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First Dirichlet Eigenvalue and Exit Time Moment Spectra Comparisons

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Abstract
We prove explicit upper and lower bounds for the Poisson hierarchy, the averaged L^1-moment spectra $\left\{\dfrac{\mathcal{A}_k\left(B_R^M\right)}{\operatorname{vol}\left(S_R^M\right)}\right\}_{k=1}^{\infty}$, and the torsional rigidity $\mathcal{A}_1(B_R^M)$ of a geodesic ball B_R^M in a Riemannian manifold M^n which satisfies that the mean curvatures of the geodesic spheres S_r^M included in it, (up to the boundary S_R^M), are controlled by the radial mean curvature of the geodesic spheres $S_r^\omega(o_\omega)$ with same radius centered at the center o_ω of a rotationally symmetric model space M_ω^n. As a consecuence, we prove a first Dirichlet eigenvalue $\lambda_1(B_R^M)$ comparison theorem and show that equality with the bound $\lambda_1(B_R^\omega(o_\omega))$, (where $B_r^\omega(o_\omega)$ is the geodesic r-ball in M_ω^n), characterizes the L^1-moment spectrum $\left\{\mathcal{A}_k(B_R^M)\right\}_{k=1}^\infty$ as the sequence $\left\{\mathcal{A}_k(B_R^\omega)\right\}_{k=1}^\infty$ and vice-versa.

Keywords Mean exit time · Torsional rigidity · Mean curvature · Geodesic ball

Mathematics Subject Classification (2010) Primary 53C20 · 58C40; Secondary 58J32

1 Introduction

Let (M^n, g) be a complete Riemannian manifold. We shall consider the Brownian motion X_t in M and, given $x \in M$, its associated family of probability measures $\mathbb{P}^x$ on the space

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of Brownian paths emanating from a point $x \in M$. Given a smoothly bounded precompact domain $D \subseteq M$, the first exit time from D is given by the quantity

$$\tau_D := \inf\{t \geq 0 : X_t \notin D\} .$$

Given $x \in D$, the function $E_D : D \to \mathbb{R}$ that assigns to the point x the expectation of the value of the first exit time τ_D with respect $\mathbb{P}^x$ is the *mean exit time function from x*, $E_D(x)$. We have the following characterization, (see [16]), of this function as a solution of the second order PDE, with Dirichlet boundary data:

$$\begin{aligned} \Delta^M E_D + 1 &= 0, \text{ in } D, \\ E_D|_{\partial D} &= 0, \end{aligned} \tag{1.1}$$

where Δ^M denotes the Laplace-Betrami operator on (M^n, g).

The mean exit time function is the first in a sequence of functions $\{E_D = u_{1,D} , u_{2,D} , ...\}$ defined in $D \subseteq M$ inductively as follows

$$\begin{aligned} \Delta^M u_{1,D} + 1 &= 0, \text{ on } D, \\ u_{1,D}|_{\partial D} &= 0, \end{aligned} \tag{1.2}$$

and, for $k \geq 2$,

$$\begin{aligned} \Delta^M u_{k,D} + k u_{k-1,D} &= 0, \text{ on } D, \\ u_{k,D}|_{\partial D} &= 0. \end{aligned} \tag{1.3}$$

This sequence is the so-called, (see [15]), *Poisson hierarchy for D*.

The Poisson hierarchy of the domain D determines the L^p-moment spectrum of D, which can be defined as the following sequence of integrals, (see e.g. [30] and references therein for a more detailed exposition of these concepts):

$$A_{p,k}(D) := \left(\int_D (u_{k,D}(x))^p dV \right)^{\frac{1}{p}}, \; k = 1, 2, ..., \infty.$$

We are going to focus our study on the L^1-moment spectrum of D, $\{\mathcal{A}_{1,k}(D)\}_{k=1}^{\infty}$ which we denote as $\{\mathcal{A}_k(D)\}_{k=1}^{\infty}$ and, in particular, in its first value, $\mathcal{A}_1(D)$, called the *torsional rigidity of D* which is as the integral

$$\mathcal{A}_1(D) = \int_D E_D(x)\, d\sigma, \tag{1.4}$$

where E_D is the smooth solution of the Dirichlet-Poisson (1.1).

The name "torsional rigidity" comes from the fact that, when $D \subseteq \mathbb{R}^2$ is a plane domain, the quantity $\mathcal{A}_1(D)$ represents the torque required when twisting an elastic beam of uniform cross section D, (see [36]). A natural question to consider is the optimization of this quantity among all the domains having the same given area/volume in a fixed space or under some other geometrical setting. This problem is known as a *Saint-Venant type problem*.

The study of this variational problem in the general context of Riemannian manifolds involves the establishment of bounds on the torsional rigidity of a given domain $D \subseteq M$, together with the identification of the domains that optimize bounds given natural constraints on the domain (for example, fixed volume) and geometric constraints on the ambient space, in a fashion analogous to the treatment of the principal eigenvalue for the Rayleigh conjecture. The techniques involved in this analysis encompasses the use of the notion of Schwarz symmetrization as well as the isoperimetric inequalities satisfied by the domains in question.

From the intrinsic point of view, the establishment of bounds for the L^p-moment spectrum and the study of the relationship between the torsional rigidity (and more generally, the L^1-moment spectrum), of a domain in a Riemannian manifold $D \subseteq M$ and its Dirichlet spectrum has been explored recently in a number of papers, (see, among others, [4–7, 15, 22–26, 29–32] and the references therein). Related with this issue and in the line of the classical Kac's question, we have the isospectrality problem, namely, to see to what extent the L^1- moment spectrum of a domain determines it up to isometry, (see [13, 14]).

From the viewpoint of submanifold theory, we can find in the papers [29], [22], and [23] upper and lower bounds for the L^1-moment spectrum of extrinsic balls $B_R^M \cap \Sigma$, (let us denote as B_R^M the geodesic R-ball in the manifold M), in submanifolds $\Sigma^m \subseteq M^n$ with controlled mean curvature H_Σ immersed in ambient Riemannian manifolds (M, g) with radial sectional curvatures $K_{(M,g)}(\frac{\partial}{\partial r},)$ bounded from above or from below. These bounds were given, on the basis of previously established isoperimetric inequalities, by the corresponding values for the torsional rigidity of the Schwartz symmetrization of the geodesic balls in rotationally symmetric spaces with a pole which are warped products of the form $M_w^n = [0, \infty) \times_w \mathbb{R}^+$, which we refer to as the *model spaces*. As we shall see in Section 2.2, the model spaces M_w^n are rotationally symmetric generalizations of the real space forms with constant sectional curvature $b \in \mathbb{R}$, denoted as $M_{w_b}^n = \mathbb{R}^n, \mathbb{S}^n(b),$ or $\mathbb{H}^n(b)$, with

$$\omega_b(r) = \begin{cases} \frac{1}{\sqrt{b}} \sin\left(\sqrt{b}\, r\right), & \text{if } b > 0 \\ r, & \text{if } b = 0 \\ \frac{1}{\sqrt{-b}} \sinh\left(\sqrt{-b}\, r\right), & \text{if } b < 0 \end{cases}$$

We shall denote as $B_r^\omega(o_\omega)$ and as $S_r^\omega(o_\omega)$ the geodesic r-ball centered at o_ω and the geodesic r-sphere, respectively, in M_ω^n.

Moreover, in [22, 23, 29], the geometry of the situation involving the case of equality in the implied bounds was characterized. On the other hand, in these papers it were given too *intrinsic* upper and lower bounds for the torsional rigidity of geodesic balls B_R^M in the ambient manifold when it was assumed that $\Sigma = M$, so the extrinsic distance became the intrinsic distance and are only assumed bounds on the radial sectional curvatures of the ambient manifold M.

The intrinsic results in [22, 23, 29] are strongly aligned with those in [30]. In this paper, the author considers a domain $D \subseteq M$, in a Riemannian manifold (M, g) which satisfies an *isoperimetric condition with comparison constant curvature space form* M_{ω_b}, (namely, that there exists a constant curvature space form M_{ω_b} such that for all smoothly bounded and precompact domains D, we have that $\mathrm{Vol}(D) = \mathrm{Vol}(B_R^{\omega_b})$ implies that $\mathrm{Vol}(\partial D) \geq \mathrm{Vol}(\partial B_R^{\omega_b})$). Then it is proved that

$$\mathcal{A}_k(D) \leq \mathcal{A}_k(B_R^{\omega_b}) \;\forall k \in \mathbb{N}, \text{ where } \quad \mathrm{Vol}(D) = \mathrm{Vol}(B_R^{\omega_b})$$

The proof of this result relies on a Talenti-type comparison theorem, (see [1, 40]), satisfied by the solutions of the Poisson problem posed on domains $D \subseteq M$, in a Riemannian manifold (M, g) which satisfies the isoperimetric condition mentioned above.

To summarize the intrinsic results obtained in [22, 23, 29] in a couple of statements, we need the following context and notation: let us consider a complete Riemannian manifold (M, g), and a geodesic R-ball $B_R^M(o)$ centered at $o \in M$. Let us denote as

$K_{(M,g)}\left(\frac{\partial}{\partial r},\right)$ its radial, (from the center o) sectional curvatures, (namely, the sectional curvatures of the planes containing the radial vector field $\frac{\partial}{\partial r}$, where r denotes the distance function from the point o). With all these notions in hand, we have the following two results. The first concerns the so-called, (see [22]), *averaged L^1-moment spectrum* of a geodesic ball:

Theorem A (see [29] and [22]) *Let (M, g) be a complete Riemannian manifold. Let us consider M^n_w a rotationally symmetric model space and let us suppose that*

$$K_{(M_w,g_w)}\left(\frac{\partial}{\partial r},\right) \geq (\leq)\, K_{(M,g)}\left(\frac{\partial}{\partial r},\right),$$

where $K_{(M_w,g_w)}\left(\frac{\partial}{\partial r},\right)$ denotes the radial sectional curvatures of M^n_w from its center point $o_\omega \in M^N_\omega$.

Then the averaged L^1-moments, $\left\{\frac{\mathcal{A}_k\left(\mathrm{B}^M_R(o)\right)}{Vol\left(\mathrm{S}^M_R(o)\right)}\right\}_{k=1}^{\infty}$ are bounded as follows

$$\frac{\mathcal{A}_k\left(\mathrm{B}^\omega_R(o_\omega)\right)}{Vol\left(\mathrm{S}^\omega_R(o_\omega)\right)} \geq (\leq)\, \frac{\mathcal{A}_k\left(\mathrm{B}^M_R(o)\right)}{Vol\left(\mathrm{S}^M_R(o)\right)}. \tag{1.5}$$

Equality in inequality (1.5) for some $k_0 \geq 1$ implies that $\mathrm{B}^M_R(o)$ and $\mathrm{B}^\omega_R(o_\omega)$ are isometric.

Concerning now the torsional rigidity, we need to assume, in addition, that the model space M^n_w is *balanced from above*, namely, that the isoperimetric quotient given by

$$q_\omega(r) = \frac{\mathrm{Vol}\left(\mathrm{B}^\omega_r(o_\omega)\right)}{\mathrm{Vol}\left(\mathrm{S}^\omega_r(o_\omega)\right)}$$

is a non-decreasing function of r. This condition is satisfied by a wide range of spaces; in particular, for all real space forms of constant sectional curvature. We then obtain the following

Theorem B *[see [22, 29]] Let (M, g) be a complete Riemannian manifold. Let us consider M^n_w a rotationally symmetric model space, balanced from above, and let us suppose that*

$$K_{(M_w,g_w)}\left(\frac{\partial}{\partial r},\right) \geq (\leq)\, K_{(M,g)}\left(\frac{\partial}{\partial r},\right),$$

where $K_{(M_w,g_w)}\left(\frac{\partial}{\partial r},\right)$ denotes the radial sectional curvatures of M^n_w from its center point.

Then the torsional rigidity $\mathcal{A}_1\left(\mathrm{B}^M_R(o)\right)$ is bounded as follows

$$\mathcal{A}_1\left(\mathrm{B}^\omega_{s(R)}(o_\omega)\right) \geq (\leq)\, \mathcal{A}_1\left(\mathrm{B}^M_R(o)\right), \tag{1.6}$$

where $\mathrm{B}^\omega_{s(R)}(o_\omega)$ is the Schwarz symmetrization of $\mathrm{B}^M_R(o)$ in the model space (M^n_ω, g_ω).

Equality in inequality (1.6) implies that $s(R) = R$ and that $\mathrm{B}^M_R(o)$ and $\mathrm{B}^\omega_R(o_\omega)$ are isometric.

As a consequence of the bounds for the L^1-moment spectrum stated in Theorem A, and the proof of Theorem 1.1 in [32], (where a formula for the first Dirichlet eigenvalue of a precompact domain D in a Riemnnian manifold M, $\lambda_1(D)$, in terms of its

L^1-moment spectrum, $\{\mathcal{A}_k(D)\}_{k=1}^{\infty}$, was developed), the following version of Cheng's eigenvalue comparison theorem was obtained:

Theorem C *[see [11, 12, 24]] Let (M, g) be a complete Riemannian manifold. Let us denote as $K_{(M,g)}(\frac{\partial}{\partial r}, \,)$ its radial sectional curvatures and as $Ricc_{(M,g)}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)$ its radial Ricci curvatures at any point.*

Let us consider M^n_w a rotationally symmetric model space and let us suppose that

$$K_{(M_w,g_w)}\left(\frac{\partial}{\partial r}, \right) \geq K_{(M,g)}\left(\frac{\partial}{\partial r}, \right),$$

$$\left(\text{or that } Ricc_{(M_w,g_w)}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \leq Ricc_{(M,g)}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)\right),$$

where $K_{(M_w,g_w)}\left(\frac{\partial}{\partial r}, \,\right)$ and $Ricc_{(M_w,g_w)}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)$ denotes the radial sectional and Ricci curvatures of M^n_w at its center point.

Then

$$\lambda_1\left(B^{\omega}_R(o_\omega)\right) \leq (\geq)\, \lambda_1\left(B^M_R(o)\right).$$

for all $R < inj(o) \leq inj(o_w)$.

Equality in any of these inequalities holds if and only if the geodesic balls $\mathrm{B}^M_R(o)$ and $\mathrm{B}^{\omega}_R(o_\omega)$ are isometric.

On the other hand, in the paper [31], P. McDonald showed that, given a precompact domain $D \subseteq M$ in a complete Riemannian manifold M that satisfies the inequalities $\mathcal{A}_k(D) \leq \mathcal{A}_k(D^*)$ where D^* is the Schwarz symmetrization of D in a constant curvature space form M_{ω_b}, then we have the inequality $\lambda_1(D^*) \leq \lambda_1(D)$, (see Theorem 1 in [31]).

Following with versions of Cheng's result, in the paper [8], the authors proved that Cheng's eigenvalue comparison is still valid assuming bounds on the mean curvature of (intrinsic) distance spheres, a weaker hypothesis (as we shall see below), than the bounds on the sectional curvatures of the manifold:

Theorem D (see [8]) *Let $B^M_R \subseteq M^n$ and $B^{w_b}_R$ be geodesic R-balls in a Riemannian manifold (M, g) and in the real space form with constant sectional curvatures $b \in \mathbb{R}$, $M^n_{w_b}$, respectively, both within the cut locus of their centers and let $(t, \overline{\theta}) \in (0, R] \times \mathbb{S}^{n-1}_1$ be the polar coordinates for B^M_R and $B^{w_b}_R$.*

Then, if $H_{S^M_t}(t, \overline{\theta})$ and $H_{S^{w_b}_t}(t)$ are the, (inward pointing), mean curvatures of the distance spheres S^M_t in M and $S^{w_b}_t$ in the real space form of constant curvature M^{w_b}, respectively, and we assume that

$$H_{S^{w_b}_t}(t) \leq (\geq)\, H_{S^M_t}(t, \overline{\theta}) \;\forall t \leq R \;\forall \overline{\theta} \in \mathbb{S}^{n-1}_1$$

we have that

$$\lambda_1(B^{\omega_b}_R(o_\omega)) \leq (\geq)\, \lambda_1(B^M_R(o)).$$

Equality in any of these inequalities holds if and only if $H_{S^M_t}(t, \overline{\theta}) = H_{S^{w_b}_t}(t) \;\forall t \leq R \;\forall \overline{\theta} \in \mathbb{S}^{n-1}_n$.

The proof relies on Barta's Lemma and the expresion of the Laplacian of the first Dirichlet eigenfunction in polar coordinates. It is precisely from this intrinsic expression that the use, as hypotheses, of bounds on the mean curvature of distance spheres comes from.

Therefore, it can be said that the results we are going to present in the following paper are inspired, on the one hand, by the intrinsic bounds for the torsional rigidity and the L^1-moment spectrum of the geodesic balls and by the estimation of $\lambda_1(B^M_R)$ obtained in the papers [22–24, 29], and on the other hand, by the weaker restrictions on the mean curvatures of geodesic spheres assumed in [8] as well as the comparisons for the L^1-moment spectrum and the first Dirichlet eigenvalue given in [30, 31].

1.1 A Glimpse at our Results

We shall consider throughout the remainder of this paper, a complete Riemannian manifold (M^n, g) and a rotationally symmetric model space (M^n_ω, g_w), with center o_w, and we shall assume that given $o \in M$, the injectivity radius of $o \in M$ satisfies $inj(o) \leq \mathrm{inj}(o_w)$. Let us fix $R < inj(o) \leq inj(o_w)$ and assume that the pointed inward mean curvatures of metric r-spheres satisfies

$$\mathrm{H}_{\mathrm{S}^\omega_r(o_\omega)} \leq \mathrm{H}_{\mathrm{S}^M_r(o)} \quad \text{for all} \quad 0 < r \leq R$$
$$\left(\text{or that} \quad \mathrm{H}_{\mathrm{S}^\omega_r(o_\omega)} \geq \mathrm{H}_{\mathrm{S}^M_r(o)} \quad \text{for all} \quad 0 < r \leq R\right)$$

These hypotheses are the same than the conditions assumed in [8], and constitutes a more general assumption than the bounds for the sectional and the Ricci curvatures in Theorems A and C, as we shall see in next Section 1.2. On the other hand, they imply the following isoperimetric conditions satisfied by the geodesic r- balls with $r \leq R$ in the complete Riemannian manifold M,

$$\frac{\mathrm{Vol}\left(\mathrm{B}^\omega_r(o_\omega)\right)}{\mathrm{Vol}\left(\mathrm{S}^\omega_r(o_\omega)\right)} \geq (\leq) \frac{\mathrm{Vol}\left(\mathrm{B}^M_r(o)\right)}{\mathrm{Vol}\left(\mathrm{S}^M_r(o)\right)} \quad \text{for all} \quad 0 < r \leq R. \tag{1.7}$$

Concerning the use of isoperimetric inequalities, (not exactly those given in Eq.1.7), in the study of the relation between the moments spectrum and the Dirichlet spectrum, we refer to the paper [30].

Under these restrictions on the mean curvatures of geodesic spheres we have obtained all the results in this paper, the most important of which are Proposition 3.2, Theorem 3.3 and Corollary 3.5 in Section 3, Theorem 4.4, Corollary 4.5 and Theorem 4.8 in Section 4, and Theorem 5.1 and Corollary 5.2 in Section 5. A technical but fundamental result, key in the proof of Theorem 4.8, is Proposition 4.6. As a consequence of this proposition, we have also obtained a Talenti-type comparison satisfied by the mean exit time function defined on geodesic balls in a Riemannian manifold satisfying our hypotheses, (Corollary 4.7).

We are going to present in the following statements of Theorem 1.1, Theorem 1.2 and Theorem 1.3 summarized versions of some of our results concerning bounds on the Poisson hierarchy, the L^1-moments spectrum and the first Dirichlet eigenvalue of the geodesic balls B^M_R (for complete results, see Sections 4 and 5 below). Our presentation is structured to make it clear that they are a generalization of those presented in Theorem A and in Theorem C.

The techniques used in the proof of these results are basically the same as those cited papers [22, 23, 29], but now with the intrinsic point of view as the main perspective. These techniques encompasses the use of the formula of the Laplacian of the mean exit time function in polar coordinates, the application of the Maximum principle, the properties of the Schwartz symmetrization of the geodesic ball B^M_R and the explicit expression of the first Dirichlet eigenvalue of a geodesic ball B^w_R in a rotationally symmetric model space

M_w^n as a limit of the sequence given by the L^1-moment spectrum of this geodesic ball, $\left\{\mathcal{A}_k(B_R^w)\right\}_{k=1}^{\infty}$, obtained in the paper [24]. This formula for $\lambda_1(B_R^\omega(o_\omega))$ was subsequently extended in the paper [7] to any precompact domain $\Omega \subseteq M$, namely

$$\lambda_1(\Omega) = \lim_{k\to\infty} \frac{k\mathcal{A}_{k-1}(\Omega)}{\mathcal{A}_k(\Omega)}. \tag{1.8}$$

In fact, the presence of the mean curvature of geodesic spheres $H_{S_r^M}$ in the expresion of the Laplacian operator in polar coordinates has played a key role in the establishment of our hypotheses and also in the analysis of the equality with the bounds in all of our comparisons.

Concerning this analysis of the equality case, an important notion which appears in Theorems 1.1, 1.2 and 1.3, (in fact, along all the equality discussions in the paper), is the concept of *determination* of a Riemannian invariant defined on geodesic balls by its L^1-moment spectrum, its L^1-averaged moment spectrum or its torsional rigidity, in a way which, though not exactly the same, has been directly inspired by P. McDonald in [31].

In the paper [31] the notion of *determination* of a Riemannian invariant $I(D)$ defined on the precompact domain $D \subseteq M$ by the L^1-moment spectrum of D is presented: we say that $\{\mathcal{A}_k(D)\}_{k=1}^{\infty}$ *determines* the invariant $I(D)$ if and only if when $\{\mathcal{A}_k(D)\}_{k=1}^{\infty} = \left\{A_k(D')\right\}_{k=1}^{\infty}$, then $I(D) = I(D')$. With this definition, in [31] it is proved that the L^1-moment spectrum of a precompact domain D determines its heat content.

