatively compact if and only if the following hold:*

(i) For each $\alpha \in (0,1]$, $B(\alpha)$ is a bounded subset in $\mathbb{R}^n$;
(ii) Given sequence $\{u_n\}_{n=1}^\infty \subset B$, there exists $r \in (0,1]$ and a subsequence, $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty \subset B$, such that $\lim_{k\to\infty}[u_{n_k}]_r = 0$.

With Theorem 1 in mind, we prove a version of Arzelà–Ascoli theorem when working with the endograph metric on $F_{USCB}(\mathbb{R}^n)$. First, some definitions:

Definition 1. *Subset $\mathcal{M} \subset C(X,(F_{USCB}(\mathbb{R}^n), H_{end}))$ is said to be* pointwise level bounded *if $\{f(x)(\alpha) \mid f \in \mathcal{M}\}$ is bounded in $\mathbb{R}^n$ for all $x \in X$ and all $\alpha \in (0,1]$.*

Definition 2. *Subset $\mathcal{M} \subset C(X,(F_{USCB}(\mathbb{R}^n), H_{end}))$ is said to be* pointwise-approached to zero *if $\{f(x) \mid f \in \mathcal{M}\}$ satisfies condition (ii) of Theorem 1 for all $x \in X$.*

Recall that $\mathcal{M} \subset C(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ is *equicontinuous* if for each $x \in X$ and each $\varepsilon > 0$, there is a neighborhood V of x such that $H_{end}(f_i(y), f_i(x)) < \varepsilon$ for all $y \in V$ and all $f \in \mathcal{M}$.

The function space $C_\tau(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ is said to satisfy the *weak fuzzy Arzelà–Ascoli theorem* if each τ-closed, pointwise-level-bounded, equicontinuous, and pointwise-approached-to-zero subset of $C_\tau(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ is τ-compact. If the converse is also valid, then we say that $C_\tau(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ satisfies the *fuzzy Arzelà-Ascoli theorem*.

We need the following:

Theorem 3. *For any space X, a subset $\mathcal{M}$ of $(F_{USCB}(\mathbb{R}^n), H_{end})^X$ is τ_p-relatively compact if and only if the following two conditions are satisfied:*

(i) $\mathcal{M}$ is pointwise-level-bounded.
(ii) $\mathcal{M}$ is pointwise-approached to zero.

Proof. To prove necessity, consider the τ_p-compact subset $\mathcal{M}$ of $(F_{USCB}(\mathbb{R}^n), H_{end})^X$. The projection map $\pi_x \colon \mathcal{M} \to (F_{USCB}(\mathbb{R}^n), H_{end})$ defined for all $x \in X$, as $\pi_x(f) = f(x)$ is continuous so that $\{f(x) : f \in \mathcal{M}\}$ is a compact subset of $(F_{USCB}(\mathbb{R}^n), H_{end})$. It suffices now to apply Theorem 1. For sufficiency, assume that $\mathcal{M}$ satisfies conditions (i)–(ii). Using Theorem 1, for each $x \in X$, the set $\{f(x) \mid f \in \mathcal{M}\}$ is relatively compact in $(F_{USCB}(\mathbb{R}^n), H_{end})$. Thus, $C = \mathrm{cl}_{\tau_p}(\prod_{x \in X}\{f(x) \mid f \in \mathcal{M}\})$ is τ_p-compact. The result now follows from the fact that the τ_p-closure of $\mathcal{M}$ is included in C. □

We now address the weak fuzzy Arzelà–Ascoli theorem. Let ρ denote the cover of X whose members are the pseudocompact subsets of X. We have:

Theorem 4. *If $\alpha \subseteq \rho$ is a cover of a space X, then $C_{\tau_\alpha}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ satisfies the weak fuzzy Arzelà–Ascoli theorem.*

Proof. Let $\mathcal{K}$ be a τ_p-closed, pointwise-level-bounded, equicontinuous, pointwise-approached-to-zero subset of $C_{\tau_\alpha}(X, F_{USCB}(\mathbb{R}^n, H_{end}))$. The previous theorem tells us that $\mathrm{cl}_{\tau_p}\mathcal{K}$ is compact. Consider now the pseudocompact subset $P \in \alpha$. Using Theorem 7.14 and Theorem 7.15 in [22], the evaluation mapping

$$\begin{aligned} e\colon \mathrm{cl}_{\tau_p}\mathcal{K} \times P &\to (F_{USCB}(\mathbb{R}^n), H_{end}) \\ (f, x) &\to f(x) \end{aligned}$$

is a continuous function. Let $\{f_i\}_{i \in I}$ be a net in $C_{\tau_\alpha}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ converging to the function f_0. Since α is a cover of X, we have $\tau_\alpha \geq \tau_p$. Therefore, $\{f_i\}_{i \in I}$ τ_p-converges to f_0. Define now the following real-valued continuous function:

$$\begin{aligned} \phi\colon \mathrm{cl}_{\tau_p}\mathcal{K} \times P &\to (F_{USCB}(\mathbb{R}^n), H_{end}) \to \mathbb{R} \\ (f, x) &\to f(x) \to H_{end}(f(x), f_0(x)) \end{aligned}$$

