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Homomorphic encoders of profinite abelian groups I ☆

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ABSTRACT

Let $\{G_i \,|\, i \in \mathbb{N}\}$ be a family of finite Abelian groups. We say that a subgroup $G \leq \prod_{i\in\mathbb{N}} G_i$ is *order controllable* if for every $i \in \mathbb{N}$ there is $n_i \in \mathbb{N}$ such that for each $c \in G$, there exists $a \in G$ satisfying that $a_{|[1,i]} = c_{|[1,i]}$, $supp(a) \subseteq [1, n_i]$, and order$(a)$ divides order$(c_{|[1,n_i]})$. In this paper we investigate the structure of order controllable subgroups. It is proved that every order controllable, profinite, abelian group contains a subset $\{g_n \,|\, n \in \mathbb{N}\}$ that topologically generates the group and whose elements g_n all have finite support. As a consequence, sufficient conditions are obtained that allow us to encode, by means of a topological group isomorphism, order controllable profinite abelian groups. Further applications of these results to group codes will appear subsequently.

1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ respectively denote the group of integers and the semigroup of natural numbers. Suppose that $\mathbb{Z}$ is given the discrete topology and $\mathbb{Z}^{\mathbb{N}}$ the corresponding product topology. Nunke proved in [10] that every infinite, closed, subgroup G of $\mathbb{Z}^{\mathbb{N}}$ is topologically isomorphic to a product of infinite cyclic groups, i.e., the group G contains a subset $\{g_n \,|\, n \in \mathbb{N}\}$ such that $G \cong \prod_{n\in\mathbb{N}} \langle g_n \rangle$. Furthermore, one can prove that the elements g_n can be selected with finite support if and only if $G \cap \mathbb{Z}^{(\mathbb{N})}$ is dense in G (here $\mathbb{Z}^{(\mathbb{N})}$ denotes the direct sum, that is, the subgroup of the product consisting of all elements with finite support). In this case, we say that $\{g_n \,|\, n \in \mathbb{N}\}$ is a *generating set* that encodes the group G.

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The main goal of this paper is to study the existence of generating sets, in a profinite abelian group G, whose elements have finite support. We prove that for order controllable groups it is always possible to find a generating set whose elements all have finite support. As a consequence, we obtain a topological group isomorphism that encodes an order controllable closed subgroup of a product of finite abelian group and describes how the subgroup is placed within the product of finite groups where it appears. Further applications of these results to group codes appear in [4,6].

Let $\{G_i \mid i \in I\}$ be a family of topological groups. As usual, its *direct product* $\prod_{i\in I} G_i$ is the set of all functions $g : I \to \bigcup\{G_i \mid i \in I\}$ such that $g(i) \in G_i$ for every $i \in I$. The group operation on $\prod_{i\in I} G_i$ is defined coordinate-wise: the product $gh \in \prod_{i\in I} G_i$ of g and h in $\prod_{i\in I} G_i$ is the function defined by $gh(i) = g(i)h(i)$ for each $i \in I$. Clearly, the identity element e of $\prod_{i\in I} G_i$ is the function that assigns the identity element e_i of G_i to every $i \in I$. We equip the product with the canonical product topology. The subgroup

$$\bigoplus_{i\in I} G_i = \{g \in \prod_{i\in I} G_i \mid g(i) = e_i \text{ for all but finitely many } i \in I\}$$

is called the *direct sum* of the family $\{G_i \mid i \in I\}$. The *support* of an element $x \in \prod_{i\in I} G_i$ is the set

$$\mathrm{supp}(x) := \{i \in I \mid x_i \neq e_i\}.$$

Given a subgroup $G \leq \prod\limits_{i\in I} G_i$ and a subset $J \subseteq I$, we denote by $G_J := \{c \in G \mid \mathrm{supp}(c) \subseteq J\}$ and $G_{|J} := \pi_J(G)$, where $\pi_J : \prod\limits_{i\in I} G_i \to \prod\limits_{i\in J} G_i$ is the canonical projection.

If S is a subset of a group G, then we denote by $\langle S\rangle$ the subgroup generated by S, that is, the smallest subgroup of G containing every element of S. However, the symbol $\langle g\rangle$ will denote the cyclic subgroup generated by $\{g\}$, $g \in G$. Since most results here concern abelian groups, we will use additive notation from here on. In particular, we will denote the identity element by 0.

The following two group-theoretic notions originate from coding theory.

Definition 1.1. A subgroup $G \leq \prod\limits_{i\in I} G_i$ is called

(1) *weakly controllable* if $G \cap \bigoplus\limits_{i\in I} G_i$ is dense in G.
(2) *weakly observable* if $G \cap \bigoplus\limits_{i\in I} G_i = \overline{G} \cap \bigoplus\limits_{i\in I} G_i$, where $\overline{G}$ stands for the closure of G in $\prod\limits_{i\in I} G_i$ for the product topology.

Although the notion of (weak) controllability was coined by Fagnani earlier in a broader context (cf. [2,3]), both notions were introduced in the area of coding theory by Forney and Trott (cf. [7]). They observed that if the groups G_i are locally compact abelian, then controllability and observability are dual properties with respect to the Pontryagin duality: If G is a closed subgroup of $\prod\limits_{i\in I} G_i$, then it is weakly controllable if and only if its annihilator $G^\perp = \{\chi \in \widehat{\prod\limits_{i\in I} G_i} \mid \chi(G) = \{0\}\}$ is a weakly observable subgroup of $\bigoplus\limits_{i\in I} \widehat{G_i} \leq \prod\limits_{i\in I} \widehat{G_i}$ (cf. [7, 4.8]).

We now describe different ways in which a subgroup is placed in a product of topological groups.

Definition 1.2. Let $\{G_i\}_{i\in\mathbb{N}}$ be a family of compact groups and let G be a closed subgroup of the product $\prod\limits_{i\in\mathbb{N}} G_i$.

(1) The subgroup G is called *rectangular* if for all $i \in \mathbb{N}$ there is a subgroup $H_i \leq G_i$ such that $G = \prod_{i\in\mathbb{N}} H_i$.

(2) The subgroup G is said to be *topologically generated* by the set $\{g_n \mid n \in \mathbb{N}\}$ if all elements $g_n,\ n \in \mathbb{N}$, have finite support and the subgroup $\bigoplus\limits_{n\in\mathbb{N}} \langle g_n\rangle$ is dense in G.

(3) If, in addition, the map

$$\Phi\colon \bigoplus_{n\in\mathbb{N}} \langle g_n\rangle \to G$$

defined by

$$\Phi((x_n)) := \sum_{n\in\mathbb{N}} x_n,$$

with $x_n \in \langle g_n\rangle$ for all $n \in \mathbb{N}$, extends to a topological, necessarily surjective, group isomorphism

$$\widehat{\Phi}\colon \prod_{n\in\mathbb{N}} \langle g_n\rangle \to G$$

we say that G is *weakly rectangular* and Φ is an *isomorphic encoder* of G. We remark that since we are only interested in topologies that are Hausdorff, the map $\widehat{\Phi}$ (the extension of the homomorphism Φ to the product) is unique. Therefore, for simplicity's sake, we will also use the same symbol Φ to denote $\widehat{\Phi}$ from here on.

