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Series expansions among weighted Lebesgue function spaces and applications to positive definite functions on compact two-point homogeneous spaces

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A B S T R A C T

Geodesically isotropic positive definite functions on compact two-point homogeneous spaces of dimension d have series representation as members of weighted Lebesgue spaces $L_1^w([-1,1])$, where the weight $w(x) = w^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ is the one related to the Jacobi orthogonal polynomials $P^{(\alpha,\beta)}(x)$ in $[-1,1]$, and the exponents α and β are related to the dimension d. We derive some recurrence relations among the coefficients of the series representations under different exponents, and we apply them to prove inheritance of positive definiteness between dimensions. Additionally, we give bounds on the curvature at the origin of such positive definite functions with compact support, extending the existing solutions from d-dimensional spheres to compact two-point homogeneous spaces.

1. Introduction

1.1. Context

The paper deals with open problems related to positive definite functions defined over compact two-point homogeneous spaces. In particular, we deal with positive definite functions that are geodesically isotropic in the sense that they solely depend on the geodesic distance between any pair of points.

Two-point homogeneous spaces have a long history that can be traced back to the Wang's *tour de force* ([39]). Compact two point homogeneous spaces are a rather broad class of manifolds that includes,

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as special cases, the unit hyper-sphere, the real/complex/quaternionic projective space, and the Cayley projective plane, that is the octonionic projective plane. These spaces have received increasing attention from the mathematical community, and the reader is referred to [7,16,17,22,26,33,37], to mention a few. Recently, there has been a renewed interest around these spaces, and we mention [9,8,10,15,32] with the references therein.

The statistical community has been recently engaged in problems related to these spaces. The book by [30] contains a comprehensive introduction. Random fields on two-point homogeneous spaces have been inspected by [28] and [29]. Generalizations have been considered by [19].

Contributions about positive definite functions on d-dimensional spheres embedded in $\mathbb{R}^{d+1}$ became ubiquitous in the last 10 years, most of this literature being inspired by the seminal paper by [24] and the list of open problems published therein. Some of these problems have been solved by [3,11,31] and by [34]. Spectral representations in more abstract contexts have been discussed in [12].

A compact two-point homogeneous space $\mathbb{M}^d$ of dimension $d \geq 1$ is both a Riemannian d-manifold and a compact symmetric space of rank 1. Each $\mathbb{M}^d$ has an invariant Riemannian (geodesic) normalized metric $d(\cdot,\cdot)$ such that all geodesics on $\mathbb{M}^d$ are closed and have length 2π. Throughout this paper we will use the notation $\theta := d(x,y) \in [0,\pi]$. Moreover, $\mathbb{M}^d$ can be endowed with the measure $d\sigma_d(x)$ induced by a normalized left Haar measure. More details are provided in Section 2.

Let $C : \mathbb{M}^d \times \mathbb{M}^d \to \mathbb{R}$ be a continuous function. Since $\mathbb{M}^d$ entails a group of motions, G_d, taking any pair of points (x,y) to (z,w) when $d(x,y) = d(z,w)$, we say that C is *geodesically isotropic* when

$$C(x,y) = C(Ax,Ay), \quad x,y \in \mathbb{M}^d, \quad A \in G_d.$$

Equivalently, the function C above can be written as

$$C(x,y) = f(\cos(d(x,y))), \quad x,y \in \mathbb{M}^d, \tag{1}$$

for some function $f : [-1,1] \to \mathbb{R}$, termed here the *radial isotropic part of* C.

The function $C : \mathbb{M}^d \times \mathbb{M}^d \to \mathbb{R}$ is *positive definite* on $\mathbb{M}^d$ if

$$\sum_{i=1}^{N}\sum_{j=1}^{N} c_i c_j C(x_i,x_j) \geq 0,$$

for any integer $N \geq 1$, for any $c_1,\dots,c_N \in \mathbb{R}$ and for any $x_1,\dots,x_N \in \mathbb{M}^d$. A positive definite function C is *strictly positive definite* if the above inequality is strict when $\sum_{j=1}^N c_j^2 \neq 0$. When C is geodesically isotropic, we shall abuse of notation when calling the corresponding function f in (1) positive definite.

For the remainder of the paper, $P_n^{(\alpha,\beta)} : [-1,1] \to \mathbb{R}$ denotes the Jacobi polynomial of degree n, and $\{P_n^{(\alpha,\beta)}\}_{n=0}^\infty$ the Jacobi family of orthogonal polynomials associated to the weight $w(x) = (1-x)^\alpha(1+x)^\beta$ for $\alpha,\beta > -1$. We refer to [38] for details about the Jacobi polynomials and some useful details are provided in Section 4.1.

Let $L_1^{\alpha,\beta} := L_1^{\alpha,\beta}([-1,1])$ be the space of the measurable functions $f : [-1,1] \to \mathbb{R}$ that are integrable with respect to the weight $(1-x)^\alpha(1+x)^\beta$, that is,

$$\int_{-1}^{1} |f(t)|(1-t)^\alpha(1+t)^\beta dt < \infty.$$

If $f \in L_1^{\alpha,\beta}$, then it has a formal Fourier-Jacobi series representation given by

$$f(x) = \sum_{n=0}^{\infty} a_n^{\alpha,\beta} \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}, \quad x \in [-1,1], \tag{2}$$

where the Fourier-Jacobi coefficient

$$a_n^{\alpha,\beta} = \frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}} \int_{-1}^{1} f(x) P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}\, dx\,, \tag{3}$$

and $h_n^{\alpha,\beta}$ is given in (16).

The correspondence between the dimension of the respective space $\mathbb{M}^d$ and the indices α and β is given in (12), and the role of these weighted Lebesgue spaces in the context of the positive definite functions on $\mathbb{M}^d$ is fundamental as the following result shows.

Theorem 1.1. *[14,22] Let $C : \mathbb{M}^d \times \mathbb{M}^d \to \mathbb{R}$ be a continuous function satisfying the identity (1) for some continuous mapping $f : [-1,1] \to \mathbb{R}$. Then, C is positive definite if and only if the mapping f can be uniquely written as*

$$f(x) = \sum_{n=0}^{\infty} a_n^{\alpha,\beta} \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}, \quad x \in [-1,1], \tag{4}$$

where

$$a_n^{\alpha,\beta} \geq 0, \quad \textit{for all } n \in \mathbb{Z}_+, \quad \sum_{n=0}^{\infty} a_n^{\alpha,\beta} < \infty. \tag{5}$$

We shall abuse of notation and refer to the Fourier-Jacobi coefficients $a_n^{\alpha,\beta}$ in (2) as (α,β)-Schoenberg coefficients of f, following [20]. This is parenthetical to the fact that the uniquely determined expansion in (4) was originally provided by [36] for the d-dimensional sphere embedded in $\mathbb{R}^{d+1}$, being a special case of $\mathbb{M}^d$ (see (12) for details). In this case, the expansion in (4) simplifies into

$$f(x) = \sum_{n=0}^{\infty} a_n^{d} \frac{\mathcal{C}_n^{(d-1)/2}(x)}{\mathcal{C}_n^{(d-1)/2}(1)}, \quad x \in [-1,1], \tag{6}$$

where $\mathcal{C}_n^{\lambda}$ defines the nth Gegenbauer polynomial of order λ and the sequence of nonnegative coefficients $\{a_n^d\}_{n=0}^{\infty}$ is summable (see [1]). The terminology adopted in [20] defines such a sequence as a d-Schoenberg sequence.

We denote $\Psi(\mathbb{M}^d)$ the class of the continuous functions $\psi : [0,\pi] \to \mathbb{R}$ such that $\psi(\cdot) = f \circ \cos(\cdot)$, where $f : [-1,1] \to \mathbb{R}$ is uniquely determined through (4). Theorem 1.1 can then be rephrased by asserting that a continuous function $\psi : [0,\pi] \to \mathbb{R}$ belongs to the class $\Psi(\mathbb{M}^d)$ if and only if it has a series representation in the form

$$\psi(\theta) = \sum_{n=0}^{\infty} a_n^{\alpha,\beta} \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(1)}, \tag{7}$$

with $a_n^{\alpha,\beta}$ as (5). Classical Fourier inversion shows that

$$a_n^{\alpha,\beta} = \frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}} \int_{0}^{\pi} \psi(\theta) P_n^{(\alpha,\beta)}(\cos\theta)(1-\cos\theta)^{\alpha}(1+\cos\theta)^{\beta} \sin\theta\, d\theta\,, \tag{8}$$

where $h_n^{\alpha,\beta}$ is given in (16).

1.2. Motivations

Our paper has both mathematical and scientific motivations. The former being triggered by some open problems in [24], related to positive definite functions defined over spheres. This paper proposes a generalized version of these problems for positive definite functions on compact two-point homogeneous spaces. The scientific motivations are instead coming from the impact of finding solutions to these problems for more applied branches of science. In particular, solutions to the problem treated here will have an impact on:

a. simplified spectral representations for positive definite functions in given compact two-points homogeneous spaces in terms of spectra defined over spaces that have a lower dimension. This fact is crucial to implementing simulation techniques for stochastic processes that are defined over these spaces. Reducing dimensionality is a major challenge in all branches of data science: to make an example, simulation of global processes (*e.g.*, climate variable) requires a major computational effort. Such a computational burden can be reduced through dimensionality reduction, and the reader is referred to [35], with the references therein, for a thorough account.

b. a better understanding of statistical phenomena that are defined over such spaces. For instance, in atmospheric data assimilation, locally supported isotropic correlation functions are used for the distance-dependent reduction of global scale covariance estimates in ensemble Kalman filter settings ([18,25]).

1.3. Our contribution

This paper contributes in four main directions:

A. We provide recurrence relations between (α, β)-Schoenberg sequences. Specifically, Theorem 3.1 provides a recursion of the type

$$a_n^{\alpha+k,\beta} = \sum_{i=0}^{k} b_{i,k,n}^{\alpha,\beta}\, a_{n+i}^{\alpha,\beta}, \quad n \in \mathbb{Z}_+,$$

with the sequence $\{b_{i,k,n}^{\alpha,\beta}\}_i$ specified therein. A similar recursion for $a_n^{\alpha,\beta+k}$ is given in Theorem 3.2. Then, Theorem 3.3 shows that

$$a_n^{\alpha+k,\beta+k'} = \sum_{i=0}^{k} \sum_{i'=0}^{k'} c_{i,i',k,k',n}^{\alpha,\beta}\, a_{n+i+i'}^{\alpha,\beta}, \quad n \in \mathbb{Z}_+,$$

with the sequence $\{c_{i,i',k,k',n}^{\alpha,\beta}\}_{i,i'}$ specified therein.

