

ON CLOSED SUBGROUPS OF PRECOMPACT GROUPS

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Dedicated to María Jesús Chasco on the occasion of her 65th birthday

ABSTRACT. It is a Theorem of W. W. Comfort and K. A. Ross that if G is a subgroup of a compact Abelian group, and S denotes those continuous homomorphisms from G to the one-dimensional torus, then the topology on G is the initial topology given by S . Assume that H is a subgroup of G . We study how the choice of S affects the topological placement and properties of H in G . Among other results, we have made significant progress toward the solution of the following specific questions: How many totally bounded group topologies does G admit such that H is a closed (dense) subgroup? If C_S denotes the poset of all subgroups of G that are S -closed, ordered by inclusion, does C_S have a greatest (resp. smallest) element? We say that a totally bounded (topological, resp.) group is an *SC-group* (*topologically simple*, resp.) if all its subgroups are closed (if G and $\{e\}$ are its only possible closed normal subgroups, resp.) In addition, we investigate the following questions. How many SC-(topologically simple totally bounded, resp.) group topologies does an arbitrary Abelian group G admit?

1. INTRODUCTION

Let (G, τ) be an Abelian Hausdorff topological group. In their 1964 seminal paper [16], Comfort and Ross assigned a Hausdorff group topology τ_S to each point-separating subgroup S of the character group \widehat{G} of G , consisting of all group-homomorphisms from G into the unit circle \mathbb{T} as follows: τ_S is the weakest topology on G that makes the elements of S continuous. (By [16, Theorem 1.9] a subgroup of the compact group \widehat{G} is point-separating if and only if it is dense.) Let $S_\tau := \{\phi \in \widehat{G} : \phi \text{ is } \tau\text{-continuous}\}$. Then S_τ is a subgroup of \widehat{G} . By [16, Theorem 1.2 and Theorem 1.3] the following holds: (i) (G, τ) is totally bounded if and only if $\tau = \tau_{S_\tau}$; (ii) if S is a point-separating subgroup of \widehat{G} , then $S_{\tau_S} = S$. (We call this assertion *the Comfort-Ross Theorem*.) Let $\mathcal{B}(G)$ be the set of all totally bounded group topologies on G . By the Comfort-Ross Theorem there is an order-preserving bijection from $\mathcal{B}(G)$ onto the set of all point-separating subgroups of \widehat{G} .

In this paper we consider τ_S for arbitrary subgroups S of \widehat{G} and S_τ if τ is not necessarily Hausdorff. For an Abelian group G let $\mathcal{PK}(G)$ be the lattice of all precompact group topologies on G , and let $\Sigma(\widehat{G})$ be the lattice of all subgroups of \widehat{G} . By using the

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Comfort-Ross Theorem, Remus in [34] showed that $f : \mathcal{PK}(G) \rightarrow \Sigma(\widehat{G})$ is a lattice-isomorphism, where $f(\tau) = S_\tau$ and $f^{-1}(\mu) = \tau_\mu$ hold. (For a generalization to arbitrary groups see [34, (3.7)] and [31].)

Fast forward to 1983 and 1985 when Remus in [34] (see also [32]) and Berhanu, Comfort and Reid in [1] proved that if G is infinite, then \widehat{G} has $2^{2^{|G|}}$ -many dense subgroups. It follows that an infinite group accepts $2^{2^{|G|}}$ -many totally bounded (precompact and Hausdorff) group topologies. Concerning the number of totally bounded (in particular pseudocompact) group topologies on non-necessarily Abelian groups we refer to [33], [9], [35], [10], [11] and [12].

Many examples regarding the interplay between precompact or totally bounded group topologies between G and subgroups of \widehat{G} have been studied by several colleagues. The following list is by no means complete or exhaustive:

- (1) The authors of [18] showed that if $\tau_1 \subsetneq \tau_2$ are (Hausdorff) group topologies on G such that (G, τ_1) is compact and (G, τ_2) is pseudocompact, then there are τ_2 -closed subgroups of G that are not τ_1 -closed. They also proved that if (G, τ_1) is a totally disconnected Abelian compact group of uncountable weight, then there is a pseudocompact group topology τ_2 such that $\tau_1 \subsetneq \tau_2$. Eventually, the authors in [15] generalize this result by removing the requirement of (G, τ_1) being totally disconnected.
- (2) The authors of [26] focus on an infinite compact (Hausdorff) totally disconnected Abelian group (G, τ) and try to obtain finer totally bounded group topologies τ' such that every τ' -closed subgroup is τ -closed.
- (3) In [38], the author focuses on totally bounded topological groups in which every subgroup is closed.
- (4) In [24, Proposition 3.4], the following is proved: Let G be a totally bounded Abelian group with character group S . If L is a subgroup of S , let L_G denote L equipped with the weakest topology that makes the elements of G (acting on L) continuous. Then, G is pseudocompact if and only if the topology inherited by each countable subgroups of S_G is its corresponding largest totally bounded group topology.
- (5) In [4], the following is proved: Let G be a precompact, bounded torsion Abelian group with character group S . If G is Baire (resp., pseudocompact), then all compact (resp., countably compact) subsets of S_G are finite. Also, G is pseudocompact if and only if all countable subgroups of S_G are closed.

In this paper we further investigate the topological properties of precompact and totally bounded abelian groups via its dual group. More precisely, if H is a subgroup of a precompact Abelian group G , what are the topologies of the form τ_S , with S a subgroup of \widehat{G} such that

- (1) H is τ_S -closed?
- (2) H is τ_S -dense?
- (3) How many subgroups S of \widehat{G} are there such that each of the above happens?
- (4) How many subgroups of \widehat{G} produce the same closed (dense) subgroups in G ?

Similarly, we want to know those subgroups of \widehat{G} producing the same closed (dense) subgroups in G .

We now formulate our main results.

Theorem A. *If G is an infinite Abelian group and $\{H_i : i \in I\}$ is a family of subgroups of G such that $|I| < 2^{|G|}$, then G admits exactly $2^{2^{|G|}}$ -many totally bounded group topologies μ such that H_i is closed in (G, μ) for all $i \in I$.*

By Proposition 6.6 this result is not true if $|I| = 2^{|G|}$.

Let G be an Abelian group and let S be a subgroup of \widehat{G} . We denote by C_S the poset of all subgroups of G that are S -closed. Hence C_S is the poset of all subgroups of G which are closed in the precompact group (G, τ_S) .

It is natural to ask whether there exists a greatest precompact group topology τ_{MS} on G with $C_{MS} = C_S$. The next result gives a positive answer, since $\mathcal{PK}(G)$ and $\Sigma(\widehat{G})$ are isomorphic as lattices.

Theorem B. *Let G be an Abelian group and let S be a subgroup of \widehat{G} . Then there exists a greatest subgroup MS containing S and such that*

$$C_{MS} = C_S.$$

We remember that a totally bounded Abelian group is an *SC group* if all its subgroups are closed (see Remark 6.9. Although this seems to be a very restrictive property, the following result shows that if G is an Abelian group that is not of bounded order, then the number of SC-group topologies is huge.

Theorem C. *The following statements hold:*

- (a) *Let G be an Abelian group which is not of bounded order. Then G admits at least $2^{\mathfrak{c}}$ -many SC-group topologies.*
- (b) *Every countable Abelian group which is not of bounded order admits exactly $2^{\mathfrak{c}}$ -many SC-group topologies.*

We remember that a topological group (G, τ) is *topologically simple* if G and $\{e\}$ are its only possible closed normal subgroups. The anti-discrete topology on a space X is defined as $\{\emptyset, X\}$. Hence, if G is a topologically simple topological group, then G is Hausdorff unless G carries the anti-discrete topology ([25, (5.4)]).

By Corollary 7.2 every infinite Abelian totally bounded group which is topologically simple is algebraically a subgroup of \mathbb{R} . The next results clarifies the question of the existence of topologically simple for subgroups of the real line.

Theorem D. *Let G be a non-trivial subgroup of \mathbb{R} . Then G admits exactly $2^{\mathfrak{c}}$ -many totally bounded group topologies τ such that (G, τ) is topologically simple. The topologies τ can be chosen such that $w(G, \tau) = \mathfrak{c}$.*

To prove the foregoing, we develop some technical results we believe are interesting on its own. For example, we characterize the subgroups of the compact character group \widehat{G} of an Abelian group G , producing the same closed subgroups in G . This is Theorem 5.1.

This paper is organized as follows: Sections 1, 2 and 3 consist of the introduction, notation and results about dense subgroups of precompact groups. In Section 4, we are concerned with the number of totally bounded group topologies on an Abelian group G such that each member of a previously fixed family of subgroups of G is closed on all those topologies. The goal of Section 5 is somewhat different as we deal with the *poset* of all subgroups S of the *big* dual group \widehat{G} having a previously fixed subgroup H of G closed in the weak topology τ_S . In Section 6, we characterize and then calculate the number of totally bounded group topologies on G such that all subgroups of G are closed. In Section 7, we go in the opposite direction as we investigate the existence of topologies without producing closed non-trivial subgroups. In Section 8, we apply previous results to one specific example: The integers. In Section 9 we conclude with final remarks.

2. NOTATION

By \mathbb{Z} we denote the group of integers and by $Z(n)$ the cyclic group of order n . Sometimes we look at \mathbb{Z} as a topological, discrete group, and some others as the underlying group of it as a topological group but without any topology. Set $\omega := \{0, 1, 2, \dots\}$ and $\mathbb{N} := \{1, 2, \dots\}$. \mathbb{P} denotes the set of prime numbers. Our model for \mathbb{T} is the group $([0, 1), + \text{ mod } 1)$ with the topology inherited from \mathbb{R} , when we need to see it as a topological group. Let G be an Abelian group. If $A \subseteq G$, the subgroup generated by A , namely $\langle A \rangle$, is the smallest subgroup containing A ; if A is the singleton $\{a\}$, we write just $\langle a \rangle$. When writing $H \leq G$, we signify that H is a subgroup of G . The symbols $r(G)$, $r_0(G)$ and $r_p(G)$ stand respectively for the *rank*, *torsion-free rank*, and *p-rank* of the group G [21, §16]. tG denotes the torsion subgroup of G . If L and M are groups which are algebraically isomorphic, then we write $L \cong M$. If the topological groups G and H are topologically isomorphic, we write $G \simeq H$. For an Abelian group G we will denote by \widehat{G} the set of all homomorphisms $\phi : G \rightarrow \mathbb{T}$, which we will also refer to as *the characters of G* . \widehat{G} becomes a group by defining $(\phi_1\phi_2)(g) := \phi_1(g) + \phi_2(g) \in \mathbb{T}$ whenever $g \in G$, and equipped with the finite-open topology $\sigma(\widehat{G}, G)$, \widehat{G} becomes a compact topological group. We know $\widehat{\mathbb{Z}} \simeq \mathbb{T}$. When (G, τ) is a topological Abelian group, then $(G, \tau)^\wedge$, or simply G^\wedge if there is no room for confusion, denotes the subgroup of \widehat{G} consisting of the τ -continuous elements. If S is a subgroup of \widehat{G} , denote by G_S the topological group obtained by equipping G with the weakest topology τ_S that makes the elements of S continuous. It follows that G_S is Hausdorff if and only if S separates the elements of G (*i.e.*, S is *point-separating*.) We have pointed out above that the Comfort-Ross Theorem establishes an order-preserving bijection between the set of all totally bounded group topologies on G and the set of all point-separating subgroups of \widehat{G} . Therefore, taking $S = \widehat{G}$, yields the finest precompact group topology on G . This topology is called the *Bohr topology* of G and it is designated by $\tau_b(G)$ here. It is a well-known fact that the Bohr topology is Hausdorff (see [6]).

As usually, if Y is a subspace of the topological space X we let \overline{Y}^X denote the closure of Y in X ; wX and χX stand respectively for the *weight* and *character* of X [6, §3]. Also, given a subgroup S of \widehat{G} , we let $H|_S$ denote the group H equipped with the topology inherited from G_S , and $\mathbb{A}(S, H) := \{\phi \in S : \phi[H] = \{0\}\}$. $\mathbb{A}(S, H)$ is called *the annihilator of H in S* and is a subgroup of S . Similarly, $\mathbb{A}(H, S) := \{g \in H : \varphi(g) = 0 \forall \varphi \in S\}$ is called *the annihilator of S in H* and is a subgroup of H .

A topological group G is *precompact* if whenever U is an open subset of G , there is a finite subset $F \subset G$ such that $G = FU$. If in addition to be precompact, G is Hausdorff, then we say that G is *totally bounded*. It is a Theorem of A. Weil [39] that the completion of the totally bounded group G is a compact group \overline{G} , which we call its *Weil completion*. (The notion *compact space* is used as in [3, Definition 1, p. 83].)

Given a topological group G , it follows that the closure N of its identity is a (normal) subgroup of G [25, (5.4)], hence G/N is a Hausdorff topological group [25, (5.21)]. We refer to the map $\phi : G \rightarrow G/N$, or simply to G/N , the *Hausdorff modification* of G . When the topology of G stems from a subgroup S of \widehat{G} , it follows that $N = \mathbb{A}(G, S)$ if G is Abelian. A topological group G is precompact if and only if its Hausdorff modification is totally bounded. By [3, chapter III, p. 248] this means that the *Hausdorff completion* of G is compact.

Often we will have to use the following.