(4) Finally, if $\Phi(\bigoplus_{n\in\mathbb{N}} \langle g_n\rangle) = G \cap \bigoplus_{i\in\mathbb{N}} G_i$, we say that G is an *implicit direct product.*

The observations below are easily verified. (cf. [9]).

(1) Weakly rectangular subgroups and rectangular subgroups of $\prod\limits_{i\in\mathbb{N}} G_i$ are weakly controllable.

(2) If each G_i is a pro-p_i-group for some prime p_i, and all p_i are distinct, then every closed subgroup of the product $\prod\limits_{i\in\mathbb{N}} G_i$ is rectangular, and thus is an implicit direct product.

(3) If each G_i is a finite simple non-abelian group, then every closed normal subgroup of the product $\prod\limits_{i\in\mathbb{N}} G_i$ is rectangular, and thus an implicit direct product.

The main goal addressed in this paper is to investigate when a profinite abelian group is weakly rectangular or an implicit direct product of finite groups. In particular we aim to know to what extent the converse of (1) above holds. Specifically, we are interested in the following (cf. [9]):

Problem 1.3. *Let $\{G_i \mid i \in \mathbb{N}\}$ be a family of finite abelian groups, and G is a closed, weakly controllable subgroup of the product $\prod\limits_{i\in\mathbb{N}} G_i$. Which additional conditions ensure that the group G is topologically generated? And when is G weakly rectangular? An implicit direct product?*

A first step in order to tackle this question, was given in [5,8], where the following result was established.

Theorem 1.4. *Let I be a countable set, $\{G_i \mid i \in \mathbb{N}\}$ be a family of finite abelian groups and $\prod_{i\in\mathbb{N}} G_i$ be its direct product. If G is a closed weakly controllable subgroup of $\prod_{i\in\mathbb{N}} G_i$, then G is topologically isomorphic to a direct product of finite cyclic groups.*

We notice that Theorem 1.4 does not answer Problem 1.3 directly, and its proof does not present any generating set for G. Incidentally, the continuity of mappings defined on weak direct sums has been investigated in [1,12]. However, the results there go in a different direction and the questions in Problem 1.3 are not addressed.

Remark 1.5. The relevance of these notions stems from coding theory where they appear in connection with the study of (convolutional) group codes [7,11]. In fact, similar concepts had been studied in symbolic dynamics previously. Thus, the notions of weak controllability and weak observability are related to the concepts of *irreducible shift* and *shift of finite type*, respectively, that appear in symbolic dynamics. Here, we are concerned with abelian profinite groups and our main interest is to clarify the overall topological and algebraic structure of abelian profinite groups that satisfy any of the properties introduced above. In a subsequent paper, we will discuss some applications of our results to the study of group codes.

We now formulate our main result.

Theorem A. *Let G be an order controllable, closed subgroup of a direct product $\prod\limits_{i\in\mathbb{N}} G_i$ of a countable family of finite abelian groups $\{G_i | i \in \mathbb{N}\}$. Then the following hold.*

(a) There is a generating set $\{y_m^{(p)} \,|\, mm \in \mathbb{N}, p \in \mathbb{P}_G\} \subseteq G \cap (\bigoplus_{i\in\mathbb{N}} G_i)$ for G such that each element $y_m^{(p)}$ has order a power of p.

(b) If, in addition, G has finite exponent, then an isomorphic encoder

$$\Phi\colon \prod_{m\in\mathbb{N},\, p\in\mathbb{P}_G} \langle y_m^{(p)}\rangle \to G$$

can be defined from the generating set and, as a consequence, G is weakly rectangular.

(c) If $\bigoplus_{m\in\mathbb{N}} \langle y_m^{(p)}\rangle[p]$ is weakly observable for each prime p, then G is an implicit direct product.

2. Basic definitions and terminology

In accordance with the general terminology, a group G is called *torsion* or *periodic* if the orders of all its elements are finite, *torsion-free* if all elements, except the identity, have infinite order. If there is a natural number n such that $ng = 0$ for all $g \in G$, we say that G has *finite exponent*. Then the smallest such n is called the *exponent* of G, denoted as $\exp(G)$. An abelian torsion group G in which the order of every element is a power of a prime number p is called *p-group*. An element g of a p-group G is said to have *finite height* if there is a largest natural number n such that the equation $p^n x = g$ has a solution in G. We denote such n as $h(g, G)$. Otherwise, we say that g has *infinite height*. Here on, the symbol $G[p]$ denotes the subgroup consisting of all elements of order dividing p. It is well known that $G[p]$ is a vector space on the field $\mathbb{Z}(p)$ of integers modulo p. In general, for every group G, we denote by $(G)_p$ the largest p-subgroup of G and $\mathbb{P}_G = \{p \in \mathbb{P} \,:\, G \text{ contains a non-tivial } p-\text{subgroup}\}$ where $\mathbb{P}$ is the set of all prime numbers.

Definition 2.1. Let $\{G_i \,|\, i \in \mathbb{N}\}$ be a family of topological groups and G a subgroup of $\prod\limits_{i\in\mathbb{N}} G_i$. We introduce the following notions:

(1) G is *controllable* if for every $i \in \mathbb{N}$ there is $n_i \in \mathbb{N}$ such that for each $c \in G$, there exists $a \in G$ such that $a_{|[1,i]} = c_{|[1,i]}$ and $\operatorname{supp}(a) \subseteq [1, n_i]$. The sequence $(n_i)_{i\in\mathbb{N}}$ is called *controllability sequence* of G when each n_i is minimal with the property.

(2) G is *order controllable* if for every $i \in \mathbb{N}$ there is $n_i \in \mathbb{N}$ such that for each $c \in G$, there exists $a \in G$ such that $a_{|[1,i]} = c_{|[1,i]}$, $supp(a) \subseteq [1, n_i]$, and $\operatorname{order}(a)$ divides $\operatorname{order}(c_{|[1,n_i]})$. The sequence $(n_i)_{i\in\mathbb{N}}$ is called *order controllability sequence* of G when each n_i is minimal with the property.

Remark 2.2. Property (1) implies the existence of $b := c - a \in G$ such that $c = a + b$, $supp(a) \subseteq [1, n_i]$ and $supp(b) \subseteq [i + 1, +\infty[$.

Property (2) implies the existence of $b := c - a \in G$ such that $c = a + b$, $supp(a) \subseteq [1, n_i]$, $supp(b) \subseteq [i+1, +\infty[$, and order$(b)$ divides order(c).

Remark 2.3.

(i) Every controllable group is weakly controllable and, if the groups G_i are finite, then the notions of controllability and weakly controllability are equivalent (see [5, Corollary 2.3], where the term uniformly controllable subgroup is used instead of controllable subgroup that we have adopted here).