B. We then focus on series expansions for given (α, β)-Schoenberg sequences. In particular, Theorem 3.4 shows that

$$a_n^{\alpha,\beta} = \sum_{j=0}^{\infty} \xi_{j,n}^{\alpha,\beta}\, a_{n+j}^{\alpha+1,\beta}, \quad n \in \mathbb{Z}_+,$$

for the sequence $\{\xi_{j,n}^{\alpha,\beta}\}_j$ as specified therein. Also, Theorems 3.5 and 3.6 show that the following identity is true:

$$a_n^{\alpha,\beta} = \sum_{j=0}^{\infty} \zeta_{j,n}^{\alpha,\beta}\, a_{n+j}^{\alpha,\beta+1} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \eta_{i,j,n}^{\alpha,\beta}\, a_{n+i+j}^{\alpha+1,\beta+1}, \quad n \in \mathbb{Z}_+,$$

where the sequences $\{\zeta_{j,n}^{\alpha,\beta}\}_j$ and $\{\eta_{i,j,n}^{\alpha,\beta}\}_{i,j}$ are specified therein.

The expansions above together with embeddings $\mathbb{M}^d \hookrightarrow \mathbb{M}^{d'}$ allow us to obtain results on strict positive definiteness. Here, the symbol $A \hookrightarrow B$ indicates the existence of an isometric embedding from the metric space A into the metric space B.

In particular, Theorem 3.7 shows that if $\mathbb{M}^d$ and $\mathbb{M}^{d'}$, $d, d' \geq 2$, belong to the same class (distinct from 16-dimensional Cayleys's elliptic plane $\mathbb{P}^{16}$) and f is a positive definite function on $\mathbb{M}^d$ and strictly positive definite on $\mathbb{M}^{d'}$, then f is strictly positive definite on $\mathbb{M}^d$. Theorems 3.8 and 3.9 deal respectively with the cases of projective spaces of even dimensions of distinct classes and of $\mathbb{P}^{16}$.

C. We consider the following problem, being a realization of Problem 1 in [23].

Problem 1.2. *Find an expression for the (α, β)-Schoenberg coefficients as a linear combination of the (α_0, β_0)-Schoenberg coefficients for some fixed $\alpha_0 < \alpha$ and $\beta_0 < \beta$.*

A solution to Problem 1.2 has been provided by [3] for the special case of the d-dimensional sphere. The solution to this problem in this more general setting is provided in Theorem 3.10.

D. We consider the following

Problem 1.3. *For any given $c \in (0, \pi]$, find the minimum curvature at the origin within the set of isotropic covariance functions vanishing beyond c (such covariances are dubbed locally supported when $c < \pi$ and globally supported when $c = \pi$). That is, find*

$$a^c(\mathbb{M}^d) := \inf\{-\psi''(0) : \psi \in \Psi^c(\mathbb{M}^d)\}, \tag{9}$$

where $\Psi^c(\mathbb{M}^d)$ is the subclass of $\Psi(\mathbb{M}^d)$ given by

$$\Psi^c(\mathbb{M}^d) = \{\psi \in \Psi(\mathbb{M}^d) : \psi(\theta) = 0,\ \theta \geq c\}.$$

This problem is a generalization for the Problem 3 in [23]. The case $\mathbb{M}^d = \mathbb{S}^d$ was considered in [3], and we extend their solution to our more general context in Theorem 3.12.

1.4. Plan of the paper

The material required for the proofs, as well as the proofs themselves, is technical and lengthy. Hence, our narrative starts with a very minimal background in Section 2. The main novel results are reported in Section 3. Section 4 engages on all the technical background, as well as on the proofs. This organization will allow any non-expert reader to focus on the main results.

2. A minimal background

Each compact two-point homogeneous space $\mathbb{M}^d$ can be considered as the orbit of some compact subgroup $\mathcal{H}$ of the orthogonal group $\mathcal{G}$, or, in other words, $\mathbb{M}^d = \mathcal{G}/\mathcal{H}$ for certain $\mathcal{G}$ and $\mathcal{H}$. On any manifold $\mathbb{M}^d$, there exists a measure $d\sigma_d(x)$ induced by the normalized (left) Haar measure on the group $\mathcal{G}$ that is invariant under the action of $\mathcal{G}$ (more information can be found in [27] and references therein). In [39,40], Wang classified compact two-point homogeneous spaces as those belonging to one of the five classes:

- the unit spheres $\mathbb{S}^d$, $d = 1, 2, \ldots$;
- the real projective spaces $\mathbb{P}^d(\mathbb{R})$, $d = 2, 3, \ldots$;
- the complex projective spaces $\mathbb{P}^d(\mathbb{C})$, $d = 4, 6, \ldots$;
- the quaternion projective spaces $\mathbb{P}^d(\mathbb{H})$, $d = 8, 12, \ldots$;
- 16-dimensional Cayley's elliptic plane $\mathbb{P}^{16}$.

The compact two-point homogeneous spaces can be isometrically embedded as follows (see [5, p. 66] and references therein):

$$\begin{array}{lll} (i) & \mathbb{S}^d \hookrightarrow \mathbb{S}^{d+1} & d=1,2,\dots \\ (ii) & \mathbb{P}^d(\mathbb{R}) \hookrightarrow \mathbb{P}^{d+1}(\mathbb{R}) & d=2,3,\dots \\ (iii) & \mathbb{P}^d(\mathbb{C}) \hookrightarrow \mathbb{P}^{d+2}(\mathbb{C}) & d=4,6,\dots \\ (iv) & \mathbb{P}^d(\mathbb{H}) \hookrightarrow \mathbb{P}^{d+4}(\mathbb{H}) & d=8,12,\dots \\ (v) & \mathbb{P}^d(\mathbb{R}) \hookrightarrow \mathbb{P}^{2d}(\mathbb{C}) & d=2,3,\dots \\ (vi) & \mathbb{P}^{2d}(\mathbb{C}) \hookrightarrow \mathbb{P}^{4d}(\mathbb{H}) & d=2,3,\dots \\ (vii) & \mathbb{P}^8(\mathbb{H}) \hookrightarrow \mathbb{P}^{16} & \end{array} \tag{10}$$

Let $L^2(\mathbb{M}^d) := L^2(\mathbb{M}^d, \sigma_d)$ be the Hilbert space of square integrable functions $f : \mathbb{M}^d \to \mathbb{C}$ with the norm $\|\cdot\|_2$ induced by the inner product

$$\langle f, g\rangle_2 = \int_{\mathbb{M}^d} f(x)\overline{g(x)}d\sigma_d(x), \quad f, g \in L^2(\mathbb{M}^d). \tag{11}$$

Let $\mathcal{B}_d$ be the Laplace-Beltrami operator on $\mathbb{M}^d$. The spectrum of $\mathcal{B}_d$ is discrete, real and non-positive. Indeed, in geodesic polar coordinates we can write $\mathcal{B}_d = \mathcal{B}_d^\theta + \mathcal{B}_d'$, where $\mathcal{B}_d'$ denotes the Laplace-Beltrami operator on the sphere in $\mathbb{M}^d$ of radius θ (for more details, see [17, p. 420]) and $\mathcal{B}_d^\theta$ is its radial part. A change of variable of the type $x = \cos(2\vartheta\theta)$ shows that the radial part $\mathcal{B}_d^\theta$ can be written, up to some positive multiple, as (see [27, Eq. (5)])

$$\mathcal{B}_d^x = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{d}{dx}(1-x)^{1+\alpha}(1+x)^{1+\beta}\frac{d}{dx},$$

where α, β and ϑ depend on the space $\mathbb{M}^d$ as follows:

$\mathbb{M}^d$	d	α	β	ϑ
$\mathbb{S}^d$	$d \geq 1$	$(d-2)/2$	$(d-2)/2$	$1/2$
$P^d(\mathbb{R})$	$d = 2, 4, \dots$	$(d-2)/2$	$(d-2)/2$	$1/4$
$P^d(\mathbb{C})$	$d = 4, 6, \dots$	$(d-2)/2$	0	$1/2$
$P^d(\mathbb{H})$	$d = 8, 12, \dots$	$(d-2)/2$	1	$1/2$
P^{16}	$d = 16$	$(d-2)/2 = 7$	3	$1/2$

(12)

The eigenfunctions of $\mathcal{B}_d^x$ are the well known Jacobi polynomials $P_n^{(\alpha,\beta)}$ ([1]) and the corresponding eigenvalues are $-n(n+\alpha+\beta+1)$. Therefore, the eigenvalues of $\Delta_d := -\mathcal{B}_d^x$ can be arranged in an increasing order, with each eigenvalue being associated to an eigenspace $\mathcal{H}_n^d := \mathcal{H}_n^d(\mathbb{M}^d)$ of Δ_d. These spaces are mutually orthogonal with respect to the inner product (11) and

$$L^2(\mathbb{M}^d) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^d.$$

The dimension N_n^d of each space $\mathcal{H}_n^d$ is given by [27, Eq. (7)]

$$N_n^d = \frac{\Gamma(\beta+1)(2n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+2)\Gamma(n+1)\Gamma(n+\beta+1)},$$

for all $\mathbb{M}^d$ except in the case where $\mathbb{M}^d = P^d(\mathbb{R})$ and n is odd. In the last case, $N_n^d = 0$. Call $\{Y_{n,j} : j = 1, 2, \ldots, N_n^d\}$ an orthonormal basis of $\mathcal{H}_n^d$. Then, the set $\{Y_{n,j} : j = 1, 2, \ldots, N_n^d; n = 0, 1, \ldots\}$ is an orthonormal basis of $L^2(\mathbb{M}^d)$. This allows to consider Fourier expansions on $L^2(\mathbb{M}^d)$ of the type

$$f = \sum_{n=0}^{\infty} \sum_{j=1}^{N_n^d} \langle f, Y_{n,j} \rangle_2 Y_{n,j}.$$

The inclusions $L_1^{\alpha,\beta} \subset L_1^{\alpha+1,\beta}$ and $L_1^{\alpha,\beta} \subset L_1^{\alpha,\beta+1}$, $\alpha, \beta > -1$, (see [9, Lemma 2.1]) guarantee that the recurrence relations and the series expansions for the (α, β)-Schoenberg sequences as described in Section 1.3 make sense.