$\mathrm{cl}_{\tau_p}\mathcal{K}$ being compact, the product space $\mathrm{cl}_{\tau_p}\mathcal{K} \times P$ is pseudocompact ([20], Theorem 3.10.26). Therefore, by a lemma of Frolík ([23], Lemma 1.3), the function G from $\mathrm{cl}_{\tau_p}\mathcal{K}$ into the reals defined as

$$G(f) = \sup_{x \in P}\phi(f, x) = \sup_{x \in P}H_{end}(f(x), f_0(x)),$$

for all $f \in cl_{\tau_p}\mathcal{K}$, is continuous. Thus, $G(f_i)$ converges to $G(f_0)$. In other words, we have just proved that $\sup_{x \in P}H_{end}(f_i(x), f_0(x))$ converges to $\sup_{x \in P}H_{end}(f_0(x), f_0(x)) = 0$. This means that $\{f_i\}_{i \in I}$ converges uniformly to f_0 on P. Since P is an arbitrary member of the cover α, we have just proved that $\{f_i\}_{i \in I}$ τ_α-converges to f_0. This fact implies that the inclusion map from $\mathrm{cl}_{\tau_p}\mathcal{K}$ into $C_{\tau_\alpha}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ is continuous so that $cl_{\tau_p}\mathcal{K}$ is τ_α-compact. The result now follows from the fact that the τ_α-closure of $\mathcal{K}$ is included in its τ_p-closure. □

Throughout what follows, we shall freely use without explicit mention the elementary fact that, being $(\mathbb{E}^1, H_{end})$ closed in $(F_{USCB}(\mathbb{R}^n), H_{end})$, the function space $C(X, (\mathbb{E}^1, H_{end}))$ is closed in $C_{\tau_\alpha}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$. Thus, we have

Corollary 1. *Let X be a space. If α is a cover of X with $\alpha \subseteq \rho$, then $C_{\tau_\alpha}(X, (\mathbb{E}^1, H_{end}))$ satisfies the weak fuzzy Arzelà–Ascoli theorem.*

Next, we will present an example of a function space that satisfies the weak fuzzy Arzelà–Ascoli theorem but fails to satisfy the fuzzy Arzelà–Ascoli theorem. Given a product space $X \times Y$, let ϕ_X denote the natural injection

$$\phi_X \colon (\mathbb{R})^{X_1 \times X_2} \to (\mathbb{R}^{X_2})^{X_1}.$$

The map ϕ_X is a homeomorphism. We denote by $\beta(Z)$ the Stone–Čech compactification of a space Z.

Example 1. Let X, Y be two pseudocompact spaces such that $X \times Y$ is not pseudocompact. According to Proposition 1.12 in [24], there exists a continuous function f on $X \times Y$ that admits a separately continuous extension to $\beta(X) \times \beta(Y)$. According to Proposition 1 in [25], the closure, say $\mathcal{K}$, of $\phi_{X_1}(f)$ in $C_p(X_2)$ is compact. Suppose now that $\mathcal{K}$ is equicontinuous for a compatible metric on $\mathbb{R}$. As in Theorem 4, the evaluation map e from $\mathcal{K} \times X_2$ into $\mathbb{R}$ is continuous. Since $f = e \circ (\phi_{X_1}(f) \times \mathrm{id}_X)$, f is a continuous function. This contradiction shows that the compact set $\mathcal{K}$ is not equicontinuous.

As a straightforward consequence of the previous example, we obtain

Theorem 5. *If a metric space (X, d) contains a closed copy of the reals, then there exists a pseudocompact space Y such that $C_p(Y, (X, d))$ contains a compact subset that is not equicontinuous.*

Corollary 2. *There exists a pseudocompact space X such that $C_{\tau_p}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ (respectively, $C_p(X, (\mathbb{E}^1, H_{end}))$) does not satisy the fuzzy Arzelà–Ascoli theorem.*

Remark 1. It is clear that we can replace *metric space* by *uniform space* in Theorem 5.

We now turn our attention to the fuzzy Arzelà–Ascoli theorem. We say that a space X is a ρ_r-space if a real-valued function (equivalently, a function into a Tychonoff space) is continuous whenever its restriction to any pseudocompact subset of X is continuous.