(ii) If $\{G_i \,|\, i \in \mathbb{N}\}$ is a family of finite, abelian, groups and G is an infinite subgroup of $\prod\limits_{i\in\mathbb{N}} G_i$ that contains an order controllable dense subgroup H, then G is order controllable as well. (To see this, take an arbitrary element $z \in G$ and let $[1, m]$ be an arbitrary finite block. By the density of H in G, there is an element $h \in H$ such that $\pi_{[1,n_m]}(z) = \pi_{[1,n_m]}(h)$, where (n_i) denotes the order controllability sequence of H. Now, applying that H is order controllable, there is $h_1 \in H$ such that $\pi_{[1,m]}(h_1) = \pi_{[1,m]}(h) = \pi_{[1,m]}(z)$, supp$(h_1) \subseteq [1, n_m]$ and order(h_1) divides order$(h_{|[1,n_m]})$ =order$(z_{|[1,n_m]})$.)

3. Profinite abelian p-groups

In this section, we describe the structure of profinite abelian p-groups.

Lemma 3.1. *Let $\{G_i \,|\, i \in \mathbb{N}\}$ be a family of finite, abelian, p-groups and let G be an infinite subgroup of $\prod\limits_{i\in\mathbb{N}} G_i$ which is order controllable. If $x \in G_{[1,n_i]}[p]$ and $\pi_{[1,i]}(x) \neq 0$, where $(n_i)_{i\in\mathbb{N}}$ is the order controllability sequence of G, then there exists $\widetilde{x} \in G_{[1,n_i]}[p]$ such that $\pi_{[1,i]}(\widetilde{x}) = \pi_{[1,i]}(x)$ and $h(x, G) = h(\widetilde{x}, G_{[1,n_i]})$. In the particular case that $\pi_{[1,i-1]}(x) = 0$ and there is j such that $n_j < i$ we can take $\widetilde{x}$ such that $h(\widetilde{x}, G_{[j+1,n_i]}) = h(x, G) = h(x, G_{[j+1,+\infty[})$. In either case, we take $\widetilde{x}$ with the maximum possible height among those elements satisfying these properties.*

Proof. Take an element $x \in G_{[1,n_i]}[p]$ with $\pi_{[1,i]}(x) \neq 0$. Since every group G_i in the product is finite and x has finite support, it follows that x has finite height. Pick an arbitrary element $y \in G$ such that $x = p^h y$ (where $h = h(x, G)$ is the maximal height), which implies that order$(y) = p^{h+1}$. Since G is order controllable, $y = \widetilde{y} + w$ where $\widetilde{y} \in G_{[1,n_i]}$, order$(\widetilde{y}) = p^{h+1}$, $w \in G_{[i+1,+\infty[}$, order$(w) \leq p^{h+1}$ and $p^h w(j) = 0$ for all $j > n_i$. Observe that $p^h w \in G_{[i+1,n_i]}[p]$, $\widetilde{x} := p^h \widetilde{y} \in G_{[1,n_i]}[p]$, $0 \neq \pi_{[1,i]}(x) = \pi_{[1,i]}(\widetilde{x})$, and $h(\widetilde{x}, G_{[1,n_i]}) = h(x, G)$.

Suppose now that $\pi_{[1,i-1]}(x) = 0$ and there is j such that $n_j < i$. Then $\pi_{[1,n_j]}(x) = \pi_{[1,n_j]}(\widetilde{x}) = 0$ and order$(\widetilde{y}_{|[1,n_j]}) \leq p^h$. Moreover, $\widetilde{y} = w_1 + w_2$, $w_1 \in G_{[1,n_j]}$, $w_2 \in G_{[j+1,n_i]}$ and order$(w_1) \leq$ order$(\widetilde{y}_{|[1,n_j]}) \leq p^h$. Then $0 \neq \widetilde{x} = p^h(w_1 + w_2) = p^h w_1 + p^h w_2 = p^h w_2$, $\pi_{[1,j]}(w_2) = 0$ and order$(w_2) = p^{h+1}$. As a consequence, $h(x, G) = h(\widetilde{x}, G_{[1,n_i]}) = h(\widetilde{x}, G_{[j+1,n_i]})$. The same argument shows that $h(x, G) = h(x, G_{[j+1,+\infty[})$. □

Next follows the main result of this section. It provides sufficient conditions for a subgroup G to be weakly rectangular or an implicit direct product.

Theorem 3.2. *Let $\{G_i \,|\, i \in \mathbb{N}\}$ be a family of finite, abelian, p-groups. If G is an (infinite) order controllable, closed, subgroup of $\prod\limits_{i\in\mathbb{N}} G_i$ then the following assertions hold true:*

(i) There is a generating set $\{y_m \,|\, m \in \mathbb{N}\} \subseteq G \cap \bigoplus_{i\in\mathbb{N}} G_i$ for G.

(ii) If G has finite exponent, then there is an isomorphic encoder

$$\Phi\colon \prod_{m\in\mathbb{N}} \langle y_m \rangle \to G.$$

As a consequence G is weakly rectangular.

(iii) Let p^{h_m+1} be the order of y_m. If the group $\sum_{m\in\mathbb{N}} \langle p^{h_m} y_m\rangle$ is weakly observable, then G is an implicit direct product.

Proof. The proof relies on the existence of two increasing sequences of natural numbers $(d_k)_{k\geq 1}$ and $(m(k))_{k\geq 0}$, where $m(0)=0$, and a sequence of finite subsets $B_k := \{x_{m(k-1)+1}, \cdots, x_{m(k)}\} \subseteq G[p] \bigcap \bigoplus_{i\in\mathbb{N}} G_i$ satisfying the following conditions:

(a) $\pi_{[d_{k-1}+1,d_k]}(B_k)$ consists of linearly independent *vectors* in $\pi_{[d_{k-1}+1,d_k]}(G[p])$;
(b) $\pi_{[d_{k-1}+1,d_k]}(B_1 \cup \cdots B_k)$ is a generating set of $\pi_{[d_{k-1}+1,d_k]}(G[p])$;
(c) $\pi_{[1,d_k]}(B_1 \cup \cdots B_k)$ forms a basis of $\pi_{[1,d_k]}(G[p])$;
(d) if $m(k-1)+1 \leq j \leq m(k)$, then $x_j \in G_{[d_{k-1}+1,n_{d_k}]}[p] \setminus \langle x_1, \cdots x_{j-1}\rangle$ and x_j has maximal height h_j in G;
(e) for each $x_j \in B_k$ there is an element $y_j \in G_{[1,n_{d_k}]}$ such that $x_j = p^{h_j} y_j$. Furthermore $y_j(i) = 0$ for all $j > m(n_i)$;
(f) $G[p] = \langle B_1\rangle \bigoplus \cdots \langle B_k\rangle \bigoplus G_{[d_k+1,+\infty[}[p]$ (here, with some notational abuse, we mean *vector space direct sum*).