3. Results

3.1. Recurrence relations involving (α, β)-Schoenberg sequences

The first result provides recurrence relations involving the $(\alpha + k, \beta)$-Schoenberg coefficient for a given integer k.

Theorem 3.1. *Let $\alpha, \beta > -1$ and $k \in \mathbb{Z}_+$, $k \geq 1$. If f belongs to $L_1^{\alpha,\beta}$, then*

$$a_n^{\alpha+k,\beta} = \sum_{i=0}^{k} b_{i,k,n}^{\alpha,\beta} a_{n+i}^{\alpha,\beta} \tag{13}$$

where

$$b_{i,k,n}^{\alpha,\beta} := \frac{(2n+\alpha+\beta+k+1)\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)} (-1)^i \binom{k}{i} \times \frac{\dfrac{\Gamma(n+i+1)}{\Gamma(n+1)} \dfrac{\Gamma(n+\beta+i+1)}{\Gamma(n+\beta+1)} \dfrac{\Gamma(n+\alpha+k+1)}{\Gamma(n+\alpha+i+1)} \dfrac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+i+1)}}{\dfrac{\Gamma(2n+\alpha+\beta+k+i+2)}{\Gamma(2n+\alpha+\beta+i+1)}}.$$

Similarly to the previous theorem, we obtain recurrence formulas involving the $(\alpha, \beta+k)$- and $(\alpha+k, \beta+k')$-Schoenberg coefficients for given k and k' integers. They are provided respectively in Theorems 3.2 and 3.3 below.

Theorem 3.2. *Let $\alpha, \beta > -1$ and $k \in \mathbb{Z}_+$, $k \geq 1$. If f belongs to $L_1^{\alpha,\beta}$, then*

$$a_n^{\alpha,\beta+k} = \sum_{i=0}^{k} d_{i,k,n}^{\alpha,\beta}\, a_{n+i}^{\alpha,\beta}, \quad n \in \mathbb{Z}_+,$$

in which

$$d_{i,k,n}^{\alpha,\beta} := (2n+\alpha+\beta+k+1) \binom{k}{i} \frac{\dfrac{\Gamma(n+i+1)}{\Gamma(n+1)} \dfrac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+i+1)}}{\dfrac{\Gamma(2n+\alpha+\beta+i+k+2)}{\Gamma(2n+\alpha+\beta+i+1)}}.$$

Theorem 3.3. *Let $\alpha, \beta > -1$ and $k, k' \in \mathbb{Z}_+$, $k, k' \geq 1$. If f belongs to $L_1^{\alpha,\beta}$, then*

$$a_n^{\alpha+k,\beta+k'} = \sum_{i=0}^{k} \sum_{i'=0}^{k'} c_{i,i',k,k',n}^{\alpha,\beta} \; a_{n+i+i'}^{\alpha,\beta}, \quad n \in \mathbb{Z}_+,$$

in which

$$c_{i,i',k,k',n}^{\alpha,\beta} = b_{i,k,n}^{\alpha,\beta+k'} \; d_{i',k',n+i}^{\alpha,\beta} = b_{i,k,n+i'}^{\alpha,\beta} \; d_{i',k',n}^{\alpha+k,\beta}$$

and $b_{i,k,n}^{\alpha,\beta}$ and $d_{i',k',n}^{\alpha,\beta}$ are given in Theorems 3.1 and 3.2 respectively.

3.2. Series expansions for (α, β)-Schoenberg sequences

Theorem 3.4. *Let $\alpha \geq -1/2$ and $\beta > -1$. If f belongs to $L_1^{\alpha,\beta}$, then*

$$a_n^{\alpha,\beta} = \sum_{j=0}^{\infty} \xi_{j,n}^{\alpha,\beta} \; a_{n+j}^{\alpha+1,\beta}$$

in which

$$\xi_{0,n}^{\alpha,\beta} = \frac{(\alpha+1)(2n+\alpha+\beta+1)}{(n+\alpha+1)(n+\alpha+\beta+1)},$$

and, for $j \geq 1$,

$$\xi_{j,n}^{\alpha,\beta} = \frac{(\alpha+1)(2n+\alpha+\beta+1)(2n+\alpha+2j+1)}{2n+\alpha+\beta+2j-1} \times \frac{\dfrac{\Gamma(n+j+1)}{\Gamma(n+1)} \dfrac{\Gamma(n+\beta+j+1)}{\Gamma(n+\beta+1)}}{\dfrac{\Gamma(n+\alpha+j+2)}{\Gamma(n+\alpha+1)} \dfrac{\Gamma(n+\alpha+\beta+j+2)}{\Gamma(n+\alpha+\beta+1)}}.$$

When we combine Theorem 3.2 ($k = 1$) to the proof of the previous theorem, we obtain the following result.

Theorem 3.5. *Let $\alpha \geq -1/2$ and $\beta > -1$. If f belongs to $L_1^{\alpha,\beta}$, then*

$$a_n^{\alpha,\beta} = \sum_{j=0}^{\infty} \zeta_{j,n}^{\alpha,\beta} \; a_{n+j}^{\alpha,\beta+1}, \quad n \in \mathbb{Z}_+,$$

in which

$$\zeta_{0,n}^{\alpha,\beta} = \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1},$$

and, for $j \geq 1$,

$$\zeta_{j,n}^{\alpha,\beta} = \frac{(2n+2j+\alpha+\beta)(2n+\alpha+\beta+3)}{(n+j+\alpha+\beta+1)(2n+2j+\alpha+\beta+3)} \frac{\Gamma(n+j+1)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+1)\Gamma(n+j+\alpha+\beta+2)}.$$

Finally, the two results above give support to the theorem below.

Theorem 3.6. *Let $\alpha \geq -1/2$ and $\beta > -1$. If f belongs to $L_1^{\alpha,\beta}$, then*

$$a_n^{\alpha,\beta} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \eta_{i,j,n}^{\alpha,\beta}\; a_{n+i+j}^{\alpha+1,\beta+1}, \quad n \in \mathbb{Z}_+,$$

where

$$\eta_{i,j,n}^{\alpha,\beta} := \xi_{i,n}^{\alpha,\beta}\; \zeta_{j,n+i}^{\alpha+1,\beta} = \zeta_{j,n}^{\alpha,\beta}\; \xi_{i,n+j}^{\alpha,\beta+1},$$

and $\xi_{i,n}^{\alpha,\beta}$ and $\zeta_{j,n}^{\alpha,\beta}$ are given in Theorems 3.4 and 3.5 respectively.

As an application of the above results together with embeddings $\mathbb{M}^d \hookrightarrow \mathbb{M}^{d'}$, we obtain the following results concerning to the strictly positive definite functions on $\mathbb{M}^{d'}$. Theorem 3.7 is related to the embeddings $(i)-(iv)$ in (10) and the compact two-point homogeneous spaces $\mathbb{M}^d$ and $\mathbb{M}^{d'}$ belong to the same class: both are either real spheres, real, complex or quaternion projective spaces. Theorems 3.8 and 3.9 are related to the embeddings (vi) and (vii) and $\mathbb{M}^d$ and $\mathbb{M}^{d'}$ belong to distinct classes.

Theorem 3.7. *Let $d, d' \geq 2$ be integers, $\mathbb{M}^d$ and $\mathbb{M}^{d'}$ compact two-point homogeneous spaces distinct of $\mathbb{P}^{16}$ and belong to the same class. If f is a positive definite function on $\mathbb{M}^d$ and strictly positive definite on $\mathbb{M}^{d'}$, then f is strictly positive definite on $\mathbb{M}^d$.*

Theorem 3.8. *Let $d \geq d' \geq 2$ be integers. If f is a positive definite function on $\mathbb{P}^{4d}(\mathbb{H})$ and strictly positive definite on $\mathbb{P}^{2d'}(\mathbb{C})$, then f is strictly positive definite on $\mathbb{P}^{4d}(\mathbb{H})$.*

Theorem 3.9. *If f is a positive definite kernel on $\mathbb{P}^{16}$ and strictly positive definite on $\mathbb{P}^{8}(\mathbb{H})$ or on $\mathbb{P}^{4}(\mathbb{C})$, then f is strictly positive definite on $\mathbb{P}^{16}$.*

3.3. Solution to Problem 1.2

Theorem 3.10. *Let $\alpha, \beta > -1$. If f belongs to $L_1^{\alpha,\beta}$, then*

$$a_n^{\alpha,\beta} = \frac{\pi P_n^{(\alpha,\beta)}(1)}{4h_n^{\alpha,\beta}} \Bigg[\tilde{G}^{\alpha,\beta}(n,n)a_n^1 + \tilde{G}^{\alpha,\beta}(n+1,n)a_{n+1}^1 + \sum_{j=2}^{\infty} \left[\tilde{G}^{\alpha,\beta}(j+n,n) - \tilde{G}^{\alpha,\beta}(j-2+n,n)\right] a_{j+n}^1 \Bigg],$$

where $\tilde{G}^{\alpha,\beta}$ is given in (32), $h_n^{\alpha,\beta}$ in (16) and a_n^1 are the 1-Schoenberg coefficients as in (28).

A more general recurrence formula can also be obtained.

Proposition 3.11. *Let $\alpha, \beta, a, b > -1$. If f belongs to $L_1^{\alpha,\beta}$, then*

$$a_n^{\alpha,\beta} = \frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}} \sum_{j=n}^{\infty} \tilde{g}_{a,b}^{\alpha,\beta}(j,n) \frac{h_j^{a,b}}{P_j^{(a,b)}(1)} a_j^{a,b}, \tag{14}$$

where $\tilde{g}_{a,b}^{\alpha,\beta}$ is given in (30) and $h_n^{a,b}$ in (16).