Lemma 2.1. *Let G be an infinite Abelian group, H a subgroup of G and S a subgroup of \widehat{G} . Then $\widehat{G_S/H}$ and $\widehat{H|_S}$, resp., are group-isomorphic to $\mathbb{A}(S, H)$ and $S/\mathbb{A}(S, H)$, resp.*

Proof: We sketch the proof that $\widehat{G_S/H}$ is group-isomorphic to $\mathbb{A}(S, H)$. Set $N := \mathbb{A}(G, S)$. We then have

$$\begin{array}{ccccc} H & \longrightarrow & NH/N & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ G_S & \longrightarrow & G_S/N & \longrightarrow & \Gamma \\ \downarrow & & \downarrow & & \downarrow \\ G_S/H & \longrightarrow & G_S/NH & \longrightarrow & \Gamma/Y \end{array}$$

where the groups in the middle column are the Hausdorff modifications of the groups on the left column, Γ is the Weil completion of G_S/N and Y the closure of NH/N in Γ . The first vertical arrows are containments whereas the second vertical arrows are projections. It follows that $\mathbb{A}(S, H) = \mathbb{A}(S, NH)$, Γ is a compact Hausdorff group and each of the groups in the middle column is dense in the corresponding group on the right column. To see, for example, that $\mathbb{A}(S, NH/N) \cong \mathbb{A}(S, NH)$, one readily sees that the map $\phi \mapsto \phi \circ \pi$ is an isomorphism, where $\pi : NH \rightarrow NH/N$ is the natural map. We then have that $\widehat{G_S/H} \cong \widehat{G_S/(NH)} \cong \widehat{\Gamma/Y} \cong \mathbb{A}(S, Y)$ (by [25, (24.5)]) $\cong \mathbb{A}(S, NH/N) \cong \mathbb{A}(S, NH) \cong \mathbb{A}(S, H)$.

That $\widehat{H|_S}$ is group-isomorphic to $S/\mathbb{A}(S, H)$ can be done in a similar fashion ■

3. DENSE SUBGROUPS

In the following let G be an Abelian group. Let H be a subgroup of G . Obviously, if H is dense as a topological group in G_S , then $\varphi[H]$ will be dense in $\varphi[G] \subseteq \mathbb{T}$, whenever $\varphi \in S$. We would like to prove that the latter condition is also sufficient.

Theorem 3.1. *Let H be a subgroup of G and S a subgroup of \widehat{G} . Then H is dense in G_S if and only if $\varphi[H]$ is dense in $\varphi[G] \subseteq \mathbb{T}$, whenever $\varphi \in S$.*

Proof: We must show that if V is an open neighborhood in G_S , there exists $h \in H \cap V$. Let $g \in V$. By definition, there exist $\phi_1, \dots, \phi_k \in S$ and $\epsilon > 0$ such that

$$\bigcap_{j=1}^k \phi_j^{-1}[V_\epsilon(\phi_j(g))] \subseteq V,$$

where $V_\epsilon(\zeta) := \{t \in \mathbb{T} : |\zeta - t| < \epsilon\}$. Consider $\overline{S_H}$, the completion of the Hausdorff modification of S equipped with the weakest topology that makes the elements of H continuous, the latter viewed as characters on \widehat{G} . Let $\phi_1, \phi_2 \in S$. If $\phi_1(h) = \phi_2(h)$ for all $h \in H$, then $\phi_1\phi_2^{-1}(h) = 0$, which implies $\phi_1\phi_2^{-1}[H] = \{0\}$. Since $\phi_1\phi_2^{-1} \in S$, our hypothesis implies that $\phi_1\phi_2^{-1}[G] = \{0\}$, hence we would have $\phi_1 = \phi_2$. Therefore, $\phi_1 \neq \phi_2$ implies there is $h \in H$ such that $\phi_1(h) \neq \phi_2(h)$, and thus it follows that $\overline{S_H}$ is compact. By [25, (26.16)], given $\phi_1, \dots, \phi_k \in S_H \subseteq \overline{S_H}$, and $g \in G$, there exists $h \in H$ such that

$$|\phi_j(h) - \phi_j(g)| < \epsilon, \quad (j = 1, 2, 3, \dots, k).$$

Hence $h \in \bigcap_{j=1}^k \phi_j^{-1}[V_\epsilon(\phi_j(g))] \subseteq V$, as required. \blacksquare

Let S be a subgroup of \widehat{G} . Obviously, $H|_S$ is closed in G_S if and only if whenever $H|_S$ is dense in $N|_S$, then $H|_S = N|_S$, for all $H \leq N \leq G$. Set $T := \{f|_N : f \in S\}$. Obviously T is a subgroup of \widehat{N} ; if S separates the points of G , then T separates the points of N and by Comfort-Ross' theorem [16, Theorem 1.2], $N|_S = N_T$.

Lemma 3.2. *The subgroup $H \leq G$ is closed in G_S if and only if*

$$\mathbb{A}(\widehat{G}, H) = \overline{\mathbb{A}(S, H)}^{\widehat{G}} = \overline{S \cap \mathbb{A}(\widehat{G}, H)}^{\widehat{G}}.$$

Proof: If H is closed in G_S , then $\widehat{G_S/H} \cong \mathbb{A}(S, H)$ by Lemma 2.1 and G_S/H is Hausdorff. Therefore $\mathbb{A}(S, H)$ is dense in $\mathbb{A}(\widehat{G}, H)$.

Conversely, since $\widehat{G/H} \cong \mathbb{A}(\widehat{G}, H)$, for every $g \in G \setminus H$, there is $\phi \in \mathbb{A}(\widehat{G}, H)$ such that $\phi(g) \neq 0$ [25, (A.7)]. Therefore, if $\mathbb{A}(S, H)$ is dense in $\mathbb{A}(\widehat{G}, H)$, there is $\chi \in \mathbb{A}(S, H)$ such that $\chi(g) \neq 0$. This implies that H is closed in G_S by Lemma 4.1. \blacksquare

Corollary 3.3. *If $\mathbb{A}(\widehat{G}, H) \subseteq S$, then H is closed in G_S . If H is of finite index in G , then the converse is true.*

Proof: The first assertion holds, since $\mathbb{A}(\widehat{G}, H)$ is closed in \widehat{G} by [25, (23.24)(c)]. Assume that H is a closed subgroup of finite index in G_S . Then H is open in G_S . If $\varphi \in \mathbb{A}(\widehat{G}, H)$, then $\varphi|_H = 0$ is continuous on H , hence continuous on G_S , hence $\varphi \in S$, as required. \blacksquare

If H is not of finite index in G , the converse may be false:

Example 3.4. Consider the group $G := \bigoplus_\omega \langle \frac{1}{4} \rangle$, where $\langle \frac{1}{4} \rangle \subset \mathbb{T}$, and its subgroup $H := \bigoplus_\omega \langle \frac{1}{2} \rangle$. Then $\widehat{G} \cong \prod_\omega Z(4)$ and $\mathbb{A}(\widehat{G}, H) \cong \prod_\omega \{0, 2\}$. Consider the subgroup $S := \bigoplus_\omega Z(4)$ of \widehat{G} . Then H is closed in G_S , yet $\mathbb{A}(\widehat{G}, H) \subseteq S$ is false. [If $g \in G \setminus H$, say $g = (g_k)$, there is $n < \omega$ such that $g_n \notin \langle \frac{1}{2} \rangle$. Consider $\phi = (t_k) \in \widehat{G}$ defined as $t_k = 0$ if $k \neq n$, and $t_n = 2$. It follows that $\phi \in S$, $\phi(g) = 2g_n = \frac{1}{2}$, yet $\phi[H] = \{0\}$.] \blacksquare

Theorem 3.5. $\mathbb{A}(\widehat{G}, H) \cap S = \{0\}$ if and only if H is dense in G_S .

Proof: (\Rightarrow) Deny. By Theorem 3.1, there would be $\phi \in S$ with $\phi[H]$ not dense in $\phi[G]$. This would imply that $|\phi[H]| < \aleph_0$, hence closed in \mathbb{T} , and there would be $g \in G \setminus H$

with $\phi(g) \notin \phi[H]$. But then, if $\pi : \mathbb{T} \longrightarrow \mathbb{T}/\phi[H] \simeq \mathbb{T}$ denotes the canonical map, then $0 \neq \pi \circ \phi \in \mathbb{A}(\widehat{G}, H) \cap S$, a contradiction.

(\Leftarrow) Let $\phi \in \mathbb{A}(\widehat{G}, H) \cap S$. By Theorem 3.1 $\{0\} = \phi[H]$ would be dense in $\phi[G]$, implying that $\{0\} = \phi[G]$, hence $\phi = 0$. \blacksquare

Remark 3.6. When H is a *maximal* proper subgroup of G , then either H is necessarily dense or closed in G_S .

A subgroup E of an Abelian (resp. Abelian topological) group A is said to be *essential* (resp. *topologically essential*) if $E \cap B \neq \{0\}$ whenever B is a non-trivial (resp. and closed) subgroup of A [21, p. 84]. By [21, Lemma 16.2] an *independent system* M is maximal if and only if $\langle M \rangle$ is essential. If M is a maximal independent system in an essential subgroup of A , then M is a maximal independent system in A . Note that in [7] and [2] topologically essential groups in our sense are called “essential”.

The following result characterizes precompact group topologies without proper dense subgroups. First we recall the following result [25, (24.10)].

Lemma 3.7. *Let S be a subgroup of the compact Abelian group \widehat{G} . Then S is closed in \widehat{G} if and only if $S = \mathbb{A}(\widehat{G}, \mathbb{A}(G, S))$.*

Proposition 3.8. *Given $S \leq \widehat{G}$ the group (G, τ_S) contains no proper dense subgroups if and only if S is topologically essential in \widehat{G} .*

Proof: Suppose that S is topologically essential in \widehat{G} and let $H \leq G$ be a proper subgroup of G . Then $\{0\} \not\subseteq \mathbb{A}(\widehat{G}, H)$ by [25, (24.12)]. Therefore $\mathbb{A}(\widehat{G}, H) \cap S \neq \{0\}$, which implies that H is not dense in (G, τ_S) by Theorem 3.5.

Conversely, let L be a non-trivial closed subgroup of \widehat{G} . Then $\mathbb{A}(G, L) \neq G$ by [25, (22.17)], which means that $\mathbb{A}(G, L)$ is not dense in (G, τ_S) . By Theorem 3.5, there is $0 \neq \phi \in S \cap \mathbb{A}(\widehat{G}, \mathbb{A}(G, L))$ which equals $S \cap L$ by the above result. This means that $S \cap L \neq \{0\}$ by Lemma 3.7. \blacksquare

Corollary 3.9. *Given $S \leq \widehat{G}$ the group (G, τ_S) is a totally bounded topological group without proper dense subgroups if and only if S is topologically essential and dense in \widehat{G} .* \blacksquare

Now we are ready to give a partial answer to the problem of finding totally bounded group topologies without proper dense subgroups.

Theorem 3.10. *If G is a torsion-free Abelian group and S an essential subgroup of \widehat{G} , then S is dense in \widehat{G} . As a consequence (G, τ_S) is a totally bounded topological group without proper dense subgroups.*

Proof: We know by the Comfort-Ross Theorem that (G, τ_S) is a totally bounded topological group if and only if S is dense in \widehat{G} . On the other hand, in order to prove that S is dense in \widehat{G} , it will suffice to show that if $g \in \mathbb{A}(G, S)$ then $g = 0$:

Take an arbitrary element $g \in \mathbb{A}(G, S)$ and define $\bar{g} : \widehat{G} \longrightarrow \mathbb{T}$ by $\bar{g}(\phi) := \phi(g)$. It follows that \bar{g} is a continuous character of \widehat{G} [25, (24.8)]. Since S is essential in \widehat{G} , for every $\phi \in \widehat{G}$ there is some $m \in \mathbb{N}$ such that $m\phi = s \in S$, which means that

$m\bar{g}(\phi) = m\phi(g) = s(g) = 0$. Therefore $\bar{g}(\widehat{G}) \subseteq t\mathbb{T}$. Now, since G is torsion-free, it follows that \widehat{G} is connected [25, (24.25)]. Thus $\bar{g}(\widehat{G})$ is a torsion compact connected subgroup of \mathbb{T} , which implies that $\bar{g}(\widehat{G}) = \{0\}$. In other words, we have that $\phi(g) = 0$ for all $\phi \in \widehat{G}$. This yields by [25, (22.17)] that $g = 0$.

That (G, τ_S) contains no dense subgroups follows from Corollary 3.9. \blacksquare

The converse is not true. If S is the torsion subgroup of $\mathbb{T} \simeq \widehat{\mathbb{Z}}$, then S is not essential in \mathbb{T} (see Corollary 8.3 below.)

Not every \widehat{G} has proper essential subgroups [21, Corollary 16.4].

Remark 3.11. Theorem 3.10 does not hold in general. The requirement that G be torsion-free cannot be dropped: Let m be an infinite cardinal, $p \in \mathbb{P}$ and $n \in \mathbb{N}$ with $n > 1$. For $G := \bigoplus_m Z(p^n)$ we get $\widehat{G} = (Z(p^n))^m$. The socle of \widehat{G} is $H := \bigoplus_{2^m} Z(p)$. Then [21, Exercise 16.10] implies that H is an essential subgroup of \widehat{G} . The closure of H is a group of order p . Hence H is not dense in \widehat{G} .

Example 3.12. For $m \geq \omega$ and $p \in \mathbb{P}$, let $G := \bigoplus_m Z(p)$. and $S \neq \widehat{G}$. Then G_S has always non-trivial dense subgroups: For, if $\phi \notin S$, then $G/\ker \phi \cong U$, where U is a subgroup of $t\mathbb{T}$ with order p . Hence $\ker \phi$ is a maximal proper subgroup of G . Since ϕ is not continuous on G_S , $\ker \phi$ is not closed in G_S . By Remark 3.6 it is dense.

On the other hand:

Example 3.13. If G is an infinite Abelian group of bounded order, then G_S has always non-trivial closed subgroups: For, if $\phi \in S$, then $\ker \phi$ is a closed subgroup of G_S . Then $G/\ker \phi \cong U$, where U is a subgroup of bounded order of $t\mathbb{T}$. Hence U is finite. Thus $\ker \phi$ is non-trivial.

Example 3.14. If $|G| > \mathfrak{c}$, then G_S has always non-trivial closed subgroups. For, if $\phi \in S$, then $\ker \phi$ is a closed subgroup of G_S that is non-trivial since it has index at most \mathfrak{c} .