Remark that (f) yields

$$\pi_{[1,d_k]}(G[p]) = \pi_{[1,d_k]}(\langle B_1\rangle \bigoplus \cdots \langle B_k\rangle)\ (\forall k \in \mathbb{N}). \tag{1}$$

As a consequence, we obtain

$$G[p] \subseteq \overline{\bigoplus_{k\in\mathbb{N}} \langle B_k\rangle} \cong \overline{\bigoplus_{m\in\mathbb{N}} \langle x_m\rangle}.$$

We proceed by induction in order to prove the existence of the sequences $(d_k)_{k\in\mathbb{N}}$, $(m(k))_{k\in\mathbb{N}}$, and $B_k := \{x_{m(k-1)+1}, \cdots, x_{m(k)}\}$.

Since G is order controllable, there is an order controllability sequence $(n_i)_{i\geq 1} \subseteq \mathbb{N}$ such that $\pi_{[1,i]}(G) = \pi_{[1,i]}(G_{[1,n_i]})$ for all $i \in \mathbb{N}$. We have further

$G = G_{[1,n_1]} + G_{[2,+\infty[} = G_{[1,n_1]} + \cdots G_{[i,n_i]} + G_{[i+1,+\infty[}$,

$G[p] = G_{[1,n_1]}[p] + G_{[2,+\infty[}[p] = G_{[1,n_1]}[p] + \cdots G_{[i,n_i]}[p] + G_{[i+1,+\infty[}[p]$.

Remark that, since every group in the product G_i is finite, all the elements in $(G \bigcap \bigoplus_{i\in\mathbb{N}} G_i)[p]$ have finite height.

Let $d_1 \in \mathbb{N}$ be the minimum element such that

$$m(1) := \dim \pi_{[1,d_1]}(G[p]) = \dim \pi_{[1,d_1]}(G_{[1,n_{d_1}]}[p]) \neq 0.$$

We select an element $x_1 \in G_{[1,n_{d_1}]}[p]$ such that $\pi_{[1,d_1]}(x_1) \neq \{0\}$ and has maximal height $h_1 := h(x_1, G) = h(x_1, G_{[1,n_{d_1}]})$, by Lemma 3.1. If $\dim \pi_{[1,d_1]}(G_{[1,n_{d_1}]}[p]) \neq 1$, we repeat the same argument in order to obtain an element $x_2 \in G_{[1,n_{d_1}]}[p]$ satisfying: (i) $\pi_{[1,d_1]}(x_2) \notin \langle \pi_{[1,d_1]}(x_1)\rangle$; and (ii) $h_1 \geq h_2 := h(x_2, G) = h(x_2, G_{[1,n_{d_1}]})$. Furthermore, we select x_2 in such a way that has maximal height among the elements in $G_{[1,n_{d_1}]}$ satisfying (i) and (ii). We go on with this procedure obtaining a finite subset $B_1 = \{x_1, x_2, \cdots x_{m(1)}\}$ such that $\pi_{[1,d_1]}(B_1)$ is a basis of $\pi_{[1,d_1]}(G[p])$ and $h_1 \geq h_2 \geq \cdots \geq h_{m(1)}$, where $h_j = h(x_j, G) = h(x_j, G_{[1,n_{d_1}]})$ is the maximal possible height, $1 \leq j \leq m(1)$. Moreover, associated to every $x_j \in B_1$ there is $y_j \in G_{[1,n_{d_1}]}$ such that $x_j = p^{h_j} y_j$. Thus the properties (a),..., (e) stated above are satisfied for $n = 1$.

We now verify (f), that is

$$G[p] = \langle B_1 \rangle \oplus G_{[d_1+1,+\infty[}[p].$$

Indeed, let $0 \neq c \in G[p]$. If $\pi_{[1,d_1]}(c) = 0$ then $c \notin \langle B_1 \rangle$ since, otherwise, we would have

$$c = \lambda_1 x_1 + \cdots + \lambda_{m(1)} x_{m(1)}$$

and

$$0 = \pi_{[1,d_1]}(c) = \lambda_1 \pi_{[1,d_1]}(x_1) + \cdots \ \lambda_{m(1)} \pi_{[1,d_1]}(x_{m(1)}),$$

which yields $\lambda_1 = \cdots = \lambda_{m(1)} = 0$ because $\pi_{[1,d_1]}(B_1)$ is an independent set.

On the other hand, if $\pi_{[1,d_1]}(c) \neq 0$, then $\pi_{[1,d_1]}(c) = \pi_{[1,d_1]}(b)$ for some $b \in \langle B_1 \rangle$. Hence $c = b + w$, and $w = c - b \in G_{[d_1+1,+\infty[}[p]$.

Now, the inductive procedure for the proof of $n \Rightarrow n+1$ is straightforward. We will only sketch the case $n = 2$, as it explains well the general case.

First, since G is infinite, for some $d_2 \in \mathbb{N}$ (take the smallest possible one), we have

$$m(2) := \dim \pi_{[1,d_2]}(G[p]) \neq \dim \pi_{[1,d_2]}(\langle B_1 \rangle).$$

Furthermore, since G is order controllable, it follows

$$\pi_{[1,d_2]}(G[p]) = \pi_{[1,d_2]}(\langle B_1 \rangle \oplus G_{[d_1+1,+\infty[}[p]) = \pi_{[1,d_2]}(\langle B_1 \rangle \oplus G_{[d_1+1,n_{d_2}]}[p]).$$

Now, we proceed as in the case $n = 1$ in order to obtain a subset

$$B_2 = \{x_{m(1)+1}, \cdots x_{m(2)}\} \subseteq G_{[d_1+1,n_{d_2}]}[p]$$

satisfying the assertions (a),...,(d) and (f) stated above. On the other hand, assertion (e) follows from Lemma 3.1. This completes the inductive argument.

Next, we prove the following

CLAIM:

$$G \cap (\oplus G_i) \subseteq \overline{\sum_{m \in \mathbb{N}} \langle y_m \rangle} = G.$$

Proof of the Claim:

First, remark that for each $x \in B_k$, we have order$(x) = p$, $supp(x) \subseteq [d_{k-1}+1, n_{d_k}]$, and $\pi_{[d_{k-1}+1,d_k]}(x) \neq 0$. Furthermore, for each $x \in B_k$, there exists $y \in G_{[1,n_{d_k}]}$ with $x = p^h y$, order$(y) = p^{h+1}$, where $h = h(x,G) = h(x, G_{[1,n_{d_k}]})$, and such that if $n_j < d_k$, for some j, then $\pi_{[1,j]}(y) = 0$ by Lemma 3.1.

Set

$$Y := \sum_{m \in \mathbb{N}} \langle y_m \rangle.$$

We first prove that every element in $\sum\limits_{m \in \mathbb{N}} \langle x_m \rangle$ has the same height in the group G as in the subgroup $Y \subseteq G$.

Indeed, let z be an arbitrary element in $\sum\limits_{m\in\mathbb{N}} \langle x_m\rangle$. Then there is some index $k \in \mathbb{N}$ such that

$$z \in \langle B_1 \cup \cdots B_k\rangle = \sum_{1\le m\le m(k)} \langle x_m\rangle.$$

Set

$$Y_k := \sum_{1\le m\le m(k)} \langle y_m\rangle,$$

since $Y_k \subseteq Y \subseteq G$, it is enough to verify that z has the same height in the group G (equivalently, in the subgroup $G_{[1,n_{d_k}]}$) as in the subgroup Y_k.