3.4. Solution to Problem 1.3

Theorem 3.12. *Let $d \in \mathbb{Z}_+$, $d \geq 1$, α, β as in (12) and $a^c(\mathbb{M}^d)$ given in (9).*

(i) If c belongs to

$$\left[\arccos\left(\frac{\alpha-\beta}{\alpha+\beta+2}\right), \pi\right],$$

then

$$a^c(\mathbb{M}^d) \geq \frac{1}{1-\cos c}.$$

(ii) If c belongs to

$$\left[\arccos\left(\frac{\beta-\alpha}{(\alpha+\beta+4)} + 2\frac{\sqrt{(\alpha+2)(\beta+2)}}{\sqrt{(\alpha+\beta+3)(\alpha+\beta+4)}}\right), \arccos\left(\frac{\alpha-\beta}{\alpha+\beta+2}\right)\right],$$

then

$$a^c(\mathbb{M}^d) \geq \frac{(\alpha+\beta+4)}{2(\alpha+1)(1-\cos c)} \times \frac{(1-\cos c)^2(\alpha+\beta+3)(\alpha+\beta+2) - 4(\alpha+1)(\alpha+2)}{2(\alpha+2)(\alpha+\beta+2) - (\alpha+\beta+3)[4(\alpha+2) + (\alpha+\beta+4)(\cos c - 1)]}.$$

4. Useful technical lemmas and proofs

4.1. Some useful properties of Jacobi and Gegenbauer polynomials

Below we list some elementary properties which will be used, we refer to [38] for more details: for $\alpha, \beta > -1$, the set $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ form a complete orthogonal system in the interval $[-1, 1]$:

$$\int_{-1}^{1} P_k^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}\, dx = \delta_{kn} h_n^{\alpha,\beta}, \tag{15}$$

where δ_{kn} is the Kronecker delta and

$$h_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}, \quad n \in \mathbb{Z}_+. \tag{16}$$

For $x \in [-1, 1]$,

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{17}$$

$$P_0^{(\alpha,\beta)}(x) = 1, \tag{18}$$

$$P_1^{(\alpha,\beta)}(x) = (\alpha+1) + (\alpha+\beta+2)\frac{(x-1)}{2}, \tag{19}$$

$$P_2^{(\alpha,\beta)}(x) = \frac{(\alpha+1)(\alpha+2)}{2} + (\alpha+2)(\alpha+\beta+3)\frac{(x-1)}{2} + \frac{(\alpha+\beta+3)(\alpha+\beta+4)}{2}\left(\frac{x-1}{2}\right)^2, \tag{20}$$

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}. \tag{21}$$

Jacobi polynomials associated to different indexes can be related as follows (see [6, Equation (1.1)]):

$$P_j^{(a,b)}(x) = \sum_n g_{a,b}^{\alpha,\beta}(j,n) P_n^{(\alpha,\beta)}(x), \tag{22}$$

where $a, b, \alpha, \beta > -1$, and $g_{a,b}^{\alpha,\beta}(j,n)$ is given by (see [6, Equation (2.5)] and [21]):

$$g_{a,b}^{\alpha,\beta}(j,n) = \frac{\Gamma(j+a+1)\Gamma(n+\alpha+\beta+1)\Gamma(j+n+a+b+1)}{\Gamma(n+a+1)\Gamma(j+a+b+1)\Gamma(2n+\alpha+\beta+1)(j-n)!} h_{a,b}^{\alpha,\beta}(j,n), \tag{23}$$

where

$$h_{a,b}^{\alpha,\beta}(j,n) = {}_3F_2(n-j, n+\alpha+1, j+n+a+b+1; n+a+1, 2n+\alpha+\beta+2; 1), \tag{24}$$

and ${}_3F_2$ is the generalized hypergeometric function.

It is known (see [6, p. 248]) that

$$h_{a,b}^{\alpha,\beta}(n,n) = 1.$$

It is clear that $g_{a,b}^{\alpha,\beta}(j,n) = 0$ for $j < n$. Theorems 1 and 2 in [6] provide conditions on α, β, a and b so that $g_{a,b}^{\alpha,\beta}(j,n) \geq 0$.

The following fact will be applied to the weight functions $w(x) = (1-x)^\alpha(1+x)^\beta$ and $w_1(x) = (1-x)^a(1+x)^b$ and to the Jacobi polynomials $p_n = P_n^{(\alpha,\beta)}$ and $q_n = P_n^{(a,b)}$ in order to obtain Lemma 4.2 which connects Jacobi polynomials with different parameters together with respective weight functions.

Remark 4.1. (See Equations (4)-(5) in [4]) Let $w(x)$ and $w_1(x)$ be positive functions on a set E. Let $\{p_n(x)\}_n$ and $\{q_n(x)\}_n$ be sequences of orthonormal polynomials associated with $w(x)$ and $w_1(x)$ respectively. Hence, if

$$q_j(x) = \sum_{n=0}^{j} c_{n,j} p_n(x) \tag{25}$$

then

$$w(x)p_n(x) = \sum_{j=n}^{\infty} c_{n,j} q_j(x) w_1(x). \tag{26}$$

The convergence of the series can be taken in the appropriate L^2 space if $[w(x)]^2/w_1(x)$ is integrable.

The Gegenbauer polynomials are multiple of the Jacobi polynomials (see [38, Eq. (4.7.1)]):

$$\mathcal{C}_n^\lambda(x) = \frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)} \frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+1/2)} P_n^{(\lambda-1/2,\lambda-1/2)}(x), \quad \lambda > -\frac{1}{2}, \tag{27}$$

and in particular (see p. 29 and Eq. (4.7.2) in [38])

$$\mathcal{C}_n^1(\cos\theta) = U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

Note that when $\alpha = \beta = (d-2)/2$, $d > 1$, we have that the coefficients in (8) are the classical d-Schoenberg coefficients a_n^d, and for $d = 1$ (see [24, Eq. (14)])

$$\begin{aligned} a_0^1 &= \frac{1}{\pi}\int_0^\pi \psi(\theta)\,d\theta\,, \\ a_n^1 &= \frac{2}{\pi}\int_0^\pi \psi(\theta)\cos(n\theta)\,d\theta\,, \quad n \geq 1. \end{aligned} \tag{28}$$

4.2. Some useful lemmas

The results presented here are used in the solution of Problem 1.2: Theorem 3.10 and Proposition 3.11. Also, Lemma 4.3 can be seen as an extension of [3, Lemma 1] which deals with Gegenbauer polynomials.

Lemma 4.2. *For $\alpha, \beta, a, b > -1$,*

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \sum_{j=n}^{\infty} \tilde{g}_{a,b}^{\alpha,\beta}(j,n) P_j^{(a,b)}(x)(1-x)^a(1+x)^b, \quad x \in [-1,1] \tag{29}$$

where

$$\begin{aligned} \tilde{g}_{a,b}^{\alpha,\beta}(j,n) = 2^{\alpha+\beta-a-b}&\frac{2j+a+b+1}{2n+\alpha+\beta+1}\frac{j!}{n!(j-n)!} \\ &\times \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(j+n+a+b+1)}{\Gamma(n+a+1)\Gamma(j+b+1)\Gamma(2n+\alpha+\beta+1)}\, h_{a,b}^{\alpha,\beta}(j,n), \end{aligned} \tag{30}$$

and

$$h_{a,b}^{\alpha,\beta}(j,n) = {}_3F_2(n-j, n+\alpha+1, j+n+a+b+1; n+a+1, 2n+\alpha+\beta+2; 1).$$

Proof. It is enough to observe that the Jacobi polynomials are not orthonormal. Then, using Remark 4.1 and the Equations (15), (22)–(24), we obtain (29) where

$$\tilde{g}_{a,b}^{\alpha,\beta}(j,n) = \frac{h_n^{\alpha,\beta}}{h_j^{a,b}} g_{a,b}^{\alpha,\beta}(j,n),$$

that coincides with (30). □

As a consequence we have

Lemma 4.3. *For $\alpha, \beta > -1$,*

$$(1-\cos\theta)^\alpha(1+\cos\theta)^\beta P_n^{(\alpha,\beta)}(\cos\theta) = \sum_{j=n}^{\infty} \tilde{G}^{\alpha,\beta}(j,n)\sin((j+1)\theta), \quad \theta \in [0,\pi], \tag{31}$$

where

$$\tilde{G}^{\alpha,\beta}(j,n) = \frac{2^{\alpha+\beta-2j-1}}{(2n+\alpha+\beta+1)} \frac{(2j+2)!}{(j+1)!n!(j-n)!} \times \\ \times \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(j+n+2)}{\Gamma(n+3/2)\Gamma(j+3/2)\Gamma(2n+\alpha+\beta+1)} h^{\alpha,\beta}(j,n), \tag{32}$$

and

$$h^{\alpha,\beta}(j,n) = {}_3F_2(n-j, n+\alpha+1, j+n+2; n+3/2, 2n+\alpha+\beta+2; 1).$$

Proof. Consider $a = b = 1/2$ in (29) and note that letting $\lambda = 1$ in (27) we obtain

$$P_j^{(1/2,1/2)}(\cos\theta) = \frac{\Gamma(2)}{\Gamma(3/2)} \frac{\Gamma(j+3/2)}{\Gamma(j+2)} C_j^1(\cos\theta) \\ = \frac{2}{\sqrt{\pi}} \frac{(2j+2)!\sqrt{\pi}}{4^{j+1}(j+1)!(j+1)!} \frac{\sin((j+1)\theta)}{\sin\theta}$$

that is,

$$P_j^{(1/2,1/2)}(\cos\theta) = \frac{(2j+2)!}{2^{2j+1}(j+1)!^2} \frac{\sin((j+1)\theta)}{\sin\theta}.$$

Thus, by Lemma 4.2, for $\theta \in (0, \pi)$ we have

$$(1-\cos\theta)^\alpha (1+\cos\theta)^\beta P_n^{(\alpha,\beta)}(\cos\theta) = \sum_{j=n}^{\infty} \tilde{G}^{\alpha,\beta}(j,n) \frac{\sin((j+1)\theta)}{\sin\theta} (1-\cos^2\theta)^{1/2} \\ = \sum_{j=n}^{\infty} \tilde{G}^{\alpha,\beta}(j,n) \sin((j+1)\theta),$$

where

$$\tilde{G}^{\alpha,\beta}(j,n) = \frac{(2j+2)!}{2^{2j+1}(j+1)!^2} \tilde{g}^{\alpha,\beta}_{1/2,1/2}(j,n) \\ = 2^{\alpha+\beta-2j-2} \frac{2j+2}{2n+\alpha+\beta+1} \frac{(2j+2)!}{(j+1)(j+1)!n!(j-n)!} \\ \times \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(j+n+2)}{\Gamma(n+3/2)\Gamma(j+3/2)\Gamma(2n+\alpha+\beta+1)} h^{\alpha,\beta}_{1/2,1/2}(j,n),$$

for $\theta = 0, \pi$ the result is trivially true. The proof is complete. □

4.3. Proofs for Section 3.1

We observe that the results of Section 3.1 extend Lemma 2.3 in [9] which gives relationship between the Fourier-Jacobi coefficients $a_n^{\alpha+1,\beta}$ and $a_n^{\alpha,\beta}$ and between $a_n^{\alpha,\beta+1}$ and $a_n^{\alpha,\beta}$. It is used in the following proofs.