Theorem 3.15. *Let (G, τ) be a compact Abelian group, and let H be a dense subgroup of (G, τ) of finite index. Then there exists a totally bounded group topology τ' on G such that:*

- (1) H is a closed subgroup of (G, τ') .
- (2) If X is the character group of (G, τ) , and Y is the character group of (G, τ') , then Y is isomorphic to $\text{Hom}(G/H, \mathbb{T}) \times X$.
- (3) The completion K of (G, τ') contains a copy of (G, τ) of index $[G : H]$.
- (4) There is a continuous epimorphism $k : K \rightarrow (G, \tau)$, extending the identity from (G, τ') onto (G, τ) .
- (5) The topological groups H as a subgroup of K , and H as a subgroup of (G, τ) are the same.

Remark 3.16. Notice that this result, which is essentially [6, (4.15 ii)], is reminiscent of Tarski's Paradox, obtaining from one compact Hausdorff group, (G, τ) , a bigger group, but no much bigger, K , in which a copy of G is dense in K , but at the same time K is the finite union of cosets of (G, τ) . We offer a different proof than the one in [6, (4.15 ii)].

Proof of Theorem 3.15. Obviously $\text{Hom}(G/H, \mathbb{T})$ is finite. Let $\pi : G \rightarrow G/H$ be the natural map. Consider the finite subgroup of $\text{Hom}(G, \mathbb{T})$ given by

$$\mathcal{F} := \{f \circ \pi : f \in \text{Hom}(G/H, \mathbb{T})\}.$$

Notice that $\psi \in \mathcal{F}$ implies $\psi[H] = \{0\}$, and since H is a dense subgroup of (G, τ) , it follows from Theorem 3.5 that $\mathcal{F} \cap X = \{0\}$. Set

$$Y := \mathcal{F} + X.$$

Finally, set $\tau' := \tau_Y$. Then Y is the character of (G, τ') which is algebraically isomorphic to the character group of its compact Weil completion K . Since G/H is finite, it is immediate to see that $H = \cap\{\ker f : f \in \mathcal{F}\}$, hence H is a closed subgroup of (G, τ') , proving (1). Since $\mathcal{F} \cap X = \{0\}$ and \mathcal{F} is (clearly) isomorphic to $\text{Hom}(G/H, \mathbb{T})$, (2) follows. Applying Pontryagin-van Kampen duality to (2), it follows that K is isomorphic to $G/H \times (G, \tau)$, yielding (3), and (4) in turn. (5) follows since $\psi \in \mathcal{F}$ implies $\psi[H] = \{0\}$. ■

Example 3.17. At the end of [37], the authors sketch a proof of the fact that, for any finite group F and every non-principal ultrafilter \mathcal{U} of ω , the (compact Hausdorff) group $G := F^\omega$ contains a dense subgroup $G_{\mathcal{U}}$ such that $G/G_{\mathcal{U}}$ is isomorphic to F , hence $G_{\mathcal{U}}$ is of finite index in G . It follows from [30] that G has $2^{\mathfrak{c}}$ -many dense subgroups of finite index.

4. CLOSURE OF SUBGROUPS

Our objective in this section is twofold:

- (a) Let G_S be a precompact Abelian group with character group S . Characterize the closure of a given subgroup of G in terms of S .
- (b) Let \mathcal{F} be a family of subgroups of G . Find the number of totally bounded group topologies τ on G such that every element of \mathcal{F} is closed in (G, τ) .

We start with a simple fact.

Lemma 4.1. *A subgroup H of an Abelian group G is closed in G_S if and only if for all $a \in G \setminus H$ there is $\phi \in \mathbb{A}(S, H)$ such that $\phi(a) \neq 0$.*

Proof: (\Leftarrow). Obvious. To see (\Rightarrow), notice that G_S/H is Hausdorff [25, (5.21)] and precompact, hence totally bounded. By the Comfort-Ross Theorem, the character group of G_S/H separates points. Therefore, there is a continuous $f : G_S/H \rightarrow \mathbb{T}$ with $f(aH) \neq 0$. Then $\phi : g \mapsto f(gH)$ is as required. ■

Notice we are not assuming that S is point-separating.

Corollary 4.2. *Let $H \leq G$ and $S \leq \widehat{G}$. Then the following assertions are equivalent:*

- (1) $g \in \overline{H}^{G_S}$.
- (2) $\varphi(g) = 0$ for all $\varphi \in \mathbb{A}(S, H)$.
- (3) $\varphi(g) \in \overline{\varphi[H]}^{\mathbb{T}}$, whenever $\varphi \in S$.

Proof: (1) \Rightarrow (2), (1) \Rightarrow (3) and (3) \Rightarrow (2) are obvious. On the other hand, (2) \Rightarrow (1) is a straightforward consequence of Lemma 4.1. ■

The next results are very similar to [25, (23.24(c)) and (24.10)] with very similar proofs.

Theorem 4.3. *Let S be a subgroup of \widehat{G} , and $N := \mathbb{A}(G, S)$.*

- (1) *If T is a subgroup of S , then $\mathbb{A}(G, T)$ is a closed subgroup of G_S . In particular, $N = \overline{\{0\}}^{G_S}$.*

(2) If $H \leq G$, then $\overline{H}^{G_S} = \mathbb{A}(G, \mathbb{A}(S, H))$.

(3) If H is a closed subgroup of G_S , then $H = H + N = \mathbb{A}(G, \mathbb{A}(S, H))$.

In particular, if H is a subgroup of G , then $H = \mathbb{A}(G, \mathbb{A}(\widehat{G}, H))$.

Proof: (1) If $x \in G \setminus \mathbb{A}(G, T)$, then there is $\varphi \in T$ such that $\varphi(x) \neq 0$. Then apply Lemma 4.1.

(2) (\subseteq) Let g and ϕ be arbitrary elements of \overline{H}^{G_S} and $\mathbb{A}(S, H)$, respectively. Then $\phi(g) \in \overline{\phi[H]} = \{0\}$, which implies $g \in \mathbb{A}(G, \mathbb{A}(S, H))$.

(\supseteq) Suppose that $g \notin \overline{H}^{G_S}$. Set $K := \overline{G_S}$, the Weil completion of G_S , which is a compact group, and $L := \overline{H}^K$. We have that $g \in K \setminus L$ and $K^\wedge = S$. Therefore, by [25, (23.26)] there is $\chi \in K^\wedge \simeq S$ such that $\chi[L] = \{0\}$ and $\chi(g) \neq 0$. Plainly, $\chi \in \mathbb{A}(S, H)$ and $g \notin \mathbb{A}(G, \mathbb{A}(S, H))$.

(3) follows from (2).

The last statement follows by taking $S = \widehat{G}$ since in this case all subgroups of G are closed in G_S (Lemma 4.6). \blacksquare

Remark 4.4. One may be tempted to believe that if S is a point-separating subgroup of \widehat{G} , then $S = \mathbb{A}(\widehat{G}, \mathbb{A}(G, S))$. This is false in general. Of course $S \subseteq \mathbb{A}(\widehat{G}, \mathbb{A}(G, S))$. But if S is any non-torsion subgroup of $\mathbb{T} \simeq \widehat{\mathbb{Z}}$, then $\mathbb{A}(\mathbb{Z}, S) = \{0\} \implies \mathbb{A}(\widehat{\mathbb{Z}}, \mathbb{A}(\mathbb{Z}, S)) = \widehat{\mathbb{Z}}$; see Corollary 8.3 below.

Corollary 4.5. *Let $H \leq N$ be subgroups of G , and let S be a subgroup of \widehat{G} . The following conditions are equivalent:*

(1) $N \subseteq \overline{H}^{G_S}$.

(2) H is τ_S -dense in N .

(3) $\mathbb{A}(G, \mathbb{A}(S, N)) = \mathbb{A}(G, \mathbb{A}(S, H))$.

In addition, $N = \overline{H}^{G_S}$ if and only if $N = \mathbb{A}(G, \mathbb{A}(S, N)) = \mathbb{A}(G, \mathbb{A}(S, H))$. \blacksquare

Comfort and Saks were the first who gave examples of SC groups.

Lemma 4.6. ([17, Lemma 2.1]) *Let G be an Abelian group with the Bohr topology τ_b . Then every subgroup of G is closed in (G, τ_b) .*

By using this result the following is shown in [4, Lemma 2.5]: Let G be an Abelian group of bounded order, and let τ be a totally bounded group topology on G . Then every subgroup of G is closed in (G, τ) if and only if $\tau = \tau_b$.

Hence for such groups G the following is false: Let $\{H_i : i \in I\}$ be a family of subgroups of G . Then there is a totally bounded group topology $\tau \neq \tau_b$ on G such that H_i is closed in (G, τ) for all $i \in I$.

In Theorem **A** we give the best possible result for $|I| < 2^{|G|}$. The corresponding proof needs some preparation.

Lemma 4.7. ([1, Theorem 4.3.]) *Let (G, τ) be an infinite totally bounded group. Then*

(a) $\chi(G, \tau) = w(G, \tau)$.

(b) If in addition G is Abelian, then $\chi(G, \tau) = (G, \tau)^\wedge$.

Lemma 4.8. ([33, Lemma (2.9)]) *Let G be a discrete group which is maximally almost periodic. Then G admits a totally bounded group topology τ with $w(G, \tau) \leq |G|$.*

Lemma 4.9. ([34, Lemma (2.16)]) *Let G be an infinite Abelian group. If $r(G)$ is finite, then G is countable. Otherwise, $r(G) = |G|$ holds.*

Lemma 4.10. *Let G be an infinite group, and let $\{\tau_i : i \in I\}$ be a family of group topologies on G such there is $i_0 \in I$ with $\chi(G, \tau_{i_0}) \geq \omega$. Set $\tau := \bigvee_{i \in I} \tau_i$ in the lattice of all group topologies on G . Then $\chi(G, \tau) \leq \max\{|I|, \sup\{\chi(G, \tau_i) : i \in I\}\}$.*

Proof: Let e be the identity of G . For all $i \in I$, let \mathcal{B}_i be a neighborhood-basis of e with respect to (G, τ_i) such that $|\mathcal{B}_i| = \chi(G, \tau_i)$. For any finite subset $M = \{i_1, \dots, i_n\}$ of I let $\mathcal{B}_M := \{\bigcap_{j=1}^n U_j : U_j \in \mathcal{B}_{i_j}\}$. Let \mathcal{F} be the set of all finite subsets of I . Then $\mathcal{B} := \bigcup_{M \in \mathcal{F}} \mathcal{B}_M$ is a neighborhood-basis of e with respect to τ . Let $\alpha := \sup\{\chi(G, \tau_i) : i \in I\}$. Then $\alpha \geq \omega$, $|\mathcal{B}_M| \leq \alpha$ for all $M \in \mathcal{F}$, and $|\mathcal{B}_{\{i_0\}}| = |\mathcal{B}_{i_0}| \geq \omega$. For $|I| \geq \omega$ we have $|\mathcal{F}| = |I|$. Since $\mathcal{B} := \bigcup_{M \in \mathcal{F}} \mathcal{B}_M$, finally [29, Proposition 4.4 and 4.5, Chapter III] implies $|\mathcal{B}| \leq \max\{|\mathcal{F}|, \alpha\} = \max\{|I|, \alpha\}$. ■

We are ready for the proof of our first main result.

Proof of Theorem A. Let $i \in I$. By Lemma 4.8, G and G/H_i admit totally bounded group topologies τ and ν_i , respectively, such that $w(G, \tau) \leq |G|$ and $w(G/H, \nu_i) \leq |G/H_I| \leq |G|$. Let $\pi_i : G \rightarrow G/H_i$ be the canonical epimorphism and $\tilde{\nu}_i$ be the initial topology on G with respect to π_i and ν_i . Surely $\tilde{\nu}_i$ is precompact. Then the quotient topology of $\tilde{\nu}_i$ on G/H_i is ν_i . Hence H_i is closed in $(G, \tilde{\nu}_i)$. Now $\chi(G, \tilde{\nu}_i) = \chi(G/H, \nu_i) \leq |G|$. Define $\mu_0 := \tau \vee (\bigvee_{i \in I} \tilde{\nu}_i)$. Then μ_0 is totally bounded, and H_i is closed in (G, μ_0) . Lemma 4.10 implies $\chi(G, \mu_0) \leq \max\{|I|, |G|\}$. By $|I| < 2^{|G|}$ we get $\chi(G, \mu_0) < 2^{|G|}$. Finally, Lemma 4.7 implies $w(G, \mu_0) = |(G, \mu_0)^\wedge| < 2^{|G|}$.

$|\widehat{G}| = 2^{|G|}$ holds by a result of Kakutani [27]. For $M := (G, \mu_0)^\wedge$ we have $|\widehat{G}/M| = 2^{|G|}$ since $|M| < 2^{|G|}$. Lemma 4.9 implies $r(\widehat{G}/M) = 2^{|G|}$. Thus \widehat{G}/M contains $2^{2^{|G|}}$ -many subgroups. Hence \widehat{G} has this number of subgroups containing M . Thus by the Comfort-Ross Theorem there are $2^{2^{|G|}}$ -many totally bounded group topologies μ on G being finer than μ_0 . Surely H_i is closed in (G, μ) . ■

The argumentation in the first part of the proof of Theorem A (for $|I| = 1$) shows:

Theorem 4.11. *Let G be an infinite group, and let H be a normal subgroup of G such that G/H is maximally almost periodic in the discrete topology. If the finest precompact group topology on G is Hausdorff, then there is a totally bounded group topology μ_0 on G with $w(G, \mu_0) \leq |G|$ such that H is closed in (G, μ_0) .* ■

By applying Theorem 4.11 and [10, Corollary 2.6(a)], we get

Theorem 4.12. *Let G be an infinite group, and let H be a normal subgroup of G such that G/H is maximally almost periodic in the discrete topology. If the finest precompact group topology τ_f on G is Hausdorff with $\alpha := w(G, \tau_f) > |G|$, then G admits at least α -many totally group topologies μ such that H is closed in (G, μ) .* ■

Remark 4.13. Remus showed in [35, Satz (2.2)] that if G is an Abelian group and H is a subgroup of G , then for every $\tau \in \mathcal{PK}(H)$ there is a finest precompact group topology τ_f^H on G which induces τ . If τ is totally bounded, then τ_f^H has the same property. This implies that all subgroups of G which contain H are closed in (G, τ_f^H) , see [35, Folgerung (2.6)(a)].