We shall see in the following Theorems 1.1 and 1.2 that, under our hypotheses, the torsional rigidity $\mathcal{A}_1(B_R^M)$ and any individual averaged moment $\dfrac{\mathcal{A}_{k_0}\left(\mathrm{B}_R^M(o)\right)}{\mathrm{Vol}\left(\mathrm{S}_R^M(o)\right)}$ determines the Poisson hierarchy, the volume, the L^1-moment spectrum and the first Dirichlet eigenvalue of the ball B_R^M, in the following sense:

When $\mathcal{A}_1(B_R^M(o)) = \mathcal{A}_1(B_{s(R)}^\omega(o_\omega))$, or there exists $k_0 \geq 1$ such that

$$\frac{\mathcal{A}_{k_0}\left(\mathrm{B}_R^M(o)\right)}{\mathrm{Vol}\left(\mathrm{S}_R^M(o)\right)} = \frac{A_{k_0}\left(\mathrm{B}_R^\omega(o_\omega)\right)}{\mathrm{Vol}\left(\mathrm{S}_R^\omega(o_\omega)\right)},$$

then $s(R) = R$ and the Poisson hierarchy, the volume, the L^1-moment spectrum and the first Dirichlet eigenvalue of the ball B_R^M is the same as the corresponding values for the geodesic ball $B_R^\omega(o_\omega)$ in the model space M_ω^n.

With all these previous considerations, we present the following:

Theorem 1.1 *[see Corollary 4.5]*

Let us consider a complete Riemannian manifold (M^n, g) and a rotationally symmetric model space (M_w^n, g_w), with center o_w, and we shall assume that given $o \in M$ a point in M, the injectivity radius of $o \in M$ satisfies $inj(o) \leq inj(o_w)$. Let us fix $R < inj(o) \leq inj(o_w)$ assuming that the pointed inward mean curvatures of metric r-spheres satisfies

$$\mathrm{H}_{\mathrm{S}_r^\omega(o_\omega)} \leq (\geq)\, \mathrm{H}_{\mathrm{S}_r^M(o)} \quad \textit{for all} \quad 0 < r \leq R. \tag{1.9}$$

Then, for all $k \geq 1$,

$$\frac{\mathcal{A}_k\left(\mathrm{B}_R^\omega(o_\omega)\right)}{Vol\left(\mathrm{S}_R^\omega(o_\omega)\right)} \geq (\leq)\, \frac{\mathcal{A}_k\left(\mathrm{B}_R^M(o)\right)}{Vol\left(\mathrm{S}_R^M(o)\right)}. \tag{1.10}$$

Equality in any of inequalities (1.10) for some $k_0 \geq 1$ implies that

$$\mathrm{H}_{\mathrm{S}_r^\omega(o_\omega)} = \mathrm{H}_{\mathrm{s}_r^M(o)} \textit{ for all } 0 < r \leq R$$

and hence, we have the equalities

(1) *Equality* $\bar{u}^{\omega}_{k,R} = u_{k,R}$ *on* $B^M_R(o)$ *for all* $k \geq 1$, *and hence, equality* $\bar{u}^{\omega}_{k,r} = u_{k,r}$ *on* $B^M 1_r(o)$ *for all* $k \geq 1$ *and for all* $0 < r \leq R$.
(2) *Equalities* $Vol\left(\mathrm{B}^{\omega}_r(o_\omega)\right) = Vol\left(\mathrm{B}^M_r(o)\right)$ *and* $Vol\left(\mathrm{S}^{\omega}_r(o_\omega)\right) = Vol\left(\mathrm{S}^M_r(o)\right)$ *for all* $0 < r \leq R$.
(3) *Equalities* $\mathcal{A}_k\left(\mathrm{B}^{\omega}_r(o_\omega)\right) = \mathcal{A}_k\left(\mathrm{B}^M_r(o)\right)$, *for all* $k \geq 1$ *and for all* $0 < r \leq R$.
(4) *Equality* $\lambda_1(B^M_r(o)) = \lambda_1(B^w_r(o_\omega))$ *for all* $0 < r \leq R$.

Namely, one value of $\dfrac{\mathcal{A}_k\left(\mathrm{B}^M_R(o)\right)}{Vol\left(\mathrm{S}^M_R(o)\right)}$ *for some* $k \geq 1$ *determines the Poisson hierarchy, the volume, the* L^1*-moment spectrum and the first Dirichlet eigenvalue of the ball* $B^M_r(o)$ *for all* $0 < r \leq R$.

Our second result is a comparison for the torsional rigidity of the ball B^M_R, and, as in Theorem B, we need that the model space M^n_ω used in the comparison to be *balanced from above*.

Theorem 1.2 *[see Theorem 4.8] Let us consider a complete Riemannian manifold* (M^n, g) *and a balanced from above rotationally symmetric model space* (M^n_w, g_w), *with center* o_w, *and we shall assume that given* $o \in M$ *a point in* M, *the injectivity radius of* $o \in M$ *satisfies* $inj(o) \leq inj(o_w)$. *Let us fix* $R < inj(o) \leq inj(o_w)$ *assuming that the pointed inward mean curvatures of metric* r*-spheres satisfies*

$$\mathrm{H}_{\mathrm{S}^{\omega}_r(o_\omega)} \leq (\geq)\, \mathrm{H}_{\mathrm{S}^M_r(o)} \quad \text{for all} \quad 0 < r \leq R. \tag{1.11}$$

Then

$$\mathcal{A}_1\left(\mathrm{B}^{\omega}_{s(R)}(o_\omega)\right) \geq (\leq)\ \mathcal{A}_1\left(\mathrm{B}^M_R(o)\right), \tag{1.12}$$

where $\mathrm{B}^{\omega}_{s(R)}(o_\omega)$ *is the Schwarz symmetrization of* $\mathrm{B}^M_R(o)$ *in the model space* (M^n_w, g_ω).

Equality in any of inequalities (1.12) implies the equality among the radius $s(R) = R$ *and that*

$$\mathrm{H}_{\mathrm{S}^{\omega}_r(o_\omega)} = \mathrm{H}_{\mathrm{S}^M_r(o)} \text{ for all } 0 < r \leq R$$

and hence, we have the equalities

(1) *Equality* $\bar{u}^{\omega}_{k,R} = u_{k,R}$ *on* $B^M_R(o)$ *for all* $k \geq 1$, *and hence, equality* $\overline{u}^{\omega}_{k,r} = u_{k,r}$ *on* $B^M_r(o)$ *for all* $k \geq 1$ *and for all* $0 < r \leq R$.
(2) *Equalities* $Vol\left(\mathrm{B}^{\omega}_r(o_\omega)\right) = Vol\left(\mathrm{B}^M_r(o)\right)$ *and* $Vol\left(\mathrm{S}^{\omega}_r(o_\omega)\right) = Vol\left(\mathrm{S}^M_r(o)\right)$ *for all* $0 < r \leq R$.
(3) *Equalities* $\mathcal{A}_k\left(\mathrm{B}^{\omega}_r(o_\omega)\right) = \mathcal{A}_k\left(\mathrm{B}^M_r(o)\right)$, *for all* $k \geq 1$ *and for all* $0 < r \leq R$.
(4) *Equality* $\lambda_1(B^M_r(o)) = \lambda_1(B^w_r(o_\omega))$ *for all* $0 < r \leq R$.

Namely, the Torsional Rigidity determines the Poisson hierarchy, the volume, the L^1*-moment spectrum and the first Dirichlet eigenvalue of the ball* $B^M_r(o)$ *for all* $0 < r \leq R$.

As a consequence of the proof of Theorem 1.1 in [32], Theorem 4.5, and volume inequalities given in 3.3, we have the following Cheng's Dirichlet eigenvalue comparison, following [8], (Theorem 5.1 in Section 5). In this case, we have proved that the first Dirichlet eigenvalue of B^M_R determines its Poisson hierarchy, its volume and its L^1-moment spectrum.

Theorem 1.3 *[see Theorem 5.1] Let* (M^n, g) *be a complete Riemannian manifold and let* (M^n_ω, g_ω) *be a rotationally symmetric model space with center* $o_\omega \in M^n_\omega$. *Let* $o \in M$ *be* *a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a metric ball $B^M_R(o)$, with $R < inj(o) \leq inj(o_w)$. Let us suppose moreover that the pointed inward mean curvatures of the geodesic spheres in M and M_ω satisfy*

$$\mathrm{H}_{\mathrm{S}^\omega_r(o_\omega)} \leq (\geq)\, \mathrm{H}_{\mathrm{S}^M_r(o)} \quad \textit{for all} \quad 0 < r \leq R. \tag{1.13}$$

Then we have the inequality

$$\lambda_1(B^\omega_R(o_\omega)) \leq (\geq)\, \lambda_1(B^M_R(o)) \,. \tag{1.14}$$

Equality in any of these inequalities implies that

$$\mathrm{H}_{\mathrm{S}^\omega_r(o_\omega)} = \mathrm{H}_{\mathrm{S}^M_r(o)} \ \textit{for all}\ 0 < r \leq R$$

and hence, we have the equalities

(1) *Equality $\bar{u}^\omega_{k,R} = u_{k,R}$ on $B^M_R(o)$ for all $k \geq 1$, and hence, equality $\bar{u}^\omega_{k,r} = u_{k,r}$ on $B^M_r(o)$ for all $k \geq 1$ and for all $0 < r \leq R$.*
(2) *Equalities $Vol\left(\mathrm{B}^\omega_r(o_\omega)\right) = Vol\left(\mathrm{B}^M_r(o)\right)$ and $Vol\left(\mathrm{S}^\omega_r(o_\omega)\right) = Vol\left(\mathrm{S}^M_r(o)\right)$ for all $0 < r \leq R$.*
(3) *Equalities $\mathcal{A}_k\left(\mathrm{B}^\omega_r(o_\omega)\right) = \mathcal{A}_k\left(\mathrm{B}^M_r(o)\right)$, for all $k \geq 1$ and for all $0 < r \leq R$.*

Namely, the first Dirichlet eigenvalue determines the Poisson hierarchy, the volume, and the L^1-moment spectrum of the ball $B^M_r(o)$ for all $0 < r \leq R$.

The characterizations of equalities in both theorems are ultimately based on a rigidity property satisfied by the Poisson hierarchy for B^M_R, $\{u_{k,R}\}_{k=1}^\infty$ in a Riemannian manifold M under the hypotheses depicted above. This rigidity property can be summarized by saying that the value at *one point* $p \in B^M_R$ of one of the functions $u_{k,R}$ of the Poisson hierarchy determines it *entirely* on the geodesic ball B^M_R, (see Proposition 3.2 and assertions (3) and (4) in Theorem 4.4 in Section 4).

1.2 Example

We remark that, under the hypothesized bounds on the sectional curvatures of the manifold, if we have the equality with the corresponding bound in the model space of any our invariants defined on the geodesic ball B^M_R, (namely, the Poisson hierarchy, the averaged L^1-moment spectrum, or the torsional rigidity), then B^M_R is isometric to the geodesic balls in the model space, $B^w_R \subseteq M^n_w$. However, the equality of the mean curvature of distance spheres in the Riemannian manifold M, with its radial bound given by the mean curvature of distance spheres in the model space M_ω does not imply the isometry among the geodesic balls, as in the previous case.

This observation is coherent with the fact that bounds on the sectional curvatures of the manifold implies bounds for the mean curvature of its geodesic spheres, namely, if (M,g) is a Riemannian manifold with radial sectional curvatures

$$K_{sec,g}\left(\frac{\partial}{\partial r},\right) \leq (\geq)\, K_{sec,g_w}\left(\frac{\partial}{\partial r},\right) = -\frac{w''(r)}{w(r)}$$

then we have that

$$H_{S^M_r} \geq (\leq)\, H_{S^w_r} = \frac{w'(r)}{w(r)}\,.$$

These implications follow from the observation that the mean curvature of geodesics spheres is the Laplacian of the distance from its center in the manifold, (see Proposition 2.3), together with the Hessian comparison analysis of the distance function as can be found in [20, 27] or [35].

However, in the paper [8], the authors exhibit in Example 3.1 in Section 3, smooth complete and rotationally symmetric metrics g on $\mathbb{R}^n$ with radial sectional curvatures bounded from below, $K_{sec,g}(\frac{\partial}{\partial r}, \,) \geq b$ outside a compact set and such that the distance spheres $S_t^{(\mathbb{R}^n,g)}$ have mean curvature $H_{S_t^{(\mathbb{R}^n,g)}} \geq H_{S_t^{w_b}}$.

In the following, we are going to present a new example which shows that bounds on the mean curvature of geodesic spheres of the manifold does not imply that the sectional curvatures of the manifold are controlled.

Let $(\mathbb{R}^2, g)$ be a Riemannian manifold such that its metric tensor expressed in polar coordinates is given by $g = dr^2 + \omega^2(r,\theta)d\theta^2$, where $\omega : \mathbb{R}^2 \to \mathbb{R}$ is a positive smooth function given by

$$\omega(r,\theta) = r\left(1 + \frac{r^2}{1 + r^2\cos^2\theta}\right). \tag{1.15}$$

On the other hand, we consider as a model space the simply connected space form $(\mathbb{R}^2, g_{\text{can}})$ of constant sectional curvature $b = 0$.

We are going to see that the mean curvatures of the geodesic spheres $\mathrm{S}_t^{(\mathbb{R}^2,g)}(\vec{0})$ of $(\mathbb{R}^2, g)$ centered at $\vec{0}$ with radius t, are bounded from below by the mean curvatures of the geodesic spheres $\mathrm{S}_t^{\omega_0}(\vec{0})$ of $(\mathbb{R}^2, g_{\text{can}})$ centered at $\vec{0}$ with the same radius, namely, that

$$\mathrm{H}_{\mathrm{S}_t^{(\mathbb{R}^2,g)}} \geq \mathrm{H}_{\mathrm{S}_t^{\omega_0}}$$

As $\mathrm{H}_{\mathrm{S}_t^{(\mathbb{R}^2,g)}}(t,\theta) = \dfrac{\frac{\partial\omega}{\partial t}(t,\theta)}{\omega(t,\theta)}$ and we have

$$\frac{\partial_\omega}{\partial_t}(t,\theta) = 1 + \frac{t^2}{1 + t^2\cos^2\theta} + \frac{2t^2}{(1 + t^2\cos^2\theta)^2}$$

we obtain

$$\mathrm{H}_{\mathrm{S}_t^{(\mathbb{R}^2,g)}}(t,\theta) = \frac{\frac{\partial\omega}{\partial t}(t,\theta)}{\omega(t,\theta)} = \frac{1}{t} + \frac{2t}{(1 + \frac{t^2}{1+t^2\cos^2\theta})(1 + t^2\cos^2\theta)^2}$$

But

$$\frac{2t}{(1 + \frac{t^2}{1+t^2\cos^2\theta})(1 + t^2\cos^2\theta)^2} \geq 0 \text{ for all } (t,\theta) \in (0, +\infty) \times [0, 2\pi)$$

Hence we have that

$$\mathrm{H}_{\mathrm{S}_t^{(\mathbb{R}^2,g)}}(t,\theta) \geq \frac{1}{t} = \mathrm{H}_{\mathrm{S}_t^{\omega_0}}(t) \text{ for all } (t,\theta) \in (0, +\infty) \times [0, 2\pi)$$

Now, let us consider the unique 2-plane tangent to a point $(t,\theta) \in \mathbb{R}^2$ generated by the coordinate vector fields $\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right\}$. We are going to compute the sectional curvature of $(\mathbb{R}^2, g)$ at this point and we will see that it is not bounded by the corresponding sectional curvature of $(\mathbb{R}^2, g_{\text{can}})$, i.e., we will show that $K_{sec,g}(t,\theta)$ is not bounded from below by 0.

As $K_{sec,g}(t,\theta) = -\dfrac{\frac{\partial^2\omega}{\partial t^2}(t,\theta)}{\omega(t,\theta)}$ then, it is straightforward to check that

$$K_{sec,g}(t,\theta) = -\frac{\frac{\partial^2\omega}{\partial t^2}(t,\theta)}{\omega(t,\theta)} = \frac{2(t^2\cos^2\theta - 3)}{(1+t^2\cos^2\theta)^2(1+t^2+t^2\cos^2\theta)}$$

Thus, for $\theta = 0$, we have that

$$K_{sec,g}(t,0) = \frac{2(t^2-3)}{(1+t^2)^2(1+2t^2)}$$

This shows that there are points $(t,\theta) \in \mathbb{R}^2$ where the sectional curvature of $(\mathbb{R}^2, g)$ is bounded either below or above by 0 which is the sectional curvature of $(\mathbb{R}^2, g_{\text{can}})$.

1.3 Outline

After the Introduction, Section 2 is devoted to the presentation of preliminary concepts, including the rotationally symmetric model spaces used to construct the bounds and the notion of Schwarz symmetrization based on these models. We have stated and proved, for the sake of completeness, all the properties of these symmetrizations we need in our context. Section 3 deals with the properties of the mean exit time function defined on the geodesic R-balls in a complete Riemannian manifold satisfying our hypotheses and its relation with its volume and the isoperimetric inequalities satisfied by these domains (Proposition 3.2, Theorem 3.3, Corollary 3.5 and Corollary 3.6) . In Section 4 we have established bounds for the Poisson hierarchy and the averaged L^1-moment spectrum of a geodesic R-ball under our restrictions (Theorem 4.4 and Corollary 4.5), and we have bounded the torsional rigidity of a geodesic R-ball by means its Schwarz symmetrization, (Theorem 4.8). Finally, in Section 5, we prove a Cheng-type comparison for the first Dirichlet eigenvalue of geodesic balls (Theorem 5.1), and we have established the relation between the first Dirichlet eigenvalue of geodesic balls, its L^1-moment spectrum and its Poisson hierarchy (Corollary 5.2).

2 Preliminaries and Comparison Setting

We are going to present some previous notions and results that will be instrumental in our work.

2.1 Polar coordinates and the Laplacian on a Riemannian Manifold

Definition 2.1 Let us consider a complete Riemannian manifold (M^n, g) and a point $o \in M$. Let us denote as $Cut(o)$ the cut locus of $o \in M$ and as $inj(o) = dist_M(o, Cut(o))$ the injectivity radius of the point $o \in M$. We shall denote by $\mathbb{S}^{n-1}_1 \subseteq \mathbb{R}^n$ the unit sphere with center $\vec{0} \in \mathbb{R}^n$.

We define, in the set $M - (Cut(o) \cup \{o\})$, the *polar coordinates* of any point $x \in M - (Cut(o) \cup \{o\})$ as the pair $(r(x), \overline{\theta}) \in (0, inj(o)) \times \mathbb{S}^{n-1}_1$, where $r(x) := r_o(x) =$

$dist_M(o,x)$ is the distance from o to x realized by the shortest geodesic between these points which starts at o with direction $\overline{\theta} \in \mathbb{S}_1^{n-1}$.

The Riemannian metric g in $M-(Cut(o)\cup\{o\})$ has in the polar coordinates the form

$$g = dr^2 + \sum_{i,j=1}^{n-1} g_{i,j}(r,\overline{\theta})d\theta^i d\theta^j\,,$$

where $\overline{\theta} \equiv (\theta_1, ..., \theta_{n-1}) \in \mathbb{S}_1^{n-1}$ is a system of local coordinates in $\mathbb{S}_1^{n-1}$ and $g_{i,j}(r,\overline{\theta}) = g\left(\frac{\partial}{\partial\theta_i}\Big|_{(r,\overline{\theta})}, \frac{\partial}{\partial\theta_j}\Big|_{(r,\overline{\theta})}\right)$.

Thus, the matrix form of the metric g in polar coordinates is a positive definite matrix given by

$$\mathfrak{G} = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & G & \\ 0 & & & \end{array}\right),$$

where G is the matrix which elements are g_{ij}, *i.e.*, $G = \left(g_{ij}\right)_{i,j\in\{1,\ldots,n-1\}}$. Hence, for any point $(r,\overline{\theta}) \in M-(Cut(o)\cup\{o\})$, we have

$$\sqrt{\det\left(\mathfrak{G}(r,\overline{\theta})\right)} = \sqrt{\det\left(G(r,\overline{\theta})\right)}.$$

Then, (see for example [9, 18]), the Laplace operator of M has the following expression in the polar coordinates

$$\Delta^M = \frac{\partial}{\partial r^2} + \frac{\partial}{\partial r}\left(\log\sqrt{\det G\left(r,\overline{\theta}\right)}\right)\frac{\partial}{\partial r} + \Delta^{S_r^M(o)}\,, \tag{2.1}$$

where $\Delta^{\S_r^M(o)}$ is the Laplace operator in the geodesic sphere $S_r^M(o) \subseteq M$.

Remark 2.2 Throughout the remainder of the paper, given $o \in M$ and as long as $R < inj(o)$, we will use interchangeably the terms *geodesic ball*, *geodesic sphere*, *metric ball*, *metric sphere*, *distance ball* and *distance sphere* to name the sets $B_R^M(o)$ and $S_R^M(o)$ respectively.

Using this result we have the following

Proposition 2.3 *Let (M^n, g) be a complete Riemannian manifold and let $o \in M$ be a point of M. Then the normalized mean curvature vector field of the geodesic sphere $S_t^M(o)$, is given by*

$$\vec{H}_{S_t^M(o)} = -H_{S_t^M(o)}\nabla^M r\,,$$

where

$$H_{S_t}^M = \frac{1}{n-1}\Delta^M r(\gamma(t)) = \frac{1}{n-1}\frac{\frac{\partial}{\partial t}\sqrt{\det G\left(t,\overline{\theta}\right)}}{\sqrt{\det G\left(t,\overline{\theta}\right)}\ \forall t>0}$$

is the pointed inward mean curvature of $S_t^M(o)$ and $\gamma(t)$ is a unit geodesic starting at the point $o \in M$.