Proof of Theorem 3.1. We will prove the statement by mathematical induction on k.

Step $k = 1$: In [9, Lemma 2.3] was proved that

$$(\alpha+1)a_n^{\alpha+1,\beta} = \frac{(n+\alpha+1)(n+\alpha+\beta+1)}{2n+\alpha+\beta+1} a_n^{\alpha,\beta} - \frac{(n+1)(n+\beta+1)}{2n+\alpha+\beta+3} a_{n+1}^{\alpha,\beta}, \quad n \in \mathbb{Z}_+.$$

Thus

$$a_n^{\alpha+1,\beta} = \frac{1}{(\alpha+1)}\left[\frac{(n+\alpha+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)}a_n^{\alpha,\beta} - \frac{(n+1)(n+\beta+1)}{(2n+\alpha+\beta+3)}a_{n+1}^{\alpha,\beta}\right],$$

which fits perfectly in Equation (13) for $k = 1$.

Induction step: let assume the expression of $b_{i,k,n}^{\alpha,\beta}$ holds for k, and let us prove it holds for $b_{i,k+1,n}^{\alpha,\beta}$. We have

$$\begin{aligned}
a_n^{\alpha+(k+1),\beta} = a_n^{(\alpha+1)+k,\beta} &= \sum_{i=0}^{k} b_{i,k,n}^{\alpha+1,\beta} a_{n+i}^{\alpha+1,\beta} \\
&= \sum_{i=0}^{k} b_{i,k,n}^{\alpha+1,\beta}\left(b_{0,1,n+i}^{\alpha,\beta}a_{n+i}^{\alpha,\beta} + b_{1,1,n+i}^{\alpha,\beta}a_{n+i+1}^{\alpha,\beta}\right) \\
&= b_{0,k,n}^{\alpha+1,\beta}b_{0,1,n}^{\alpha,\beta}a_n^{\alpha,\beta} + b_{k,k,n}^{\alpha+1,\beta}b_{1,1,n+k}^{\alpha,\beta}a_{n+k+1}^{\alpha,\beta} \\
&\qquad + \sum_{i=1}^{k}\left(b_{i,k,n}^{\alpha+1,\beta}b_{0,1,n+i}^{\alpha,\beta} + b_{i-1,k,n}^{\alpha+1,\beta}b_{1,1,n+i-1}^{\alpha,\beta}\right)a_{n+i}^{\alpha,\beta}.
\end{aligned}$$

Thus we just need to prove that:

- $b_{0,k,n}^{\alpha+1,\beta}b_{0,1,n}^{\alpha,\beta} = b_{0,k+1,n}^{\alpha,\beta}$ (it is the case $i = 0$);
- $b_{i,k,n}^{\alpha+1,\beta}b_{0,1,n+i}^{\alpha,\beta} + b_{i-1,k,n}^{\alpha+1,\beta}b_{1,1,n+i-1}^{\alpha,\beta} = b_{i,k+1,n}^{\alpha,\beta}$ for $i = 1, 2, \ldots, k$;
- $b_{k,k,n}^{\alpha+1,\beta}b_{1,1,n+k}^{\alpha,\beta} = b_{k+1,k+1,n}^{\alpha,\beta}$ (it is the case $i = k+1$).

For $i = 0$, by well known property $a\Gamma(a) = \Gamma(a+1)$ of the Gamma function, we obtain:

$$\begin{aligned}
b_{0,k,n}^{\alpha+1,\beta}b_{0,1,n}^{\alpha,\beta} &= \frac{(2n+\alpha+\beta+k+2)\Gamma(\alpha+2)}{\Gamma(\alpha+k+2)} \\
&\qquad \times \frac{\dfrac{\Gamma(n+\alpha+k+2)}{\Gamma(n+\alpha+2)}\dfrac{\Gamma(n+\alpha+\beta+k+2)}{\Gamma(n+\alpha+\beta+2)}}{\dfrac{\Gamma(2n+\alpha+\beta+k+3)}{\Gamma(2n+\alpha+\beta+2)}} \\
&\qquad \times \frac{1}{(\alpha+1)}\frac{(n+\alpha+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)} \\
&= \frac{(2n+\alpha+\beta+k+2)\Gamma(\alpha+1)}{\Gamma(\alpha+k+2)} \\
&\qquad \times \frac{\dfrac{\Gamma(n+\alpha+k+2)}{\Gamma(n+\alpha+1)}\dfrac{\Gamma(n+\alpha+\beta+k+2)}{\Gamma(n+\alpha+\beta+1)}}{\dfrac{\Gamma(2n+\alpha+\beta+k+3)}{\Gamma(2n+\alpha+\beta+1)}} \\
&= b_{0,k+1,n}^{\alpha,\beta}.
\end{aligned}$$

For $i = k + 1$:

$$b^{\alpha+1,\beta}_{k,k,n} b^{\alpha,\beta}_{1,1,n+k} = \frac{(2n+\alpha+\beta+k+2)\Gamma(\alpha+2)}{\Gamma(\alpha+k+2)} \times (-1)^k \frac{\frac{\Gamma(n+k+1)}{\Gamma(n+1)} \frac{\Gamma(n+\beta+k+1)}{\Gamma(n+\beta+1)}}{\frac{\Gamma(2n+\alpha+\beta+2k+3)}{\Gamma(2n+\alpha+\beta+k+2)}} \times \left(-\frac{1}{(\alpha+1)} \frac{(n+k+1)(n+\beta+k+1)}{(2n+\alpha+\beta+2k+3)}\right)$$

$$= \frac{(2n+\alpha+\beta+k+2)\Gamma(\alpha+1)}{\Gamma(\alpha+k+2)} \times (-1)^{k+1} \frac{\frac{\Gamma(n+k+2)}{\Gamma(n+1)} \frac{\Gamma(n+\beta+k+2)}{\Gamma(n+\beta+1)}}{\frac{\Gamma(2n+\alpha+\beta+2k+4)}{\Gamma(2n+\alpha+\beta+k+2)}}$$

$$= b^{\alpha,\beta}_{k+1,k+1,n}.$$

Finally for $i = 1, 2, \ldots, k$, we have:

$$b^{\alpha+1,\beta}_{i,k,n} b^{\alpha,\beta}_{0,1,n+i} + b^{\alpha+1,\beta}_{i-1,k,n} b^{\alpha,\beta}_{1,1,n+i-1} = \frac{(2n+\alpha+\beta+k+2)\Gamma(\alpha+2)}{\Gamma(\alpha+k+2)} (-1)^i \binom{k}{i} \times \frac{\frac{\Gamma(n+i+1)}{\Gamma(n+1)} \frac{\Gamma(n+\beta+i+1)}{\Gamma(n+\beta+1)} \frac{\Gamma(n+\alpha+k+2)}{\Gamma(n+\alpha+i+2)} \frac{\Gamma(n+\alpha+\beta+k+2)}{\Gamma(n+\alpha+\beta+i+2)}}{\frac{\Gamma(2n+\alpha+\beta+k+i+3)}{\Gamma(2n+\alpha+\beta+i+2)}} \times \frac{1}{(\alpha+1)} \frac{(n+\alpha+i+1)(n+\alpha+\beta+i+1)}{(2n+\alpha+\beta+2i+1)}$$

$$+ \frac{(2n+\alpha+\beta+k+2)\Gamma(\alpha+2)}{\Gamma(\alpha+k+2)} (-1)^{i-1} \binom{k}{i-1} \times \frac{\frac{\Gamma(n+i)}{\Gamma(n+1)} \frac{\Gamma(n+\beta+i)}{\Gamma(n+\beta+1)} \frac{\Gamma(n+\alpha+k+2)}{\Gamma(n+\alpha+i+1)} \frac{\Gamma(n+\alpha+\beta+k+2)}{\Gamma(n+\alpha+\beta+i+1)}}{\frac{\Gamma(2n+\alpha+\beta+k+i+2)}{\Gamma(2n+\alpha+\beta+i+1)}} \times \left(-\frac{1}{(\alpha+1)} \frac{(n+i)(n+\beta+i)}{(2n+\alpha+\beta+2i+1)}\right).$$

Using the well known property of the Gamma function, after some algebraic manipulation, we obtain

$$b^{\alpha+1,\beta}_{i,k,n} b^{\alpha,\beta}_{0,1,n+i} + b^{\alpha+1,\beta}_{i-1,k,n} b^{\alpha,\beta}_{1,1,n+i-1} = \frac{(2n+\alpha+\beta+k+2)\Gamma(\alpha+1)}{\Gamma(\alpha+k+2)} (-1)^i \times \frac{\frac{\Gamma(n+i+1)}{\Gamma(n+1)} \frac{\Gamma(n+\beta+i+1)}{\Gamma(n+\beta+1)} \frac{\Gamma(n+\alpha+k+2)}{\Gamma(n+\alpha+i+1)} \frac{\Gamma(n+\alpha+\beta+k+2)}{\Gamma(n+\alpha+\beta+i+1)}}{\frac{\Gamma(2n+\alpha+\beta+k+i+3)}{\Gamma(2n+\alpha+\beta+i+1)}} \times \left[\binom{k}{i} \frac{(2n+\alpha+\beta+i+1)}{(2n+\alpha+\beta+2i+1)} + \binom{k}{i-1} \frac{(2n+\alpha+\beta+k+i+2)}{(2n+\alpha+\beta+2i+1)}\right].$$

Now, it is not difficult to see that:

$$\left[\binom{k}{i}\frac{(2n+\alpha+\beta+i+1)}{(2n+\alpha+\beta+2i+1)}+\binom{k}{i-1}\frac{(2n+\alpha+\beta+k+i+2)}{(2n+\alpha+\beta+2i+1)}\right]$$
$$=\binom{k+1}{i}\left[\frac{(k-i+1)}{(k+1)}\frac{(2n+\alpha+\beta+i+1)}{(2n+\alpha+\beta+2i+1)}\right.$$
$$\left.+\frac{i}{(k+1)}\frac{(2n+\alpha+\beta+k+i+2)}{(2n+\alpha+\beta+2i+1)}\right]$$
$$=\binom{k+1}{i}.$$