In summary: If G is an Abelian group and H is an infinite subgroup of G , then there are at least $2^{2^{|H|}}$ -many totally bounded group topologies τ on G such that every subgroup N of G with $H \subseteq N$ is closed in (G, τ) .

Let G be an infinite countable Abelian group with only countably many subgroups. These groups are classified in [36]. Groups like $\bigoplus_m \mathbb{Z} \oplus \bigoplus_n Z(p^\infty)$ with $m, n < \omega$ are examples. By Theorem **A** these groups have exactly 2^c many SC-group topologies.

5. SUBGROUPS OF \widehat{G} PRODUCING THE SAME CLOSED SUBGROUPS IN G

In this section, we study the poset of all subgroups $S \leq \widehat{G}$ having a previously fixed subgroup $H \leq G$ closed in the weak topology τ_S . We prove that there always exists a greatest element (Theorem **B**) but not a smallest one in general.

We first characterize the subgroups of \widehat{G} producing the same closed subgroups of G .

Theorem 5.1. *Let G be an Abelian group and let S_1, S_2 be subgroups of \widehat{G} . Then S_1 and S_2 produce the same closed subgroups in G if and only if*

$$\overline{L \cap S_1}^{\widehat{G}} = \overline{L \cap S_2}^{\widehat{G}}$$

for all closed subgroups L of \widehat{G} , when equipped with $\sigma(\widehat{G}, G)$.

Proof: (\Leftarrow) Suppose that

$$\overline{L \cap S_1}^{\widehat{G}} = \overline{L \cap S_2}^{\widehat{G}}$$

for all closed subgroups L of \widehat{G} and let H be a τ_{S_1} -closed subgroup of G . Then $S_1 \cap \mathbb{A}(\widehat{G}, H) = \mathbb{A}(S_1, H)$ is dense in $\mathbb{A}(\widehat{G}, H)$ by Lemma 3.2. As a consequence, $S_2 \cap \mathbb{A}(\widehat{G}, H) = \mathbb{A}(S_2, H)$ is dense in $\mathbb{A}(\widehat{G}, H)$ as well, which implies by Lemma 3.2 that H is τ_{S_2} -closed. (\Rightarrow) Reasoning by contradiction, suppose there is a closed subgroup L of \widehat{G} such that $\overline{S_1 \cap L}^{\widehat{G}} \not\subseteq \overline{S_2 \cap L}^{\widehat{G}}$. Observe that, by Lemma 3.7,

$$\overline{S_i \cap L}^{\widehat{G}} = \mathbb{A}(\widehat{G}, \mathbb{A}(G, \overline{S_i \cap L}^{\widehat{G}})), \quad 1 \leq i \leq 2.$$

Set $H := \mathbb{A}(G, \overline{S_1 \cap L}^{\widehat{G}})$. Then H is τ_{S_1} -closed by Theorem 4.3. However, Lemma 3.2 implies

$$\begin{aligned} \overline{\mathbb{A}(S_2, H)}^{\widehat{G}} &= \overline{S_2 \cap \mathbb{A}(\widehat{G}, H)}^{\widehat{G}} = \overline{S_2 \cap \mathbb{A}(\widehat{G}, \mathbb{A}(G, \overline{S_1 \cap L}^{\widehat{G}}))}^{\widehat{G}} = \overline{S_2 \cap \overline{S_1 \cap L}^{\widehat{G}}}^{\widehat{G}} \subseteq \\ &\overline{S_2 \cap L}^{\widehat{G}} \not\subseteq \overline{S_1 \cap L}^{\widehat{G}} = \mathbb{A}(\widehat{G}, \mathbb{A}(G, \overline{S_1 \cap L}^{\widehat{G}})) = \mathbb{A}(\widehat{G}, H). \end{aligned}$$

As a consequence, $\mathbb{A}(S_2, H)$ is not dense in $\mathbb{A}(\widehat{G}, H)$, which means by Lemma 3.2 that H is not τ_{S_2} -closed. This completes the proof. \blacksquare

The latter result implies the following consequences. In the Corollary below, note that $G/\mathbb{A}(G, R)$ is Hausdorff.

Corollary 5.2. *Let $S \leq \widehat{G}$ and let $R := \overline{S}^{\widehat{G}}$. The following assertions are equivalent:*

- (1) $\overline{L \cap S}^{\widehat{G}} = L \cap R$ for every closed subgroup L of \widehat{G} .

- (2) S and R produce the same closed subgroups in G .
 (3) S and R produce the same closed subgroups in $G/\mathbb{A}(G, R)$.

Proof: (1) \Leftrightarrow (2) is a straightforward consequence of Theorem 5.1.

(2) \Leftrightarrow (3) By Theorem 4.3 the closure of $\{0\}$ in G_S resp. G_R is $\mathbb{A}(G, R)$. Now apply [25, (5.34)] . \blacksquare

Example 5.3. Fix different prime numbers p and q and consider $S := Z(p^\infty)$. Of course, $R := \overline{S}^{\widehat{\mathbb{Z}}} = \mathbb{T}$. We will see below in Corollary 8.2 (or see [19, (3.5.4) and (3.5.5)]), that the closed subgroups of \mathbb{Z}_S have the form $p^k\mathbb{Z}$. On the other hand, every subgroup of \mathbb{Z} is closed in \mathbb{Z}_R (Lemma 4.6. If $L := \langle 1/q \rangle$, then L is a closed subgroup of $\widehat{\mathbb{Z}}$, but $\overline{L \cap S}^{\widehat{\mathbb{Z}}} = \{0\} \neq L = L \cap R$.

Corollary 5.4. *Let $R \leq \widehat{G}$ be a closed subgroup of \widehat{G} . Then $H \leq G$ is τ_R -closed if and only if $H = H + \mathbb{A}(G, R)$. In other words, $\overline{H}^{G_R} = H + \mathbb{A}(G, R)$.*

Proof: If R separates points in G , then $R = \widehat{G}$ and the result follows from [17, Lemma 2.1]. So we may assume that $R \neq \widehat{G}$. By Theorem 4.3, $\overline{\{0\}}^{G_R} = \mathbb{A}(G, R)$. Take the canonical continuous epimorphism $\pi : (G, \tau_R) \rightarrow (G/\mathbb{A}(G, R), \tau_R^q)$, where τ_R^q is the Hausdorff quotient topology. Now $\mathbb{A}(G, R) \subseteq \overline{H}^{G_R}$ implies $H \subseteq H + \mathbb{A}(G, R) \subseteq \overline{H}^{G_R}$. By [25, (23.25)] the character group of $G/\mathbb{A}(G, R)$ endowed with the discrete topology is topologically isomorphic to $\mathbb{A}(\widehat{G}, \mathbb{A}(G, R))$. Lemma 3.7 implies $R = \mathbb{A}(\widehat{G}, \mathbb{A}(G, R))$. The topological isomorphism is defined by $\rho(\psi) = \psi \circ \pi$. Thus τ_R^q is the Bohr topology on $G/\mathbb{A}(G, R)$. As a consequence, every subgroup of the latter group is τ_R -closed by Lemma 4.6. In particular, the subgroup $\pi[H]$ is τ_R -closed. Being the quotient map obviously τ_R -continuous, it follows that

$$H + \mathbb{A}(G, R) = \pi^{-1}[\pi[H]]$$

is τ_R -closed in G . \blacksquare

We can now use Theorem 5.1 as a main tool to prove that every subgroup $S \leq \widehat{G}$ is contained in a greatest subgroup MS defining the same set of τ_S -closed subgroups.

Proof of Theorem B. Set

$$\mathcal{S} := \{T \leq \widehat{G} : S \leq T \text{ and } C_T = C_S\}.$$

We claim that the pair (\mathcal{S}, \subseteq) is inductive if it is ordered by inclusion.

Indeed, let $\{S_i : i \in I\}$ be a chain in (\mathcal{S}, \subseteq) . Take $S_0 := \cup\{S_i : i \in I\}$. If $L \leq \widehat{G}$ is closed, we have

$$\overline{S \cap L}^{\widehat{G}} \subseteq \overline{S_0 \cap L}^{\widehat{G}}.$$

On the other hand

$$\begin{aligned} \overline{S_0 \cap L}^{\widehat{G}} &= \overline{(\cup\{S_i : i \in I\}) \cap L}^{\widehat{G}} = \overline{\cup\{S_i \cap L : i \in I\}}^{\widehat{G}} \subseteq \\ &\overline{\cup\{\overline{S_i \cap L} : i \in I\}}^{\widehat{G}} = \overline{\cup\{\overline{S \cap L} : i \in I\}}^{\widehat{G}} = \overline{S \cap L}^{\widehat{G}}. \end{aligned}$$

The second to last equality above follows from Theorem 5.1. This implies that $S_0 \in \mathcal{S}$, which completes the proof of the claim. Therefore, we have verified the existence of

maximal elements in (\mathcal{S}, \subseteq) by Zorn's Lemma. In order to demonstrate that there is a greatest element, it will suffice to prove that if S_1 and S_2 are in \mathcal{S} , so is $S_1 + S_2$.

It is clear that every subgroup τ_S -closed is also $\tau_{(S_1+S_2)}$ -closed. Therefore $C_S \subseteq C_{(S_1+S_2)}$.

Conversely, suppose that $H \leq G$ is $\tau_{(S_1+S_2)}$ -closed. By Lemma 4.1 we have that if $g \notin H$, there is $\chi_1 \in S_1$ and $\chi_2 \in S_2$ such that $(\chi_1 + \chi_2)[H] = \{0\}$ and $(\chi_1 + \chi_2)(g) \neq 0$. Therefore

$$\chi_1|_H = -\chi_2|_H.$$

Set

$$K = \overline{H}^{G_S}$$

and observe that, since $S_i \in \mathcal{S}$, $1 \leq i \leq 2$, it follows

$$K = \overline{H}^{G_S} = \overline{H}^{G_{S_i}}, \quad 1 \leq i \leq 2.$$

Therefore both characters χ_1 and χ_2 can be extended to K continuously. Denoting these extensions by χ_1 and χ_2 again for simplicity's sake, we have that $(\chi_1 + \chi_2)[H] = \{0\}$, which implies $(\chi_1 + \chi_2)[K] = \{0\}$. This entails that $g \notin K$. Hence $K = H$, which completes the proof. \blacksquare

Remark 5.5. Given an Abelian group G , it is not true in general that for each subgroup $S \leq \widehat{G}$, there exists a minimum subgroup $mS \leq \widehat{G}$ such that

$$C_{mS} = C_S$$

(see Theorem 8.10 below).

6. TOTALLY BOUNDED GROUPS TOPOLOGIES IN WHICH EVERY SUBGROUP IS CLOSED

In this section we approach the following

Question 6.1. *Let G be an infinite Abelian group.*

- (a) *Characterize the subgroups S of \widehat{G} such that all subgroups of G are τ_S -closed.*
- (b) *Find the number of SC-group topologies on G .*

Consider the torsion subgroup $t\widehat{G}$ of \widehat{G} , take $\phi \in t\widehat{G}$ and $H := \ker \phi$. We have then that H is of finite index in G , hence if it were closed in G_S , then it would be also open and this would imply $\phi \in S$. It follows

Lemma 6.2. *If all the subgroups of G are τ_S -closed, then $t\widehat{G} \leq S$.* \blacksquare

As a consequence, we have

Corollary 6.3. *If G is of bounded order and all the subgroups of G are τ_S -closed, then $S = \widehat{G}$ and $G_S = (G, \tau_b(G))$.* \blacksquare

After this result, one might conjecture that all the subgroups of G are τ_S -closed if and only if $t\widehat{G} \leq S$. However this is wrong. Indeed, consider the group $G = Z(p^\infty)$, p prime. Since all proper subgroups of G are finite, we have that all subgroups are also closed on every totally bounded group topology of G . Nevertheless, we have that $\widehat{G} = \Delta_p$, which is a torsion-free group. This shows that $t\widehat{G}$ can be trivial even though every totally bounded topology on G has all its subgroups closed.

Definition 6.4. ([19], p. 133) A subgroup H of a topological group (G, τ) is *totally dense* in G if $H \cap K$ is dense in K for every closed normal subgroup K of G .

Now Question 6.1(a) can be solved.

Proposition 6.5. *Let G be an Abelian group and S be a subgroup of \widehat{G} . Then every subgroup of (G, τ_S) is closed if and only if S is totally dense in \widehat{G} .*

Proof: By Lemma 4.6 every subgroup of G is closed in $(G, \tau_{\widehat{G}})$. Then apply Theorem 5.1 if we take $S_1 = S$ and $S_2 = \widehat{G}$. ■

Now [28, Corollary 2] and Proposition 6.5 imply

Proposition 6.6. *Let G be an Abelian group equipped with the finest precompact group topology, that is, the Bohr topology $\tau_b(G)$. Then the following assertions are equivalent:*

- (1) $\tau_b(G)$ is the only precompact group topology τ on G such that all subgroups of (G, τ) are closed;
- (2) G is of bounded order.

We notice that Corollary 6.3 yields the implication (2) \Rightarrow (1) in the above proposition that, incidentally, is proven in [4, Lemma 2.5]. On the other hand, Proposition 6.5 and [19, Exercise 5.5.6] imply

Proposition 6.7. *Let G be an infinite Abelian group. Then the following assertions are equivalent:*

- (1) For every totally bounded group (G, τ) each subgroup of (G, τ) is closed;
- (2) There is a prime number p such that G is isomorphic to $Z(p^\infty)$.