Proof The proof is straightforward taking $\{\vec{e}_i(t)\}_{i=1}^n$ an orthonormal basis of $T_{\gamma(t)}\mathrm{S}_t^M$, with $\vec{e}_n(t) = \nabla^M r(\gamma(t))$, the unit normal to S_t^M at $\gamma(t)$, pointed outward. Then, after some computations,

$$\begin{aligned}\vec{H}_{S_t^M} &= \frac{1}{n-1}\left(\mathrm{tr}L_{\nabla^M r}\right)\nabla^M r = -\frac{1}{n-1}\mathrm{div}^M(\nabla^M r)\\ &= -\frac{1}{n-1}\Delta^M r(\gamma(t))\nabla^M r(\gamma(t))\end{aligned} \tag{2.2}$$

so

$$H_t = \langle \vec{H}_{S_t^M}, -\nabla^M r(\gamma(t))\rangle = \frac{1}{n-1}\Delta^M r(\gamma(t)).$$

The result follows now using equation (2.1). □

Given a domain (connected open set) D in M, a function $u \in C^2(D)$ is *harmonic* (resp. *subharmonic*) if $\Delta^M u = 0$ (resp. $\Delta^M u \geq 0$) on D. We gather the strong maximum principle and the Hopf boundary point lemma for subharmonic functions in the next statement.

Theorem 2.4 *Let D be a smooth domain of a Riemannian manifold M. Consider a subharmonic function $u \in C^2(D) \cap C(\overline{D})$. Then, we have:*

(i) if u achieves its maximum in D then u is constant,
(ii) if there is $p_0 \in \partial D$ such that $u(p) < u(p_0)$ for any $p \in D$ then $\frac{\partial u}{\partial \nu}(p_0) > 0$, where ν denotes the outer unit normal along ∂D.

Proof The proof of (i) can be found in [19, Corollary 8.15]. The proof of (ii) can be derived from (i) as in the Euclidean case [21, Lemma 3.4]. □

2.2 Model Spaces

The model spaces M_ω^n are rotationally symmetric spaces defined as follows:

Definition 2.5 (See [18, 20]) A ω-model M_ω^n is a smooth warped product with base $B^1 = [0, R[\subset \mathrm{R}$ (on $0 < R \leq \infty$), fiber $F^{n-1} = \mathrm{S}_1$ (i.e., the unit $(n-1)$-sphere with standard metric), and warping function $\omega : [0, R[\to \mathrm{R}_+ \cup \{0\}$ with $\omega(0) = 0$, $\omega'(0) = 1$, $\omega^{(2k)}(0) = 0$ and $\omega(r) > 0$ for all $k \in \mathbb{N}^*$ and for all $r > 0$, where $\omega^{(2k)}$ denotes the even derivatives of the warping function.

The point $o_\omega = \pi^{-1}(0)$, where π denotes the natural projection onto the base B^1, is called *center point* of the model space. If $R = +\infty$, then o_ω is a pole of M_ω^n. We denote as $r = r(x)$ the distance to the center o_ω of the point $x \in M_\omega^n$.

Remark 2.6 The simply connected space forms $M_{\omega_b}^n$ of constant sectional curvature b can be constructed as ω-models with any given point as center point using the warping functions

$$\omega_b(r) = \begin{cases} \frac{1}{\sqrt{b}}\sin\left(\sqrt{b}\,r\right), & \text{if } b > 0,\\ r, & \text{if } b = 0,\\ \frac{1}{\sqrt{-b}}\sinh\left(\sqrt{-b}\,r\right), & \text{if } b < 0.\end{cases} \tag{2.3}$$

Note that for $b > 0$, the warped metric $g_{\omega_b} = dr^2 + \omega_b^2(r)g_{\mathbb{S}_1^{n-1}}$ determined by the function $\omega_b(r)$ admits smooth extension to $r = \pi/\sqrt{b}$. For $b \leq 0$ any center point is a pole.

In [18, 20, 29, 33], we have a complete description of these model spaces, including the computation of their sectional curvatures K_{o_ω, M^n_ω} in the radial directions from the center point o_ω. They are determined by the radial function $K_{o_\omega, M^n_\omega}(\sigma_x) = K_\omega(r) = -\frac{\omega''(r)}{\omega(r)}$. Moreover, the normalized inward mean curvature of the distance sphere $S^\omega_r(o_\omega)$ of radius r from the center point, is, at the point $p = \gamma(r) \in S^\omega_r(o_\omega)$, where $\gamma(t)$ is the normal geodesic parametrized by arclength joining o_ω and p

$$H_{S^w_r}(p) = \eta_\omega(r) = \frac{\omega'(r)}{\omega(r)} = \frac{d}{dr}\ln(\omega(r)). \tag{2.4}$$

In particular, in [29] we introduce, for any given warping function $\omega(r)$, the *isoperimetric quotient* $q_\omega(r)$ for the corresponding ω-model space M^n_ω as follows:

$$q_\omega(r) = \frac{\mathrm{Vol}\left(\mathrm{B}^\omega_r(o_\omega)\right)}{\mathrm{Vol}\left(\mathrm{S}^\omega_r(o_\omega)\right)} = \frac{\int_0^r \omega^{n-1}(t)\,dt}{\omega^{n-1}(r)}. \tag{2.5}$$

On the other hand, using equation (2.1), the Laplace operator in M^n_w is given by

$$\Delta^{M^n_w} = \frac{\partial}{\partial r^2} + (n-1)\frac{w'(r)}{w(r)}\frac{\partial}{\partial r} + \Delta^{S^\omega_r(o_\omega)}. \tag{2.6}$$

Then, we have the following results concerning the mean exit time function of the geodesic R-ball $\mathrm{B}^\omega_r(o_\omega) \subseteq \mathrm{M}^n_\omega$, (see [29]):

Proposition 2.7 *Let E^ω_R the solution of the Poisson Problem (1.1), defined on the geodesic R-ball $\mathrm{B}^\omega_R(o_\omega)$ in the model space M^n_ω. Then E^ω_R is a non-increasing radial function given by*

$$E^\omega_R(x) = E^\omega_R(r_{o_\omega}(x)) = \int_{r_{o_\omega}(x)}^R q_\omega(t)\,dt\,, \tag{2.7}$$

where $r \equiv r_{o_\omega}(x) = dist_{M^n_\omega}(o_\omega, x)$ denotes the distance to the center point. Hence, it attains its maximum at $r = 0$, with $E^{\omega\prime}_R(0) = 0$ and $E^{\omega\prime}_R(r) < 0\ \forall r \in]0, R]$.

Proof Using the expresion of the Laplace operator given in Eq. 2.6, it is straightforward to check that $E_R(r) = \int_r^R q_\omega(t)\,dt$ satisfies the equation

$$\Delta^{M^n_w} E_R = -1$$

with boundary condition $E_R(R) = 0$. □

2.3 Balance Conditions

We present now a purely intrinsic condition on the general model spaces M^n_ω, (see [29]), which will play a key role in the last section of the paper:

Definition 2.8 A given ω-model space M^n_ω is *balanced from above* if we have the inequality

$$q_\omega(r)\eta_\omega(r) \le \frac{1}{n-1}, \quad \text{for all } r \ge 0. \tag{2.8}$$

In [29] the following characterization of the balance condition was established:

Proposition 2.9 *Let us consider the ω-model space M^n_ω. Then M^n_ω is balanced from above if and only if the following equivalent conditions hold:*

$$\frac{d}{dr}\left(q_\omega(r)\right) \geq 0, \tag{2.9}$$

$$\omega^n(r) \geq (n-1)\,\omega'(r)\int_0^r \omega^{n-1}(t)\,dt. \tag{2.10}$$

Several examples of balanced from above ω-model spaces M^n_ω were given in [29]. We enumerate some of them:

Example 2.10

(1) Every ω_b-model space $M^n_{\omega_b} = [0, R[\times_{\omega_b}\mathbb{S}_1$ of constant positive sectional curvature $b > 0$ and $R < \frac{\pi}{2\sqrt{b}}$ is balanced from above. In fact, when $b > 0$ and for $r > 0$ we have that (2.10) is a strict inequality, (see Lemma 2.4 in [28]).

(2) The ω_b-model spaces $M^n_{\omega_b}$ of constant non-positive sectional curvature $b \leq 0$ are balanced from above. In fact, when $b < 0$, we have that inequality (2.10) is equivalent to inequality

$$\int_0^r \sinh^{n-1}(\sqrt{-b}t) \leq \frac{\sinh^n(\sqrt{-b}r)}{\sqrt{-b}(n-1)\cosh(\sqrt{-b}r)}$$

which holds for all $r > 0$ because $\tanh^2(\sqrt{-b}r) \leq 1$ $\forall r > 0$. The case $b = 0$ is trivial.

(3) The ω-model space M^n_ω with $\omega(t) := t + t^3$, $t \in [0, \infty)$ is balanced from above.

2.4 Symmetrization into Model Spaces

As in [29] we use the concept of *Schwarz-symmetrization* as considered in e.g., [1, 37], or, more recently, in [10, 30]. The Schwarz-symmetrization it is also known in the literature as the *symmetric decreasing rearrangement*, (see e.g. the works [2, 3, 39, 40]). For the sake of completeness, we review and show some facts about this instrumental concept, in the context of Riemannian manifolds.

Definition 2.11 Let (M^n, g) be a complete Riemannian manifold. Suppose that $\mathrm{D} \subseteq M$ is a precompact open connected domain in M^n. Let (M^n_ω, g_ω) be a rotationally symmetric model space, with pole $o_\omega \in M^n_\omega$. Then the *ω-model space symmetrization of* D is denoted by D^{*_ω} and is defined to be the unique $L(D)$-ball in M^n_ω, centered at o_ω

$$D^{*_\omega} := B^\omega_{L(D)}(o_\omega)$$

satisfying

$$\mathrm{Vol}(D) = \mathrm{Vol}\left(B^\omega_{L(D)}(o_\omega)\right).$$

In the particular case that D is a geodesic R-ball $\mathrm{B}^M_R(o)$ in M centered at $o \in M$, then the radius $L(\mathrm{B}^M_R(o))$ is some increasing function $s(R) = L(\mathrm{B}^M_R(o))$ which depends on the geometry of M, so we can write

$$\mathrm{B}^M_R(o)^{*_\omega} = B^\omega_{s(R)}(o_\omega)$$

and this symmetrization $B^\omega_{s(R)}(o_\omega)$ satisfies

$$\mathrm{Vol}(\mathrm{B}^M_R(o)) = \mathrm{Vol}(B^\omega_{s(R)}(o_\omega)). \tag{2.11}$$

Remark 2.12 When it is clear from the context, we write D^* instead of $D^{*\omega}$.

In the remainder of the paper, and if there is not confusion, we shall omit the centers $o \in M$ and $o_\omega \in M^n_\omega$ when we refer to the balls $B^M_r(o)$ and $B^\omega_r(o_\omega)$ and the spheres $S^M_r(o)$ and $S^\omega_r(o_\omega)$.

Given $f : D \to \mathbb{R}^+$ a smooth non-negative function on D, we are going to introduce the notion of *ω- symmetrization* $f^{*\omega} : D^{*\omega} \to \mathbb{R}^+$. But first, we will show some useful facts.

Definition 2.13 Let (M^n, g) be a complete Riemannian manifold, $D \subseteq M$ a precompact domain in M and $f : D \subseteq M \longrightarrow \mathbb{R}^+$ a smooth non-negative function on D. For $t \geq 0$ we define the sets

$$D(t) := \{x \in D \mid f(x) \geq t\} \subseteq M$$

and

$$\Gamma(t) := \{x \in D \mid f(x) = t\}.$$

Remark 2.14

(1) The set $D(t)$ is precompact for all $t \geq 0$ and moreover, $\partial D(t) = \Gamma(t) \subseteq D(t)$.
(2) Note too that $D(0) = D$ and that if $t_1 \leq t_2$ then $D(t_2) \subseteq D(t_1)$.
(3) If $T := \sup_{x\in D} f(x)$, then $D(t) = \emptyset\ \forall t > T$, and hence, $\mathrm{Vol}(D(t)) = 0\ \forall t \geq T$.
(4) Therefore, we have a family of nested sets $\{D(t)\}_{t\in[0,T]}$ that covers D.

Now, we define the *symmetrization* of a function:

Definition 2.15 Let (M^n, g) be a complete Riemannian manifold, $D \subseteq M$ a precompact domain in M and $f : D \subseteq M \longrightarrow \mathbb{R}^+$ a smooth non-negative function on D. Let (M^n_ω, g_ω) be a rotationally symmetric model space. Then the *ω-symmetrization of f* is the function $f^{*\omega} : D^{*\omega} \longrightarrow \mathbb{R}$ defined, for all $x^* \in D^{*\omega}$, by

$$f^{*\omega}(x^*) = \sup\left\{t \geq 0 \mid x^* \in D(t)^{*\omega}\right\}.$$

Note that the symmetrization $f^{*\omega}$ ranges on $[0, T]$, namely, $f^{*\omega} : D^{*\omega} \to [0, T]$, where $T := \sup_{x\in D} f(x)$.

Remark 2.16

(1) When it is clear from the context, we write f^* instead of $f^{*\omega}$ and D^* instead of $D^{*\omega}$.
(2) By Sard's theorem, if $D_f \subseteq D$ denotes the set of critical points of f, the set $S_f = f(D_f) \subseteq [0, T]$ of critical values of f has null measure, and the set of regular values of f, $R_f = [0, T] \sim S_f$ is open and dense in $[0, T]$. In particular, for any $t \in R_f$, the set $\Gamma(t) = \{x \in D \mid f(x) = t\}$ is a smooth embedded hypersurface in D and $\|\nabla^M f\|$ does not vanish along $\Gamma(t)$.
(3) Given $f_1, f_2 : D \to \mathbb{R}$ smooth and non-negative, if $f_1 \leq f_2$, then $f_1^* \leq f_2^*$.

With these observations in hand, we have the following

Definition 2.17 Let (M^n, g) be a complete Riemannian manifold and let (M^n_ω, g_ω) be a rotationally symmetric model space. Given the precompact domain $D \subseteq M$ and $f : D \subseteq M \longrightarrow \mathbb{R}^+$ a smooth non-negative function on D, let $D(t)^*$ and T be as in Definition 2.15 and define the function

$$\widetilde{r} : [0, T] \to [0, L(D)]$$

such that, for all $t \in [0, T]$, $\widetilde{r}(t)$ is defined as the radius of the symmetrization

$$D(t)^* = \mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega)$$

satisfying

$$\mathrm{Vol}\,(\mathrm{D}(t)) = \mathrm{Vol}\left(\mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega)\right).$$

Remark 2.18 Note that, as $D(0) = D$, then $D(0)^* = D^*$, i.e., $\widetilde{r}(0) = L(D)$, the radius defined in Definition 2.11, and $D^* = \mathrm{B}^{\omega}_{\widetilde{r}(0)}(o_\omega)$. On the other hand, as $\mathrm{Vol}(D(t)) = 0\ \forall t \geq T$, then $\widetilde{r}(t) = 0\ \forall t \geq T$.

Concerning this last definition, we have the following result, which will play an important rôle in the proof of Proposition 4.6:

Lemma 2.19 *The function $\widetilde{r} : [0, T] \to [0, L(D)]$ is non-increasing. In particular, for all regular values $t \in R_f$, the function $\widetilde{r}|_{R_f} : R_f \subseteq [0, T] \to [0, L(D)]$ satisfies*

$$\widetilde{r}'(t) = -\frac{\int_{\partial D(t)} \left\|\nabla^M f\right\|^{-1} d\mu_t}{Vol\left(\mathrm{S}^{\omega}_{\widetilde{r}(t)}\right)} < 0$$

so $\widetilde{r}$ is strictly decreasing in R_f, and hence, injective (and bijective onto its image).

Remark 2.20 Note that when $R_f = [0, T]$, then $\widetilde{r} : [0, T] \to [0, L(D)]$ is bijective.

Proof When $t_1 \leq t_2$, then $\mathrm{D}(t_2) \subseteq \mathrm{D}(t_1)$ and hence $\mathrm{Vol}(\mathrm{D}(t_2)) \leq \mathrm{Vol}(\mathrm{D}(t_1))$, so $\mathrm{Vol}(\mathrm{B}^{\omega}_{\widetilde{r}(t_2)}(o_\omega)) \leq \mathrm{Vol}(\mathrm{B}^{\omega}_{\widetilde{r}(t_1)}(o_\omega))$ and hence $\widetilde{r}(t_2) \leq \widetilde{r}(t_1)$.

On the other hand, given $t \in R_f$, let us denote as:

$$\mathrm{V}(t) = \mathrm{Vol}\,(\mathrm{D}(t)) = \mathrm{Vol}\left(\mathrm{B}^{\omega}_{\widetilde{r}(t)}\right).$$

Then,

$$\mathrm{V}'(t) = \mathrm{Vol}\left(\mathrm{S}^{\omega}_{\widetilde{r}(t)}\right)\widetilde{r}'(t)$$

and as $\partial\mathrm{D}(t) = \Gamma(t) = \{x \in \mathrm{D} \,|\, f(x) = t\}$, by the co-area formula (see [9], [38]), and as $t \in R_f$, we have

$$\widetilde{r}'(t) = -\frac{\int_{\partial D(t)} \left\|\nabla^M f\right\|^{-1} d\mu_t}{\mathrm{Vol}\left(\mathrm{S}^{\omega}_{\widetilde{r}(t)}\right)} < 0$$

for all $t \in R_f$. Therefore, $\widetilde{r}|_{R_f}$ is strictly decreasing. □

To finish this subsection, we are going to prove in Theorem 2.21 that, given $f : \mathrm{D} \subseteq M \longrightarrow \mathbb{R}^+$ a non-negative function defined on the precompact domain D, the symmetrized function $f^* : D^{*_\omega} \longrightarrow \mathbb{R}$ is a radial function, and that f and f^* are both equimeasurable.

Theorem 2.21 *Let (M^n, g) be a complete Riemannian manifold, $D \subseteq M$ a precompact domain in M and $f : \mathrm{D} \subseteq M \longrightarrow \mathbb{R}^+$ a non-negative and smooth function on* D. *Let (M^n_ω, g_ω) a rotationally symmetric model space such that its center o_ω is a pole. The symmetrized objects f^* and* D^* *satisfy the following properties:*

(1) *The function f^* depends only on the geodesic distance to the center o_ω of the ball* D^* *in M^n_ω and is non-increasing.*

(2) *The functions f and f^* are equimeasurable in the sense that*

$$Vol_M\left(\{x \in \mathrm{D} \mid f(x) \geq t\}\right) = Vol_{M^n_\omega}\left(\left\{x^* \in \mathrm{D}^* \mid f^*(x^*) \geq t\right\}\right) \tag{2.12}$$

for all $t \geq 0$.

Proof To prove the first statement, let us consider $x_1^*, x_2^* \in \mathrm{D}^* = \mathrm{B}^{\omega}_{\widetilde{r}(0)}(o_\omega)$ such that $r_{o_\omega}(x_1^*) = r_{o_\omega}(x_2^*)$. Then it is evident that $x_1^* \in \mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega)$ if and only if $x_2^* \in \mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega)$ for all $t \in [0, T]$. Hence,

$$f^*(x_1^*) = \sup\left\{t \geq 0 \mid x_1^* \in \mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega)\right\} = \sup\left\{t \geq 0 \mid x_2^* \in \mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega)\right\} = f^*(x_2^*)$$

which means that f^* is a radial function. Namely, f^* depends only on the geodesic distance to the center o_ω, $f^*(x^*) = f^*(r_{o_\omega}(x^*))$.

To see that f^* is non-increasing, let us consider $x_1^*, x_2^* \in \mathrm{D}^*$ such that $r_{o_\omega}(x_1^*) \leq r_{o_\omega}(x_2^*)$. We are going to see that $t_1 := f^*(x_1^*) \geq t_2 := f^*(x_2^*)$.

As

$$f^*(x_2^*) = \sup\left\{t \geq 0 \mid x_2^* \in \mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega)\right\} = \sup\left\{t \geq 0 \mid r_{o_\omega}(x_2^*) \leq \widetilde{r}(t)\right\} = t_2$$

then, if $t \leq t_2$, we have that $x_2^* \in \mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega)$, so $r_{o_\omega}(x_2^*) \leq \widetilde{r}(t)\ \forall t \leq t_2$. In particular, $r_{o_\omega}(x_1^*) \leq r_{o_\omega}(x_2^*) \leq \widetilde{r}(t_2)$, so $x_1^* \in \mathrm{B}^{\omega}_{\widetilde{r}(t_2)}(o_\omega)$ and therefore, $t_1 = f^*(x_1^*) = \sup\left\{t \geq 0 \mid x_1^* \in \mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega)\right\} \geq t_2 = f^*(x_2^*)$.

To prove the second statement, note that, for all $t > 0$, we have, by Definitions 2.15 and 2.17,

$$\mathrm{D}(t)^* = \mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega) = \{x^* \in \mathrm{D}^* \mid f^*(x^*) \geq t\}.$$

In fact, if $x^* \in \mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega)$, then $f^*(x^*) = \sup\left\{t \geq 0 \mid x^* \in \mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega)\right\} \geq t$ and, conversely, if $f^*(x^*) = \sup\left\{t \geq 0 \mid x^* \in \mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega)\right\} \geq t$, then $x^* \in \mathrm{B}^{\omega}_{\widetilde{r}(t)}(o_\omega)$.

Therefore, since $\mathrm{D}(t) = \{x \in \mathrm{D} \mid f(x) \geq t\}$, we obtain that

$$\mathrm{Vol}\left(\{x \in \mathrm{D} \mid f(x) \geq t\}\right) = \mathrm{Vol}\left(\mathrm{D}(t)\right) = \mathrm{Vol}\left(\mathrm{D}(t)^*\right) = \mathrm{Vol}\left(\left\{x^* \in \mathrm{D}^* \mid f^*(x^*) \geq t\right\}\right).$$

□

3 Mean Exit Time Comparison

We start this section with the notion of *transplanted mean exit time*.