Assume for the moment that

$$0 \neq z = \lambda_{m(k-1)+1} x_{m(k-1)+1} + \cdots \lambda_r x_r \in \langle B_k\rangle, \tag{2}$$

$0 \le \lambda_j < p$, $\lambda_r \neq 0$, $m(k-1) < j \le r \le m(k)$, where the terms appearing in (2) are displayed with decreasing height, that is, in the same order as they are listed in B_k. Thus

$$h_j = h(x_j, G) \ge h(x_{j+1}, G) = h_{j+1},$$

$m(k-1) < j \le r \le m(k)$. We also have $\pi_{[1,d_{k-1}]}(z) = 0$ and $\pi_{[d_{k-1}+1,d_k]}(z) \neq 0$.

Set

$$H_k := \sum_{m(k-1)<m\le m(k)} \langle y_m\rangle.$$

Remark that, since the elements $x_j \in B_k$ are taken with decreasing height, it follows that each $\lambda_j x_j \neq 0$ has the same height in G as in H_k. Furthermore, the height of z in G is

$$h := h(z, G) = h(x_r, G) = h_r = \min\{h_j \mid \lambda_j \neq 0, m(k-1) < j \le r\} = h(z, H_k). \tag{3}$$

Indeed, if we had $h > h_r$, then we would have selected z (or another vector of the same height) in place of x_r when defining B_k. Thus $h(z, G) = h(z, H_k) \le h(z, Y_k) \le h(z, G)$, and we are done in this case.

The general case is proved by induction. Assume that whenever

$$0 \neq z \in \langle B_i \cup \cdots \cup B_k\rangle,$$

where i is the first index such that $\pi_{[1,d_i]}(z) \neq 0$, we have that $h(z, G) = h(z, Y_k)$.

Reasoning by induction, take an arbitrary element $0 \neq z \in \langle B_{i-1} \cup \cdots B_k\rangle$, where $i-1$ is the first index such that $\pi_{[1,d_{i-1}]}(z) \neq 0$.

Then $z = z_{i-1} + z_i + \cdots z_k$, $z_j \in \langle B_j\rangle$, $i-1 \le j \le k$, where

$$\pi_{[1,d_{i-1}]}(z_{i-1}) = \pi_{[1,d_{i-1}]}(z)$$

and, from the argument in the paragraph above, the height of z_j in H_j is the same as in G, $i-1 \le j \le k$.

If $h(z_{i-1}, G) < h(z_i + \cdots z_k, G)$, then

$$h(z, Y_k) \le h(z, G) = h(z_{i-1}, G) = h(z_{i-1}, H_{i-1}) \le h(z_{i-1}, Y_k) \le h(z_{i-1}, G)$$

by (3). On the other hand, by the inductive hypothesis, we have

$$h(z_i + \cdots z_k, G) = h(z_i + \cdots z_k, Y_k).$$

Hence

$$h(z_{i-1}, Y_k) = h(z_{i-1}, G) < h(z_i + \cdots z_k, G) = h(z_i + \cdots z_k, Y_k),$$

which yields

$$h(z, Y_k) = h(z_{i-1}, Y_k) = h(z_{i-1}, G) = h(z, G).$$

This completes the proof when $h(z_{i-1}, G) < h(z_i + \cdots z_k, G)$. The case $h(z_i + \cdots z_k, G) < h(z_{i-1}, G)$ is analogous.

Therefore, we may assume without loss of generality that

$$h(z_{i-1}, G) = h = h(z_i + \cdots z_k, G).$$

Moreover, by the inductive hypothesis, we also have

$$h(z_{i-1}, Y_k) = h(z_{i-1}, G) = h = h(z_i + \cdots z_k, G) = h(z_i + \cdots z_k, Y_k).$$

Reasoning by contradiction, suppose that

$$h(z, G) = r > h(z, Y_k) \geq h.$$

Since G is order controllable we can decompose

$$z = p^r y = p^r v_{i-1} + p^r w_{i-1},$$

where

$$\begin{gathered} y \in G_{[1,n_{d_k}]},\ v_{i-1} \in G_{[1,n_{d_{i-1}}]},\ w_{i-1} \in G_{[d_{i-1}+1,n_{d_k}]}, \\ \operatorname{order}(p^r y) = \operatorname{order}(p^r v_{i-1}) = p, \\ \pi_{[1,d_{i-2}]}(p^r y) = \pi_{[1,d_{i-2}]}(p^r v_{i-1}) = 0, \end{gathered}$$

and

$$\pi_{[d_{i-2}+1,d_{i-1}]}(p^r y) = \pi_{[d_{i-2}+1,d_{i-1}]}(p^r v_{i-1}) = \pi_{[d_{i-2}+1,d_{i-1}]}(z) = \pi_{[d_{i-2}+1,d_{i-1}]}(z_{i-1}) \neq 0.$$

Let $\lambda_l x_l$ be the last term in the sum of z_{i-1}, then the height of x_l in G coincides with the height of z_{i-1} in G, which is h by (3). Furthermore, $p^r v_{i-1} \in G_{[d_{i-2}+1,n_{d_{i-1}}]}[p]$ and

$$\pi_{[d_{i-2}+1,d_{i-1}]}(p^r v_{i-1}) \notin \pi_{[d_{i-2}+1,d_{i-1}]}(\langle x_{m(i-2)+1}, \cdots x_{l-1} \rangle).$$

This is a contradiction with the previous choice of x_l because the height of $p^r v_{i-1}$ in G is $r > h$ and x_l was selected with maximal possible height in G. Therefore, we have proved $h(z, G) = h = h(z, Y_k) = h(z, Y)$.

We now prove that for every $z \in \operatorname{tor}(G)$ (the torsion subgroup of G) there is a sequence

$$(\lambda_m) \in \prod_{m \in \mathbb{N}} \mathbb{Z}(p^{h_m+1})$$

such that

$$z = \lim_{k\to\infty} \sum_{1\le m\le m(k)} \lambda_m y_m \tag{4}$$

which is tantamount to

$$\pi_{[1,d_k]}(z) = \pi_{[1,d_k]}\left(\sum_{1\le m\le m(k)} \lambda_m y_m\right)$$

for every $k \in \mathbb{N}$.

We proceed by induction on the order p^s of z.

Take any element $z \in G[p]$. By Equation (1), we know that

$$\pi_{[1,d_k]}(G[p]) = \pi_{[1,d_k]}(\langle B_1\rangle \bigoplus \cdots \langle B_k\rangle)$$

holds for all $k \in \mathbb{N}$. Therefore, there is a sequence

$$(\alpha_m) \in \prod_{m\in\mathbb{N}} \mathbb{Z}(p)$$

such that

$$\pi_{[1,d_k]}(z) = \pi_{[1,d_k]}\left(\sum_{1\le m\le m(k)} \alpha_m x_m\right) = \pi_{[1,d_k]}\left(\sum_{1\le m\le m(k)} \alpha_m p^{h_m} y_m\right)$$

for all $k \in \mathbb{N}$. This means

$$z = \lim_{k\to\infty} \sum_{1\le m\le m(k)} \alpha_m p^{h_m} y_m.$$

This completes the proof for $s = 1$ if we set $\lambda_m := \alpha_m p^{h_m}$ for all $m \in \mathbb{N}$.