From Proposition 6.5, the solution to Question 6.1(b) for a given group G reduces to the search of totally dense subgroups of \widehat{G} . In this direction, Comfort and Dikranjan [7] have proven that the smallest (under inclusion) totally dense subgroup of a compact Abelian group K is tK , when it is itself totally dense. Furthermore, according to their Theorem 4.1, this happens if and only if K has no copies of Δ_p . Therefore, we have the following:

Proposition 6.8. *Let G be an Abelian group such that \widehat{G} has copies of Δ_p for no prime p . Let S be a subgroup of \widehat{G} . All the subgroups of G are τ_S -closed if and only if $t\widehat{G} \leq S \leq \widehat{G}$.*

Every totally bounded group topology on $G = Z(p^\infty)$ has all its subgroups closed. In this case of course there exist $2^{\mathfrak{c}}$ -many such topologies. Hence we see that the number of totally bounded group topologies making all the subgroups of G closed depends on G and goes from 1 (groups of bounded order) all the way to $2^{2^{|G|}}$ ($G = Z(p^\infty)$).

In relation with the above questions, we first list some observations.

Remark 6.9.

- (a) Let G be an Abelian group which is not of bounded order. Then G admits at least \mathfrak{c} -many totally bounded group topologies τ such that every subgroup of G is closed in (G, τ) . For, $\widehat{G} = B(\widehat{G})$, the set of compact elements of \widehat{G} . If \widehat{G} were not *admissible*, i.e., if $B(\widehat{G}) = t\widehat{G}$, then \widehat{G} would be a torsion group, implying that G is of bounded order [25, (25.9)], a contradiction. By [28, Corollary 3] \widehat{G}

contains a totally dense subgroup H such that $\widehat{G}/H \cong \mathbb{Q}$. Since \mathbb{Q} contains \mathfrak{c} -many subgroups, we see that \widehat{G} contains \mathfrak{c} -many subgroups, each containing the totally dense subgroup H . Now apply Proposition 6.5.

- (b) In [38, p. 170] *SC groups* (see Section 1) and *DSC groups* are introduced: A compact Abelian group having a dense SC group is called a DSC group, in other words: it is the completion of a SC group.

Let G be any group. Then the finite-index topology τ_l on G is the finest linear precompact topology on G . If G is Abelian, the character group of (G, τ_l) is $t\widehat{G}$. Proposition 6.5 implies: (G, τ_l) is a SC group if and only if $t\widehat{G}$ is totally dense in \widehat{G} . Note that τ_l is the only possible linear precompact group topology μ such that (G, μ) is a SC group.

Now the proof of [38, Theorem 1.15] implies the following: (G, τ_l) is a SC group if and only if G has a free Abelian subgroup F of finite rank such that G/F is a torsion group and for each prime p the p -component of G/F is of bounded order. For the proof, take $H = (G, \tau_l)$, and \widehat{G} the completion of this H in the proof of [38, Theorem 1.15]. Note that in the proof of [38, Theorem 1.15] it is allowed that the rank of F is zero [Look at the end of the proof of [38, Proposition 1.14].].

First we will show that Abelian groups G with $r_0(G) > 0$ have many SC-group topologies.

Lemma 6.10. *Let $m > 0$ be a cardinal. Then \mathbb{T}^m contains $(2^{\mathfrak{c}} \cdot 2^{2^m})$ -many totally dense subgroups.*

Proof: By [20, Theorem 3] or [23], the group $(t\mathbb{T})^m$ is totally dense in \mathbb{T}^m . Let $H := \mathbb{T}^m / (t\mathbb{T})^m$. Then H is algebraically isomorphic to \mathbb{R}^m and, therefore, has torsion-free rank $|H|$. Hence there are $2^{|H|}$ -many subgroups S of \mathbb{T}^m containing the totally dense group $(t\mathbb{T})^m$. Thus each such S is a totally dense subgroup of \mathbb{T}^m . Finally, it suffices to notice that $|H| = \mathfrak{c}^m = \mathfrak{c} \cdot 2^m$. ■

Theorem 6.11. *Let G be an Abelian group with $r_0(G) > 0$. Then there are $(2^{\mathfrak{c}} \cdot 2^{2^{r_0(G)}})$ -many SC-group topologies on G .*

Proof: Let $K := \widehat{G}$. By [9, Lemma 5.4], there is a continuous homomorphism from K onto $\mathbb{T}^{r_0(G)}$. Now it suffices to apply [18, Lemma 4.1 (c)], Lemma 6.10 and Proposition 6.5 in order to complete the proof. ■

Corollary 6.12. *Let G be an infinite Abelian group. If $r_0(G) = |G|$, then there are exactly $2^{2^{|G|}}$ -many SC-group topologies on G .* ■

Next we consider some Abelian torsion groups that are not of bounded order.

Theorem 6.13. *Let G be an Abelian torsion group such that $I = \{p \in \mathbb{P} : r_p(G) \neq 0\}$ is infinite. Then G admits at least $2^{\mathfrak{c}}$ -many SC-group topologies.*

Proof: Let $H := \prod_{p \in I} Z(p)$. By [18, Lemma 5.5], $tH = \bigoplus_{p \in I} Z(p)$ is totally dense in H . Since $|tH| = \omega < |H| = \mathfrak{c}$, the group H/tH has $2^{\mathfrak{c}}$ many subgroups by Lemma 4.8. Hence H possesses the same number of totally dense subgroups. Let $K := \widehat{G}$. Then by [9, Lemma 5.4] there is a continuous epimorphism from K onto $\prod_{p \in \mathbb{P}} (Z(p))^{r_p(G)}$. In

particular, there is a continuous epimorphism from K onto H . Now [18, Lemma 4.1(c)] implies that K has 2^c -many totally dense subgroups. Finally apply Proposition 6.5 to complete the proof. \blacksquare

We observe that Proposition 6.6 implies that the above result fails if I is finite.

Proposition 6.14. *Let G be an infinite Abelian torsion group of cardinality α . Assume $\text{cf}(\alpha) = \omega$ and $\log(2^\alpha) = \alpha$. Let $I := \{p \in \mathbb{P} : r_p(G) \geq \omega\}$ be infinite. If $r_p(G) < |G|$ for all $p \in I$ and all $r_p(G)$ with $p \in I$ are distinct, then G admits exactly $2^{2^{|G|}}$ -many SC-group topologies.*

Proof: Let $\alpha_p := r_p(G)$ for all $p \in \mathbb{P}$. Consider $K := \prod_{p \in I} Z(p)^{\alpha_p}$. Then $tK = \bigoplus_{p \in I} (\bigoplus_{2^{\alpha_p}} Z(p))$. Hence $\beta := |tK| = \sum_{p \in I} 2^{\alpha_p}$. Since G has infinite rank, by Lemma 4.9 we get $\alpha = \sum_{p \in I} \alpha_p$. Then [14, Lemma 5.4] implies $\beta < 2^\alpha$. K is the character group of $H := \bigoplus_{p \in I} (\bigoplus_{\alpha_p} Z(p))$. Thus $|K| = 2^\alpha$. Hence $|tK| < |K|$. By [18, Lemma 5.5], tK is totally dense in K . Since $|K/tK| = |K|$, it follows that K has $2^{|K|} = 2^{2^{|G|}}$ many totally dense subgroups. By [9, Lemma 5.4], there is a continuous epimorphism from G onto K . Apply [18, Lemma 4.1(c)] and Proposition 6.5 to complete the proof. \blacksquare

Remark 6.15. (a) We give an example for an application of Theorem 6.14:

Let m be an infinite cardinal. Define $\alpha_0 := m$ and $\alpha_{n+1} := 2^{\alpha_n}$ for all $n \in \omega$. Let $(p_n)_{n \in \omega}$ be a sequence of prime numbers which are pairwise different. Set $G := \bigoplus_{n \in \omega} (\bigoplus_{\alpha_n} Z(p_n))$. Then $|G| = \alpha := \sum_{n \in \omega} \alpha_n > m$. Clearly $\text{cf}(\alpha) = \omega$, and [14, Lemma 5.4] implies $\alpha = \log(2^\alpha)$. Then Proposition 6.14 implies that G admits exactly $2^{2^{|G|}}$ -many SC-group topologies.

(b) Let K be the Abelian compact totally disconnected group of weight α defined in the proof of the above Theorem. Then $\text{cf}(\alpha) = \omega$ and $\alpha = \log(2^\alpha)$ imply the crucial fact $|tK| < |K|$. Now let L be any Abelian compact totally disconnected group of infinite weight α with $|tL| < |L|$. Then [14, Theorem 5.8] implies $\text{cf}(\alpha) = \omega$ and $\alpha = \log(2^\alpha)$.

Now it is natural to pose the following

Question 6.16. *Let G be an infinite Abelian group which is not of bounded order. Does G admit exactly $2^{2^{|G|}}$ -many SC-group topologies in the following cases:*

- (a) G is a torsion group,
- (b) $0 < r_0(G) < |G|$.

We will show that in both cases the answer is “no”. For that we need several lemmas.

Lemma 6.17. *For an infinite cardinal m and $p \in \mathbb{P}$ let $G := \bigoplus_m Z(p^\infty)$. Then G admits at least 2^{2^m} -many SC-group topologies.*

Proof: We have $\widehat{G} = \Delta_p^m$. By [28, Lemma 5] Δ_p contains a dense subgroup H with $\Delta_p/H \cong \mathbb{Q}$. Then H^m is dense in \widehat{G} , and $V := \widehat{G}/H^m \cong \mathbb{Q}^m \cong \bigoplus_{2^m} \mathbb{Q}$. Let

$$f: \Delta_p^m \longrightarrow \bigoplus_{2^m} \mathbb{Q}$$

be the algebraic epimorphism

$$\Delta_p^m \longrightarrow V \longrightarrow \mathbb{Q}^m \longrightarrow \bigoplus_{2^m} \mathbb{Q}.$$

If $S \subseteq 2^m$ is nonempty, let $\pi_S: \bigoplus_{2^m} \mathbb{Q} \rightarrow \bigoplus_S \mathbb{Q}$ be the natural projection and let $\Sigma_S: \bigoplus_S \mathbb{Q} \rightarrow \mathbb{Q}$ be the homomorphism that adds up the coordinates of elements in $\bigoplus_S \mathbb{Q}$:

$$\Sigma_S(q_j) = \sum_{j \in S} q_j < \infty.$$

We then have

$$\Delta_p^m \xrightarrow{f} \bigoplus_{2^m} \mathbb{Q} \xrightarrow{\pi_S} \bigoplus_S \mathbb{Q} \xrightarrow{\Sigma_S} \mathbb{Q}.$$

Let $H_S := \ker \Sigma_S \circ \pi_S \circ f$. Note that $H^m \subseteq H_S$ and by [28, Cor. 3] we have that H_S is totally dense in Δ_p^m . We now show that if S_1 and S_2 are different subsets of 2^m , then $H_{S_1} \neq H_{S_2}$. Suppose, say, that $\sigma \in S_1 \setminus S_2$. Consider $q \in \mathbb{Q}$, $q \neq 0$, and define $g \in \bigoplus_{2^m} \mathbb{Q}$ by

$$g(t) := \begin{cases} q & \text{if } t = \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have that

$$\pi_{S_1}(g(t)) = \begin{cases} q & \text{if } t = \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

while

$$\pi_{S_2}(g(t)) = 0.$$

Pick $x \in f^{-1}(g)$. Then

$$\Sigma \circ \pi_{S_1} \circ f(x) = \Sigma \circ \pi_{S_1}(g) = q \neq 0 \Rightarrow x \notin H_{S_1},$$

while

$$\Sigma \circ \pi_{S_2} \circ f(x) = \Sigma \circ \pi_{S_2}(g) = 0 \Rightarrow x \in H_{S_2}.$$

Hence $H_{S_1} \neq H_{S_2}$.

In summary, we have constructed 2^{2^m} -many dense subgroups H_S of \widehat{G} such that $\widehat{G}/W_i \cong \mathbb{Q}$. Now, applying Proposition 6.5, we obtain that G admits at least 2^{2^m} -many SC-group topologies. \blacksquare

Lemma 6.18. *Let G be an infinite Abelian group and H an infinite subgroup of it. If H has m -many SC group topologies, then G admits at least m -many such group topologies.*

Proof: By duality theory there is a continuous epimorphism from \widehat{G} onto \widehat{H} . Then Proposition 6.5 implies that \widehat{H} has m -many totally dense subgroups. Thus \widehat{G} has also (at least) m -many totally dense subgroups by [18, Lemma 4.1(c)]. Now apply again Proposition 6.5. \blacksquare

Lemma 6.19. *Let $\alpha \geq \mathfrak{c}$ be a cardinal, B an Abelian group of bounded order with $|B| = \alpha$, and H an infinite Abelian divisible group with $2^{|H|} \leq \alpha$. Then the Abelian group $G := B \times H$ with $|G| = \alpha$ admits at most $2^{2^{|H|}} < 2^{2^\alpha}$ many SC-group topologies.*

Proof: We have $\widehat{G} = \widehat{B} \times \widehat{H}$. By duality theory we get that \widehat{B} is of bounded order, and \widehat{H} is torsion-free. Hence $t\widehat{G} \cong \widehat{B}$ and $\widehat{G}/t\widehat{G} \cong \widehat{H}$. Now let τ be a totally bounded group topology such that every subgroup of G is closed in (G, τ) . Then Lemma 6.2 gives $t\widehat{G} \subseteq (G, \tau)^\wedge$. A result of Kakutani [27] implies $|\widehat{H}| = 2^{|H|} \geq \mathfrak{c}$. By Lemma 4.9 we get $r(\widehat{H}) = 2^{|H|}$. Hence \widehat{H} has exactly $2^{2^{|H|}}$ -many subgroups. Thus there is the same number of subgroups of \widehat{G} containing $t\widehat{H}$. ■

Now we are ready to give the announced negative answer to Question 6.16.

Theorem 6.20. *Let α and β be infinite cardinals with $2^\beta \leq \alpha$. Then there is an Abelian group G with $|G| = \alpha$ such that $m = 2^{2^\beta} < 2^{2^{|\mathfrak{G}|}}$ holds for the number m of SC-group topologies on G which can be chosen in the following ways:*

- (a) G is a torsion group.
- (b) $r_0(G) = \beta < \alpha$.