Definition 3.1 Let (M, g) a complete Riemannian manifold and (M^n_ω, g_ω) a model space with center o_ω. Given $o \in M$, let us consider a geodesic R-ball $B^M_R(o)$, with $0 < R < inj(o)$ and the geodesic R-ball in M^n_ω, centered at the center o_ω, $B^\omega_R(o_\omega)$. Let E^M_R and E^ω_R be the mean exit time functions defined on $B^M_R(o)$ and $B^\omega_R(o_\omega)$, respectively.

Now, we *transplant* the radial mean exit time function of M^n_ω to M by defining the function $\mathrm{E}^\omega_R : \mathrm{B}^M_R \to \mathbb{R}$ as $\mathrm{E}^\omega_R(x) := E^\omega_R(r_o(x))\ \forall x \in \mathrm{B}^M_R$ where r_o is the distance function to o, the center of the ball $\mathrm{B}^M_R(o)$.

The function E^ω_R is a radial function called the *transplanted mean exit time* function of B^M_R.

We can compare the transplanted mean exit time function E^{ω}_R defined in a geodesic ball B^M_R with the mean exit time function E^M_R corresponding with this ball. Remember that, throughout the text, we can omit the centers $o \in M$ and $o_\omega \in M^n_\omega$ when we refer to the balls $\mathrm{B}^M_r(o)$ and $\mathrm{B}^{\omega}_r(o_\omega)$ and the spheres $\mathrm{S}^M_r(o)$ and $\mathrm{S}^{\omega}_r(o_\omega)$.

Our first result in this regard is following:

Proposition 3.2 *Let (M^n, g) be a complete Riemannian manifold and let (M^n_ω, g_ω) be a rotationally symmetric model space with center $o_\omega \in M^n_\omega$. Let $o \in M$ be a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a geodesic ball $B^M_R(o)$, with $R < inj(o) \leq inj(o_w)$. Then the following assertions are equivalent:*

(1) $E^M_R = \mathbb{E}^{\omega}_R$ *on* $B^M_R(o)$.
(2) $H_{S^w_r(o_\omega)} = H_{S^M_r(o)}$ $\forall r \in]0, R]$.

where $\mathrm{H}_{\mathrm{S}^M_r(o)}$ denotes the mean curvature of the geodesic r- sphere $\mathrm{S}^M_r(o) \subseteq M$ and $\mathrm{H}_{\mathrm{S}^{\omega}_r(o_\omega)}$ is the corresponding mean curvature of the geodesic r-sphere $\mathrm{S}^{\omega}_r(o_\omega) \subseteq M^n_\omega$.

Proof Using polar coordinates $(r, \overline{\theta})$ in $M - (Cut(o) \cup \{o\})$, Eqs. 2.1 and 2.6 and applying Maximum Principle, equality $E^M_R = \mathbb{E}^{\omega}_R$ on $B^M_R(o)$ is equivalent to equality

$$\Delta^M \mathbb{E}^{\omega}_R(r, \overline{\theta}) = \Delta^M E^M_R(r, \overline{\theta}) = -1 = \Delta^{M^n_\omega} E^{\omega}_R \quad \forall (r, \overline{\theta})$$

which, in its turn, applying Proposition 2.3 and Eqs. 2.4 and 2.6, is equivalent to equality, for all $(r, \overline{\theta}) \in [0, R] \times \mathbb{S}_1$:

$$\mathbb{E}^{\omega\prime\prime}_R(r) + (n-1)H_{S^M_r(o)}\, \mathbb{E}^{\omega\prime}_R(r) = E^{\omega\prime\prime}_R(r) + (n-1)H_{S^w_r(o_\omega)}\, E^{\omega\prime}_R(r)$$

and, as for all $r \in]0, R]$, $\mathbb{E}^{\omega\prime\prime}_R(r) = E^{\omega\prime\prime}_R(r)$ and $\mathbb{E}^{\omega'}_R(r) = E^{\omega'}_R(r) < 0 \, \forall r \in]0, R]$, this last equality is equivalent to equality

$$H_{S^M_r(o)} = H_{S^w_r(o_\omega)} \; \forall r \in]0, R].$$

□

Now, we can state the following comparison theorem:

Theorem 3.3 *Let (M^n, g) be a complete Riemannian manifold and let (M^n_ω, g_ω) be a rotationally symmetric model space with center $o_\omega \in M^n_\omega$. Let $o \in M$ be a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a metric ball $B^M_R(o)$, with $R < inj(o) \leq inj(o_w)$. Let us suppose moreover that*

$$\mathrm{H}_{\mathrm{S}^{\omega}_r(o_\omega)} \leq \; (\geq)\, \mathrm{H}_{\mathrm{S}^M_r(o)} \quad \textit{for all} \quad 0 < r \leq R \tag{3.1}$$

where $\mathrm{H}_{\mathrm{S}^M_r}$ denotes the mean curvature of the metric r- sphere $\mathrm{S}^M_r(o) \subseteq M$ and $\mathrm{H}_{\mathrm{S}^{\omega}_r}$ is the corresponding mean curvature of the metric r-sphere $\mathrm{S}^{\omega}_r(o_\omega) \subseteq M^n_\omega$.

Then, we have the inequality

$$\mathbb{E}^{\omega}_R \geq \; (\leq)\, E^M_R \quad in \quad \mathrm{B}^M_R(o)\,, \tag{3.2}$$

where $\mathbb{E}^{\omega}_R(x) := E^{\omega}_R\,(r_o(x))$ is the transplanted mean exit time function in $\mathrm{B}^M_R(o)$.

Moreover, if there exists $p \in B^M_R(o)$ such that $\mathbb{E}^{\omega}_R(p) = E^M_R(p)$, then

$$\mathbb{E}^{\omega}_R = E^M_R \quad in \;\; \mathrm{B}^M_R(o)$$

and hence,

$$H_{S_r^w(o_\omega)} = H_{S_r^M(o)} \ \forall r \in]0, R].$$

Proof To prove first assertion, let us consider polar coordinates $(r, \overline{\theta}) \in [0, inj(o)) \times \mathbb{S}_1^{n-1}$ centered at the center $o \in M$ of the geodesic ball B_R^M, with $R < inj(o)$, (as before and throughout the rest of the paper, we shall omit the center point of the ball o if there is not confusion). By definition of $\mathbb{E}_R^\omega$ and Eq. 2.7, we have that this radial function satisfies

$$\mathbb{E}_R^{\omega\prime}(r) = E_R^{\omega\prime}(r) < 0, \quad \text{for all } r \in]0, R]. \tag{3.3}$$

Since $\Delta^{M_\omega^n} E_R^\omega = -1$ on B_R^ω,

$$\mathbb{E}_R^{\omega\prime\prime}(r) = E_R^{\omega\prime\prime}(r) = -1 - (n-1)\,\frac{\omega'(r)}{\omega(r)}\,E_R^{\omega\prime}(r).$$

Therefore, using Eq. 2.1 and applying Proposition 2.3 and Eqs. 2.4 and 2.6, we have, for all $(r, \overline{\theta}) \in]0, R] \times \mathbb{S}_1^{n-1}$:

$$\Delta^M \mathbb{E}_R^\omega(r, \overline{\theta}) = -1 + (n-1)\left(\mathrm{H}_{\mathrm{S}_r^M} - \mathrm{H}_{\mathrm{S}_r^\omega}\right) E_R^{\omega\prime}(r)\,. \tag{3.4}$$

Then, from Eqs. 3.3 and 3.4, and assuming inequality $\mathrm{H}_{\mathrm{S}_r^\omega} \leq \mathrm{H}_{\mathrm{S}_r^M}$ for all $r > 0$ we obtain that

$$\Delta^M \mathbb{E}_R^\omega(r, \overline{\theta}) \leq -1 = \Delta^M E_R^M(r, \overline{\theta}), \quad \text{for all } (r, \overline{\theta}) \in]0, R] \times \mathbb{S}_1^{N-1}\,. \tag{3.5}$$

Thus

$$\Delta^M \left(E_R^M - \mathbb{E}_R^\omega\right)(r, \overline{\theta}) \geq 0 \text{ on } B_R^M$$

and since $\left(E_R^M - \mathbb{E}_R^\omega\right)(R) = 0$ we have, applying the strong maximum principle

$$\mathbb{E}_R^\omega \geq E_R^M \quad \text{on } B_R^M$$

as we wanted to prove. We obtain opposite inequalities with same arguments, assuming that $\mathrm{H}_{\mathrm{S}_r^\omega} \geq \mathrm{H}_{\mathrm{S}_r^M}$ for all $r > 0$.

To prove the second assertion, assume that

$$\mathrm{H}_{\mathrm{S}_r^\omega(o_\omega)} \leq \mathrm{H}_{\mathrm{S}_r^M(o)} \quad \text{for all} \quad 0 < r \leq R\,.$$

Suppose that there exists $p \in B_R^M$ such that $\mathbb{E}_R^\omega(p) = E_R^M(p)$. Therefore, we have that $\Delta^M \left(E_R^M - \mathbb{E}_R^\omega\right) \geq 0$ on B_R^M and that $E_R^M - \mathbb{E}_R^\omega \leq 0 = (E_R^M - \mathbb{E}_R^\omega)(p)$ on B_R^M. Hence, $E_R^M - \mathbb{E}_R^\omega$ attains its maximum in B_R^M. Applying the strong maximum principle, the difference function $E_R^M - \mathbb{E}_R^\omega = C$ is constant on B_R^M and, by continuity, as $E_R^M - \mathbb{E}_R^\omega = 0$ on $\partial B_R^M = S_R^M$, then $C = 0$. Equality of the mean curvatures follows from Proposition 3.2. □

Remark 3.4 In Theorem 3.3, we have compared the functions $E_R^M : B_R^M \to \mathbb{R}$ and $\mathbb{E}_R^\omega : B_R^M \to \mathbb{R}$, both defined in $B_R^M \subseteq M$. Note that, using the properties of the symmetrized functions, (see [30] and observation (3) in Remark 2.16) and under the assumptions of Theorem above, we conclude from inequality (3.2) that

$$\mathbb{E}_R^{\omega *} \geq (\leq)\, E_R^{M *} \quad \text{in} \quad B_{s(R)}^w(o_w)\,, \tag{3.6}$$

where $B_{s(R)}^w(o_w)$ is the Schwarz symmetrization of B_R^M.

As in a Talenti-type result, we have bounded the symmetrized function E_R^{M*} with another radial function $\mathbb{E}_R^{\omega*}$ defined on $B^w_{s(R)}(o_w)$, the Schwarz symmetrization of B_R^M. We shall see in Corollary 4.7 the relation among $\mathbb{E}_R^{\omega*}$ and $E^M_{s(R)}$, the solution of the symmetrized Poisson problem which is the natural bound of E_R^{M*} in a Talenti-type comparison theorem.

Corollary 3.5 *Let (M^n, g) be a complete Riemannian manifold and let (M^n_ω, g_ω) be a rotationally symmetric model space with center $o_\omega \in M^n_\omega$. Let $o \in M$ be a point in M and let us suppose that $inj(o) \le inj(o_\omega)$. Let us consider a metric ball $B_R^M(o)$, with $R < inj(o) \le inj(o_w)$. Let us suppose moreover that*

$$\mathrm{H}_{\mathrm{S}^\omega_r} \le (\ge)\, \mathrm{H}_{\mathrm{S}^M_r} \quad \textit{for all} \quad 0 < r \le R. \tag{3.7}$$

Then we have the isoperimetric inequalities

$$\frac{Vol\left(\mathrm{B}^\omega_r(o_\omega)\right)}{Vol\left(\mathrm{S}^\omega_r(o_\omega)\right)} \ge (\le) \frac{Vol\left(\mathrm{B}^M_r(o)\right)}{Vol\left(\mathrm{S}^M_r(o)\right)} \quad \textit{for all} \quad 0 < r \le R. \tag{3.8}$$

Moreover, equality in inequalites (3.8) for some radius $r_0 \in]0, R]$ implies that

$$H_{S^w_r(o_\omega)} = H_{S^M_r(o)} \ \ \forall r \in]0, r_0].$$

As a consequence of inequalities (3.8), for all $0 < r \le R$, we have

$$\begin{aligned} Vol\left(\mathrm{B}^\omega_r(o_\omega)\right) &\le (\ge)\, Vol\left(\mathrm{B}^M_r(o)\right), \\ Vol(S^\omega_r(o_\omega)) &\le (\ge)\, Vol(S^M_r(o)). \end{aligned} \tag{3.9}$$

Finally, equality

$$Vol\left(\mathrm{B}^\omega_{r_0}(o_\omega)\right) = Vol\left(\mathrm{B}^M_{r_0}(o)\right)$$

for some radius $r_0 \in]0, R]$ implies that

$$H_{S^w_r(o_\omega)} = H_{S^M_r(o)} \ \ \forall r \in]0, r_0].$$

Proof Let us fix one radius $r \in]0, R]$. The proof follows the lines of the proof of Theorem 1.1 and Corollary 1.2 in [34], adapting it to this intrinsic context and using the new hypotheses.

First, let us assume that $\mathrm{H}_{\mathrm{S}^\omega_r} \le \mathrm{H}_{\mathrm{S}^M_r}$, for all $0 < r \le R$. If we fix $r \in]0, R]$, then we have, in particular, that $\mathrm{H}_{\mathrm{S}^\omega_s} \le \mathrm{H}_{\mathrm{S}^M_s}$, for all $0 < s \le r$. We can apply Theorem 3.3 to obtain

$$\Delta^M \mathbb{E}^\omega_r \le (\ge)\, \Delta^M E^M_r = -1 \text{ on } B^M_r.$$

Therefore, since $\left\|\nabla^M r\right\| = 1$, and using the *Divergence Theorem*, we have

$$\begin{aligned} \mathrm{Vol}\left(\mathrm{B}^M_r\right) &\le \int_{\mathrm{B}^M_r} -\Delta^M \mathbb{E}^\omega_r \, d\widetilde{\sigma} = -\int_{\mathrm{B}^M_r} \mathrm{div}\left(\nabla^M \mathbb{E}^\omega_r\right) d\widetilde{\sigma} & (3.10) \\ &= -\int_{\mathrm{S}^M_r} \left\langle \nabla^M \mathbb{E}^\omega_r, \nabla^M r \right\rangle d\sigma = -\mathbb{E}^{\omega\prime}_r(r) \mathrm{Vol}\left(\mathrm{S}^M_r\right). & (3.11) \end{aligned}$$

Thus, we obtain, using Proposition 2.7,

$$\mathrm{Vol}\left(\mathrm{B}^M_r\right) \le -\mathbb{E}^{\omega\prime}_r(r)\mathrm{Vol}\left(\mathrm{S}^M_r\right) = q_\omega(r)\mathrm{Vol}\left(\mathrm{S}^M_r\right) = \frac{\mathrm{Vol}\left(\mathrm{B}^\omega_r\right)}{\mathrm{Vol}\left(\mathrm{S}^\omega_r\right)}\mathrm{Vol}\left(\mathrm{S}^M_r\right)$$

and therefore

$$\frac{\mathrm{Vol}\left(\mathrm{B}^{\omega}_r\right)}{\mathrm{Vol}\left(\mathrm{S}^{\omega}_r\right)} \geq \frac{\mathrm{Vol}\left(\mathrm{B}^{M}_r\right)}{\mathrm{Vol}\left(\mathrm{S}^{M}_r\right)}.$$

We are going to discuss the equality assertion: we are still assuming that $\mathrm{H}_{\mathrm{S}^{\omega}_r} \leq \mathrm{H}_{\mathrm{S}^{M}_r}$, for all $0 < r \leq R$. If there exists $r_0 \in]0, R]$ such that we have

$$\frac{\mathrm{Vol}\left(\mathrm{B}^{\omega}_{r_0}\right)}{\mathrm{Vol}\left(\mathrm{S}^{\omega}_{r_0}\right)} = \frac{\mathrm{Vol}\left(\mathrm{B}^{M}_{r_0}\right)}{\mathrm{Vol}\left(\mathrm{S}^{M}_{r_0}\right)}$$

then the inequality in Eq. 3.10 becomes an equality with the radius r_0.

In particular,

$$\mathrm{Vol}\left(\mathrm{B}^{M}_{r_0}\right) = \int_{\mathrm{B}^{M}_{r_0}} -\Delta^M \mathbb{E}^{\omega}_{r_0}\, d\widetilde{\sigma}$$

and hence, as $1 + \Delta^M \mathbb{E}^{\omega}_{r_0} \leq 0$ on $B^M_{r_0}$, we conclude that $1 + \Delta^M \mathbb{E}^{\omega}_{r_0} = 0$ on $B^M_{r_0}$ and hence, as $\Delta^M \mathbb{E}^{\omega}_{r_0} = \Delta^M E^M_{r_0}$ on $B^M_{r_0}$ then, applying the maximum principle, $\mathbb{E}^{\omega}_{r_0} = E^M_{r_0}$ on $B^M_{r_0}$ and hence, by Proposition 3.2, $\mathrm{H}_{\mathrm{S}^{\omega}_r} = \mathrm{H}_{\mathrm{S}^{M}_r}\ \forall r \in]0, r_0]$.

When we assume that $\mathrm{H}_{\mathrm{S}^{\omega}_r} \geq \mathrm{H}_{\mathrm{S}^{M}_r}\ \forall r \in]0, R]$, we argue as before, inverting all the inequalities to conclude the opposite isoperimetric inequality. The equality discussion is the same, mutatis mutandi.

To prove statement (3.9), and as in Corollary 1.2 in [34], let us define, given $0 < R < inj(o)$ the function

$$G : [0, R] \to \mathbb{R}$$

as

$$G(s) := \begin{cases} \ln\left(\dfrac{\mathrm{Vol}\left(\mathrm{B}^{M}_s\right)}{\mathrm{Vol}\left(\mathrm{B}^{\omega}_s\right)}\right), & \text{if } s > 0, \\ 0, & \text{if } s = 0. \end{cases}$$

Then, if $\mathrm{H}_{\mathrm{S}^{\omega}_s} \leq \mathrm{H}_{\mathrm{S}^{M}_s}\ \forall s \in]0, R]$, we have, applying inequality (3.8), that

$$G'(s) = \frac{\mathrm{Vol}\left(\mathrm{S}^{M}_s\right)}{\mathrm{Vol}\left(\mathrm{B}^{M}_s\right)} - \frac{\mathrm{Vol}\left(\mathrm{S}^{\omega}_s\right)}{\mathrm{Vol}\left(\mathrm{B}^{\omega}_s\right)} \geq 0\ \forall s \in]0, R].$$

Hence, G is non-decreasing in $]0, R]$. The rest of the proof follows as in [34], using in this case the asymptotic expansion around $s = 0$ for the volume of a geodesic s-ball, (see Theorem 9.12 in [17]) to conclude with a straightforward computation, that $\lim_{s\to 0} G(s) = 0 = G(0)$, and hence, that $G(s)$ is continuous and $G(s) \geq G(0)\ \forall s \in [0, R]$, so, given $s = r \in]0, R]$, we have

$$\mathrm{Vol}\left(\mathrm{B}^{\omega}_r\right) \leq \mathrm{Vol}\left(\mathrm{B}^{M}_r\right)\ \forall r \in]0, R].$$

Moreover, isoperimetric inequality (3.8), together with the above inequality implies that

$$\mathrm{Vol}(S^{\omega}_r) \leq \mathrm{Vol}(S^{M}_r)\ \forall r \leq R.$$

We are going to discuss the equality assertion: let us assume that $\mathrm{H}_{\mathrm{S}^{\omega}_r} \leq \mathrm{H}_{\mathrm{S}^{M}_r}\ \forall r \in]0, R]$ and that there exists $r_0 \in]0, R]$ such that $\mathrm{Vol}\left(\mathrm{B}^{\omega}_{r_0}\right) = \mathrm{Vol}\left(\mathrm{B}^{M}_{r_0}\right)$. Then, $G(0) = G(r_0) = 0$ and, as G in non-decreasing, for all $r \in [0, r_0]$, we have

$$0 = G(0) \leq G(r) \leq G(r_0) = 0$$

so $G(r) = 0\ \forall r \in [0, r_0]$ and therefore, $G'(r) = 0\ \forall r \in [0, r_0]$ which implies that $\mathrm{H}_{\mathrm{S}_r^\omega} = \mathrm{H}_{\mathrm{S}_r^M}\ \forall r \in]0, r_0]$.

When we assume that $\mathrm{H}_{\mathrm{S}_r^\omega} \geq \mathrm{H}_{\mathrm{S}_r^M}\ \forall r \in]0, R]$, we argue as before, inverting all the inequalities to conclude that G is non-increasing in $]0, R]$ and hence

$$\mathrm{Vol}\left(\mathrm{B}_r^\omega\right) \geq \mathrm{Vol}\left(\mathrm{B}_r^M\right)\ \forall r \in]0, R]$$

and $\mathrm{Vol}(S_r^\omega) \geq \mathrm{Vol}(S_r^M)\ \forall r \leq R$.

The equality discussion is the same as above, mutatis mutandi. □

Corollary 3.6 *Let (M^n, g) be a complete Riemannian manifold and let (M_ω^n, g_ω) be a rotationally symmetric model space with center $o_\omega \in M_\omega^n$. Let $o \in M$ be a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a metric ball $B_R^M(o)$, with $R < inj(o) \leq inj(o_w)$. Let us suppose moreover that*

$$\mathrm{H}_{\mathrm{S}_r^\omega} \leq (\geq)\, \mathrm{H}_{\mathrm{S}_r^M} \quad \textit{for all} \quad 0 < r \leq R. \tag{3.12}$$

Then, if there exists $p \in B_R^M(o)$ such that equality $\mathbb{E}_R^\omega(p) = E_R^M(p)$ holds, we have, for all $r \in]0, R]$:

(1) *The equalities $\mathbb{E}_r^\omega = E_r^M$ on $B_r^M(o)$.*
(2) *The isoperimetric equalities*

$$\frac{Vol\left(\mathrm{B}_r^\omega(o_\omega)\right)}{Vol\left(\mathrm{S}_r^\omega(o_\omega)\right)} = \frac{Vol\left(\mathrm{B}_r^M(o)\right)}{Vol\left(\mathrm{S}_r^M(o)\right)}.$$

(3) *The volume equalities $Vol\left(\mathrm{B}_r^\omega\right) = Vol\left(\mathrm{B}_r^M\right)$ and $Vol(S_r^\omega) = Vol(S_r^M)$.*

Proof First of all, equality assertion in Theorem 3.3 states that, as we are assuming one of the inequalities in (3.12), then if there exists $p \in B_R^M(o)$ such that equality $\mathbb{E}_R^\omega(p) = E_R^M(p)$ holds, we conclude the equality $\mathbb{E}_R^\omega = E_R^M$ on $B_R^M(o)$. Applying Proposition 3.2, from this equality, we have equality $\mathrm{H}_{\mathrm{S}_r^\omega} = \mathrm{H}_{\mathrm{S}_r^M}\ \forall r \in]0, R]$. This last equality implies that, given any fixed $r \in]0, R]$, we have the equalities $\mathrm{H}_{\mathrm{S}_s^\omega} = \mathrm{H}_{\mathrm{S}_s^M}\ \forall s \in]0, r]$ and hence, by Proposition 3.2 again, we obtain $\mathbb{E}_r^\omega = E_r^M$ on $B_r^M(o)$.