Now, suppose that the assertion is true when $\mathrm{order}(z) \le p^s$ and pick an arbitrary element $z \in G$ with $\mathrm{order}(z) = p^{s+1}$. Then $p^s z$ has order p and therefore belongs to $G[p]$. By Equation (1) again, we know that there is a sequence

$$(\alpha_m) \in \prod_{m\in\mathbb{N}} \mathbb{Z}(p)$$

such that

$$\pi_{[1,d_k]}(p^s z) = \pi_{[1,d_k]}\left(\sum_{1\le m\le m(k)} \alpha_m x_m\right) = \pi_{[1,d_k]}\left(\sum_{1\le m\le m(k)} \alpha_m p^{h_m} y_m\right)$$

for all $k \in \mathbb{N}$.

Now, since we have chosen each element x_m with the maximal possible height, it follows that $s \le h_m$ for all $1 \le m \le m(k)$, and $k \in \mathbb{N}$. Therefore

$$\pi_{[1,d_k]}(p^s z) = \pi_{[1,d_k]}\left(p^s \sum_{1\le m\le m(k)} \alpha_m p^{h_m - s} y_m\right),$$

for all $k \in \mathbb{N}$, which yields

$$\pi_{[1,d_k]}\left(p^s(z - \sum_{1\leq m\leq m(k)} \alpha_m p^{h_m - s} y_m)\right) = 0$$

for all $k \in \mathbb{N}$.

Set

$$v = \lim_{k\to\infty} \sum_{1\leq m\leq m(k)} \alpha_m p^{h_m - s} y_m \in G$$

where the limit exists, and therefore v is well defined, because $y_m(i) = 0$ for all $m > m(n_i)$. Then we have $z = v + (z - v)$, where order$(z - v) \leq p^s$. By the inductive hypothesis, there is a sequence

$$(\mu_m) \in \prod_{m\in\mathbb{N}} \mathbb{Z}(p^{h_m+1})$$

such that

$$z - v = \lim_{k\to\infty} \sum_{1\leq m\leq m(k)} \mu_m y_m.$$

Therefore

$$z = \lim_{k\to\infty} \sum_{1\leq m\leq m(k)} (\alpha_m p^{h_m - s} + \mu_m) y_m.$$

This completes the proof of the inductive argument. Therefore, it is proved that

$$G \cap (\oplus G_i) \subseteq \operatorname{tor}(G) \subseteq \overline{\sum_{m\in\mathbb{N}} \langle y_m\rangle}.$$

Since G is closed and order controllable, it follows

$$\overline{G \cap (\oplus G_i)} = G = \overline{\sum_{m\in\mathbb{N}} \langle y_m\rangle}.$$

This completes the proof of the Claim.

We now proceed with the proof of the three assertions formulated in this theorem.

(i) We will now prove that G is topologically generated by the set $\{y_m \mid m \in \mathbb{N}\}$.

First, observe that the finite subgroup $\langle y_m\rangle$, generated by y_m in G, is isomorphic to $\mathbb{Z}(p^{h_m+1})$ for every $m \geq 1$. Thus, without loss of generality, we may replace the group $\langle y_m\rangle$ by $\mathbb{Z}(p^{h_m+1})$ in the sequel. Consider now the group $\prod\limits_{m\in\mathbb{N}} \mathbb{Z}(p^{h_m+1})$, equipped with the product topology and its dense subgroup $\bigoplus\limits_{m\in\mathbb{N}} \mathbb{Z}(p^{h_m+1})$. Set

$$\Phi : \bigoplus_{m\in\mathbb{N}} \mathbb{Z}(p^{h_m+1}) \longrightarrow G \cap (\oplus G_i) \leq G$$

defined by

$$\Phi[(k_1, \ldots, k_m, \ldots)] = \sum_{m\in\mathbb{N}} k_m y_m.$$

Since only finitely many k_m are non-null, the map Φ is clearly well defined. We will prove that Φ is also a topological group isomorphism on its image.

In order to verify that Φ is one-to-one, suppose there is a sequence

$$(k_1, \cdots, k_r, 0, \cdots) \in \ker f,\ 0 \leq k_j < p^{h_j+1},$$

with some $k_j \neq 0$. Then we have

$$k_1 y_1 + \cdots + k_r y_r = 0.$$

Expressing every $k_j \neq 0$ in base p, we obtain $k_j = a^{(j)}_{h_j} p^{h_j} + \cdots + a^{(j)}_1 p + a^{(j)}_0$, $0 \leq a^{(j)}_i < p$, $0 \leq i \leq h_j$, $1 \leq j \leq r$. Let p^{s_j} the minimal power of p that appears in the expression of $k_j \neq 0$. Since y_j has order p^{h_j+1} the order of $k_j y_j$ is $p^{h_j - s_j + 1}$.

Defining $d := \max\{h_j - s_j \,|\, k_j \neq 0, 1 \leq j \leq r\}$ and multiplying by p^d the equality above, we obtain an expression as follows

$$p^d \left((a^{(i_1)}_{h_{i_1}} p^{h_{i_1}} + \cdots + a^{(i_1)}_{h_{i_1-d}} p^{h_{i_1}-d}) y_{i_1} + \cdots (a^{(i_l)}_{h_{i_l}} p^{h_{i_l}} + \cdots + a^{(i_l)}_{h_{i_l - s}} p^{h_{i_l}-d}) y_{i_l} \right) = 0,$$

where we have only considered those elements $\{y_{i_j}\}_{j=1}^{l}$ such that $h_{i_1} - s_{i_1} = \cdots h_{i_l} - s_{i_l} = d$. Since $p^{h_{i_j}} y_{i_j} = x_{i_j}$ has order p, we have

$$a^{(i_1)}_{s_{i_1}} x_{i_1} + \cdots a^{(i_l)}_{s_{i_l}} x_{i_l} = 0.$$

Since the elements $\{x_{i_1}, \cdots, x_{i_l}\}$ are all independents, it follows that

$$a^{(i_1)}_{s_{i_1}} = \cdots a^{(i_l)}_{s_{i_l}} = 0.$$

This is a contradiction which completes the proof. Therefore Φ is 1-to-1.