Therefore,

$$b^{\alpha+1,\beta}_{i,k,n}b^{\alpha,\beta}_{0,1,n+i}+b^{\alpha+1,\beta}_{i-1,k,n}b^{\alpha,\beta}_{1,1,n+i-1}=\frac{(2n+\alpha+\beta+k+2)\Gamma(\alpha+1)}{\Gamma(\alpha+k+2)}(-1)^i\binom{k+1}{i}$$
$$\times\frac{\frac{\Gamma(n+i+1)}{\Gamma(n+1)}\frac{\Gamma(n+\beta+i+1)}{\Gamma(n+\beta+1)}\frac{\Gamma(n+\alpha+k+2)}{\Gamma(n+\alpha+i+1)}\frac{\Gamma(n+\alpha+\beta+k+2)}{\Gamma(n+\alpha+\beta+i+1)}}{\frac{\Gamma(2n+\alpha+\beta+k+i+3)}{\Gamma(2n+\alpha+\beta+i+1)}}$$
$$=b^{\alpha,\beta}_{i,k+1,n}.\quad\square$$

Proof of Theorem 3.2. Follows using mathematical induction on k. In [9, Lemma 2.3] was proved that

$$a^{\alpha,\beta+1}_n=\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}a^{\alpha,\beta}_n+\frac{n+1}{2n+\alpha+\beta+3}a^{\alpha,\beta}_{n+1},\quad n\in\mathbb{Z}_+,$$

which coincides with the statement in the step $k=1$. The continuation of the proof is very similar to the previous one and we will omit the calculations here. □

Proof of Theorem 3.3. Follows by applying sequentially Theorems 3.1 and 3.2. □

4.4. Proofs for Section 3.2

Proof of Theorem 3.4. Theorem 3.1 for $k=1$ and $n\geq 0$ gives

$$a^{\alpha+1,\beta}_n=b^{\alpha,\beta}_{0,1,n}\;a^{\alpha,\beta}_n+b^{\alpha,\beta}_{1,1,n}\;a^{\alpha,\beta}_{n+1},$$

whence we can isolate

$$a^{\alpha,\beta}_n=\gamma^{\alpha,\beta}_n\;a^{\alpha+1,\beta}_n+\omega^{\alpha,\beta}_n a^{\alpha,\beta}_{n+1},\tag{33}$$

where

$$\gamma^{\alpha,\beta}_n:=(b^{\alpha,\beta}_{0,1,n})^{-1}=\frac{\Gamma(\alpha+2)}{(2n+\alpha+\beta+2)\Gamma(\alpha+1)}\frac{\frac{\Gamma(2n+\alpha+\beta+3)}{\Gamma(2n+\alpha+\beta+1)}}{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+1)}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+\beta+1)}}$$

and

$$\omega_n^{\alpha,\beta} := -\frac{(b_{1,1,n}^{\alpha,\beta})}{(b_{0,1,n}^{\alpha,\beta})} = \frac{\frac{\Gamma(n+2)}{\Gamma(n+1)}\frac{\Gamma(n+\beta+2)}{\Gamma(n+\beta+1)}}{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+1)}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+\beta+1)}}\frac{(2n+\alpha+\beta+1)}{(2n+\alpha+\beta+3)}.$$

Now we can use Eq. (33) recursively with $a_{n+1}^{\alpha,\beta}$, $a_{n+2}^{\alpha,\beta}$, ..., $a_{n+k}^{\alpha,\beta}$, for any $k \geq 1$, in order to get:

$$\begin{aligned}
a_n^{\alpha,\beta} &= \gamma_n^{\alpha,\beta}\ a_n^{\alpha+1,\beta} + \omega_n^{\alpha,\beta} a_{n+1}^{\alpha,\beta} \\
&= \gamma_n^{\alpha,\beta}\ a_n^{\alpha+1,\beta} + \omega_n^{\alpha,\beta}\left[\gamma_{n+1}^{\alpha,\beta}\ a_{n+1}^{\alpha+1,\beta} + \omega_{n+1}^{\alpha,\beta} a_{n+2}^{\alpha,\beta}\right] \\
&= \gamma_n^{\alpha,\beta}\ a_n^{\alpha+1,\beta} + \omega_n^{\alpha,\beta}\gamma_{n+1}^{\alpha,\beta}\ a_{n+1}^{\alpha+1,\beta} + \omega_n^{\alpha,\beta}\omega_{n+1}^{\alpha,\beta} a_{n+2}^{\alpha,\beta} \\
&= \gamma_n^{\alpha,\beta}\ a_n^{\alpha+1,\beta} + \omega_n^{\alpha,\beta}\gamma_{n+1}^{\alpha,\beta}\ a_{n+1}^{\alpha+1,\beta} + \omega_n^{\alpha,\beta}\omega_{n+1}^{\alpha,\beta}\left[\gamma_{n+2}^{\alpha,\beta}\ a_{n+2}^{\alpha+1,\beta} + \omega_{n+2}^{\alpha,\beta} a_{n+3}^{\alpha,\beta}\right] \\
&= \gamma_n^{\alpha,\beta}\ a_n^{\alpha+1,\beta} + \omega_n^{\alpha,\beta}\gamma_{n+1}^{\alpha,\beta}\ a_{n+1}^{\alpha+1,\beta} + \omega_n^{\alpha,\beta}\omega_{n+1}^{\alpha,\beta}\gamma_{n+2}^{\alpha,\beta}\ a_{n+2}^{\alpha+1,\beta} + \omega_n^{\alpha,\beta}\omega_{n+1}^{\alpha,\beta}\omega_{n+2}^{\alpha,\beta} a_{n+3}^{\alpha,\beta} \\
&= \cdots \\
&= \gamma_n^{\alpha,\beta}\ a_n^{\alpha+1,\beta} + \sum_{j=1}^{k}\left(\prod_{l=0}^{j-1}\omega_{n+l}^{\alpha,\beta}\right)\gamma_{n+j}^{\alpha,\beta} a_{n+j}^{\alpha+1,\beta} + \left(\prod_{l=0}^{k}\omega_{n+l}^{\alpha,\beta}\right) a_{n+k+1}^{\alpha,\beta}.
\end{aligned}$$

Let us analyze the last term for $k \to \infty$. On the one hand, $\lim_{k\to\infty} a_{n+k+1}^{\alpha,\beta} = 0$, because $f \in L_1^{\alpha,\beta}$, and then $\sum_{n=0}^{\infty} a_n^{\alpha,\beta}\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} < \infty$ for all $x \in [-1,1]$ (in particular $\sum_{n=0}^{\infty} a_n < \infty$).

On the other hand, we shall prove that $\prod_{l=0}^{k}\omega_{n+l}^{\alpha,\beta}$ converges to 0 as $k \to \infty$ for all fixed $n, \alpha > -1/2$ and $\beta > -1$.

We have that the product

$$\prod_{l=0}^{k}\omega_{n+l}^{\alpha,\beta} = \prod_{l=0}^{k}\frac{\frac{\Gamma(n+2+l)}{\Gamma(n+1+l)}\frac{\Gamma(n+\beta+2+l)}{\Gamma(n+\beta+1+l)}}{\frac{\Gamma(n+\alpha+2+l)}{\Gamma(n+\alpha+1+l)}\frac{\Gamma(n+\alpha+\beta+2+l)}{\Gamma(n+\alpha+\beta+1+l)}}\frac{(2n+\alpha+\beta+1+2l)}{(2n+\alpha+\beta+3+2l)}$$

is a telescoping product and then

$$\prod_{l=0}^{k}\omega_{n+l}^{\alpha,\beta} = \frac{\frac{\Gamma(n+k+2)}{\Gamma(n+1)}\frac{\Gamma(n+\beta+k+2)}{\Gamma(n+\beta+1)}}{\frac{\Gamma(n+\alpha+k+2)}{\Gamma(n+\alpha+1)}\frac{\Gamma(n+\alpha+\beta+k+2)}{\Gamma(n+\alpha+\beta+1)}}\frac{2n+\alpha+\beta+1}{2n+\alpha+\beta+2k+1}. \tag{34}$$

Rearranging the factors

$$\prod_{l=0}^{k}\omega_{n+l}^{\alpha,\beta} = \frac{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\beta+1)}}{\frac{\Gamma(n+\alpha+k+2)}{\Gamma(n+k+2)}\frac{\Gamma(n+\alpha+\beta+k+2)}{\Gamma(n+\beta+k+2)}}\frac{2n+\alpha+\beta+1}{2n+\alpha+\beta+2k+1}.$$

Using that $\dfrac{\Gamma(k+a)}{\Gamma(k+b)} \approx k^{a-b}$ when $k \to \infty$ for a, b fixed (see [2, p. 20]) and omitting constant factors with respect to k, we obtain

$$\prod_{l=0}^{k} \omega_{n+l}^{\alpha,\beta} \approx k^{-\alpha}k^{-\alpha}k^{-1} = \frac{1}{k^{1+2\alpha}}.$$

Hence we have that the sequence $\left\{\prod_{l=0}^{k} \omega_{n+l}^{\alpha,\beta}\right\}_{k=0}^{\infty}$ converges to zero when $\alpha > -1/2$, it is bounded when $\alpha = -1/2$, and it diverges to $+\infty$ or to $-\infty$, depending on the sign of the factor $\Gamma(n+\alpha+\beta+1)$ when $-1 < \alpha < -1/2$.