On the other hand, equality $\mathbb{E}_R^\omega = E_R^M$ on $B_R^M(o)$ implies that $\Delta^M \mathbb{E}_R^\omega = -1 = \Delta^M E_R^M$ on $B_R^M(o)$, which implies in its turn that

$$\mathrm{Vol}\left(\mathrm{B}_R^M\right) = \int_{\mathrm{B}_R^M} -\Delta^M \mathbb{E}_R^\omega\, d\sigma = -\mathbb{E}_R^{\omega'}(R)\mathrm{Vol}\left(\mathrm{S}_R^M\right)$$

and hence, by Proposition 2.7

$$\frac{\mathrm{Vol}\left(\mathrm{B}_R^\omega(o_\omega)\right)}{\mathrm{Vol}\left(\mathrm{S}_R^\omega(o_\omega)\right)} = \frac{\mathrm{Vol}\left(\mathrm{B}_R^M(o)\right)}{\mathrm{Vol}\left(\mathrm{S}_R^M(o)\right)}.$$

Moreover, fixing $r \in]0, R]$, we know that, as $\mathbb{E}_R^\omega = E_R^M$ on $B_R^M(o)$, then $\mathbb{E}_r^\omega = E_r^M$ on $B_r^M(o)$. Applying Proposition 3.2 and using this equality implies

$$\frac{\mathrm{Vol}\left(\mathrm{B}_r^\omega(o_\omega)\right)}{\mathrm{Vol}\left(\mathrm{S}_r^\omega(o_\omega)\right)} = \frac{\mathrm{Vol}\left(\mathrm{B}_r^M(o)\right)}{\mathrm{Vol}\left(\mathrm{S}_r^M(o)\right)}$$

with the same argument as above.

Finally, as equality $\mathbb{E}_R^\omega = E_R^M$ on $B_R^M(o)$ implies that $\mathbb{E}_r^\omega = E_r^M$ on $B_r^M(o)$ $\forall r \in]0, R]$, then, if we define

$$\begin{cases} G(r) := \ln\left(\dfrac{\operatorname{Vol}\left(\mathrm{B}_r^M\right)}{\operatorname{Vol}\left(\mathrm{B}_r^\omega\right)}\right), & \text{if } r \in]0, R], \\ 0, & \text{if } r = 0, \end{cases}$$

then $G'(r) = 0\,\forall r \in]0, R]$, and hence, $G(r) = 0\,\forall r \in]0, R]$, so $\operatorname{Vol}\left(\mathrm{B}_r^\omega\right) = \operatorname{Vol}\left(\mathrm{B}_r^M\right)$ $\forall r \in]0, R]$ and differentiating with respect the parameter r, $\operatorname{Vol}(S_r^\omega) = \operatorname{Vol}(S_r^M)$ $\forall r \leq R$. □

To finish this section, we present the following property satisfied by the symmetrization of the transplanted mean exit time function $\mathbb{E}_R^\omega$. This result is an intrinsic corollary of Theorem 4.4 in [22], (see too Section 6 in [22]).

Theorem 3.7 *Let (M^n, g) be a complete Riemannian manifold and let (M_ω^n, g_ω) be a rotationally symmetric model space with center $o_\omega \in M_\omega^n$. Let $o \in M$ be a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a metric ball $B_R^M(o)$, with $R < inj(o) \leq inj(o_w)$, and let us assume that there exists $\mathrm{B}_{s(R)}^\omega(o_\omega)$, the Schwarz symmetrization of B_R^M in M_ω^n. Let $\mathbb{E}_R^{\omega^*} : \mathrm{B}_{s(R)}^\omega \longrightarrow \mathbb{R}$ be the symmetrization of the transplanted mean exit time function $\mathbb{E}_R^\omega : \mathrm{B}_R^M \longrightarrow \mathbb{R}$. Then*

$$\int_{\mathrm{B}_R^M} \mathbb{E}_R^\omega \, d\sigma = \int_{\mathrm{B}_{s(R)}^\omega} \mathbb{E}_R^{\omega^*} \, d\widetilde{\sigma}. \tag{3.13}$$

4 Moment Spectrum Comparison

We are going to apply the Mean Exit comparisons obtained in Section 3 to obtain estimates of the moment spectrum, and the torsional rigidity of a geodesic ball in a Riemannian manifold with bounds on the mean curvature of its extrinsic spheres.

4.1 Estimates for the Poisson Hierarchy and the Moment Spectrum of a Geodesic Ball

We shall start by defining the so called Poisson hierarchy of a domain in a Riemannian manifold, (see [15]).

Definition 4.1 Let (M^n, g) be a complete Riemannian manifold and let $D \subset M$ be a smooth precompact domain. We define the *Poisson hierarchy for D* as the sequence $\left\{u_{k,D}\right\}_{k=1}^\infty$ of solutions of the following recurrence of boundary value problems

$$\begin{aligned} \Delta^M u_{k,D} + k u_{k-1,D} &= 0, \text{ on } D, \\ u_{k,D}|_{\partial D} &= 0, \end{aligned} \tag{4.1}$$

with $u_{0,D} = 1$ on D.

Let us note that $u_{1,D} = E_D^M$, i.e. the mean exit time function from D.

As we did in Definition 3.1, we transplant the Poisson hierarchy for the geodesic balls in a model space to the geodesic balls in a Riemanian manifold in the following way:

Definition 4.2 Let (M^n, g) be a complete Riemannian manifold and let (M^n_ω, g_ω) be a rotationally symmetric model space with center $o_\omega \in M^n_\omega$. Let $o \in M$ be a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a metric ball $B^M_R(o)$, with $R < inj(o) \leq inj(o_w)$.

Let us consider the Poisson hierarchy for $B^\omega_R(o_\omega)$, namely, the sequence $\left\{u^\omega_{k,R}\right\}_{k=1}^\infty$ which, for $k \geq 1$, are the solutions of

$$\begin{aligned} \Delta^{M^n_\omega} u^\omega_{k,R} + k u^\omega_{k-1,R} &= 0, \text{ on } \mathrm{B}^\omega_R, \\ u^\omega_{k,R}|_{\mathrm{S}^\omega_R} &= 0, \end{aligned}$$

with $u^\omega_{0,R} = 1$ on B^ω_R.

It is known, for all $k \geq 1$ that $u^\omega_{k,R}(x) = u^\omega_{k,R}\left(r_{o_\omega}(x)\right)$, i.e. $u^\omega_{k,R}$ is radial, and that $u^{\omega'}_{k,R} \leq 0$ (see Proposition 3.1 of [23]).

Thus, for all $k \geq 1$, we can transplant these functions to $\mathrm{B}^M_R(o) \subseteq M$ by defining

$$\bar{u}^\omega_{k,R} : \mathrm{B}^M_R(o) \to \mathbb{R}$$

as $\bar{u}^\omega_{k,R}(x) := u^\omega_{k,R}(r_o(x)) \;\; \forall x \in \mathrm{B}^M_R(o)$, where r_o is the distance function to the center of $\mathrm{B}^M_R(o)$.

The sequence $\left\{\bar{u}^\omega_{k,R}\right\}_{k=1}^\infty$ is the *transplanted Poisson hierarchy for* $B^M_R(o)$.

Associated to the Poisson hierarchy of a domain $D \subseteq M$, the exit time moment spectrum of this domain is defined in the following way:

Definition 4.3 Let $D \subseteq M$ a smooth precompact domain. We define the *moment spectrum of D* as the sequence of integrals $\{\mathcal{A}_k(D)\}_{k=1}^\infty$ given by:

$$\mathcal{A}_k(D) := \int_D u_{k,D}\, d\sigma$$

where $\left\{u_{k,D}\right\}_{k=1}^\infty$ is the Poisson hierarchy for D.

Let us note that $\mathcal{A}_1(D)$ is the torsional rigidity of D.

We have the following comparison for the Poisson hierarchy of a geodesic ball in a Riemannian manifold:

Theorem 4.4 *Let (M^n, g) be a complete Riemannian manifold and let (M^n_ω, g_ω) be a rotationally symmetric model space with center $o_\omega \in M^n_\omega$. Let $o \in M$ be a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a metric ball $B^M_R(o)$, with $R < inj(o) \leq inj(o_w)$. Let us suppose moreover that the mean curvatures of the geodesic spheres in M and M^n_ω satisfies*

$$\mathrm{H}_{\mathrm{S}^\omega_r} \leq (\geq)\, \mathrm{H}_{\mathrm{S}^M_r} \quad \textit{for all} \quad 0 < r \leq R. \tag{4.2}$$

Then the Poisson hierarchy for $\mathrm{B}^M_R(o) \subseteq M$, $\left\{u_{k,R}\right\}_{k=1}^\infty$, and its transplanted Poisson hierarchy for $B^M_R(o)$, $\left\{\bar{u}_{k,R}\right\}_{k=1}^\infty$ (and, for any fixed $r \in]0, R]$, the corresponding Poisson hierarchies for $B^M_r(o)$), satisfies

(1) $\bar{u}^\omega_{1,R} \geq (\leq)\, u_{1,R}$ *on* B^M_R.

(2) *For all $k \geq 2$, $\bar{u}^\omega_{k,R} \geq (\leq)\, u_{k,R}$ on* B^M_R.

(3) *If there exists* $p \in B^M_R$ *and* $k_0 \geq 1$ *such that* $\bar{u}^\omega_{k_0,R}(p) = u_{k_0,R}(p)$*, then*

$$\mathrm{H}_{\mathrm{S}^\omega_r} = \mathrm{H}_{\mathrm{S}^M_r} \; \forall r \in]0, R]$$

and

$$\bar{u}^\omega_{k,R} = u_{k,R} \;\; in \;\; B^M_R \;\; \forall k \geq 1.$$

(4) *If there exists* $p \in B^M_R$ *and* $k_0 \geq 1$ *such that* $\bar{u}^\omega_{k_0,R}(p) = u_{k_0,R}(p)$*, then*

$$\bar{u}^\omega_{k,r} = u_{k,r} \;\; in \;\; B^M_r \;\; \forall k \geq 1 \;\; and \;\; \forall r \in [0, R].$$

and hence,

$$\mathcal{A}_k(B^\omega_r) = \mathcal{A}(B^M_r) \; \forall r \in \;]0, R \;] \;\; and \;\; k \geq 1\,.$$

Proof Statement (1) is proved in Theorem 3.3

The proof of statement (2) follows using induction on k, as it is done in [23]. Indeed, assuming that $\mathrm{H}_{\mathrm{S}^\omega_r} \leq \mathrm{H}_{\mathrm{S}^M_r} \;\; \forall r \in]0, R]$ and as $\bar{u}^{\omega\prime}_{k,R}(r) \leq 0 \; \forall r \in]0, R]$, we have that

$$\bar{u}^{\omega\prime}_{k,R}(r)\mathrm{H}_{\mathrm{S}^\omega_r} \geq \bar{u}^{\omega\prime}_{k,R}(r)\mathrm{H}_{\mathrm{S}^M_r} \;\; \forall r \in]0, R]$$

and then, by Eqs. 2.4 and 2.6 and Proposition 2.3, we have, for all $k \geq 2$:

$$\begin{aligned}\Delta^M \bar{u}^\omega_{k,R} &= \bar{u}^{\omega\prime\prime}_{k,R}(r) + (n-1)\mathrm{H}_{\mathrm{S}^M_r}\, \bar{u}^{\omega\prime}_{k,R}(r) \leq \bar{u}^{\omega\prime\prime}_{k,R}(r) + (n-1)\mathrm{H}_{\mathrm{S}^\omega_r}\bar{u}^{\omega\prime}_{k,R}(r) \\ &= \Delta^{M^n_\omega} u^\omega_{k,R} = -k u^\omega_{k-1}(r) = -k\bar{u}^\omega_{k-1}(r) \;\; \forall r \in]0, R].\end{aligned} \tag{4.3}$$

Now remember that $\bar{u}^\omega_1 \geq u_1$ on B^M_R and let us suppose that

$$\bar{u}^\omega_{k,R} \geq u_{k,R} \text{ on } \mathrm{B}^M_R.$$

Then, by induction with $k+1$ and using equation (4.3), we have that

$$\Delta^M \bar{u}^\omega_{k+1,R} \leq -(k+1)\bar{u}^\omega_{k,R} \leq -(k+1)u_{k,R} = \Delta^M u_{k+1,R} \text{ on } \mathrm{B}^M_R. \tag{4.4}$$

Thus, $\Delta^M \left(u_{k+1,R} - \bar{u}^\omega_{k+1,R}\right) \geq 0$ on B^M_R and, applying the Maximum Principle, we obtain that

$$\bar{u}^\omega_{k+1,R} \geq u_{k+1,R}\,.$$

When we assume that $\mathrm{H}_{\mathrm{S}^\omega_r} \geq \mathrm{H}_{\mathrm{S}^M_r} \;\; \forall r \in]0, R]$, the argument is exactly the same, inverting all the inequalities. All this proves (2).

To prove assertion (3), let us suppose that, as hypothesis, $\mathrm{H}_{\mathrm{S}^\omega_r} \leq \mathrm{H}_{\mathrm{S}^M_r} \;\; \forall r \in]0, R]$, and that there exists $p \in B^M_R$ and $k_0 \geq 1$ such that

$$\bar{u}^\omega_{k_0,R}(p) = u_{k_0,R}(p).$$

We know that, for all $k \geq 1$, $\bar{u}^\omega_{k,R} \geq u_{k,R}$ on B^M_R. Then, as, on $\mathrm{B}^M_R(o)$, $\Delta^M \bar{u}^\omega_{k,R} \leq -k\bar{u}^\omega_{k-1} \leq -k\; u_{k-1} = \Delta^M u_{k,R}$ for all $k \geq 1$, we have, in particular,

$$\Delta^M \left(u_{k_0,R} - \bar{u}^\omega_{k_0,R}\right) \geq 0$$

on $\mathrm{B}^M_R(o)$.

Moreover, as $\bar{u}^\omega_{k_0,R} \geq u_{k_0,R}$ on $\mathrm{B}^M_R(o)$, then $u_{k_0,R} - \bar{u}^\omega_{k_0,R} \leq 0$ on $\mathrm{B}^M_R(o)$ and there exists $p \in B^M_R$ such that $(u_{k_0,R} - \bar{u}^\omega_{k_0,R})(p) = 0$. Then, applying the strong maximum principle, $\bar{u}^\omega_{k_0,R} = u_{k_0,R}$ on B^M_R, because $u_{k_0,R} - \bar{u}^\omega_{k_0,R}$ is constant on $\mathrm{B}^M_R(o)$, continuous in $\overline{\mathrm{B}^M_R(o)}$ and $u_{k_0,R} - \bar{u}^\omega_{k_0,R} = 0$ on $\mathrm{S}^M_R(o)$.

On the other hand, as $\bar{u}^\omega_{k_0-1} \geq u_{k_0-1}$ on B^M_R, we have, on $B^M_R(o)$:

$$\begin{aligned} \Delta^M \bar{u}^\omega_{k_0,R} &= \Delta^M u_{k_0,R} = -k_0 u_{k_0-1,R} \geq \\ & -k_0 \bar{u}^\omega_{k_0-1,R} = -k_0 u^\omega_{k_0-1,R} = \Delta^{M^n_\omega} u^\omega_{k_0,R} \end{aligned} \tag{4.5}$$

so, for all $r \in]0, R]$:

$$\bar{u}^{\omega''}_{k_0,R}(r) + (n-1)\mathrm{H}_{\mathrm{S}^M_r}\, \bar{u}^{\omega'}_{k_0,R}(r) \geq u^\omega_{k_0,R''}(r) + (n-1)\mathrm{H}_{\mathrm{S}^\omega_r} u^{\omega'}_{k_0,R}(r). \tag{4.6}$$

As $\bar{u}^{\omega''}_{k_0,R}(r) = u^{\omega''}_{k_0,R}(r)$ and $\bar{u}^{\omega'}_{k_0,R}(r) = u^{\omega'}_{k_0,R}(r)$ for all $r \in]0, R]$, we conclude that

$$\mathrm{H}_{\mathrm{S}^M_r} \bar{u}^{\omega'}_{k_0,R}(r) \geq \mathrm{H}_{\mathrm{S}^\omega_r} u^{\omega'}_{k_0,R}(r) \;\forall r \in]0, R]$$

and hence, as $u^{\omega'}_{k_0,R}(r) < 0 \;\forall r \in]0, R]$, then

$$\mathrm{H}_{\mathrm{S}^\omega_r} \geq \mathrm{H}_{\mathrm{S}^M_r} \;\; \forall r \in]0, R].$$

As, by hypothesis, $\mathrm{H}_{\mathrm{S}^\omega_r} \leq \mathrm{H}_{\mathrm{S}^M_r} \;\forall r \in]0, R]$, we have finally that

$$\mathrm{H}_{\mathrm{S}^\omega_r} = \mathrm{H}_{\mathrm{S}^M_r} \;\; \forall r \in]0, R].$$

Now, to prove that $\bar{u}^\omega_{k,R} = u_{k,R}$ on $B^M_R(o)$, we argue as follows: as we know that $\mathrm{H}_{\mathrm{S}^\omega_r} = \mathrm{H}_{\mathrm{S}^M_r} \;\forall r \in]0, R]$, let us apply Proposition 3.2, to have that $\bar{u}^\omega_{1,R} = u_{1,R}$ on $B^M_R(o)$, and we procceed by induction. Let us suppose that $\bar{u}^\omega_{k,R} = u_{k,R}$ on $B^M_R(o)$. To see that $\bar{u}^\omega_{k+1,R} = u_{k+1,R}$ on $B^M_R(o)$, we compute

$$\begin{aligned} \Delta^M \bar{u}^\omega_{k+1,R} &= \bar{u}^{\omega''}_{k+1,R}(r) + \mathrm{H}_{\mathrm{S}^M_r}\, \bar{u}^{\omega'}_{k+1,R}(r) = \bar{u}^{\omega''}_{k+1,R}(r) + \mathrm{H}_{\mathrm{S}^\omega_r} \bar{u}^{\omega'}_{k+1,R}(r) \\ &= \Delta^{M^n_\omega} u^\omega_{k+1,R} = -(k+1) u^\omega_{k,R} = -(k+1)\bar{u}^\omega_{k,R} \\ &= -(k+1) u_{k,R} = \Delta^M u_{k+1,R} \text{ on } B^M_R(o). \end{aligned} \tag{4.7}$$

Hence $\Delta^M(\bar{u}^\omega_{k+1,R} - u_{k+1,R}) = 0$ on $B^M_R(o)$ and as $\bar{u}^\omega_{k+1,R} - u_{k+1,R} = 0$ on $S^M_R(o)$, then, appliyng Maximum Principle again, we conclude that $\bar{u}^\omega_{k+1,R} = u_{k+1,R}$ on $B^M_R(o)$.

Finally, to prove assertion (4), let us assume that that $\mathrm{H}_{\mathrm{S}^\omega_r} \leq \mathrm{H}_{\mathrm{S}^M_r} \;\forall r \in]0, R]$, and that there exists $p \in B^M_R$ and $k_0 \geq 1$ such that

$$\bar{u}^\omega_{k_0,R}(p) = u_{k_0,R}(p).$$

As before, we conclude that

$$\mathrm{H}_{\mathrm{S}^\omega_r} = \mathrm{H}_{\mathrm{S}^M_r} \;\; \forall r \in]0, R],$$

and hence, fixing $r \in]0, R]$, that

$$\mathrm{H}_{\mathrm{S}^\omega_s} = \mathrm{H}_{\mathrm{S}^M_s} \;\; \forall s \in]0, r].$$

Now, to prove that $\bar{u}^\omega_{k,r} = u_{k,r}$ on $B^M_r(o)$, we argue as in the proof of (3): as we know that $\mathrm{H}_{\mathrm{S}^\omega_s} = \mathrm{H}_{\mathrm{S}^M_s} \;\forall s \in]0, r]$, let us apply Proposition 3.2, to have that $\bar{u}^\omega_{1,r} = u_{1,r}$ on $B^M_r(o)$, and we procceed by induction, as in the proof of assertion (3). □

As a consequence of the Theorem 4.4 we have the following result, where it is proved that, under our hypotheses, any of the averaged moments of the geodesic balls determines its first Dirichlet eigenvalue:

Corollary 4.5 *Let (M^n, g) be a complete Riemannian manifold and let (M^n_ω, g_ω) be a rotationally symmetric model space with center $o_\omega \in M^n_\omega$. Let $o \in M$ be a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a metric ball $B^M_R(o)$, with $R < inj(o) \leq$*

$inj(o_w)$. Let us suppose moreover that the mean curvatures of the geodesic spheres in M and M^n_ω satisfies

$$\mathrm{H}_{\mathrm{S}^\omega_r} \leq\ (\geq)\, \mathrm{H}_{\mathrm{S}^M_r} \quad \textit{for all} \quad 0 < r \leq R. \tag{4.8}$$

Then, for all $k \geq 1$,

$$\frac{\mathcal{A}_k\left(\mathrm{B}^\omega_R\right)}{Vol\left(\mathrm{S}^\omega_R\right)} \geq\ (\leq)\ \frac{\mathcal{A}_k\left(\mathrm{B}^M_R\right)}{Vol\left(\mathrm{S}^M_R\right)}. \tag{4.9}$$

Equality in any of inequalities (4.9) for some $k \geq 1$ implies that

$$\mathrm{H}_{\mathrm{S}^\omega_R(o_\omega)} = \mathrm{H}_{\mathrm{S}^M_R(o)} \textit{ for all } 0 < r \leq R$$

and hence, we have the equalities

(1) *Equality $\bar{u}^\omega_{k,R} = u_{k,R}$ on $B^M_R(o)$ for all $k \geq 1$, and hence, equality $\bar{u}^\omega_{k,r} = u_{k,r}$ on $B^M_r(o)$ for all $k \geq 1$ and for all $0 < r \leq R$.*
(2) *Equalities $Vol\left(\mathrm{B}^\omega_r\right) = Vol\left(\mathrm{B}^M_r\right)$ and $Vol\left(\mathrm{S}^\omega_r\right) = Vol\left(\mathrm{S}^M_r\right)$ for all $0 < r \leq R$.*
(3) *Equalities $\mathcal{A}_k\left(\mathrm{B}^\omega_r\right) = \mathcal{A}_k\left(\mathrm{B}^M_r\right)$, for all $k \geq 1$ and for all $0 < r \leq R$.*
(4) *Equalities $\lambda_1(B^\omega_r) = \lambda_1(B^M_r)$ for all $0 < r \leq R$.*

Namely, one value of $\dfrac{\mathcal{A}_k\left(\mathrm{B}^M_R(o)\right)}{Vol\left(\mathrm{S}^M_R(o)\right)}$ for some $k \geq 1$ determines the Poisson hierarchy, the volume, the L^1-moment spectrum and the first Dirichlet eigenvalue of the ball $B^M_r(o)$ for all $0 < r \leq R$.