Proof: We apply Lemma 6.19: Let B as defined there and (a) $H := \bigoplus_\beta Z(p^\infty)$ resp. (b) $H := \bigoplus_\beta \mathbb{Q}$. Then, in both cases (a) and (b), the subgroup H admits 2^{2^β} -many SC-group topologies. Indeed, it suffices to apply Lemma 6.17 in case (a), and Corollary 6.12 in case (b), respectively. Finally, applying Lemma 6.18, the proof is complete. ■

Lemma 6.18 is a main tool for the proof of

Proposition 6.21. *The following holds:*

- (a) *Let G be a countable Abelian group which is not reduced. Then G admits exactly $2^\mathfrak{c}$ -many SC-group topologies.*
- (b) *Every infinite countable divisible Abelian group admits exactly $2^\mathfrak{c}$ -many SC-group topologies.*
- (c) *Every Abelian group which is not reduced admits at least $2^\mathfrak{c}$ -many SC-group topologies.*
- (d) *Let G be a divisible Abelian group with $\text{cf}(|G|) > \omega$. Then G admits exactly $2^{2^{|\mathfrak{G}|}}$ -many SC-group topologies.*

Proof: (a) G contains \mathbb{Q} or $Z(p^\infty)$ for some $p \in \mathbb{P}$. Then apply Lemma 6.18, Proposition 6.7 and Theorem 6.11.

(b) follows from (a).

(c) Apply Lemma 6.18 and (b).

(d) By Corollary 6.12 we may assume that $r_0(G) < |G|$. Then [21, p. 85] implies $r(G) = \sum_p r_p(G)$. Hence $|G| = \sum_p r_p(G)$ by Lemma 4.9. Since $\text{cf}(|G|) > \omega$, there is $p_0 \in \mathbb{P}$ such that $|G| = r_{p_0}(G)$. By [21, p. 105], $|G|$ contains a subgroup isomorphic to $\bigoplus_{|G|} \mathbb{Z}(p_0^\infty)$.

Finally, it suffices to apply Lemma 6.17 and Lemma 6.18. ■

Concerning the problem whether a countable Abelian group which is not of bounded order admits exactly $2^\mathfrak{c}$ -many SC-group topologies we notice the following facts.

Set $K := \prod_{n \in \mathbb{N}} Z(p^n)$ and let \mathbf{p} be a free ultrafilter on \mathbb{N} or, equivalently, a point in the remainder $\beta\mathbb{N} \setminus \mathbb{N}$ of the Stone-Čech compactification $\beta\mathbb{N}$ of \mathbb{N} furnished with the discrete topology. Every continuous function $f: \mathbb{N} \rightarrow \mathbb{T}$ can be extended to a continuous function $\bar{f}: \beta\mathbb{N} \rightarrow \mathbb{T}$ (for more details see [8, §2]). If \mathcal{F} is a filter on \mathbb{N} such that the filter with the basis $\{f(F) : F \in \mathcal{F}\}$ converges in \mathbb{T} to t , we write $t = \lim_{\mathcal{F}} f(n)$. Now \mathbf{p} converges in $\beta\mathbb{N}$ to itself ([22, section 6.5]). Thus we have

$$\bar{f}(\mathbf{p}) = \lim_{\mathbf{p}} f(n).$$

For $n \in \mathbb{N}$ we denote by Z_{p^n} the copy of $Z(p^n)$ placed as a subgroup of \mathbb{T} and

$$X_n: Z(p^n) \rightarrow \mathbb{T}$$

designates the canonical isomorphism of $Z(p^n)$ onto Z_{p^n} .

For all $x := (x(n)) \in K$ let $f_x: \mathbb{N} \rightarrow \mathbb{T}$ with $f_x(n) := X_n(x(n))$. We define the function

$$\chi_{\mathbf{p}}: K \rightarrow \mathbb{T}$$

by

$$\chi_{\mathbf{p}}[(x(n))] := \lim_{\mathbf{p}} f_x(n)$$

for all $(x(n)) \in K$. This means $\chi_{\mathbf{p}}[(x(n))] = \bar{f}_x(\mathbf{p})$. Now $f_x(n) + f_y(n) = f_{x+y}(n)$ holds for all $n \in \mathbb{N}$. Since \mathbb{N} is dense in $\beta\mathbb{N}$, we get $\bar{f}_x(\mathbf{u}) + \bar{f}_y(\mathbf{u}) = \bar{f}_{x+y}(\mathbf{u})$ for all $\mathbf{u} \in \beta\mathbb{N}$. Hence $\chi_{\mathbf{p}}$ is a character.

Let $U := \bigoplus_{n \in \mathbb{N}} Z(p^n)$. Then $\chi_{\mathbf{p}}[U] = \{0\}$ holds. Since U is dense in K , the character $\chi_{\mathbf{p}}$ is discontinuous.

Lemma 6.22. ([3, Proposition 5, p. 60]) *Let \mathbf{u} be an ultrafilter on a set X . If A and B are two subsets of X such that $A \cup B \in \mathbf{u}$, then either $A \in \mathbf{u}$ or $B \in \mathbf{u}$.*

Lemma 6.23. $\chi_{\mathbf{p}}[K] = \mathbb{T}$.

Proof: Let $\alpha \in \mathbb{T}$ be arbitrary. Since Z_{p^∞} is dense in \mathbb{T} , there is a sequence (k_n/p^n) converging to α (here $k_n/p^n \in Z_{p^n}$ for all $n \in \mathbb{N}$). Define $x(n) = X_n^{-1}(k_n/p^n)$ for all $n \in \mathbb{N}$. By using Lemma 6.22 we have

$$\chi_{\mathbf{p}}(x(n)) = \lim_{\mathbf{p}} X_n(x(n)) = \lim_{\mathbf{p}} k_n/p^n = \alpha.$$

Hence $\alpha \in \chi_{\mathbf{p}}[K]$. ■

The following assertion is well-known.

Lemma 6.24. *There is a copy Q_p of \mathbb{Q} such that $\mathbb{T} \cong Q_p \oplus L_p$ for some subgroup L_p of \mathbb{T} .*

Lemma 6.25. *Set $S_{\mathbf{p}} := \chi_{\mathbf{p}}^{-1}[L_p]$. Then $S_{\mathbf{p}}$ is a totally dense subgroup of K .*

Proof: By Lemma 6.23 and Lemma 6.24 we have $K/S_{\mathbf{p}} \cong \mathbb{Q}$. $\chi_{\mathbf{p}}[U] = \{0\}$ implies that $S_{\mathbf{p}}$ is dense in K . Thus it suffices to apply [28, Corollary 3]. ■

Lemma 6.26. $S_{\mathbf{p}} \neq S_{\mathbf{q}}$ if $\mathbf{p} \neq \mathbf{q}$.

Proof: There is $A_{\mathbf{p}} \in \mathbf{p}$ with $A_{\mathbf{p}} \notin \mathbf{q}$. Hence $A_{\mathbf{q}} := \mathbb{N} \setminus A_{\mathbf{p}} \in \mathbf{q}$ by Lemma 6.22. We take an element $(x(n)) \in K$ satisfying the following requirements:

- (1) $x(n) = 0$ for all $n \in A_{\mathbf{q}}$;
- (2) Take $\alpha \neq 0$ in Q_p . Since Z_{p^∞} is dense in \mathbb{T} , there is a sequence (k_n/p^n) converging to α (here $k_n/p^n \in Z_{p^n}$ for all $n \in \mathbb{N}$). Define $x(n) = X_n^{-1}(k_n/p^n)$ for all $n \in \mathbb{N} \setminus A_{\mathbf{q}}$.

We have

$$\chi_{\mathbf{q}}[(x(n))] = \lim_{\mathbf{q}} X_n(x(n)) = 0.$$

This implies that $(x(n)) \in \chi_{\mathbf{q}}^{-1}(0) \subseteq S_{\mathbf{q}}$. On the other hand Lemma 6.22 implies

$$\chi_{\mathbf{p}}[(x(n))] = \lim_{\mathbf{p}} X_n(x(n)) = \lim_{\mathbf{p}} k_n/p^n = \alpha \in Q_p.$$

Hence $(x(n)) \notin \chi_{\mathbf{p}}^{-1}(L_p) = S_{\mathbf{p}}$. This proves that $S_{\mathbf{p}} \neq S_{\mathbf{q}}$. ■

Theorem 6.27. *K contains 2^c -many totally dense subgroups.*

Proof: \mathbb{N} has 2^c -many free ultrafilters by [8, Corollary 7.4]. Then apply Lemma 6.25 and Lemma 6.26. ■

Now Proposition 6.5 yields

Corollary 6.28. $\bigoplus_{n \in \mathbb{N}} Z(p^n)$ admits exactly 2^c -many SC-group topologies.

We obtain

Theorem 6.29. *Let G be a torsion group such that G has only finitely many p -components G_p and is not of bounded order. Then G admits at least 2^c -many SC-group topologies.*

Proof: There exists $p_0 \in \mathbb{P}$ with $|G_{p_0}| \geq \omega$ such that $L := G_{p_0}$ is not of bounded order. By [25, (A.24)] L contains a subgroup B such that

- (a) B is isomorphic with a direct sum of cyclic p_0 -groups;
- (b) B is a pure subgroup of L ;
- (c) the quotient group L/B is divisible.

We consider the following cases:

- (1) B is not of bounded order. Then B is isomorphic with a direct sum of cyclic p_0 -groups, where infinitely many summands are pairwise different. Hence it contains a subgroup U being isomorphic with $H_{p_0} := \bigoplus_{n \in \mathbb{N}} Z(p_0^n)$. Then Corollary 6.28 implies that H_{p_0} has 2^c -many SC-group topologies. By Lemma 6.18 the same holds for G .
- (2) B is of bounded order. Thus it is a proper subgroup of L . Since B is a pure subgroup of L , then [21, Theorem 27.5] implies that B is a direct summand of L . By (c) it follows that L is not reduced. Thus Proposition 6.21(c) gives that L admits 2^c -many SC-group topologies. Finally apply again Lemma 6.18 to see that G admits also 2^c -many SC-group topologies. ■

Now we can prove one of the main results in this section.

Proof of Theorem C. (a) Apply Theorem 6.11, Theorem 6.13, Theorem 6.29 and Lemma 6.18. (b) follows from (a). ■

7. TOPOLOGICALLY SIMPLE GROUPS

We now consider the opposite direction from the section before, namely the existence of totally bounded group topologies without producing closed non-trivial subgroups. Topologically simple groups were introduced in Section 1. The following result gives a characterization of infinite Abelian totally bounded groups which are topologically simple.

Theorem 7.1. *Let (G, τ) be an infinite Abelian totally bounded group and let $S \leq \widehat{G}$ be its character group. Then the following assertions are equivalent:*

- (1) (G, τ) is topologically simple.
- (2) $L \cap S = \{0\}$ for all proper closed subgroups L of \widehat{G} .
- (3) All group topologies coarser than τ are Hausdorff or anti-discrete.
- (4) Let μ be the totally bounded group topology on S induced by the compact group topology of \widehat{G} . Then (S, μ) is topologically simple.

Proof: (1) \Rightarrow (2): Let L be a proper closed subgroup of \widehat{G} . By Lemma 3.7, we have $L = \mathbb{A}(\widehat{G}, \mathbb{A}(G, L))$, where $\mathbb{A}(G, L)$ is not trivial. Since (G, τ) is topologically simple, $\mathbb{A}(G, L)$ is dense in it. By Theorem 3.5 the intersection of L and S is trivial.

(2) \Rightarrow (1): Let H be a non-trivial subgroup of G . Then $\mathbb{A}(\widehat{G}, H)$ is a proper closed subgroup of \widehat{G} by [25] (23.24, Remarks (b) and (c)). Hence (2) implies that the intersection of $\mathbb{A}(\widehat{G}, H)$ and S is trivial. By Theorem 3.5, H is dense in (G, τ) .

(1) \Rightarrow (3) is trivial.

(3) \Rightarrow (1). Let $N \subsetneq G$ be a closed subgroup of (G, τ) . Since N is closed, the group G/N , equipped with the quotient topology τ_q defined by the canonical map $\pi: G \rightarrow G/N$, is Hausdorff. Let $\mu := \pi^{-1}(\tau_q)$ be the initial topology on G defined by π . Since N is μ -closed, it follows that μ is precompact, coarser than τ and not anti-discrete. By (3), this means that $N = \{0\}$.

(2) \Rightarrow (4). Let H be a closed subgroup of (S, μ) with $H \neq S$. Let N be the closure of H in \widehat{G} . By $H \neq S$ and $H = N \cap S$ we have $N \neq \widehat{G}$. Now (2) implies $H = \{0\}$.

(4) \Rightarrow (2). Let L be a proper closed subgroup of \widehat{G} . Then $H := L \cap S$ is a closed subgroup of (S, μ) . Assume $H = S$. Then $S \subseteq L$ and hence $L = \overline{L} = \widehat{G}$, which is impossible. Thus H is a proper closed subgroup of (S, μ) . This implies $H = \{0\}$, since (S, μ) is topologically simple. \blacksquare

Corollary 7.2. *If (G, τ) is an infinite Abelian totally bounded topologically simple group, then:*

- (1) It is monothetic and torsion-free,
- (2) its character group is monothetic and torsion-free,
- (3) algebraically, it is a subgroup of the real numbers \mathbb{R} ,
- (4) its weight is at most \mathfrak{c} .