Theorem 6. *If X is a ρ_r-space, then $C_{\tau_\rho}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ satisfies the fuzzy Arzelà–Ascoli theorem.*

Proof. According to Theorem 4, we only need to prove that a compact subset $\mathcal{K}$ of $C_{\tau_\rho}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ is τ_ρ-closed, pointwise-level-bounded, pointwise-approached to zero and equicontinuous. To do this, consider $P \in \rho$ and the evaluation map

$$\begin{aligned} e_P \colon \mathcal{K} \times P &\to (F_{USCB}(\mathbb{R}^n), H_{end}) \\ (f, x) &\to f(x) \end{aligned}$$

Since τ_ρ induces the topology of uniform convergence on P, a standard argument by using triangle inequality shows that e_P is continuous. Now, since the product of a compact space and a ρ_r-space is a ρ_r-space ([26]), the evaluation map from $\mathcal{K} \times X$ into $(F_{USCB}(\mathbb{R}^n), H_{end}))$ is continuous. Thus, $\mathcal{K}$ is equicontinuous ([22], Theorem 7.19, Theorem 7.20). Since $\tau_\alpha \geq \tau_p$, the results now follows from Theorem 3. □

Corollary 3. *If X is a ρ_r-space, then $C_{\tau_\rho}(X, (\mathbb{E}^n, H_{end}))$ satisfies the fuzzy Arzelà–Ascoli theorem.*

Recall that space S is said to be locally pseudocompact (respectively, locally compact) if every $x \in X$ has a pseudocompact (respectively, compact) neighborhood.

Corollary 4. *If X is a locally pseudocompact space, then $C_{\tau_\rho}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ (respectively, $C_{\tau_\rho}(X, (\mathbb{E}^n, H_{end}))$) satisfies the fuzzy Arzelà–Ascoli theorem.*

Corollary 5. *If X is a locally compact space, then $C_{\tau_{co}}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ (respectively, $C_{\tau_{co}}(X, (\mathbb{E}^n, H_{end}))$) satisfies the fuzzy Arzelà–Ascoli theorem.*

An argument similar to the one used in Theorem 6 yields

Theorem 7. *If X is a k_r-space, then $C_{\tau_{co}}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ (respectively, $C_{\tau_{co}}(X, (\mathbb{E}^n, H_{end}))$) satisfies the fuzzy Arzelà–Ascoli theorem.*

Corollary 6. *If X is a locally compact space, then $C_{\tau_{co}}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ (respectively, $C_{\tau_\rho}(X, (\mathbb{E}^n, H_{end}))$) satisfies the fuzzy Arzelà–Ascoli theorem.*

Corollary 7. *If X is a compact space, then $C_{\tau_u}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ (respectively, $C_{\tau_u}(X, (\mathbb{E}^n, H_{end}))$) satisfies the fuzzy Arzelà–Ascoli theorem.*

This last result can be improved by characterizing spaces X such that the topology of uniform convergence τ_u on $C(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ satisfies the fuzzy Arzelà–Ascoli theorem. If X is a pseudocompact space, we will use the notion that $C_{\tau_u}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ is metrizable. Indeed, the set of the form

$$\left\{(f,g) \in C(X,(F_{USCB}(\mathbb{R}^n),H_{end})) \times C(X,(F_{USCB}(\mathbb{R}^n),H_{end})) \mid \sup_{a\in P} H_{end}(f(a),g(a)) < \tfrac{1}{n}\right\},$$

is a subbase for a uniformity $\mathcal{U}$ on $C(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ inducing the uniform convergence topology τ_u. Thus, the uniformity $\mathcal{U}$ has a countable base, and, consequently, it is metrizable (see, e.g., [20], Theorem 8.1.12).

Theorem 8. *For a space X, the following conditions are equivalent:*

(i) X is pseudocompact.
(ii) $C_{\tau_u}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ satisfies fuzzy Arzelà–Ascoli theorem.
(iii) $C_{\tau_u}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ satisfies the weak fuzzy Arzelà–Ascoli theorem.