Proof In the model spaces we have that $\Delta^{M_\omega} u^\omega_{k+1,R} = -(k+1)u^\omega_k$ on the geodesic ball $\mathrm{B}^{M_\omega}_R(o_\omega)$, so, applying Divergence theorem in this setting , we obtain

$$\mathcal{A}_k\left(\mathrm{B}^\omega_R\right) = \int_{\mathrm{B}^\omega_R} u^\omega_{k,R}\, d\widetilde{\sigma} = -\frac{1}{k+1}\int_{\mathrm{B}^\omega_R} \Delta^{M^n_\omega} u^\omega_{k+1,R}\, d\widetilde{\sigma} = -\frac{1}{k+1}\, u^{\omega'}_{k+1,R}(R)\mathrm{Vol}\left(\mathrm{S}^\omega_R\right).$$

Therefore, for all $k \geq 1$,

$$-\frac{1}{k+1} u^{\omega\prime}_{k+1,R}(R) = \frac{\mathcal{A}_k\left(\mathrm{B}^\omega_R\right)}{\mathrm{Vol}\left(\mathrm{S}^\omega_R\right)}. \tag{4.10}$$

Assuming now as hypothesis one of the inequalities in (4.8), we obtain correspondingly the inequalities

$$\Delta^M \bar{u}^\omega_{k+1,R} \leq\ (\geq)\, \Delta^M u_{k+1,R} \text{ on } \mathrm{B}^M_R.$$

Then, using the Divergence theorem and that $\bar{u}^\omega_{k+1,R}$ is radial in B^M_R, we have

$$\begin{aligned} \mathcal{A}_k\left(\mathrm{B}^M_R\right) &= \textstyle\int_{\mathrm{B}^M_R} u_{k,R}\, d\sigma = -\dfrac{1}{k+1}\textstyle\int_{\mathrm{B}^M_R} \Delta^M u_{k+1,R} d\sigma \\ &\leq\ (\geq)\,.\ -\dfrac{1}{k+1}\textstyle\int_{\mathrm{B}^M_R} \Delta^M \bar{u}^\omega_{k+1,R}\, d\sigma \\ &= -\dfrac{1}{k+1}\textstyle\int_{\mathrm{S}^M_R} \left\langle \nabla^M \bar{u}^\omega_{k+1,R}, \nabla^M r\right\rangle d\sigma_r \\ &= -\dfrac{1}{k+1}\, \bar{u}^{\omega\prime}_{k+1,R}(R)\mathrm{Vol}\left(\mathrm{S}^M_R\right). \end{aligned} \tag{4.11}$$

Then, using (4.10), and that $\bar{u}^{\omega\prime}_{k+1,R}(R) = u^{\omega\prime}_{k+1,R}(R)$, we finally obtain that

$$\mathcal{A}_k\left(\mathrm{B}^M_R\right) \leq (\geq) \frac{\mathcal{A}_k\left(\mathrm{B}^\omega_R\right)}{\mathrm{Vol}\left(\mathrm{S}^\omega_R\right)} \mathrm{Vol}\left(\mathrm{S}^M_R\right).$$

To discuss the case of equality, assume that $\mathrm{H}_{\mathrm{S}^\omega_r} \leq \mathrm{H}_{\mathrm{S}^M_r}$, for all $0 < r \leq R$, and that

$$\frac{\mathcal{A}_{k_0}\left(\mathrm{B}^\omega_R\right)}{\mathrm{Vol}\left(\mathrm{S}^\omega_R\right)} = \frac{\mathcal{A}_{k_0}\left(\mathrm{B}^M_R\right)}{\mathrm{Vol}\left(\mathrm{S}^M_R\right)} \tag{4.12}$$

for some $k_0 \geq 1$. Then the inequality in (4.11) is an equality for this fixed k_0, so $\bar{u}^\omega_{k_0+1,R} = u_{k_0+1,R}$ on $B^M_R(o)$. Applying assertion (3) in Theorem 4.4, we have that $\mathrm{H}_{\mathrm{S}^\omega_r} = \mathrm{H}_{\mathrm{S}^M_r}$ for all $0 < r \leq R$ and that $\bar{u}^\omega_{k,R} = u_{k,R}$ on $B^M_R(o)$ for all $k \geq 1$. In particular, $\bar{u}^\omega_{1,R} = u_{1,R}$ on $B^M_R(o)$, so, by Corollary 3.6, $\mathrm{Vol}\left(\mathrm{S}^\omega_r\right) = \mathrm{Vol}\left(\mathrm{S}^M_r\right)$ and $\mathrm{Vol}\left(\mathrm{B}^\omega_r\right) = \mathrm{Vol}\left(\mathrm{B}^M_r\right)$ for all $r \in]0, R]$ and, hence, for all $k \geq 1$,

$$\begin{aligned}
\mathcal{A}_k\left(\mathrm{B}^M_R\right) &= \int_{\mathrm{B}^M_R} u_{k,R}\, d\sigma = -\frac{1}{k+1} \int_{\mathrm{B}^M_R} \Delta^M u_{k+1,R} d\sigma \\
&= -\frac{1}{k+1} \int_{\mathrm{B}^M_R} \Delta^M \bar{u}^\omega_{k+1,R}\, d\sigma = -\frac{1}{k+1} \bar{u}^{\omega\prime}_{k+1,R}(R) \mathrm{Vol}\left(\mathrm{S}^M_R\right) \\
&= \frac{\mathcal{A}_k\left(\mathrm{B}^\omega_R\right)}{\mathrm{Vol}\left(\mathrm{S}^\omega_R\right)} \mathrm{Vol}\left(\mathrm{S}^M_R\right) = \mathcal{A}_k\left(\mathrm{B}^\omega_R\right).
\end{aligned} \tag{4.13}$$

Moreover, applying assertion (4) in Theorem 4.4, from equality $\bar{u}^\omega_{k_0+1,R} = u_{k_0+1,R}$ on $B^M_R(o)$ we can deduce that $\bar{u}^\omega_{k,r} = u_{k,r}$ on $B^M_r(o)$ for all $k \geq 1$ and for all $r \in]0, R]$, so given $r \in]0, R]$, and for all $k \geq 1$,

$$\begin{aligned}
\mathcal{A}_k\left(\mathrm{B}^M_r\right) &= \int_{\mathrm{B}^M_r} u_{k,r}\, d\sigma = -\frac{1}{k+1} \int_{\mathrm{B}^M_r} \Delta^M u_{k+1,r} d\sigma \\
&= -\frac{1}{k+1} \int_{\mathrm{B}^M_r} \Delta^M \bar{u}^\omega_{k+1,r}\, d\sigma = -\frac{1}{k+1} \bar{u}^{\omega\prime}_{k+1,r}(r) \mathrm{Vol}\left(\mathrm{S}^M_r\right) \\
&= \frac{\mathcal{A}_k\left(\mathrm{B}^\omega_r\right)}{\mathrm{Vol}\left(\mathrm{S}^\omega_r\right)} \mathrm{Vol}\left(\mathrm{S}^M_r\right) = \mathcal{A}_k\left(\mathrm{B}^\omega_r\right).
\end{aligned} \tag{4.14}$$

Finally, to prove the last assertion of the Theorem, we know that, assuming that $\mathrm{H}_{\mathrm{S}^\omega_r} \leq \mathrm{H}_{\mathrm{S}^M_r}$ for all $0 < r \leq R$, the equality

$$\frac{\mathcal{A}_{k_0}\left(\mathrm{B}^\omega_R\right)}{\mathrm{Vol}\left(\mathrm{S}^\omega_R\right)} = \frac{\mathcal{A}_{k_0}\left(\mathrm{B}^M_R\right)}{\mathrm{Vol}\left(\mathrm{S}^M_R\right)}$$

implies equalities $\mathcal{A}_k\left(\mathrm{B}^\omega_r\right) = \mathcal{A}_k\left(\mathrm{B}^M_r\right)$, for all $k \geq 1$, and for all $r \in]0, R]$. Then, given $B^M_r \subseteq M$ in a Riemannian manifold (M, g), (see [24] and [7]):

$$\begin{aligned}
\lambda_1(B^M_r) &= \lim_{k\to\infty} \frac{k\mathcal{A}_{k-1}\left(\mathrm{B}^M_r\right)}{\mathcal{A}_k\left(\mathrm{B}^M_r\right)} \\
&= \lim_{k\to\infty} \frac{k\mathcal{A}_{k-1}\left(\mathrm{B}^\omega_r\right)}{\mathcal{A}_k\left(\mathrm{B}^\omega_r\right)} = \lambda_1(B^\omega_r).
\end{aligned} \tag{4.15}$$

□

4.2 An Estimate for the Torsional Rigidity of a Geodesic R-ball

We are going to bound the torsional rigidity of a metric ball B^M_R in a Riemannian manifold (M, g) in Theorem 4.8, assuming that the mean curvature of the geodesic spheres in this Riemnnian manifold is bounded from above or from below by the corresponding mean curvature of the geodesic spheres in a symmetric model space (M^n_ω, g_ω) which is balanced from above.

This result can be considered as a continuation of the intrinsic comparison done in Section 6 of the paper [22]. In that paper upper and lower bounds for the torsional rigidity of a metric ball $\mathrm{B}^M_R(o)$ in a Riemannian manifold (M, g) with a pole $o \in M$ were obtained under more restrictive conditions, namely, assuming that the radial sectional curvatures were bounded above or below by the corresponding radial sectional curvatures of a suitable model space.

The proof of Theorem 4.8 relies on Proposition 4.6. Let us consider a symmetric model space rearrangement of the metric ball B^M_R as described in Definition 2.11 and Definition 2.15, namely, a symmetrization of B^M_R which is a geodesic $s(R)$-ball in the model space M^n_ω such that $\mathrm{Vol}\left(\mathrm{B}^M_R(o)\right) = \mathrm{Vol}\left(\mathrm{B}^\omega_{s(R)}(o_\omega)\right)$, together with the symmetrization $\mathbb{E}^{\omega^*}_R : \mathrm{B}^\omega_{s(R)} \longrightarrow \mathbb{R}$ of the transplanted mean exit time function $\mathbb{E}^\omega_R : \mathrm{B}^M_R \longrightarrow \mathbb{R}$. It is evident that Proposition 4.6, Theorem 4.8 and Corollary 4.9 make sense for those geodesic balls $B^M_R(o)$ which posses a Schwarz symmerization $\mathrm{B}^\omega_{s(R)}(o_\omega)$.

Then, in Proposition 4.6 we compare the symmetrized function $\mathbb{E}^{\omega^*}_R : B^\omega_{s(R)}(o_\omega) \to \mathbb{R}$ and the solution of the Poisson problem in $B^\omega_{s(R)}(o_\omega)$, $E^\omega_{s(R)} \to \mathbb{R}$. Its proof follows closely the lines of the proof of Propositions 5.2 and 5.4 in [22], we have included it because the changes due to its intrinsic character, the different assumptions on the curvatures we have assumed here and the new analysis of the case of equality.

Proposition 4.6 *Let (M^n, g) be a complete Riemannian manifold and let (M^n_ω, g_ω) be a rotationally symmetric model space with center $o_\omega \in M^n_\omega$, balanced from above. Let $o \in M$ be a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a metric ball $B^M_R(o)$, with $R < inj(o) \leq inj(o_w)$. Let us suppose moreover that the mean curvatures of the geodesic spheres in M and M_ω satisfies*

$$\mathrm{H}_{\mathrm{S}^\omega_r} \leq (\geq)\, \mathrm{H}_{\mathrm{S}^M_r} \quad \textit{for all} \quad 0 < r \leq R. \tag{4.16}$$

Then

$$\mathbb{E}^{\omega^*\prime}_R(\widetilde{r}) \geq (\leq)\, E^{\omega\prime}_{s(R)}(\widetilde{r}) \quad \textit{for all } \widetilde{r} \in (0, s(R)) \tag{4.17}$$

and hence,

$$\mathbb{E}^{\omega^*}_R(\widetilde{r}) \leq (\geq)\, E^\omega_{s(R)}(\widetilde{r}) \quad \textit{for all } \widetilde{r} \in [0, s(R)]. \tag{4.18}$$

Equality in any of the inequalities (4.18) implies the equality among the radius $s(R) = R$ and the equality

$$\mathrm{H}_{\mathrm{S}^\omega_r} = \mathrm{H}_{\mathrm{S}^M_r}, \quad \textit{for all } 0 < r \leq R$$

and hence, we have the equalities

(1) *Equality $\bar{u}^\omega_{k,R} = u_{k,R}$ on $B^M_R(o)$ for all $k \geq 1$, and hence, equality $\bar{u}^\omega_{k,r} = u_{k,r}$ on $B^M_r(o)$ for all $k \geq 1$ and for all $0 < r \leq R$.*
(2) *Equalities $Vol\left(\mathrm{B}^\omega_r\right) = Vol\left(\mathrm{B}^M_r\right)$ and $Vol\left(\mathrm{S}^\omega_r\right) = Vol\left(\mathrm{S}^M_r\right)$ for all $0 < r \leq R$.*
(3) *Equalities $\mathcal{A}_k\left(\mathrm{B}^\omega_r\right) = \mathcal{A}_k\left(\mathrm{B}^M_r\right)$, for all $k \geq 1$ and for all $0 < r \leq R$.*

(4) *Equalities* $\lambda_1(B_r^\omega) = \lambda_1(B_r^M)$ *for all* $0 < r \leq R$.

Proof We are going to analyze first the symmetrization $\mathbb{E}_R^{\omega^*}$. The transplanted function

$$\mathbb{E}_R^\omega : B_R^M(o) \longrightarrow \mathbb{R}$$

satisfies that $\mathbb{E}_R^\omega \in C^\infty(B_R^M(o) - \{o\}) \cap C^0(\overline{B}_R^M(o))$, and, moreover, that $\mathbb{E}_R^\omega|_{S_R^M(o)} = 0$.

Let us consider the radial function $\psi = E_R^\omega$ defined on the interval $[0, R]$ in Eq. 2.7 of Proposition 2.7. Let us denote by $T = \max_{[0,R]}\psi$. Thus, as ψ is monotone, (strictly decreasing, with $\psi(0) = T$ and $\psi(R) = 0$), we have that $\frac{d}{dr}\psi < 0$ on $]0, R]$ and that $\psi : [0, R] \longrightarrow [0, T]$ is bijective.

Now, let us define the function $a : [0, T] \longrightarrow [0, R]$ as $a(t) := \psi^{-1}(t)$, satisfying $a(0) = \psi^{-1}(0) = R$ and $a(T) = \psi^{-1}(T) = 0$. We know that

$$a'(t) = \frac{1}{\psi'(a(t))} < 0 \ \forall t \in (0, T)$$

so $a(t)$ is strictly decreasing in $(0, T)$.

Let us denote, for all $x \in B_R^M(o)$,

$$\varphi(x) = \mathbb{E}_R^\omega(x) := E_R^\omega(r_o(x)) = \psi(r_o(x)).$$

We have that $\varphi(B_R^M(o)) = \psi([0, R]) = [0, T]$, so the function $\varphi : B_R^M(o) \to [0, T]$ satisfies $\|\nabla^M\varphi\| = |\frac{d}{dr}\psi|\|\nabla^M r_o\| \neq 0$ for all $x \in B_R^M(o) - \{o\}$. Therefore, the set of regular values of φ is $R_\varphi = (0, T)$.

On the other hand, and given $t \in [0, T]$, let us consider the sets

$$\begin{aligned} D(t) &= \left\{x \in \mathrm{B}_R^M \mid \varphi(x) \geq t\right\} = \left\{x \in \mathrm{B}_R^M \mid \mathbb{E}_R^\omega(x) \geq t\right\} \\ &= \left\{x \in \mathrm{B}_R^M \mid r_o(x) \leq \psi^{-1}(t)\right\} = \mathrm{B}_{a(t)}^M \end{aligned}$$

and

$$\Gamma(t) = \left\{x \in \mathrm{B}_R^M \mid \varphi(x) = t\right\} = \left\{x \in \mathrm{B}_R^M \mid \psi\,(r_o(x)) = t\right\} = \mathrm{S}_{a(t)}^M.$$

We have too that $D(0) = B_{a(0)}^M = B_R^M$ and $D(T) = B_{a(T)}^M = \{o\}$, where o is the center of the geodesic ball B_R^M.

We consider the symmetrization in M_ω^n of the sets $\mathrm{D}(t) = B_{a(t)}^M \subseteq \mathrm{B}_R^M \subseteq M$, namely, the geodesic balls $\mathrm{D}(t)^* = \mathrm{B}_{\widetilde{r}(t)}^\omega(o_\omega)$ in M_ω^n such that

$$\mathrm{Vol}\,(\mathrm{D}(t)) = \mathrm{Vol}\left(\mathrm{B}_{\widetilde{r}(t)}^\omega(o_\omega)\right).$$

For each $t \in [0, T]$, let us consider the function $\widetilde{r}(t)$, defined in Definition 2.17. Then, in this particular context, we have that $\widetilde{r} : [0, T] \longrightarrow [0, s(R)]$ is strictly decreasing and hence, bijective. In fact, note that if $t_1, t_2 \in [0, T]$ such that $t_1 < t_2$, then, as $a(t)$ is strictly decreasing, $a(t_1) > a(t_2)$, so

$$\mathrm{Vol}\left(\mathrm{B}_{\widetilde{r}(t_1)}^\omega(o_\omega)\right) = \mathrm{Vol}\left(\mathrm{B}_{a(t_1)}^M\right) > \mathrm{Vol}\left(\mathrm{B}_{a(t_2)}^M\right) = \mathrm{Vol}\left(\mathrm{B}_{\widetilde{r}(t_2)}^\omega(o_\omega)\right)$$

and hence $\widetilde{r}(t_1) > \widetilde{r}(t_2)$.

On the other hand, applying Lemma 2.19, we have that for all $t \in R_\varphi = (0, T)$,

$$\widetilde{r}\prime(t) = \frac{-1}{\mathrm{Vol}\left(\mathrm{S}_{\widetilde{r}(t)}^\omega\right)} \int_{\Gamma(t)} \left\|\nabla^M\varphi\right\|^{-1} d\sigma_t. \tag{4.19}$$

The inverse of $\widetilde{r}$ is the decreasing function

$$\phi \,:\, [0, s(R)] \longrightarrow [0, T]; \quad \phi := \phi(\widetilde{r}),$$

such that $\phi'(\widetilde{r}(t)) = \frac{1}{\widetilde{r}'(t)}$ for all $t \in [0, T]$, $\phi(0) = T$ and $\phi(s(R)) = 0$.