Proof: (1) follows directly from the definitions, and (2) from Theorem 7.1. [25] (24.32) implies that G is isomorphic to a torsion-free subgroup of the (discrete) torus group \mathbb{T} . Thus the torsion-free rank of G is at most \mathfrak{c} . Considering its divisible hull we get (3). By Theorem 7.1, the same holds for S . This implies the last assertion. \blacksquare

Remark 7.3. Let G be the group of integers. Then \widehat{G} is the compact torus group \mathbb{T} . By Corollary 7.2 we have for every dense subgroup S of \mathbb{T} : (G, τ_S) is topologically simple if

and only if S (as a subgroup of \mathbb{T}) is topologically simple if and only if S is a torsion-free subgroup of \mathbb{T} . Hence, there is a one-to-one correspondence between the totally bounded group topologies on G which produce topologically simple groups and the (dense) torsion-free subgroups of \mathbb{T} . Therefore, there are exactly $2^{\mathfrak{c}}$ -many totally bounded group topologies on G which produce topologically simple groups. The weight can be chosen to be \mathfrak{c} . (See Theorem **D** and Corollaries 8.3 and 8.4, below.)

Remark 7.4. Let G be a subgroup of the real numbers. Then G can be considered as a subgroup of the torus \mathbb{T} . Since \mathbb{T} has only finite proper closed subgroups, the topology of \mathbb{T} induces a metrizable totally bounded group topology τ on G such that (G, τ) is topologically simple.

Lemma 7.5. *Let G be an infinite Abelian group. S is a subgroup of \widehat{G} such that G_S is topologically simple if and only if every non-zero element of S is injective.*

Proof: (\Rightarrow) If $g_1, g_2 \in G \setminus \{0\}, \phi \in S \setminus \{0\}$ with $g_1 \neq g_2$, yet $\phi(g_1) = \phi(g_2)$, then $0 \neq g_1 - g_2 \in H := \phi^{-1}[\{0\}] \implies H$ is a non-trivial proper subgroup of G_S , a contradiction.

(\Leftarrow) Let H be a closed subgroup of G_S , different than G . Then G_S/H is totally bounded with character group $\mathbb{A}(S, H)$ (Lemma 2.1). Notice that $\mathbb{A}(S, H) \neq \{0\}$. If there were a non-zero element $h \in H$, then $\phi(h) = 0$ for every non-zero element $\phi \in \mathbb{A}(S, H)$, contradicting that all non-zero elements of S are injective. It follows that $H = \{0\}$, as required. \blacksquare

Lemma 7.6. *Assume $1 \leq \kappa \leq \mathfrak{c}$ and let $G := \bigoplus_{\kappa} \mathbb{Z}$. Then there is a family \mathcal{F} of homomorphism $\phi : G \rightarrow \mathbb{R}$ such that the following holds: $|\mathcal{F}| = \mathfrak{c}$ and $\langle \mathcal{F} \rangle \setminus \{0\}$ consists of injective functions.*

Proof: Let $\mathcal{B} := \{e_t : t \in \kappa\}$ be a generating independent subset of G , and $\mathcal{B}' := \{u_t : t \in \mathfrak{c}\}$ a base of the vector space $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ over \mathbb{Q} . Since $\mathfrak{c} = \sum_{i \in I} \kappa_i$ with $|I| = \mathfrak{c}$ and $\kappa_i = \kappa$ for all $i \in I$ by [29, Proposition 4.4, Chapter III], \mathbb{R} is a union of \mathfrak{c} -many pairwise disjoint subsets of size κ . Thus we get a family $\mathcal{E} = \{f_i : i \in I\}$ with the following property (*): For all $i \in I$ the function $f_i : \kappa \rightarrow \mathfrak{c}$ is injective with $f_i[\kappa] \cap f_j[\kappa] = \emptyset$ for $i \neq j$. It follows that $|\mathcal{E}| = \mathfrak{c}$. For each $f \in \mathcal{E}$ define $\phi_f : G \rightarrow \mathbb{R}$ by $\phi_f(e_t) := u_{f(t)}$. Now let $\mathcal{F} := \{\phi_f : f \in \mathcal{E}\}$. Then $|\langle \mathcal{F} \rangle| = \mathfrak{c}$. $\langle \mathcal{F} \rangle \setminus \{0\}$ consists of injective functions: For, let $\psi = \sum_j a_j \phi_{f_j} \in \langle \mathcal{F} \rangle \setminus \{0\}$ and $g = \sum_t b_t e_t \in G \setminus \{0\}$. Then $\psi(g) = (\sum_j a_j \phi_{f_j})(\sum_t b_t e_t) = \sum_j \sum_t a_j b_t u_{f_j(t)} \neq 0$, since by property (*) $\{u_{f_j(t)}\}$ is linearly independent with respect to the vector space $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ over \mathbb{Q} . Hence ψ is injective. \blacksquare

Lemma 7.7. *Let G be a non-trivial subgroup of \mathbb{R} . Then G admits a totally bounded group topology τ such that $w(G, \tau) = \mathfrak{c}$ and (G, τ) is topologically simple.*

Proof: Let m be the rank of G , and let $L := \bigoplus_m \mathbb{Z}$. Since algebraically $\mathbb{R} \subset \mathbb{T}$, by Lemma 7.6 there is a family \mathcal{F} of homomorphisms $\phi : L \rightarrow \mathbb{T}$ such that $|\mathcal{F}| = \mathfrak{c}$ and $\langle \mathcal{F} \rangle \setminus \{0\}$ consists of injective functions. By [25, (A.7)] each ϕ can be extended to a character $\hat{\phi} : G \rightarrow \mathbb{T}$. Let $\mathcal{M} := \{\hat{\phi} : \phi \in \mathcal{F}\}$. Let $S := \langle \mathcal{M} \rangle$. Then $|\mathcal{M}| = \mathfrak{c}$, and S is a subgroup of \widehat{G} with $|S| = \mathfrak{c}$. Now each $\phi \in S \setminus \{0\}$ is injective: For, let $g \in G \setminus \{0\}$. Since L is an essential subgroup of G , we have $\langle g \rangle \cap L \neq \{0\}$. Hence, there is $n \in \mathbb{Z} \setminus \{0\}$ with $ng \in L \setminus \{0\}$. Since $\langle \mathcal{F} \rangle \setminus \{0\}$ consists of injective functions, we get $0 \neq \phi(ng) = n\phi(g)$. Hence $\phi(g) \neq 0$. Let τ be the precompact group topology on G with character group S .

By Lemma 7.5 it is topologically simple, in particular totally bounded. Since $|S| = \mathfrak{c}$, Lemma 4.7 implies $w(G, \tau) = \mathfrak{c}$. \blacksquare

Now we are ready for the

Proof of Theorem D. Let (G, τ) be defined as in Lemma 7.7. Then $|S| = \mathfrak{c}$ holds for its character group S . By Theorem 7.1, S is torsion-free. Lemma 4.9 implies $r(S) = \mathfrak{c}$. Then $\bigoplus_{\mathfrak{c}} \mathbb{Z} \subseteq S$. \mathfrak{c} has $2^{\mathfrak{c}}$ -many subsets of size \mathfrak{c} by [5, Proposition 5.2.14]. Thus S has $2^{\mathfrak{c}}$ -many subgroups U of cardinality \mathfrak{c} . Let τ_U be the precompact group topology on G with character group U . By Theorem 7.1(3), it is totally bounded. Since $\tau_U \subseteq \tau$, (G, τ_U) is also topologically simple. $|U| = \mathfrak{c}$ implies $w(G, \tau_U) = \mathfrak{c}$ by Lemma 4.7.

Let μ be any totally bounded group topology on G such that (G, μ) is topologically simple. By Corollary 7.2(4), $w(G, \mu) \leq \mathfrak{c}$. Let S' be the character group of (G, μ) . Then $S' \subseteq \widehat{G}$ with $|S'| \leq \mathfrak{c}$ and $|\widehat{G}| = 2^{|G|}$. By [5, Proposition 5.2.14], \widehat{G} has exactly $(2^{|G|})^{\mathfrak{c}} = 2^{\mathfrak{c}}$ many subsets of size $\leq \mathfrak{c}$. Hence G has at most $2^{\mathfrak{c}}$ -many totally bounded group topologies μ such that (G, μ) is topologically simple. \blacksquare

Corollary 8.4 below is an example of this Theorem.

Remark 7.8. The proof of Theorem **D** shows the following: Let \mathcal{P} be the power set of \mathfrak{c} , and let G be a non-trivial subgroup of \mathbb{R} . Let $\mathcal{TS}(G)$ be the set of all topologically simple totally bounded group topologies on G . Then there exists an order-preserving injection $f : \mathcal{P} \rightarrow \mathcal{TS}(G)$ such that $w(G, f[M]) = |M|$ for all $M \in \mathcal{P}$. There is $\tau \in \mathcal{TS}(G)$ with $w(G, \tau) = \mathfrak{c}$, and there is $\mathcal{L} \subseteq \mathcal{TS}(G)$ with $|\mathcal{L}| = 2^{\mathfrak{c}}$ such that all $\mu \in \mathcal{L}$ are coarser than τ .

In [13] the following notation is introduced: Let κ and λ be cardinals, and let $\mathcal{P}(\kappa)$ be the power set of κ . Then $C(\kappa, \lambda)$ means that there is in $\mathcal{P}(\kappa)$ a chain of length λ . By [13] we have the following:

- (1) $C(\mathfrak{c}, \mathfrak{c}^+)$ holds by [13, Corollary 1.7],
- (2) $C(\mathfrak{c}, 2^{\mathfrak{c}})$ is not a theorem of ZFC by [13, Corollary 1.3].

Let G be a non-trivial subgroup of \mathbb{R} . Then:

- (1) In $\mathcal{TS}(G)$ there is a chain of length \mathfrak{c}^+ ,
- (2) The following is not a theorem of ZFC : There is $\tau \in \mathcal{TS}(G)$ with $w(G, \tau) = \mathfrak{c}$ and a chain $\mathcal{C} \subseteq \mathcal{TS}(G)$ with $|\mathcal{C}| = 2^{\mathfrak{c}}$ such that all $\mu \in \mathcal{C}$ are coarser than τ .

8. AN EXAMPLE: THE INTEGERS

Let us consider the special case when $G = \mathbb{Z}$, the group of the integers. If $n \in \omega$, then $n\mathbb{Z}$ is a subgroup of \mathbb{Z} , and conversely, if H is a subgroup of \mathbb{Z} , there is a unique $n \in \omega$ such that $H = n\mathbb{Z}$.

Lemma 8.1. *Let S be a subgroup of $\widehat{\mathbb{Z}}$. $k\mathbb{Z}$ is τ_S -closed if and only if $\frac{1}{k} \in S$.*

Proof: (\Leftarrow) $(\frac{1}{k})^{-1}[\{0\}] = k\mathbb{Z}$.
 (\Rightarrow) Having finite index in \mathbb{Z} , it follows that $k\mathbb{Z}$ is τ_S -open. Hence the map $\phi : \mathbb{Z}_S \rightarrow \mathbb{Z}_S/k\mathbb{Z} \simeq \mathbb{Z}_k \rightarrow \langle \frac{1}{k} \rangle \subset \widehat{\mathbb{Z}}$, $1 \mapsto \frac{1}{k}$ is τ_S -continuous since it sends $k\mathbb{Z}$ to 0. Since $\phi = \frac{1}{k}$, it follows that $\frac{1}{k} \in S$. \blacksquare

The following has been noticed in [19, (3.5.4) and (3.5.5)].

Corollary 8.2. *Let S be a subgroup of $\widehat{\mathbb{Z}}$. $p^k\mathbb{Z}$ is τ_S -closed for all $k \in \omega$ if and only if $Z(p^\infty) \subseteq S$.*

Corollary 8.3. *Let S be a subgroup of $\widehat{\mathbb{Z}}$ such that $S \cap t\widehat{\mathbb{Z}} = \{0\}$. Then \mathbb{Z}_S has no non-trivial closed subgroups.*

Since algebraically $\mathbb{T} = t\mathbb{T} \oplus (\oplus_{\mathfrak{c}} \mathbb{Q})$ ([25, (A.14)]), we have the following two results.

Corollary 8.4. *There exist $2^{\mathfrak{c}}$ -many point-separating subgroups S of $\widehat{\mathbb{Z}} = \mathbb{T}$ such that every subgroup of \mathbb{Z}_S is dense.*

Proof: If $X \subseteq \mathfrak{c}$, set $S_X := \{0\} \oplus (\oplus_X \mathbb{Q})$. Obviously, S_X is dense in \mathbb{T} , hence it is point-separating. By the above corollary, \mathbb{Z}_{S_X} has no non-trivial closed subgroups and $X_1 \neq X_2$ implies $S_{X_1} \neq S_{X_2}$. ■

The following result is a special case of Theorem 6.13, but we give a short independent proof.

Corollary 8.5. *There exist $2^{\mathfrak{c}}$ -many point-separating subgroups S of $\widehat{\mathbb{Z}} = \mathbb{T}$ such that every subgroup of \mathbb{Z}_S is closed.*

Proof: If $X \subseteq \mathfrak{c}$, set $S_X := t\mathbb{T} \oplus (\oplus_X \mathbb{Q})$. Obviously, S_X is dense in \mathbb{T} , hence it is point-separating. By Corollaries 3.3 or 8.2, every subgroup of \mathbb{Z}_S is closed, and $X_1 \neq X_2 \implies S_{X_1} \neq S_{X_2}$. ■

The acronym lcm stands for *least common multiple*.