The sequence (y_m) that we have defined above verifies that $y_m(i) = 0$ for all $m > m(n_i)$. As a consequence, we have that $\lim\limits_{m\to\infty} y_m(i) = 0$ for all $i \in \mathbb{N}$, which implies the continuity of Φ. Indeed, let (z_α) be a sequence in $\bigoplus\limits_{m\in\mathbb{N}} \mathbb{Z}(p^{h_m+1})$ converging to 0. If $V_i = (0, \cdots 0) \times \prod\limits_{j>i} G_j$ is an arbitrary basic neighborhood of 0 in $\prod\limits_{j\in\mathbb{N}} G_j$, since $(0, \cdots 0) \times \prod\limits_{j>m(n_i)} \mathbb{Z}(p^{h_j+1})$ is a neighborhood of 0 in $\prod\limits_{j\in\mathbb{N}} \mathbb{Z}(p^{h_j+1})$, then there is α_i such that $z_{\alpha|[1,m(n_i)]} = 0$ for all $\alpha \geq \alpha_i$. Therefore $z_\alpha = (0, \cdots, 0, k_{m(n_i)+1,\alpha}, \cdots)$ and $\Phi(z_\alpha) = \sum\limits_{m>m(n_i)} k_{m,\alpha} y_m \in V_i$, for all $\alpha \geq \alpha_i$ and for all $i \in \mathbb{N}$. Thus, the sequence $(\Phi(z_\alpha))$ converges to $\Phi(0) = 0$, which verifies the continuity of Φ.

As a consequence, there is a continuous extension

$$\Phi : \prod_{m\in\mathbb{N}} \mathbb{Z}(p^{h_m+1}) \longrightarrow G$$

that we still denote by Φ for short, which is continuous and onto. Furthermore, it is easily seen that it holds

$$\Phi[(k_m)] = \sum_{m\in\mathbb{N}} k_m y_m.$$

Remark that, since $y_m(i) = 0$ for all $m > m(n_i)$, it follows that

$$\sum_{m\in\mathbb{N}} k_m y_m(i)$$

reduces to a finite sum for all $i \in \mathbb{N}$. Therefore Φ is well defined. This proves that $\{y_m \,|\, m \in \mathbb{N}\}$ is a generating set for G.

(ii) Next we prove that if G has finite exponent then Φ is 1-to-1 on $\prod\limits_{m\in\mathbb{N}} \mathbb{Z}(p^{h_m+1})$ and, as a consequence, that Φ is an isomorphic encoder and G is weakly rectangular.

For that purpose, it will suffice to check that $\ker\Phi = \{0\}$.

We proceed by induction on the order p^s of the elements $\mathbf{v} := (\lambda_m) \in \ker\Phi$.

Suppose order$(\mathbf{v}) = p$, which means $\lambda_m = \alpha_m p^{h_m}$, $0 \le \alpha_m < p$, for all $m \in \mathbb{N}$. We have

$$\Phi(\mathbf{v}) = \sum_{m\in\mathbb{N}} \alpha_m p^{h_m} y_m = \sum_{m\in\mathbb{N}} \alpha_m x_m = 0.$$

For every $l \in \mathbb{N}$, set $\mathbf{v}_l := (\mu_m)$, where $\mu_m = \lambda_m$ if $1 \le m \le m(l)$ and $\mu_m = 0$ if $m > m(l)$. It follows that $\lim\limits_{l\to\infty} \mathbf{v}_l = \mathbf{v}$. By the continuity of Φ we obtain

$$\lim_{l\to\infty} \sum_{1\le m\le m(l)} \alpha_m x_m = \sum_{m\in\mathbb{N}} \alpha_m x_m = 0.$$

Thus, for every $k \in \mathbb{N}$, there is $l_k \in \mathbb{N}$ such that

$$\pi_{[1,d_k]}(\Phi(\mathbf{v}_l)) = \pi_{[1,d_k]}\left(\sum_{1\le m\le m(l)} \alpha_m x_m\right) = \sum_{1\le m\le m(l)} \alpha_m \pi_{[1,d_k]}(x_m) = 0$$

for all $l \ge l_k$. On the other hand

$$\Phi(\mathbf{v}_l) = \sum_{1\le m\le m(l)} \alpha_m x_m + \underbrace{\sum_{m=m(1)+1}^{m(2)}}_{\pi_{[1,d_1]}(x_m)=0} \alpha_m x_m + \cdots \underbrace{\sum_{m=m(k)+1}^{m(l)}}_{\pi_{[1,d_k]}(x_m)=0} \alpha_m x_m,$$

where $0 \le \alpha_m < p$, then for $l \ge l_k$, we have

$$\pi_{[1,d_1]}(\Phi(\mathbf{v}_l)) = \sum_{1\le m\le m(1)} \alpha_m \pi_{[1,d_1]}(x_m) = 0$$

and since $\pi_{[1,d_1]}(B_1) = \{\pi_{[1,d_1]}(x_1), \cdots, \pi_{[1,d_1]}(x_{m(1)})\}$ is a basis for $\pi_{[1,d_1]}(G[p])$ we obtain that $\alpha_1 = \cdots = \alpha_{m(1)} = 0$.

In like manner, from

$$\pi_{[1,d_2]}(\Phi(\mathbf{v}_l)) = \sum_{m=m(1)+1}^{m(2)} \alpha_m \pi_{[1,d_2]}(x_m) = 0,$$

we deduce that $\alpha_{m(1)+1} = \cdots = \alpha_{m(2)} = 0$. Therefore, iterating this argument, we obtain $\alpha_1 = \cdots = \alpha_{m(k)} = 0$. Since $\lambda_m = \alpha_m p^{h_m}$ for all $1 \le m \le m(l)$, it follows that $\lambda_m = 0$ for all $1 \le m \le m(k)$. Since this holds for every $k \in \mathbb{N}$, it follows that $\lambda_m = 0$ for all $m \in \mathbb{N}$. This completes the proof for $s = 1$.

Now, suppose that the assertion is true when order$(\mathbf{v}) \le p^s$ and pick an arbitrary element $\mathbf{v} = (\lambda_m) \in \ker\Phi$ such that order$(\mathbf{v}) = p^{s+1}$. Then $p^s\mathbf{v} \in \ker\Phi$ and has order p. Therefore, the arguments above applied to $p^s\mathbf{v}$ yield that $p^s\mathbf{v} = 0$, which is a contradiction. By the inductive assumption, it follows that $\mathbf{v} = 0$, which completes the proof.

Therefore, we have proved that

$$\Phi : \prod_{m\in\mathbb{N}} \mathbb{Z}(p^{h_m+1}) \longrightarrow G$$

is 1-to-1. The compactness of the domain implies that Φ is a topological group isomorphism onto G.

(iii) Assume that $\sum\limits_{m\in\mathbb{N}} \langle x_m\rangle$ is weakly observable. This means

$$\overline{\sum_{m\in\mathbb{N}} \langle x_m\rangle} \cap \bigoplus G_i = \sum_{m\in\mathbb{N}} \langle x_m\rangle.$$

We have to verify that the map

$$\Phi : \bigoplus_{m\in\mathbb{N}} \mathbb{Z}(p^{h_m+1}) \longrightarrow G \cap \bigoplus G_i$$

is onto. Reasoning by contradiction, suppose there is an element

$$z \in G \cap \bigoplus G_i \setminus \sum_{m\in\mathbb{N}} \langle y_m\rangle,$$

which has the smallest possible order, p^{s+1}, $s \geq 0$, of an element with this property. Since, by the foregoing Claim, we have that

$$G[p] \cap \bigoplus G_i \subseteq \overline{\sum_{m\in\mathbb{N}} \langle x_m\rangle},$$

it follows that

$$G[p] \cap \bigoplus G_i = \sum_{m\in\mathbb{N}} \langle x_m\rangle.$$

Therefore, since $p^s z \in G[p] \cap \bigoplus G_i$, there must be a finite subset $J \subseteq \mathbb{N}$ such that

$$p^s z = \sum_{m\in J} \alpha_m x_m = \sum_{m\in J} \alpha_m p^{h_m} y_m,\ 0 < \alpha_m < p.$$

We have already proved that $s \leq h_m$ for all $m \in J$, which yields

$$p^s z = p^s \sum_{m\in J} \alpha_m p^{h_m - s} y_m.$$

Set

$$v := \sum_{m\in J} \alpha_m p^{h_m - s} y_m = f((\alpha_m p^{h_m - s})).$$

Then $p^s(z - v) = 0$.