With all this background, we can say now that, in accordance with Definition 2.15 and Theorem 2.21, the symmetrization of $\varphi = \mathbb{E}^\omega_R \,:\, \mathrm{B}^M_R \longrightarrow \mathbb{R}$ is a radial function $\varphi^* = \mathbb{E}^{\omega^*}_R \,:\, \mathrm{B}^\omega_{s(R)} \longrightarrow \mathbb{R}$ which satisfies the following equality

$$\varphi^*(x^*) = \mathbb{E}^{\omega^*}_R(x^*) = \mathbb{E}^{\omega^*}_R\left(r_{o_\omega}(x^*)\right) = t_0 = \phi\left(\widetilde{r}(t_0)\right) = \phi\left(\widetilde{r}\right). \tag{4.20}$$

To see Eq. 4.20, we argue as follows: given $x^* \in B^M_R(o)^* = \mathrm{B}^\omega_{s(R)}(o_\omega) = \cup_{t\in[0,T]}\mathrm{S}^\omega_{\widetilde{r}(t)}(o_\omega)$, (concerning the second equality, recall that $\widetilde{r} \,:\, [0, T] \rightarrow [0, s(R)]$ is bijective), there exists some biggest value t_0 such that $r_{o_\omega}(x^*) = \widetilde{r}(t_0)$ and, hence, $x^* \in \mathrm{B}^\omega_{\widetilde{r}(t_0)} = \mathrm{D}(t_0)^*$. We then have that

$$\varphi^*(x^*) = \varphi^*(r_{o_\omega}(x^*)) = \sup\left\{t \geq 0 / x^* \in B^w_{\widetilde{r}(t)}(o)\right\} = t_0 = \phi\left(\widetilde{r}(t_0)\right) \tag{4.21}$$

and hence, for all $t \in (0, T)$, $\varphi^* \equiv \varphi^*(\widetilde{r}(t))$ and we have, applying Eq. 4.19:

$$\begin{aligned}\frac{d}{d\widetilde{r}}|_{\widetilde{r}=\widetilde{r}(t)}\varphi^*(\widetilde{r}) &= \varphi^{*\prime}(\widetilde{r}(t)) = \mathbb{E}^{\omega^*\prime}_R(\widetilde{r}(t)) = \phi'(\widetilde{r}(t)) \\ &= \frac{1}{\widetilde{r}'(t)} = -\frac{\mathrm{Vol}\left(\mathrm{S}^\omega_{\widetilde{r}(t)}\right)}{\int_{\Gamma(t)}\left\|\nabla^M\varphi\right\|^{-1}\,d\sigma_t}.\end{aligned} \tag{4.22}$$

But, as $\|\nabla^M\varphi(x)\| = |\psi'(r_o(x))| \neq 0$ for all $x \in B^M_R(o) - \{o\}$ and $\Gamma(t) = \mathrm{S}^M_{a(t)}$ for all $t \in R_\varphi = (0, T)$, we conclude that

$$\int_{\Gamma(t)}\left\|\nabla^M\varphi\right\|^{-1}\,d\sigma_t = \frac{1}{|\psi'(a(t))|}\mathrm{Vol}\left(\mathrm{S}^M_{a(t)}\right) \tag{4.23}$$

and hence, Eq. (4.22) becomes, using Eq. (4.23), and the fact that $\psi = E^\omega_R$, in the following expression, for all $t \in [0, T]$:

$$\begin{aligned}\varphi^{*\prime}(\widetilde{r}(t)) &= -\,|\psi'(a(t))|\,\frac{\mathrm{Vol}\left(\mathrm{S}^\omega_{\widetilde{r}(t)}\right)}{\mathrm{Vol}\left(\mathrm{S}^M_{a(t)}\right)} \\ &= -\frac{\mathrm{Vol}\left(\mathrm{B}^\omega_{a(t)}\right)}{\mathrm{Vol}\left(\mathrm{S}^\omega_{a(t)}\right)}\,\frac{\mathrm{Vol}\left(\mathrm{S}^\omega_{\widetilde{r}(t)}\right)}{\mathrm{Vol}\left(\mathrm{S}^M_{a(t)}\right)}.\end{aligned} \tag{4.24}$$

On the other hand, let us assume that $\mathrm{H}_{\mathrm{S}^\omega_r} \leq \mathrm{H}_{\mathrm{S}^M_r}$, for all $r \in (0, R]$. Then by Corollary 3.5 we know that $\mathrm{Vol}\left(\mathrm{B}^\omega_r\right) \leq \mathrm{Vol}\left(\mathrm{B}^M_r\right)$ for all $r \in [0, R]$. Therefore,

$$\mathrm{Vol}\left(\mathrm{B}^\omega_{\widetilde{r}(t)}\right) = \mathrm{Vol}\left(\mathrm{B}^M_{a(t)}\right) \geq \mathrm{Vol}\left(\mathrm{B}^\omega_{a(t)}\right), \quad \text{for all } t \in [0, T]. \tag{4.25}$$

Then, since $\mathrm{Vol}\left(\mathrm{B}^\omega_r\right)$ is an increasing function, because $\frac{d}{dr}\mathrm{Vol}\left(\mathrm{B}^\omega_r\right) = \mathrm{Vol}\left(\mathrm{S}^\omega_r\right) \geq 0$, we have that

$$\widetilde{r}(t) \geq a(t), \quad \text{for all } t \in [0, T]. \tag{4.26}$$

so, since M^n_ω is balanced from above, $q_{\omega'}(r) \geq 0$, we obtain:

$$\frac{\operatorname{Vol}\left(\mathrm{B}^\omega_{\widetilde{r}(t)}\right)}{\operatorname{Vol}\left(\mathrm{S}^\omega_{\widetilde{r}(t)}\right)} \geq \frac{\operatorname{Vol}\left(\mathrm{B}^\omega_{a(t)}\right)}{\operatorname{Vol}\left(\mathrm{S}^\omega_{a(t)}\right)}, \quad \text{for all } t \in [0, T]. \tag{4.27}$$

Therefore, using Eq. 4.24 and the fact that $\operatorname{Vol}\left(\mathrm{B}^\omega_{\widetilde{r}(t)}\right) = \operatorname{Vol}\left(\mathrm{B}^M_{a(t)}\right)$, we have

$$\mathbb{E}^{\omega *}_R{}'(\widetilde{r}(t)) = \varphi^*{}'(\widetilde{r}(t)) \geq -\frac{\operatorname{Vol}\left(\mathrm{B}^M_{a(t)}\right)}{\operatorname{Vol}\left(\mathrm{S}^M_{a(t)}\right)}, \quad \text{for all } t \in [0, T]. \tag{4.28}$$

Now, we apply Proposition 2.7, the isoperimetric inequality (3.8) of Corollary 3.5, the fact that $\widetilde{r}(t) \geq a(t)$, and that $q'_\omega \geq 0$, to obtain finally

$$\mathbb{E}^{\omega *}_R{}'(\widetilde{r}(t)) \geq -\frac{\operatorname{Vol}\left(\mathrm{B}^M_{a(t)}\right)}{\operatorname{Vol}\left(\mathrm{S}^M_{a(t)}\right)} \geq -\frac{\operatorname{Vol}\left(\mathrm{B}^\omega_{\widetilde{r}(t)}\right)}{\operatorname{Vol}\left(\mathrm{S}^\omega_{\widetilde{r}(t)}\right)} = E^{\omega'}_{s(R)}(\widetilde{r}(t)) \ \forall t \in (0, T). \tag{4.29}$$

Now, as $\mathbb{E}^{\omega *\prime}_R(\widetilde{r}) \geq E^{\omega'}_{s(R)}(\widetilde{r}) \ \forall \widetilde{r} \in (0, s(R))$, we have, integrating along $[0, s(R)]$, and taking into account that $\mathbb{E}^{\omega *}_R(s(R)) = E^\omega_{s(R)}(s(R)) = 0$,

$$\begin{aligned} -\mathbb{E}^{\omega *}_R(\widetilde{r}) = \int_{\widetilde{r}}^{s(R)} \mathbb{E}^{\omega *\prime}_R(u)du \geq \\ \int_{\widetilde{r}}^{s(R)} E^{\omega\prime}_{s(R)}(u)du = -E^\omega_{s(R)}(\widetilde{r}) \ \ \forall \widetilde{r} \in [0, s(R)] \end{aligned} \tag{4.30}$$

so

$$\mathbb{E}^{\omega *}_R(\widetilde{r}) \leq E^\omega_{s(R)}(\widetilde{r}) \ \ \forall \widetilde{r} \in [0, s(R)].$$

If we assume that $\mathrm{H}_{\mathrm{S}^\omega_r} \geq \mathrm{H}_{\mathrm{S}^M_r}$, for all $r \in [0, R]$, we use the same argument, changing all the inequalities, to obtain

$$\begin{aligned} \mathbb{E}^{\omega *\prime}_R(\widetilde{r}(t)) \leq E^{\omega\prime}_{s(R)}(\widetilde{r}(t)) \ \forall t \in (0, T) \text{ and hence} \\ \mathbb{E}^{\omega *}_R(\widetilde{r}) \geq E^\omega_{s(R)}(\widetilde{r}) \ \ \forall \widetilde{r} \in [0, s(R)]. \end{aligned} \tag{4.31}$$

We are going to study the case of equality, when we assume the hypothesis $\mathrm{H}_{\mathrm{S}^\omega_r} \leq \mathrm{H}_{\mathrm{S}^M_r}$, for all $r \in [0, R]$, (the discussion of equality if we assume $\mathrm{H}_{\mathrm{S}^\omega_r} \geq \mathrm{H}_{\mathrm{S}^M_r}$, for all $r \in [0, R]$ is the same, mutatis mutandi).

Equality $\mathbb{E}^{\omega *}_R(\widetilde{r}) = E^\omega_{s(R)}(\widetilde{r}) \ \forall r \in]0, s(R)]$ implies equality $\mathbb{E}^{\omega *\prime}_R(\widetilde{r}) = E^{\omega\prime}_{s(R)}(\widetilde{r}) \ \forall \widetilde{r} \in (0, s(R))$, which in its turn implies that inequalities in (4.29) and hence, in (4.28) and (4.27) become equalities for all $t \in [0, T]$. In particular, from equality in Eq. 4.27 and inequality (4.25), we deduce that

$$\frac{\operatorname{Vol}\left(\mathrm{S}^\omega_{a(t)}\right)}{\operatorname{Vol}\left(\mathrm{S}^\omega_{\widetilde{r}(t)}\right)} \leq 1, \quad \text{for all } t \in [0, T]. \tag{4.32}$$

On the other hand, using again equality in inequality (4.27) and taking into account that $\mathrm{H}_{\mathrm{S}_r^\omega} \leq \mathrm{H}_{\mathrm{S}_r^M}$ for all $r \in [0, R]$, and then using the isoperimetric inequality (3.8), we obtain:

$$\frac{\operatorname{Vol}\left(\mathrm{B}^\omega_{\widetilde{r}(t)}\right)}{\operatorname{Vol}\left(\mathrm{S}^\omega_{\widetilde{r}(t)}\right)} = \frac{\operatorname{Vol}\left(\mathrm{B}^\omega_{a(t)}\right)}{\operatorname{Vol}\left(\mathrm{S}^\omega_{a(t)}\right)} \geq \frac{\operatorname{Vol}\left(\mathrm{B}^M_{a(t)}\right)}{\operatorname{Vol}\left(\mathrm{S}^M_{a(t)}\right)}, \quad \text{for all } t \in [0, T]. \tag{4.33}$$

Using (4.32) and $\operatorname{Vol}\left(\mathrm{B}^\omega_{\widetilde{r}(t)}\right) = \operatorname{Vol}\left(\mathrm{B}^M_{a(t)}\right)$, we have

$$\operatorname{Vol}\left(\mathrm{S}^\omega_{a(t)}\right) \leq \operatorname{Vol}\left(\mathrm{S}^\omega_{\widetilde{r}(t)}\right) \leq \operatorname{Vol}\left(\mathrm{S}^M_{a(t)}\right). \tag{4.34}$$

Now, differentiating the equality

$$\operatorname{Vol}\left(\mathrm{B}^\omega_{\widetilde{r}(t)}\right) = \operatorname{Vol}\left(\mathrm{B}^M_{a(t)}\right), \quad \text{for all } t \in [0, T], \tag{4.35}$$

we obtain

$$\operatorname{Vol}\left(\mathrm{S}^\omega_{\widetilde{r}(t)}\right) \widetilde{r}'(t) = \operatorname{Vol}\left(\mathrm{S}^M_{a(t)}\right) a'(t), \quad \text{for all } t \in (0, T), \tag{4.36}$$

and hence, using inequality (4.34),

$$\frac{\widetilde{r}'(t)}{a'(t)} = \frac{\operatorname{Vol}\left(\mathrm{S}^M_{a(t)}\right)}{\operatorname{Vol}\left(\mathrm{S}^\omega_{\widetilde{r}(t)}\right)} \geq 1, \quad \text{for all } t \in (0, T), \tag{4.37}$$

so $\widetilde{r}'(t) \geq a'(t)\ \forall t \in (0, T)$, and therefore, as $\widetilde{r}(T) = a(T) = 0$, we finally obtain, integrating along $[0, T]$, that $\widetilde{r}(t) \leq a(t)\ \forall t \in [0, T]$. Hence, as we know, (see inequality (4.26)), that $\widetilde{r}(t) \geq a(t)\ \forall t \in [0, T]$, we obtain

$$\widetilde{r}(t) = a(t)\ \forall t \in [0, T].$$

Therefore, $s(R) = \widetilde{r}(0) = a(0) = R$ and, moreover, $\operatorname{Vol}\left(\mathrm{B}^\omega_{\widetilde{r}(t)}\right) = \operatorname{Vol}\left(\mathrm{B}^M_{\widetilde{r}(t)}\right)$ for all $t \in [0, T]$, so

$$\operatorname{Vol}\left(\mathrm{B}^\omega_r\right) = \operatorname{Vol}\left(\mathrm{B}^M_r\right)\ \forall r \in [0, R]$$

and hence

$$\operatorname{Vol}\left(\mathrm{S}^\omega_r\right) = \operatorname{Vol}\left(\mathrm{S}^M_r\right)\ \forall r \in [0, R].$$

Now, we apply the equality assertion in Corollary 3.5 to conclude that

$$\mathrm{H}_{\mathrm{S}_r^\omega} = \mathrm{H}_{\mathrm{S}_r^M}, \quad \text{for all } 0 \leq r \leq R$$

To prove equality assertions (1), (3) and (4), the argument follows as in the proofs of Theorem 4.4 and Corollary 4.5. Namely: to prove (1), and as we know that $\mathrm{H}_{\mathrm{S}_r^\omega} = \mathrm{H}_{\mathrm{S}_r^M}\ \forall r \in]0, R]$, let us apply Proposition 3.2, to have that $\bar{u}^\omega_{1,R} = u_{1,R}$ on $B^M_R(o)$, and we procceed by induction, as in the proof of assertion (3) in Theorem 4.4, to obtain the equalities $\bar{u}^\omega_{k,R} = u_{k,R}$ on B^M_R and for all $k \in \mathbb{N}$. From these equalities we also have the equalities $\bar{u}^\omega_{k,r} = u_{k,r}$ on B^M_r, for all $r \in]0, R]$ and for all $k \in \mathbb{N}$. To prove (3), as we have that $\operatorname{Vol}(B^M_R) = \operatorname{Vol}(B^\omega_R)$ and $\operatorname{Vol}(S^M_R) = \operatorname{Vol}(S^\omega_R)$, we proceed as in the proof of Corollary 4.5, (see Eq. 4.13), to conclude that, for all $k \geq 1$,

$$\mathcal{A}_k\left(\mathrm{B}^M_R\right) = \frac{\mathcal{A}_k\left(\mathrm{B}^\omega_R\right)}{\operatorname{Vol}\left(\mathrm{S}^\omega_R\right)} \operatorname{Vol}\left(\mathrm{S}^M_R\right) = \mathcal{A}_k\left(\mathrm{B}^\omega_R\right).$$

Finally, to prove (4) the argument is the same than in the proof of Corollary 4.5. □

A first consequence of Proposition 4.6, as we previously announced in Remark 3.4 in Section 3, is the following two-sided Riemannian version of Talenti's comparison theorem, (see [39], [40]), restricted to the mean exit time function defined on geodesic balls in Riemannian manifolds satisfying our assumptions:

Corollary 4.7 *Let (M^n, g) be a complete Riemannian manifold and let (M^n_ω, g_ω) be a rotationally symmetric model space with center $o_\omega \in M^n_\omega$, balanced from above. Let $o \in M$ be a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a metric ball $B^M_R(o)$, with $R < inj(o) \leq inj(o_w)$. Let us suppose moreover that the mean curvatures of the geodesic spheres in M and M_ω satisfies*

$$\mathrm{H}_{\mathrm{S}^\omega_r} \leq (\geq)\, \mathrm{H}_{\mathrm{S}^M_r} \quad \text{for all} \quad 0 < r \leq R. \tag{4.38}$$

Then

$$E^{M*}_R(r) \leq (\geq)\, E^\omega_{s(R)}(r) \quad \text{for all} \;\; r \in [0, s(R)]. \tag{4.39}$$

Equality in inequality (4.39) implies the equality among the radius $s(R) = R$ and the equality

$$\mathrm{H}_{\mathrm{S}^\omega_r} = \mathrm{H}_{\mathrm{S}^M_r}, \;\; \text{for all}\; 0 < r \leq R$$

and hence, we have the equalities

(1) *Equality $\bar{u}^\omega_{k,R} = u_{k,R}$ on $B^M_R(o)$ for all $k \geq 1$, and hence, equality $\bar{u}^\omega_{k,r} = u_{k,r}$ on $B^M_r(o)$ for all $k \geq 1$ and for all $0 < r \leq R$.*
(2) *Equalities $Vol\left(\mathrm{B}^\omega_r\right) = Vol\left(\mathrm{B}^M_r\right)$ and $Vol\left(\mathrm{S}^\omega_r\right) = Vol\left(\mathrm{S}^M_r\right)$ for all $0 < r \leq R$.*
(3) *Equalities $\mathcal{A}_k\left(\mathrm{B}^\omega_r\right) = \mathcal{A}_k\left(\mathrm{B}^M_r\right)$, for all $k \geq 1$ and for all $0 < r \leq R$.*
(4) *Equalities $\lambda_1(B^\omega_r) = \lambda_1(B^M_r)$ for all $0 < r \leq R$.*

Proof Applying Theorem 3.3 and the properties of the symmetrized functions, (see [30] and assertion (3) of Remark 2.16) we have the inequality

$$\mathbb{E}^{\omega*}_R \geq (\leq)\, E^{M*}_R \quad \text{in} \quad B^w_{s(R)}(o_w)\,,$$

where $B^w_{s(R)}(o_w)$ is the Schwarz symmetrization of B^M_R, namely, $\mathrm{Vol}(B^w_{s(R)}(o_w)) = \mathrm{Vol}(B^M_R)$.
Now, from Proposition 4.6, we have that

$$\mathbb{E}^{\omega^*}_R(r) \leq (\geq)\, E^\omega_{s(R)}(r) \quad \text{for all} \;\; r \in [0, s(R)].$$

and the result is done.

Equality in inequality (4.39) implies equality in inequality (4.18) of Proposition 4.6 and we conclude the list of equality assertions as in Proposition 4.6. □

Also as a consequence of the Proposition 4.6 we have the following result, where it is proved that, under our hypotheses, the torsional rigidity of the geodesic balls determines its first Dirichlet eigenvalue:

Theorem 4.8 *Let (M^n, g) be a complete Riemannian manifold and let (M^n_ω, g_ω) be a rotationally symmetric model space with center $o_\omega \in M^n_\omega$, balanced from above. Let $o \in M$ be a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a metric ball*

$B^M_R(o)$, with $R < inj(o) \leq inj(o_w)$. Let us suppose moreover that the mean curvatures of the geodesic spheres in M and M_ω satisfies

$$\mathrm{H}_{\mathrm{S}^\omega_r} \leq (\geq)\, \mathrm{H}_{\mathrm{S}^M_r} \quad \textit{for all} \quad 0 < r \leq R. \tag{4.40}$$

Then

$$\mathcal{A}_1\left(\mathrm{B}^\omega_{s(R)}\right) \geq (\leq)\, \mathcal{A}_1\left(\mathrm{B}^M_R\right) \tag{4.41}$$

where $\mathrm{B}^\omega_{s(R)}$ is the Schwarz symmetrization of B^M_R in the model space (M^n_ω, g_ω).

Equality in any of inequalities (4.41) implies the equality among the radius $s(R) = R$ and that

$$\mathrm{H}_{\mathrm{S}^\omega_r(o_\omega)} = \mathrm{H}_{\mathrm{S}^M_r(o)} \textit{ for all } 0 < r \leq R$$

and hence, we have the equalities

(1) *Equality $\bar{u}^\omega_{k,R} = u_{k,R}$ on $B^M_R(o)$ for all $k \geq 1$, and hence, equality $\bar{u}^\omega_{k,r} = u_{k,r}$ on $B^M_r(o)$ for all $k \geq 1$ and for all $0 < r \leq R$.*
(2) *Equalities $Vol\left(\mathrm{B}^\omega_r(o_\omega)\right) = Vol\left(\mathrm{B}^M_r(o)\right)$ and $Vol\left(\mathrm{S}^\omega_r(o_\omega)\right) = Vol\left(\mathrm{S}^M_r(o)\right)$ for all $0 < r \leq R$.*
(3) *Equalities $\mathcal{A}_k\left(\mathrm{B}^\omega_r(o_\omega)\right) = \mathcal{A}_k\left(\mathrm{B}^M_r(o)\right)$, for all $k \geq 1$ and for all $0 < r \leq R$.*
(4) *Equality $\lambda_1(B^w_r(o_\omega)) = \lambda_1(B^M_r(o))$ for all $0 < r \leq R$.*

Namely, the torsional rigidity determines the Poisson hierarchy, the volume, the L^1-moment spectrum and the first Dirichlet eigenvalue of the ball $B^M_r(o)$ for all $0 < r \leq R$.

Proof Let us consider a symmetric model space rearrangement of the metric ball B^M_R as it has been described in Definition 2.11 and Definition 2.15, namely, a symmetrization of B^M_R which is a geodesic $s(R)$-ball in the model space M^n_ω such that $\mathrm{Vol}\left(\mathrm{B}^M_R\right) = \mathrm{Vol}\left(\mathrm{B}^\omega_{s(R)}\right)$, together with the symmetrization $\mathbb{E}^{\omega^*}_R : \mathrm{B}^\omega_{s(R)} \longrightarrow \mathbb{R}$ of the transplanted mean exit time function $\mathbb{E}^\omega_R : \mathrm{B}^M_R \longrightarrow \mathbb{R}$.

Applying Theorems 3.3 and 3.7 and Proposition 4.6, we have that

$$\begin{aligned}\mathcal{A}_1\left(\mathrm{B}^M_R\right) &= \int_{\mathrm{B}^M_R} E^M_R\, d\sigma \leq (\geq) \int_{\mathrm{B}^M_R} \mathbb{E}^\omega_R\, d\sigma \\ &= \int_{\mathrm{B}^\omega_{s(R)}} \mathbb{E}^{\omega^*}_R\, d\widetilde{\sigma} \leq (\geq) \int_{\mathrm{B}^\omega_{s(R)}} E^\omega_{s(R)}\, d\widetilde{\sigma} = \mathcal{A}_1\left(\mathrm{B}^\omega_{s(R)}\right)\end{aligned} \tag{4.42}$$

and Eq. 4.41 is proved.