Lemma 8.6. *If $n_1, n_2 \in \mathbb{N}$, then $n_1\mathbb{Z} \cap n_2\mathbb{Z} = \text{lcm}(n_1, n_2)\mathbb{Z}$.*

Proof: Obviously \supseteq holds. To see \subseteq , set $M = \text{lcm}(n_1, n_2)$ and assume that $n_1\mathbb{Z} \cap n_2\mathbb{Z} = t\mathbb{Z}$. Then there exist $z_1, z_2 \in \mathbb{Z}$ such that $n_1z_1 = t = n_2z_2 \implies M|t$, as required. ■

Lemma 8.7. *Let $F' = \{n_1, \dots, n_k\} \subset \mathbb{N}$, $n \in \mathbb{N}$ and $F = F' \cup \{n\}$. Then*

$$\text{lcm } F = \text{lcm}(\text{lcm } F', n).$$

Proof: Set $M := \text{lcm}(\text{lcm } F', n)$, $N := \text{lcm } F$, and $x := \text{lcm } F'$. Then $M = \text{lcm}(x, n)$. Accordingly, $x|M$ and $n|M$. But $x = \text{lcm } F' \implies n_j|x$ ($j = 1, \dots, k$) $\implies n_j|M$ ($j = 1, \dots, k$). Hence, $N|M$. On the other hand, $N = \text{lcm } F \implies n_j|N$ ($j = 1, \dots, k$) $\implies x|N$. Since $n|N$, it follows that $M|N$, as required. ■

Lemma 8.8. *Let C be a non-empty subset of \mathbb{N} . Then*

$$\left\langle \frac{1}{n} : n \in C \right\rangle = \left\langle \frac{1}{\text{lcm } F} : F \in [C]^{<\infty} \right\rangle,$$

as subgroups of $\widehat{\mathbb{Z}}$, where $[C]^{<\infty}$ stands for all the finite subsets of C .

Proof: \subseteq Obvious (take $F = \{n\}$).

\supseteq We prove that $\frac{1}{\text{lcm } F} \in \text{LHS}$ whenever $F := \{n_1, \dots, n_k\} \subset C$. This obviously will prove the required contention. We use induction on k . Obviously true if $k = 1$. Assume that $\frac{1}{\text{lcm } F} \in \text{LHS}$, whenever F has t elements or less. Assume then that $F = \{n_1, \dots, n_t, n\}$. Set $F' = \{n_1, \dots, n_t\}$. As in the proof of Lemma 8.7, set $N := \text{lcm}(\text{lcm } F', n) = \text{lcm } F$, and $x := \text{lcm } F'$. By the induction hypothesis, $\frac{1}{x} \in \text{LHS}$. In addition, let d stand for the greatest common divisor of x and n . Recall that there exists integers a and b such that $d = ax + bn$. Since $xn = Nd = N(ax + bn)$ we have that $\frac{1}{N} = \frac{ax+bn}{nx} = \frac{a}{n} + \frac{b}{x} \in \text{LHS}$. ■

Remark 8.9. Obviously, if H_1 and H_2 are closed subgroups of a topological group G , so is $H_1 \cap H_2$. When $G = \mathbb{Z}$, there are $n_1, n_2 \in \omega$ such that $H_1 = n_1\mathbb{Z}$ and $H_2 = n_2\mathbb{Z}$. It follows from Lemma 8.6 that $H_1 \cap H_2 = \text{lcm}(n_1, n_2)\mathbb{Z}$. If S is a subgroup of $\widehat{\mathbb{Z}}$, set $C_S := \{n \in \omega : n\mathbb{Z} \text{ is } \tau_S\text{-closed}\}$. By Lemma 8.8 we have that C_S is closed under taking lcms of finite subsets of C_S . Given $C \subseteq \mathbb{N}$, set $S_C := \langle \frac{1}{n} : n \in C \rangle \subset \widehat{\mathbb{Z}}$. If $\overline{C} := \{n \in \mathbb{N} : \frac{1}{n} \in S_C\}$, then $C \subseteq \overline{C}$, and since $S_C = S_{\overline{C}}$ (Lemma 8.8), we have that \overline{C} is closed under taking lcms of its finite subsets. When $C \subseteq \mathbb{N}$ is such that $C = \overline{C}$ we say that C is *lcm-closed*. In Theorem **B** we saw the existence of a greatest subgroup MS containing S and such that $C_{MS} = C_S$. In the following result, we explicitly build C_{MS} when $G = \mathbb{Z}$.

Theorem 8.10. *Let S be a subgroup of $\widehat{\mathbb{Z}}$. Then there exists a greatest dense subgroup MS containing S and such that $C_{MS} = C_S$.*

Similarly, there exists a smallest subgroup mS contained in S such that $C_{mS} = C_S$. The group mS is dense in $\widehat{\mathbb{Z}}$ if and only if C_S is infinite. If not, there are no smallest dense subgroups $D \subseteq \widehat{\mathbb{Z}}$ with $C_D = C_S$. Moreover, if S is a torsion group, then $mS = S$.

Proof: Set $mS := \langle \frac{1}{n} : n \in C_S \rangle \subset \widehat{\mathbb{Z}}$. By Lemma 8.1 $mS \subseteq S$, and by Remark 8.9, $C_{mS} = C_S$, hence mS is as required. Obviously, mS is finite if and only if C_S is so. Hence, mS is dense in $\widehat{\mathbb{Z}}$ if and only if C_S is infinite. Notice that $mS \subseteq t\widehat{\mathbb{Z}}$. If $\widehat{\mathbb{Z}} \cong t\widehat{\mathbb{Z}} \times F$ (of course, algebraically, $F \cong \mathbb{R}$), then $MS := mS \times F$ is a maximal dense subgroup of $\widehat{\mathbb{Z}}$ containing S and such that $C_{MS} = C_S$.

Assume C_S is finite and suppose D is a dense subgroup of $\widehat{\mathbb{Z}}$ such that $C_D = C_S$. Then D/mS is infinite and contains an element $x + mS$ ($x \in D$) of infinite order. It follows that $\langle x \rangle \cap mS = \{0\}$. Then $D' := \langle mS \cup \{2x\} \rangle$ is a proper infinite subgroup of D with $C_{D'} = C_S$.

Assume that S is a torsion group and let $s \in S$. If $s \neq 0$, there are $m, n \in \mathbb{N}$ relatively prime such that $s = \frac{m}{n}$. But then there is $k \in \mathbb{N}$ such that $\frac{1}{n} = ks \in S \implies n \in C_S \implies \frac{1}{n} \in mS \implies s \in mS$. Hence, $mS = S$. \blacksquare

Compare this result with Theorem **B**.

We can rewrite the above in terms of precompact group topologies:

Corollary 8.11. *Let τ be a precompact group topology on \mathbb{Z} . Then there exists a greatest totally bounded group topology $T\tau$ on \mathbb{Z} producing precisely the same τ -closed subgroups.*

Similarly, there exists a smallest precompact group topology $t\tau$ on \mathbb{Z} producing precisely the same τ -closed subgroups. Moreover, $t\tau$ is Hausdorff if and only if there exist infinitely many τ -closed subgroups. If not, then there are no smallest totally bounded group topologies t with the same τ -closed subgroups.

Proof: By the Comfort-Ross Theorem, there is a subgroup S of $\widehat{\mathbb{Z}}$ such that $\tau = \tau_S$. Applying Theorem 8.10 to S , we obtain the result. \blacksquare

Remark 8.12. Given $C \subseteq \mathbb{N}$, set $S_C := \langle \frac{1}{n} : n \in C \rangle \subset \widehat{\mathbb{Z}}$. Then we have $C_{S_C} = \overline{C}$ (see Remark 8.9) and $mS_{\overline{C}} = S_{\overline{C}} = S_C$ (Lemma 8.8). The partially ordered set (under inclusion) $\mathcal{S}_C := [mS_C, MS_C] := \{S \subseteq \widehat{\mathbb{Z}} : S \text{ is a subgroup of } \widehat{\mathbb{Z}} \text{ with } mS_C \subseteq S \subseteq MS_C\}$ (see the proof of Theorem **B**) has cardinality $2^{\mathfrak{c}}$ and maximal chains of length \mathfrak{c} . As

mentioned before, C is finite if and only if mS_C is finite. Moreover, mS_C is the only torsion group in (the poset) $[mS_C, MS_C]$. We also notice that $S_1, S_2 \in \mathcal{S}_C$, then $mS_C \subseteq S_1 \cap S_2$ *i.e.*, \mathcal{S}_C has the finite intersection property (fip), and $S_1, S_2 \in \mathcal{S}_C \implies \langle S_1 \cup S_2 \rangle \in \mathcal{S}_C$.

We have thus assigned to any collection \mathcal{C} of subgroups of \mathbb{Z} a subset C of \mathbb{N} . \mathcal{C} has fip if and only if C is lcm-closed. If this is the case, we have assigned a poset \mathcal{S}_C of subgroups of $\widehat{\mathbb{Z}}$, such that if $S \in \mathcal{S}_C$, then τ_S has precisely \mathcal{C} as its system of closed subgroups. The smallest element is a torsion-group which is infinite if and only if C is infinite. All of the elements in \mathcal{S}_C , except the smallest one, have torsion-free elements.

Remark 8.13. We can see the above in terms of precompact group topologies. Let τ be such a topology on \mathbb{Z} . By the Comfort-Ross Theorem, there is a subgroup S of $\widehat{\mathbb{Z}}$ such that $\tau = \tau_S$, and τ is Hausdorff if and only if S is infinite. Using the notation in Corollary 8.11, the poset (under inclusion) $I_\tau := [t\tau, T\tau] := \{\text{precompact group topologies on } \mathbb{Z} \text{ between } t\tau \text{ and } T\tau\}$ has cardinality $2^{\mathfrak{c}}$ and maximal chains of length \mathfrak{c} . $t\tau$ is the only element in I_τ that could not be Hausdorff; it is if and only if the system of τ -closed subgroups is infinite. We also notice that if $\tau_1, \tau_2 \in I_\tau$, then $t\tau \subseteq \tau_1 \cap \tau_2$.

Remark 8.14. Let S be a subgroup of $\widehat{\mathbb{Z}}$. It should be obvious by now that it is its torsion part, tS , the one that produces the system of τ_S -closed subgroups, *i.e.*, $k\mathbb{Z}$ is τ_S -closed if and only if $k\mathbb{Z}$ is τ_{tS} -closed. Hence, setting $\tau = \tau_S$ in Remark 8.13, there is a poset I_τ of precompact group topologies on \mathbb{Z} producing the same system of τ_S -closed subgroups. Again, I_τ has cardinality $2^{\mathfrak{c}}$, maximal chains of length \mathfrak{c} and τ_{tS} is its smallest element which is Hausdorff if and only if tS is infinite. By now we can see that given subgroups S_1, S_2 of $\widehat{\mathbb{Z}}$, they may or may not produce the same system of closed subgroups. They produce the same system of closed subgroups if and only if $tS_1 = tS_2$, *i.e.*, if they have the same torsion subgroups.

Given a precompact group topology τ , and a subgroup of \mathbb{Z} , what is its τ -closure? The following results are to be compared with Theorem 4.3, Corollary 5.4 and Theorem 3.5.

Lemma 8.15. *Let S be a subgroup of $\widehat{\mathbb{Z}}$ and let $k\mathbb{Z}$ be a subgroup of \mathbb{Z} . Then the τ_S -closure of $k\mathbb{Z}$ equals $t\mathbb{Z}$ where $t \in C_S, t|k$, and $s \in C_S, s > t \implies s$ does not divide k .*

Proof: Assume that the τ_S -closure of $k\mathbb{Z}$ equals $t\mathbb{Z}$. Then $t \in C_S$ and $k\mathbb{Z} \subseteq t\mathbb{Z} \implies t|k$. If $s \in C_S$, then $s\mathbb{Z}$ is τ_S -closed, and if $s|k$, then $k\mathbb{Z} \subseteq s\mathbb{Z} \implies t\mathbb{Z} \subseteq s\mathbb{Z} \implies s|t \implies s \leq t$, as required. ■

Theorem 8.16. *Let S be a subgroup of $\widehat{\mathbb{Z}}$ and let $k\mathbb{Z}$ be a proper subgroup of \mathbb{Z} . Then $k\mathbb{Z}$ is τ_S -dense in \mathbb{Z} if and only if $k \notin C_S$, and if $s \in C_S, s > 1$, then s does not divide k .*

Proof: (\implies) Obviously $k \notin C_S$. If there is $s \in C_S, s > 1$ with $s|k$, then $k\mathbb{Z} \subseteq s\mathbb{Z}$ implies the τ_S -closure of $k\mathbb{Z}$ is contained in $s\mathbb{Z} \neq \mathbb{Z}$, hence $k\mathbb{Z}$ wouldn't be τ_S -dense in \mathbb{Z} .

(\impliedby) Suppose $k\mathbb{Z}$ is not τ_S -dense in \mathbb{Z} , and let $s\mathbb{Z}$ be its τ_S -closure. Then $s \in C_S, s|k$, and $s > 1$. If $s = k$, then $k \in C_S$, a contradiction. Then $s \neq k, k \notin C_S$ and $s|k$, a contradiction. ■

Some extreme cases:

Corollary 8.17. *Let S be a subgroup of $\widehat{\mathbb{Z}}$. S is a torsion-free subgroup of $\widehat{\mathbb{Z}}$ if and only if every proper subgroup of \mathbb{Z} is τ_S -dense in \mathbb{Z} .*

Corollary 8.18. *Let S be a subgroup of $\widehat{\mathbb{Z}}$. $t\widehat{\mathbb{Z}} \subseteq S$ if and only if every proper subgroup of \mathbb{Z} is τ_S -closed in \mathbb{Z} .*

9. CONCLUSION

In this paper, we have dealt with the topological structure of totally bounded Abelian groups via their dual groups. Every Abelian group G has associated a big dual group \widehat{G} that consists of the homomorphisms of G into the one-dimensional torus (the unit circle of the complex plane) in such a way that each totally bounded Abelian group topology defined on G coincides with the weak or initial topology defined by a *separating* subgroup of \widehat{G} . Thus, each separating subgroup S of \widehat{G} defines a totally bounded group topology on G and vice versa. We have exploited this pairing: *topology versus subgroup of \widehat{G}* in order to obtain real progress in the understanding of totally bounded group topologies on Abelian groups. Among other results, we have calculated the number of totally bounded group topologies that have a determined family of subgroups as closed subsets, and the number of totally bounded group topologies that have all its subgroups closed. We have also proved that the family of subgroups of \widehat{G} that define the same collection of closed subgroups of G always contain a greatest element but in general not a smallest one. Finally we have calculated the number of topologically simple totally bounded group topologies for non-trivial subgroups of the real line, the only possible groups having this property.

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