If $s = 0$, we obtain $z = v$ and we are done.

So, assume that $s > 0$. In this case, we have an element $z - v \in G \cap \bigoplus G_i$, whose order is p^s. By our initial assumption, this means that $z - v = f((\lambda_m))$ for some element $(\lambda_m) \in \bigoplus\limits_{m\in\mathbb{N}} \mathbb{Z}(p^{h_m+1})$. Therefore

$$z = v + f((\lambda_m)) = f((\alpha_m p^{h_m - s} + \lambda_m)),$$

which is a contradiction. This completes the proof. □

Example 3.3. Let $G \subseteq \prod\limits_{i\in\mathbb{N}} G_i$, where $G_i = \mathbb{Z}(2^2)$, be the subgroup generated by the set $\{y_n \mid n \in \mathbb{N}\}$, where $y_1 \in G_{[1,2]}$ with $y_1(1) = 2$ and $y_1(2) = 1$, and $y_n \in G_{[n,n+1]}$ with $y_n(n) = y_n(n+1) = 1$ for $n > 1$.

The group G is not order controllable. Indeed, for any block $[1, m]$, pick $y \in G$ such that $y(n) = 2$ for all $1 \leq n \leq m+1$, which only admits the sum $y = z_m + z$ with the first part $z_m \in G_{[1,m+1]}$, where $z_m(n) = y(n) = 2$, $1 \leq n \leq m$ and $z_m(m+1) = 1$, $m \geq 1$. Then $\mathrm{order}(y_{|[1,m+1]}) = 2$ but $\mathrm{order}(z_m) = 4$.

On the other hand, it is easily seen that $\overline{G}^{\prod G_i}$ is an implicit direct product of the family $\{G_i \mid i \in \mathbb{N}\}$. Therefore, the choice of an appropriate generating set is essential in order to determine whether a subgroup of a product is weakly rectangular or an implicit direct product.

4. Main result

Let G be a closed subgroup of $X = \prod\limits_{i\in\mathbb{N}} G_i$ (a countable product of finite abelian groups). Since each group G_i is finite and abelian, by the fundamental structure theorem of finite abelian groups, we have that every group G_i is a finite sum of finite p-groups, that is $G_i \cong \bigoplus\limits_{p\in\mathbb{P}_i} (G_i)_p$ and $\mathbb{P}_i = \mathbb{P}_{G_i}$ is finite, $i \in \mathbb{N}$. Note that $\mathbb{P}_X = \cup \mathbb{P}_i$. We have

$$\prod_{i\in\mathbb{N}} G_i \cong \prod_{i\in\mathbb{N}} (\prod_{p\in\mathbb{P}_i} (G_i)_p) \cong \prod_{p\in\mathbb{P}_X} (\prod_{i\in\mathbb{N}_p} (G_i)_p)$$

where $\mathbb{N}_p = \{i \in \mathbb{N} \mid G_i \text{ has a nontrivial } p-\text{subgroup}\}$.

Thus

$$(X)_p \cong \prod_{i\in\mathbb{N}_p} (G_i)_p.$$

Consider the embedding

$$j : G \hookrightarrow \prod_{p\in\mathbb{P}_G} (\prod_{i\in\mathbb{N}_p} (G_i)_p)$$

and the canonical projection

$$\pi_p : \prod_{p\in\mathbb{P}_G} (\prod_{i\in\mathbb{N}_p} (G_i)_p) \to \prod_{i\in\mathbb{N}_p} (G_i)_p.$$

Set $G^{(p)} = (\pi_p \circ j)(G)$, that is a compact group. We have

$$(G)_p \cong G^{(p)}.$$

Now, it is easily seen that if G is order controllable then $(G)_p$ has this property for each $p \in \mathbb{P}_G$. Taking this fact into account, we obtain the following result that answers to Question 1.3 for products of finite abelian groups.

We can now prove Theorem **A**.

Proof of Theorem A. . Since $G \cap (\bigoplus\limits_{i\in\mathbb{N}} G_i)$ is dense in G, we have that

$$(\pi_p \circ j)(G \cap (\bigoplus_{i\in\mathbb{N}} G_i)) = G^{(p)} \cap \bigoplus_{i\in\mathbb{N}_p} (G_i)_p$$

is dense in $G^{(p)}$. Thus (a) is a direct consequence of Theorem 3.2. That is, for each $p \in \mathbb{P}_G$, there is a sequence

$$\{y_m^{(p)} \,|\, m \in \mathbb{N}\} \subseteq G^{(p)} \cap \bigoplus_{i\in\mathbb{N}_p} (G_i)_p$$

such that $\{y_m^{(p)} \,|\, m \in \mathbb{N}\}$ is a generating set for $G^{(p)}$. Furthermore, observe that if $p \in \mathbb{P}_G$, then $G^{(p)} \cap \bigoplus\limits_{i\in\mathbb{N}_p} (G_i)_p \cong (G \cap (\bigoplus\limits_{i\in\mathbb{N}} G_i))_p$. Thus, using this isomorphism, we may assume with some notational abuse that

$$\{y_m^{(p)} \,|\, m \in \mathbb{N}\} \subseteq (G \cap (\bigoplus_{i\in\mathbb{N}} G_i))_p$$

Therefore, the sequence

$$\{y_m^{(p)} \,|\, m \in \mathbb{N}, p \in \mathbb{P}_G, p \in \mathbb{P}_G\} \subseteq G \cap (\bigoplus_{i\in\mathbb{N}} G_i)$$

is a generating set for G.

In order to prove (b), we apply Theorem 3.2 again and, since G has finite exponent, for each $p \in \mathbb{P}_G$, we have that $G^{(p)} \cong \prod\limits_{m\in\mathbb{N}} \langle y_m^{(p)} \rangle$, which yields (b).

Finally, If $\bigoplus\limits_{m\in\mathbb{N}} \langle y_m^{(p)} \rangle[p]$ is weakly observable for each $p \in \mathbb{P}_G$, then $G^{(p)}$ is an implicit direct product for every $p \in \mathbb{P}_G$, which again implies that G is an implicit direct product. $\square$

Question 4.1. *Under what conditions is it possible to extend Theorem A to non-Abelian groups?*

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