We are going to study the case of equality, when we assume the hypothesis $\mathrm{H}_{\mathrm{S}^\omega_r} \leq \mathrm{H}_{\mathrm{S}^M_r}$, for all $r \in [0, R]$, (the discussion of equality if we assume $\mathrm{H}_{\mathrm{S}^\omega_r} \geq \mathrm{H}_{\mathrm{S}^M_r}$, for all $r \in [0, R]$ is the same, mutatis mutandi). Equality in (4.42) implies that all the inequalities contained in this expression become equalities. In particular, we have that $\int_{\mathrm{B}^M_R} E^M_R\, d\sigma = \int_{\mathrm{B}^M_R} \mathbb{E}^\omega_R\, d\sigma$ and that $\int_{\mathrm{B}^\omega_{s(R)}} \mathbb{E}^{\omega^*}_R = \int_{\mathrm{B}^\omega_{s(R)}} E^\omega_{s(R)}\, d\widetilde{\sigma}$.

From this second equality and inequality (4.18) in Proposition 4.6, we have that $\mathbb{E}^{\omega^*}_R = E^\omega_{s(R)}$ on $[0, s(R)]$. Applying again Proposition 4.6, we deduce that $s(R) = R$, and that $\mathrm{H}_{\mathrm{S}^\omega_r} = \mathrm{H}_{\mathrm{S}^M_r}$ for all $0 < r \leq R$.

On the other hand, equality $\int_{B_R^M} E_R^M \, d\sigma = \int_{B_R^M} \mathbb{E}_R^\omega \, d\sigma$ implies, using Theorem 3.3, that $E_R^M = \mathbb{E}_R^\omega$ on B_R^M. Hence we conclude equality $\bar{u}_{k,R}^\omega = u_{k,R}$ on $B_R^M(o)$ for all $k \geq 1$ using assertion (3) in Theorem 4.4 and that $\bar{u}_{k,r}^\omega = u_{k,r}$ on $B_r^M(o)$ for all $k \geq 1$ and for all $r \in [0, R]$ using assertion (4) in Theorem 4.4.

Moreover, equality $E_R^M = \mathbb{E}_R^\omega$ on B_R^M implies, using equality conclusions in Corollary 3.6, that, for all $r \in]0, R]$,

$$\begin{aligned} \frac{\operatorname{Vol}\left(B_r^\omega(o_\omega)\right)}{\operatorname{Vol}\left(S_r^\omega(o_\omega)\right)} &= \frac{\operatorname{Vol}\left(B_r^M(o)\right)}{\operatorname{Vol}\left(S_r^M(o)\right)}, \\ \operatorname{Vol}\left(B_r^\omega\right) &= \operatorname{Vol}\left(B_r^M\right), \\ \operatorname{Vol}(S_r^\omega) &= \operatorname{Vol}(S_r^M). \end{aligned} \tag{4.43}$$

Hence, as we are assuming that $\mathcal{A}_1\left(B_R^M\right) = \mathcal{A}_1\left(B_{s(R)}^\omega\right)$ and we have deduced $s(R) = R$, then we obtain the equality

$$\frac{\mathcal{A}_1\left(B_R^\omega\right)}{\operatorname{Vol}\left(S_R^\omega\right)} = \frac{\mathcal{A}_1\left(B_R^M\right)}{\operatorname{Vol}\left(S_R^M\right)}$$

and hence, by Corollary 4.5, $\mathcal{A}_k\left(B_r^M\right) = \mathcal{A}_k\left(B_r^\omega\right)$ for all $k \geq 1$ and for all $0 < r \leq R$.

Finally, as equality $\mathcal{A}_1\left(B_R^M\right) = \mathcal{A}_1\left(B_{s(R)}^\omega\right)$ implies equalities $\mathcal{A}_k\left(B_r^\omega\right) = \mathcal{A}_k\left(B_r^M\right)$, for all $k \geq 1$ and for all $r \in]0, R]$, we have that, given $B_r^M \subseteq M$ in a Riemannian manifold (M, g) with $r \in]0, R]$, (see [24] and [7]):

$$\begin{aligned} \lambda_1(B_r^M) &= \lim_{k \to \infty} \frac{k\mathcal{A}_{k-1}\left(B_r^M\right)}{\mathcal{A}_k\left(B_r^M\right)} \\ &= \lim_{k \to \infty} \frac{k\mathcal{A}_{k-1}\left(B_r^\omega\right)}{\mathcal{A}_k\left(B_r^\omega\right)} = \lambda_1(B_r^\omega). \end{aligned} \tag{4.44}$$

□

Corollary 4.9 *Let (M^n, g) be a complete Riemannian manifold and let (M_ω^n, g_ω) be a rotationally symmetric model space with center $o_\omega \in M_\omega^n$, balanced from above. Let $o \in M$ be a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a metric ball $B_R^M(o)$, with $R < inj(o) \leq inj(o_w)$. Let us suppose moreover that the mean curvatures of the geodesic spheres in M and M_ω satisfies*

$$H_{S_r^\omega} \leq (\geq) H_{S_r^M} \quad \textit{for all} \quad 0 < r \leq R. \tag{4.45}$$

Then

$$\mathcal{A}_1\left(B_R^M\right) \leq E_{s(R)}^\omega(0) Vol(B_R^M). \tag{4.46}$$

Proof Assuming that $H_{S_r^\omega} \leq H_{S_r^M}$ for all $0 < r \leq R$, we use Eq. 4.42 to obtain, having into account that $E_{s(R)}^\omega(r) \leq E_{s(R)}^\omega(0) \, \forall r \in]0, s(R)]$,

$$\begin{aligned} \mathcal{A}_1\left(\mathrm{B}_R^M\right) &= \int_{\mathrm{B}_R^M} E_R^M \, d\sigma \;\leq\; \int_{\mathrm{B}_R^M} \mathbb{E}_R^\omega \, d\sigma \\ &= \int_{\mathrm{B}_{s(R)}^\omega} \mathbb{E}_R^{\omega^*} d\widetilde{\sigma} \leq \int_{\mathrm{B}_{s(R)}^\omega} E_{s(R)}^\omega \, d\widetilde{\sigma} \leq E_{s(R)}^\omega(0)\mathrm{Vol}(\mathrm{B}_R^M). \end{aligned} \tag{4.47}$$

□

Remark 4.10 As (M_ω^n, g_ω) is balanced from above, then $\frac{d}{dr}\left(q_\omega(r)\right) \geq 0$, so $q_\omega(r)$ is non-decreasing with r. Then, as $E_{s(R)}^\omega(r(x)) = \psi(r(x)) = \int_{r(x)}^{s(R)} q_\omega(t)\, dt$, we have that

$$E_{s(R)}^\omega(0) = \int_0^{s(R)} q_\omega(t)\, dt \leq s(R) q_\omega(s(R)) = s(R) \frac{\mathrm{Vol}\left(\mathrm{B}_{s(R)}^\omega(o_\omega)\right)}{\mathrm{Vol}\left(\mathrm{S}_{s(R)}^\omega(o_\omega)\right)}$$

so

$$\mathcal{A}_1\left(\mathrm{B}_R^M\right) \leq \mathbb{E}_{s(R)}^\omega(0)\mathrm{Vol}(\mathrm{B}_R^M) \leq s(R) \frac{\mathrm{Vol}\left(\mathrm{B}_{s(R)}^\omega(o_\omega)\right)}{\mathrm{Vol}\left(\mathrm{S}_{s(R)}^\omega(o_\omega)\right)} \mathrm{Vol}(\mathrm{B}_R^M).$$

5 Cheng-type First Dirichlet Eigenvalue Comparison and the Determination of the Moment Spectrum of a Geodesic Ball

Finally, as a corollary of the previous results, we have in Theorem 5.1 a Cheng-type first Dirichlet eigenvalue comparison, (see [8]). On the other hand, in Corollary 5.2, we have been able to show that, under our hypotheses, the first Dirichlet eigenvalue of geodesic balls determines its exit time moment spectrum and its Poisson hierarchy.

Theorem 5.1 *Let (M^n, g) be a complete Riemannian manifold and let (M_ω^n, g_ω) be a rotationally symmetric model space with center $o_\omega \in M_\omega^n$. Let $o \in M$ be a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a metric ball $B_R^M(o)$, with $R < inj(o) \leq inj(o_w)$. Let us suppose moreover that the mean curvatures of the geodesic spheres in M and M_ω satisfies*

$$\mathrm{H}_{\mathrm{S}_r^\omega} \leq (\geq)\, \mathrm{H}_{\mathrm{S}_r^M} \quad \text{for all} \quad 0 < r \leq R. \tag{5.1}$$

Then we have the inequalities

$$\lambda_1(B_R^\omega) \leq (\geq)\, \lambda_1(B_R^M) \tag{5.2}$$

where B_R^ω is the geodesic ball in M_ω^m.

Equality in any of these inequalities implies that

$$\mathrm{H}_{\mathrm{S}_r^\omega} = \mathrm{H}_{\mathrm{S}_r^M} \text{ for all } 0 < r \leq R$$

and hence, we have the equalities

(1) *Equality $\bar{u}_{k,R}^\omega = u_{k,R}$ on $B_R^M(o)$ for all $k \geq 1$, and hence, equality $\bar{u}_{k,r}^\omega = u_{k,r}$ on $B_r^M(o)$ for all $k \geq 1$ and for all $0 < r \leq R$.*
(2) *Equalities Vol $\left(\mathrm{B}_r^\omega\right) =$ Vol $\left(\mathrm{B}_r^M\right)$ and Vol $\left(\mathrm{S}_r^\omega\right) =$ Vol $\left(\mathrm{S}_r^M\right)$ for all $0 < r \leq R$.*
(3) *Equalities $\mathcal{A}_k\left(\mathrm{B}_r^\omega\right) = \mathcal{A}_k\left(\mathrm{B}_r^M\right)$, for all $k \geq 1$ and for all $0 < r \leq R$.*

Namely, the first Dirichlet eigenvalue determines the Poisson hierarchy, the volume, and the L^1*-moment spectrum of the ball* $B_r^M(o)$ *for all* $0 < r \leq R$.

Proof The proof follows the lines of the proof of Theorems 6 and 7 in [24]. This technique is based in the the description of the first Dirichlet eigenvalue of a smooth precompact domain D in a Riemannian manifold given by P. McDonald and R. Meyers in [32].

When $D = B_R^M$, we have

$$\lambda_1(B_R^M) = \sup\left\{\eta \geq 0 \,:\, \lim_{k\to\infty}\sup\left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k(B_R^M)}{\Gamma(k+1)} < \infty\right\} \quad . \tag{5.3}$$

Let us assume first that $\mathrm{H}_{S_r^\omega} \leq \mathrm{H}_{S_r^M}$, for all $0 < r \leq R$. Then, we have, by Corollary 4.5, that

$$\frac{\mathcal{A}_k(B_R^M)}{\mathrm{Vol}(S_R^M)} \leq \frac{\mathcal{A}_k(B_R^\omega)}{\mathrm{Vol}(S_R^\omega)} \quad \text{for all } k \in \mathbb{N}. \tag{5.4}$$

On the other hand, by Corollary 3.5:

$$\frac{\mathrm{Vol}(S_R^M)}{\mathrm{Vol}(S_R^\omega)} \geq \frac{\mathrm{Vol}(B_R^M)}{\mathrm{Vol}(B_R^\omega)} \geq 1. \tag{5.5}$$

Then, using inequality (5.4) the set

$$\mathcal{F}_2 := \left\{\eta \geq 0 \,:\, \lim_{k\to\infty}\sup\left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k(B_R^\omega)}{\Gamma(k+1)}\frac{\mathrm{Vol}(S_R^M)}{\mathrm{Vol}(S_R^\omega)} < \infty\right\}$$

is included in the set

$$\mathcal{F}_1 := \left\{\eta \geq 0 \,:\, \lim_{k\to\infty}\sup\left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k(B_R^M)}{\Gamma(k+1)} < \infty\right\} \quad ,$$

so we have, using this last observation and inequality (5.5),

$$\begin{aligned}
\lambda_1(B_R^M) &= \sup\left\{\eta \geq 0 \,:\, \lim_{k\to\infty}\sup\left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k(B_R^M)}{\Gamma(k+1)} < \infty\right\}\\
&\geq \sup\left\{\eta \geq 0 \,:\, \lim_{k\to\infty}\sup\left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k(B_R^\omega)}{\Gamma(k+1)}\frac{\mathrm{Vol}(S_R^M)}{\mathrm{Vol}(S_R^\omega)} < \infty\right\}\\
&= \frac{\mathrm{Vol}(S_R^M)}{\mathrm{Vol}(S_R^\omega)}\sup\left\{\eta \geq 0 \,:\, \lim_{k\to\infty}\sup\left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k(B_R^\omega)}{\Gamma(k+1)} < \infty\right\}\\
&= \frac{\mathrm{Vol}(S_R^M)}{\mathrm{Vol}(S_R^\omega)}\lambda_1(B_R^\omega) \geq \lambda_1(B_R^\omega).
\end{aligned} \tag{5.6}$$

If we assume $\mathrm{H}_{S_r^\omega} \geq \mathrm{H}_{S_r^M}$, for all $0 < r \leq R$, then we obtain $\lambda_1(B_R^\omega) \geq \lambda_1(B_R^M)$ with the same argument, inverting all the inequalities.

Finally, equality $\lambda_1(B_R^\omega) = \lambda_1(B_R^M)$ implies that all the inequalities in (5.6) are equalities, so we have the equality in the inequality (5.5), (namely, the equality in the isoperimetric inequality (3.8) in Corollary 3.5), and moreover the equality between the volumes $\mathrm{Vol}(B^\omega_R) = \mathrm{Vol}(B^M_R)$ and $\mathrm{Vol}\left(\mathrm{S}^\omega_R\right) = \mathrm{Vol}\left(\mathrm{S}^M_R\right)$. Hence, we have, by Corollary 3.5, the equalities

$$\mathrm{H}_{\mathrm{S}^\omega_r} = \mathrm{H}_{\mathrm{S}^M_r} \text{ for all } 0 < r \leq R$$

and, in its turn, equalities $\mathrm{Vol}(B^\omega_r) = \mathrm{Vol}(B^M_r)$ and $\mathrm{Vol}\left(\mathrm{S}^\omega_r\right) = \mathrm{Vol}\left(\mathrm{S}^M_r\right)$ $\forall r \in]0, R]$. Assertions (1) and (3) follows from Proposition 3.2 and Theorem 4.4. □

We finish the paper with a consequence of Theorems 5.1 and 4.4 which summarizes the relation between the first Dirichlet eigenvalue, the L^1-moment spectrum and the Poisson hierarchy of the geodesic balls $B^M_R(o)$ of a Riemannian manifold which satisfies our restriction on the mean curvatures of the geodesic spheres included in it, $S^M_r(o)$, $r \leq R$.

Corollary 5.2 *Let (M^n, g) be a complete Riemannian manifold and let (M^n_ω, g_ω) be a rotationally symmetric model space with center $o_\omega \in M^n_\omega$. Let $o \in M$ be a point in M and let us suppose that $inj(o) \leq inj(o_\omega)$. Let us consider a metric ball $B^M_R(o)$, with $R < inj(o) \leq inj(o_w)$. Let us suppose moreover that the mean curvatures of the geodesic spheres in M and M_ω satisfies*

$$\mathrm{H}_{\mathrm{S}^\omega_r} \leq (\geq)\, \mathrm{H}_{\mathrm{S}^M_r} \quad \textit{for all} \quad 0 < r \leq R. \tag{5.7}$$

Then, the following equalities are equivalent:

(1) $\lambda_1(B^\omega_R) = \lambda_1(B^M_R)$.
(2) $\mathcal{A}_k(B^\omega_R) = \mathcal{A}_k(B^M_R)$ $\forall k \geq 1$.
(3) $\bar{u}^w_{k,R} = u_{k,R}$ $\forall k \geq 1$ *in* B^M_R.

Moreover, equality $\mathrm{H}_{\mathrm{S}^\omega_r} = \mathrm{H}_{\mathrm{S}^M_r}$ for all $0 < r \leq R$ implies any, (and hence, all), of the equalities (1), (2) and (3).

Proof We are going to prove these equivalences. We first assume that

$$\mathrm{H}_{\mathrm{S}^\omega_r} \leq \mathrm{H}_{\mathrm{S}^M_r} \quad \text{for all} \quad 0 < r \leq R.$$

We see first that equality (1) implies equalities (3), namely, that the first Dirichlet eigenvalue of the geodesic ball B^M_R determines its Poisson hierarchy. To do that, we start with the last observation in Theorem 5.1, namely, that equality $\lambda_1(B^\omega_R) = \lambda_1(B^M_R)$ implies that all the inequalities in (5.6) are equalities, so we have the equality in the inequality (5.5), (namely, the equality in the isoperimetric inequality (3.8) in Corollary 3.5), and moreover the equality between the volumes $\mathrm{Vol}(B^\omega_r) = \mathrm{Vol}(B^M_r)$ and $\mathrm{Vol}\left(\mathrm{S}^\omega_r\right) = \mathrm{Vol}\left(\mathrm{S}^M_r\right)$ for all $r \in [0, R]$. Hence, we have, by Corollary 3.5, the equalities

$$\mathrm{H}_{\mathrm{S}^\omega_r} = \mathrm{H}_{\mathrm{S}^M_r} \text{ for all } 0 < r \leq R.$$

Then, by Proposition 3.2, we have that $\bar{u}^w_{1,R} = u_{1,R}$ on B^M_R. Hence we conclude equality $\bar{u}^\omega_{k,R} = u_{k,R}$ on $B^M_R(o)$ for all $k \geq 1$ using assertion (3) in Theorem 4.4. We have concluded that (1) implies (3).

To see that equality (1) implies equalities (2), we compute now as in Corollary 4.5: for all $k \geq 1$, we have that, as $\mathrm{Vol}\left(\mathrm{S}^\omega_r\right) = \mathrm{Vol}\left(\mathrm{S}^M_r\right)$ for all $r \in [0, R]$,

$$\begin{aligned}\mathcal{A}_k\left(\mathrm{B}^M_R\right) &= \int_{\mathrm{B}^M_R} u_{k,R}\, d\sigma = -\frac{1}{k+1}\int_{\mathrm{B}^M_R} \Delta^M u_{k+1,R} d\sigma\\ &= -\frac{1}{k+1}\int_{\mathrm{B}^M_R} \Delta^M \bar{u}^{\omega}_{k+1,R}\, d\sigma = -\frac{1}{k+1}\,\bar{u}^{\omega\prime}_{k+1,R}(R)\mathrm{Vol}\left(\mathrm{S}^M_R\right)\\ &= \frac{\mathcal{A}_k\left(\mathrm{B}^{\omega}_R\right)}{\mathrm{Vol}\left(\mathrm{S}^{\omega}_R\right)}\mathrm{Vol}\left(\mathrm{S}^M_R\right) = \mathcal{A}_k\left(\mathrm{B}^{\omega}_R\right)\end{aligned} \tag{5.8}$$

and hence we have equalities (2).

To see that equalities (2) implies equality (1), i.e., that the exit time moment spectrum of B^M_R determines its first Dirichlet eigenvalue, we compute, using Theorem A in [24] and as $\mathcal{A}_k\left(\mathrm{B}^M_R\right) = \mathcal{A}_k\left(\mathrm{B}^{\omega}_R\right)\ \forall k \geq 1$:

$$\begin{aligned}\lambda_1(B^M_R) &= \lim_{k\to\infty}\frac{k\mathcal{A}_{k-1}\left(\mathrm{B}^M_R\right)}{\mathcal{A}_k\left(\mathrm{B}^M_R\right)}\\ &= \lim_{k\to\infty}\frac{k\mathcal{A}_{k-1}\left(\mathrm{B}^{\omega}_R\right)}{\mathcal{A}_k\left(\mathrm{B}^{\omega}_R\right)} = \lambda_1(B^{\omega}_R).\end{aligned} \tag{5.9}$$

To see that equalities (3) implies equality (1), namely, that the Poisson hierarchy of the ball B^M_R determines its first Dirichlet eigenvalue, we will see first that equalities (3) implies equalities (2). Assuming that (3) is satisfied, we have that $\bar{u}^{\omega}_{k,R} = u_{k,R}$ on $B^M_R(o)$ for all $k \geq 1$. In particular, $\bar{u}^{\omega}_{1,R} = u_{1,R}$ on $B^M_R(o)$, so, by Corollary 3.6, $\mathrm{Vol}\left(\mathrm{S}^{\omega}_r\right) = \mathrm{Vol}\left(\mathrm{S}^M_r\right)$ and $\mathrm{Vol}\left(\mathrm{B}^{\omega}_r\right) = \mathrm{Vol}\left(\mathrm{B}^M_r\right)$ for all $r \in]0, R]$ and, hence, given $r \in]0, R]$, and for all $k \geq 1$,

$$\begin{aligned}\mathcal{A}_k\left(\mathrm{B}^M_R\right) &= \int_{\mathrm{B}^M_R} u_{k,R}\, d\sigma = -\frac{1}{k+1}\int_{\mathrm{B}^M_R} \Delta^M u_{k+1,R} d\sigma\\ &= -\frac{1}{k+1}\int_{\mathrm{B}^M_R} \Delta^M \bar{u}^{\omega}_{k+1,R}\, d\sigma = -\frac{1}{k+1}\,\bar{u}^{\omega\prime}_{k+1,R}(R)\mathrm{Vol}\left(\mathrm{S}^M_R\right)\\ &= \frac{\mathcal{A}_k\left(\mathrm{B}^{\omega}_R\right)}{\mathrm{Vol}\left(\mathrm{S}^{\omega}_R\right)}\mathrm{Vol}\left(\mathrm{S}^M_R\right) = \mathcal{A}_k\left(\mathrm{B}^{\omega}_R\right)\end{aligned}$$

so we have inequalities (2). Now, we use equation (5.9) to obtain (1). □

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