Proof. (1)$\Longrightarrow$(2) follows from Theorem 6 and (2)$\Longrightarrow$(3) is obvious. We show (3)$\Longrightarrow$(1). We take advantage of the fact that $(\mathbb{E}^1, H_{end})$ and, a posteriori, $(F_{USCB}(\mathbb{R}^n), H_{end})$ contains a (closed) copy of the reals. Suppose that X is not pseudocompact. Then, we can find an infinite sequence $\{U_n\}_{n=1}^\infty$ of pairwise disjoint open sets that is locally finite. For each $n \in \mathbb{N}$, pick $x_n \in U_n$. X being a Tychonoff space, there exists a continuous function f_n from X into the reals such that $f_n|_{X \setminus U_n} \equiv 0$ and $f_n(x_n) = n$. It is clear that the sequence $\{f_n\}_{n=1}^\infty$ pointwise-converges to the zero function. Thus, $\{f_n\}_{n=1}^\infty$ is relatively τ_p- compact. Since the τ_u-closure of our sequence is contained in its τ_p-closure, Theorem 3 tells us that the τ_u-closure of $\{f_n\}_{n=1}^\infty$ is pointwise-level-bounded and pointwise-approached to zero. We shall prove that the τ_u-closure of $\{f_n\}_{n=1}^\infty$ is equicontinuous. Notice that, since $\tau_u \geq \tau_p$, it suffices to prove that the sequence $\{f_n\}_{n=1}^\infty$ is equicontinuous (see [20], Theorem 7.14). Let $x \in X$. Since all the functions f_n vanish outside $\bigcup_{n=1}^\infty U_n$ and the sequence $\{U_n\}_{n=1}^\infty$ is locally finite, it is an easy matter to prove that $\{f_n\}_{n=1}^\infty$ is equicontinuous at x. Assume now that there is $n \in \mathbb{N}$ with $x \in U_n$. Given $\varepsilon > 0$, there exists a neighborhood $W \subseteq U_n$ such that $H_{end}(f(y), f(x)) < \varepsilon$ for all $y \in W$. The results now follows from the fact that $f_m|_{X \setminus U_n} \equiv 0$ for all $m \neq n$. We conclude the proof by showing that the τ_u-closure of $\{f_n\}_{n=1}^\infty$ is not τ_u-compact. Taking into account that the topology τ_u is metrizable and our sequence pointwise-converges to zero, it suffices to prove that any subsequence of $\{f_n\}_{n=1}^\infty$ does not converge to zero. But this fact is a straightforward consequence of the definition of f_n for each $n \in \mathbb{N}$. The proof is complete. □

Corollary 8. *For a space X, the following conditions are equivalent:*

(i) X *is pseudocompact.*
(ii) $C_{\tau_u}(X, (\mathbb{E}^1, H_{end}))$ *satisfies fuzzy Arzelà–Ascoli theorem.*
(iii) $C_{\tau_u}(X, (\mathbb{E}^1, H_{end}))$ *satisfies the weak fuzzy Arzelà–Ascoli theorem.*

With each compact space being pseudocompact, the implication (1)⟹(2) of the previous theorem implies Corollary 7.

We close the paper with some applications of the previous results. In the outstanding paper [23], Frolík introduced the class $\mathfrak{B}$ of all pseudocompact spaces X such that the product space $X \times Y$ is also pseudocompact for every pseudocompact space Y. Noble [27], later, showed that $\mathfrak{B}$ is closed by taking arbitrary products. Moreover, by a result of Tkachenko [28], every pseudocompact topological group belongs to $\mathfrak{B}$ (see also [29]). Thus, we have

Corollary 9. *If* $X = \prod_{i \in I} X_i$ *is an arbitrary product of pseudocompact spaces in class* $\mathfrak{B}$ *(in particular, if X is an arbitray product of pseudocompact groups), then X satisfies Theorem 8 and Corollary 8.*

k_r-spaces play an important role in funcional analysis. A special class of this kind of spaces comprises k_r-pseudocompact spaces. Noble showed [30] that an arbitrary product of k_r-pseudocompact spaces is a k_r-pseudocompact space as well. Therefore, we have

Corollary 10. *If* $X = \prod_{i \in I} X_i$ *is an arbitrary product of* k_r*-pseudocompact spaces, then the following conditions hold:*

(i) *X satisfies Theorem 8 and Corollary 8.*
(ii) $C_{\tau_{co}}(X, (F_{USCB}(\mathbb{R}^n), H_{end}))$ *and* $C_{\tau_{co}}(X, (\mathbb{E}^1, H_{end}))$ *satisfy fuzzy Arzelà–Ascoli theorem.*

4. Conclusions

In classical analysis, Arzelà–Ascoli-type theorem characterizes compactness in function spaces. The aim of this manuscript is to investigate a fuzzy-type Arzelà–Ascoli theorem. The authors introduce weak fuzzy Arzelà-Ascoli theorem and fuzzy Arzelà–Ascoli theorem, and provide several sufficient conditions for a function space to satisfy (weak) fuzzy Arzelà–Ascoli theorem. Using a counterexample, it is shown that weak fuzzy Arzelà–Ascoli theorem is not equivalent to fuzzy Arzelà–Ascoli theorem, and it is proved that the function space in question satisfies (weak) fuzzy Arzelà–Ascoli theorem if and only if X is pseudocompact. Some applications of these results are also given. Our results are expected to be applied in future research in fields such as fuzzy differential equations and optimization theory.

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