Therefore, for $\alpha \geq -1/2$, we have that $\lim_{k\to\infty} \left(\prod_{l=0}^{k} \omega_{n+l}^{\alpha,\beta}\right) a_{n+k+1}^{\alpha,\beta} = 0$ and then we can write

$$a_n^{\alpha,\beta} = \gamma_n^{\alpha,\beta}\ a_n^{\alpha+1,\beta} + \sum_{j=1}^{\infty} \left(\prod_{l=1}^{j} \omega_{n+l-1}^{\alpha,\beta}\right) \gamma_{n+j}^{\alpha,\beta}\ a_{n+j}^{\alpha+1,\beta}.$$

Now, using (34) and the definition of $\gamma_{n+j}^{\alpha,\beta}$, if $j \geq 1$, we obtain

$$\left(\prod_{l=1}^{j} \omega_{n+l-1}^{\alpha,\beta}\right) \gamma_{n+j}^{\alpha,\beta} = \frac{\dfrac{\Gamma(n+j+1)}{\Gamma(n+1)} \dfrac{\Gamma(n+\beta+j+1)}{\Gamma(n+\beta+1)}}{\dfrac{\Gamma(n+\alpha+j+2)}{\Gamma(n+\alpha+1)} \dfrac{\Gamma(n+\alpha+\beta+j+2)}{\Gamma(n+\alpha+\beta+1)}} \times \frac{(\alpha+1)(2n+\alpha+\beta+1)(2n+\alpha+2j+1)}{2n+\alpha+\beta+2j-1}.$$

The proof is complete. □

Proof of Theorem 3.5. Follows combining Theorem 3.2 ($k = 1$) to the previous proof. □

Proof of Theorem 3.6. Follows by applying sequentially Theorems 3.4 and 3.5. □

Proof of Theorem 3.7. The case $\mathbb{M}^d = \mathbb{S}^d$ was proved in [13, Theorem 4.1]. So, we may assume that $\mathbb{M}^d$ and $\mathbb{M}^{d'}$ are one of the projective spaces. If $d' \geq d$, the assertion follows from the inclusions in (10). Suppose that $d' < d$, that is, $d = d' + k$, for some $k > 0$. If we consider (α, β) and (α', β') the indexes related with the dimension d and d' respectively, then $\alpha = \alpha' + k$. Initially, consider the case $k = 1$. By Theorem 3.4, since $\xi_{j,n} > 0$, $a_n^{\alpha,\beta} > 0$ if and only if $a_{n+j}^{\alpha+1,\beta}$ for at least one $j \geq 0$. Thus, the set $\{n \in \mathbb{Z}_+ : a_n^{\alpha,\beta} > 0\}$ is infinite if and only if $\{n \in \mathbb{Z}_+ : a_n^{\alpha+1,\beta} > 0\}$ is also infinite. Now Theorems 2 and 3 in [8] show that f is strictly positive definite on $\mathbb{M}^d$. If $k > 1$, an iterated process completes the proof. □

Note that in the proof of Theorem 3.7 we apply Theorem 3.4, which presents a recurrence formula for the α index. When applying Theorem 3.6, we get Theorem 3.8:

Proof of Theorem 3.8. If $(\alpha, 1)$ and $(\alpha', 0)$ are the indexes related with $\mathbb{P}^{4d}(\mathbb{H})$ and $\mathbb{P}^{2d'}(\mathbb{C})$ respectively, there exists $k \geq 0$ such that $\alpha = \alpha' + k$. Hence, by Theorem 3.6, the same arguments presented in the proof of Theorem 3.7 guarantee that the set $\{n \in \mathbb{Z}_+ : a_n^{\alpha,0} > 0\}$ is infinite if and only if $\{n \in \mathbb{Z}_+ : a_n^{\alpha+k,1} > 0\}$ is also infinite. Thus, by Theorem 3 in [8], we have that f is strictly positive definite on $\mathbb{P}^{2d'}(\mathbb{C})$. □

Proof of Theorem 3.9. Follows by similar arguments used in the preview proof applied to the particular case $\mathbb{M}^d = \mathbb{P}^{16}$. □

4.5. Proofs for Section 3.3

Proof of Theorem 3.10. By (3) and Lemma 4.3 we obtain

$$\begin{aligned} a_n^{\alpha,\beta} &= \frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}} \int_0^\pi f(\theta) P_n^{(\alpha,\beta)}(\cos\theta)(1-\cos\theta)^\alpha (1+\cos\theta)^\beta \sin\theta \, d\theta \\ &= \frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}} \int_0^\pi \left[\sum_{j=n}^\infty \tilde{G}^{\alpha,\beta}(j,n) \sin((j+1)\theta) \right] f(\theta) \sin\theta \, d\theta \\ &= \frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}} \int_0^\pi \left[\sum_{j=n}^\infty \tilde{G}^{\alpha,\beta}(j,n) \sin((j+1)\theta) \sin\theta \right] f(\theta) \, d\theta \\ &= \frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}} \int_0^\pi \left[\sum_{j=0}^\infty \tilde{G}^{\alpha,\beta}(j+n,n) \sin((j+n+1)\theta) \sin\theta \right] f(\theta) \, d\theta. \end{aligned}$$

This implies

$$\begin{aligned} a_n^{\alpha,\beta} &= \frac{1}{2} \frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}} \int_0^\pi \Bigg[\sum_{j=0}^\infty \tilde{G}^{\alpha,\beta}(j+n,n) \left[\cos((j+n)\theta) \right. \\ &\qquad \left. - \cos((j+n+2)\theta) \right] \Bigg] f(\theta) \, d\theta \\ &= \frac{1}{2} \frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}} \int_0^\pi \Bigg[\tilde{G}^{\alpha,\beta}(n,n) \cos(n\theta) + \sum_{j=1}^\infty \tilde{G}^{\alpha,\beta}(j+n,n) \cos((j+n)\theta) \\ &\qquad - \sum_{j=0}^\infty \tilde{G}^{\alpha,\beta}(j+n,n) \cos((j+n+2)\theta) \Bigg] f(\theta) \, d\theta. \end{aligned}$$

Letting $\nu = j+2$ in the last sum,

$$\begin{aligned} a_n^{\alpha,\beta} &= \frac{1}{2} \frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}} \int_0^\pi \Bigg[\tilde{G}^{\alpha,\beta}(n,n) \cos(n\theta) + \tilde{G}^{\alpha,\beta}(n+1,n) \cos((n+1)\theta) \\ &\qquad + \sum_{j=2}^\infty \tilde{G}^{\alpha,\beta}(j+n,n) \cos((j+n)\theta) \\ &\qquad - \sum_{\nu=2}^\infty \tilde{G}^{\alpha,\beta}(\nu-2+n,n) \cos((\nu+n)\theta) \Bigg] f(\theta) \, d\theta \end{aligned}$$

$$= \frac{1}{2}\frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}}\int_0^\pi \Bigg[\tilde{G}^{\alpha,\beta}(n,n)\cos(n\theta)+\tilde{G}^{\alpha,\beta}(n+1,n)\cos((n+1)\theta)$$

$$+\sum_{j=2}^{\infty}[\tilde{G}^{\alpha,\beta}(j+n,n)-\tilde{G}^{\alpha,\beta}(j-2+n,n)]\cos((j+n)\theta)]\Bigg] f(\theta)\,d\theta.$$

Thus, by (28),

$$a_n^{\alpha,\beta} = \frac{\tilde{G}^{\alpha,\beta}(n,n)}{2}\frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}}\int_0^\pi \cos(n\theta)f(\theta)\,d\theta$$

$$+\frac{\tilde{G}^{\alpha,\beta}(n+1,n)}{2}\frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}}\int_0^\pi \cos((n+1)\theta)f(\theta)\,d\theta$$

$$+\sum_{j=2}^{\infty}\Bigg[\frac{\tilde{G}^{\alpha,\beta}(j+n,n)-\tilde{G}^{\alpha,\beta}(j-2+n,n)}{2}$$

$$\times\frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}}\int_0^\pi \cos((j+n)\theta)f(\theta)\,d\theta\Bigg]$$

$$= \tilde{G}^{\alpha,\beta}(n,n)\frac{\pi P_n^{(\alpha,\beta)}(1)}{4h_n^{\alpha,\beta}}a_n^1+\tilde{G}^{\alpha,\beta}(n+1,n)\frac{\pi P_n^{(\alpha,\beta)}(1)}{4h_n^{\alpha,\beta}}a_{n+1}^1$$

$$+\sum_{j=2}^{\infty}\left[\tilde{G}^{\alpha,\beta}(j+n,n)-\tilde{G}^{\alpha,\beta}(j-2+n,n)\right]\frac{\pi P_n^{(\alpha,\beta)}(1)}{4h_n^{\alpha,\beta}}a_{j+n}^1,$$

which concludes the proof. □

Proof of Proposition 3.11. By (3) and Lemma 4.2 we obtain

$$a_n^{\alpha,\beta} = \frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}}\int_{-1}^{1} f(x)P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\,dx$$

$$= \frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}}\int_{-1}^{1}\sum_{j=n}^{\infty}\tilde{g}_{a,b}^{\alpha,\beta}(j,n)P_j^{(a,b)}(x)f(x)(1-x)^a(1+x)^b\,dx.$$

Hence,

$$a_n^{\alpha,\beta} = \frac{P_n^{(\alpha,\beta)}(1)}{h_n^{\alpha,\beta}}\sum_{j=n}^{\infty}\tilde{g}_{a,b}^{\alpha,\beta}(j,n)\int_{-1}^{1} P_j^{(a,b)}(x)f(x)(1-x)^a(1+x)^b\,dx$$

which concludes the proof. □

4.6. Proofs for Section 3.4

Let ψ be a function belongs to the class $\Psi(\mathbb{M}^d)$ and its representation in series as in (7). It is not difficult to see that if $\sum_{n=1}^{\infty} n^5 a_n^{\alpha,\beta} < \infty$, then

$$\psi''(\theta) = \sum_{n=1}^{\infty} \frac{a_n^{\alpha,\beta}}{P_n^{(\alpha,\beta)}(1)} \frac{d^2}{d\theta^2}\left(P_n^{(\alpha,\beta)}(\cos\theta)\right), \quad \theta \in [0,\pi].$$

In particular,

$$\psi''(0) = -\sum_{n=1}^{\infty} \frac{n(n+\alpha+\beta+1)}{2(\alpha+1)} a_n^{\alpha,\beta}. \tag{35}$$

Proof of Theorem 3.12. As in [3] in order to find a bound to $a^c(\mathbb{M}^d)$ we consider the more suitable set

$$\widetilde{\Psi}^c(\mathbb{M}^d) = \{\psi \in \Psi(\mathbb{M}^d) : \psi(c) = 0\},$$

and

$$\tilde{a}^c(\mathbb{M}^d) = \inf\{-\psi''(0) :\ \psi \in \widetilde{\Psi}^c(\mathbb{M}^d)\}.$$

Since $\Psi^c(\mathbb{M}^d) \subset \widetilde{\Psi}^c(\mathbb{M}^d)$, then $a^c(\mathbb{M}^d) \geq \tilde{a}^c(\mathbb{M}^d)$. Moreover, as the sequence $\{a_n^{\alpha,\beta}\}_n$ form a probability mass system, then $-\psi''(0)$ shall be smaller for functions ψ whose mass is concentrated in lower index coefficients.

Thus we need to solve the system

$$\begin{cases} \displaystyle\sum_{n=0}^{\infty} a_n^{\alpha,\beta} \frac{P_n^{(\alpha,\beta)}(\cos c)}{P_n^{(\alpha,\beta)}(1)} = 0 \\ \displaystyle\sum_{n=0}^{\infty} a_n^{\alpha,\beta} = 1 \end{cases} \tag{36}$$

where $a_n^{\alpha,\beta} \geq 0$ for all $n \in \mathbb{Z}_+$.

Note that by (18) and (21), we have

$$\frac{P_0^{(\alpha,\beta)}(\cos c)}{P_0^{(\alpha,\beta)}(1)} = 1,$$

and then the non-null constant function $\psi = a_0^{\alpha,\beta}$ ($a_n^{\alpha,\beta} = 0$, for $n \geq 1$) does not belong to $\widetilde{\Psi}^c(\mathbb{M}^d)$.

Now we first consider $a_n^{\alpha,\beta} = 0$, for $n \geq 2$. Solving the system below

$$\begin{cases} a_0^{\alpha,\beta} + a_1^{\alpha,\beta} = 1 \\ a_0^{\alpha,\beta} + a_1^{\alpha,\beta} \dfrac{P_1^{(\alpha,\beta)}(\cos c)}{P_1^{(\alpha,\beta)}(1)} = 0 \end{cases} \tag{37}$$

we get

$$a_0^{\alpha,\beta} = 1 - a_1^{\alpha,\beta}, \qquad a_1^{\alpha,\beta} = \frac{1}{1 - \dfrac{P_1^{(\alpha,\beta)}(\cos c)}{P_1^{(\alpha,\beta)}(1)}}. \tag{38}$$

Using (19) and (21), we obtain

$$a_0^{\alpha,\beta} = \frac{(\alpha-\beta) - (\alpha+\beta+2)\cos c}{(\alpha+\beta+2)(1-\cos c)}, \qquad a_1^{\alpha,\beta} = \frac{2(\alpha+1)}{(\alpha+\beta+2)(1-\cos c)}. \tag{39}$$

The possible values for α and β given in (12) imply $a_1^{\alpha,\beta} > 0$ for every $c \in (0, \pi]$. Then the conditions $a_0^{\alpha,\beta}, a_1^{\alpha,\beta} \geq 0$ hold true if and only if

$$a_0^{\alpha,\beta} = \frac{(\alpha-\beta) - (\alpha+\beta+2)\cos c}{(\alpha+\beta+2)(1-\cos c)} \geq 0,$$

which is equivalent to

$$c \geq \arccos\left(\frac{\alpha-\beta}{\alpha+\beta+2}\right).$$

In order to find $\tilde{a}^c(\mathbb{M}^d)$ it is sufficient to calculate $-\psi''(0)$ given in (35) considering $a_0^{\alpha,\beta}, a_1^{\alpha,\beta}$ as in (39) and $a_n^{\alpha,\beta} = 0$ for $n \geq 2$. Thus,

$$\tilde{a}^c(\mathbb{M}^d) = \frac{1}{1-\cos c}, \quad \text{for } c \in \left[\arccos\left(\frac{\alpha-\beta}{\alpha+\beta+2}\right), \pi\right]. \tag{40}$$

Thus, for $c \in \left[0, \arccos\left(\frac{\alpha-\beta}{\alpha+\beta+2}\right)\right]$, there are no functions on $\widetilde{\Psi}^c(\mathbb{M}^d)$ with $a_n^{\alpha,\beta} = 0$, for $n \geq 2$. Hence, consider $a_n^{\alpha,\beta} = 0$, for $n \geq 3$ and

$$0 \leq c \leq \arccos\left(\frac{\alpha-\beta}{\alpha+\beta+2}\right). \tag{41}$$

In order to solve the following system

$$\begin{cases} a_0^{\alpha,\beta} + a_1^{\alpha,\beta} + a_2^{\alpha,\beta} = 1 \\ a_0^{\alpha,\beta} + a_1^{\alpha,\beta} \dfrac{P_1^{(\alpha,\beta)}(\cos c)}{P_1^{(\alpha,\beta)}(1)} + a_2^{\alpha,\beta} \dfrac{P_2^{(\alpha,\beta)}(\cos c)}{P_2^{(\alpha,\beta)}(1)} = 0 \end{cases} \tag{42}$$

we will consider a coefficient as a parameter, we say $a_2^{\alpha,\beta} = \gamma$. Then, (42) is equivalent to

$$\begin{cases} a_0^{\alpha,\beta} = 1 - a_1^{\alpha,\beta} - \gamma, \\ a_1^{\alpha,\beta} = \left[1 - \gamma\left(1 - \dfrac{P_2^{(\alpha,\beta)}(\cos c)}{P_2^{(\alpha,\beta)}(1)}\right)\right]\left(1 - \dfrac{P_1^{(\alpha,\beta)}(\cos c)}{P_1^{(\alpha,\beta)}(1)}\right)^{-1}. \end{cases}$$

Thus, using (38)-(39) and (20)-(21),

$$\begin{aligned} a_1^{\alpha,\beta} = {} & \frac{2(\alpha+1)}{(\alpha+\beta+2)(1-\cos c)} \\ & - \gamma\left(\frac{(\alpha+\beta+3)[4(\alpha+2) + (\alpha+\beta+4)(\cos c - 1)]}{2(\alpha+2)(\alpha+\beta+2)}\right). \end{aligned} \tag{43}$$

Consequently,

$$\begin{aligned} a_0^{\alpha,\beta} = {} & \frac{\overbrace{\beta - \alpha - (\alpha+\beta+2)\cos c}^{(\alpha+\beta+2)(1-\cos c) - 2(\alpha+1)}}{(\alpha+\beta+2)(1-\cos c)} \\ & - \gamma\left(\frac{2(\alpha+2)(\alpha+\beta+2) - (\alpha+\beta+3)[4(\alpha+2) + (\alpha+\beta+4)(\cos c - 1)]}{2(\alpha+2)(\alpha+\beta+2)}\right). \end{aligned} \tag{44}$$

The condition $a_0^{\alpha,\beta} \geq 0$ provides

$$\gamma \geq \frac{2(\alpha+2)}{1-\cos c} \times \frac{\beta-\alpha-(\alpha+\beta+2)\cos c}{2(\alpha+2)(\alpha+\beta+2)-(\alpha+\beta+3)[4(\alpha+2)+(\alpha+\beta+4)(\cos c-1)]} \tag{45}$$

since

$$2(\alpha+2)(\alpha+\beta+2)-(\alpha+\beta+3)[4(\alpha+2)+(\alpha+\beta+4)(\cos c-1)] < 0, \tag{46}$$

which is equivalent to

$$c < \arccos\left(\frac{\beta-\alpha-1}{\beta+\alpha+3}\right). \tag{47}$$

On the other hand, the condition $a_1^{\alpha,\beta} \geq 0$ provides

$$\gamma \leq \frac{4(\alpha+1)(\alpha+2)}{[4(\alpha+2)+(\alpha+\beta+4)(\cos c-1)](\alpha+\beta+3)(1-\cos c)}, \tag{48}$$

since

$$4(\alpha+2)+(\alpha+\beta+4)(\cos c-1) > 0, \tag{49}$$

which is equivalent to

$$c < \arccos\left(\frac{\beta-3\alpha-4}{\alpha+\beta+4}\right). \tag{50}$$

By (45) and (48) we have:

$$\frac{2(\alpha+2)(\beta-\alpha-(\alpha+\beta+2)\cos c)}{(1-\cos c)(2(\alpha+2)(\alpha+\beta+2)-(\alpha+\beta+3)[4(\alpha+2)+(\alpha+\beta+4)(\cos c-1)])} \leq \frac{4(\alpha+1)(\alpha+2)}{[4(\alpha+2)+(\alpha+\beta+4)(\cos c-1)](\alpha+\beta+3)(1-\cos c)}.$$

Using (46) and (49), the equation above becomes

$$\begin{aligned} &4(\alpha+1)(\alpha+2)(\alpha+\beta+2)-8(\alpha+1)(\alpha+\beta+3)(\alpha+2) \\ &\qquad -2(\alpha+1)(\alpha+\beta+3)(\alpha+\beta+4)(\cos c-1) \\ &\leq [4(\alpha+2)(\alpha+\beta+3)+(\alpha+\beta+3)(\alpha+\beta+4)(\cos c-1)] \\ &\qquad \times((1-\cos c)(\alpha+\beta+2)-2(\alpha+1)). \end{aligned}$$

Regrouping the terms of $(\cos c-1)$, $(\cos c-1)^2$ and the terms not depending on $(\cos c-1)$, we have:

$$(\alpha+\beta+3)(\alpha+\beta+4)(\cos c-1)^2+4(\alpha+2)(\alpha+\beta+3)(\cos c-1)+4(\alpha+1)(\alpha+2) \leq 0. \tag{51}$$

Now the solution of the equation of second degree from (51) is:

$$\cos c = \frac{\beta-\alpha}{(\alpha+\beta+4)} \pm 2\frac{\sqrt{(\alpha+2)(\beta+2)}}{\sqrt{(\alpha+\beta+3)}(\alpha+\beta+4)}.$$

Considering the initial condition (41) of this step and the fact that $\beta-\alpha \leq 0$ implies

$$\frac{\beta-\alpha}{(\alpha+\beta+4)} - 2\frac{\sqrt{(\alpha+2)(\beta+2)}}{\sqrt{(\alpha+\beta+3)}(\alpha+\beta+4)} \leq \frac{\beta-\alpha}{(\alpha+\beta+4)} \leq \frac{\alpha-\beta}{\alpha+\beta+2},$$

the solution of (51) is

$$\frac{\alpha-\beta}{\alpha+\beta+2} \leq \cos c \leq \frac{\beta-\alpha}{(\alpha+\beta+4)} + 2\frac{\sqrt{(\alpha+2)(\beta+2)}}{\sqrt{(\alpha+\beta+3)}(\alpha+\beta+4)}. \tag{52}$$

We observe that the conditions (47) and (50) are both satisfied because

$$\frac{\alpha-\beta}{\alpha+\beta+2} \geq \frac{\beta-\alpha-1}{\beta+\alpha+3} \geq \frac{\beta-3\alpha-4}{\alpha+\beta+4},$$

for all α, β as in (12), and by (41).

Therefore the non negativity of the coefficients $a_n^{\alpha,\beta}$ turns into the inequalities

$$\begin{cases} \gamma \geq \dfrac{2(\alpha+2)}{1-\cos c} \\ \quad \times \dfrac{\beta-\alpha-(\alpha+\beta+2)\cos c}{2(\alpha+2)(\alpha+\beta+2)-(\alpha+\beta+3)[4(\alpha+2)+(\alpha+\beta+4)(\cos c-1)]} \\ \gamma \leq \dfrac{4(\alpha+1)(\alpha+2)}{[4(\alpha+2)+(\alpha+\beta+4)(\cos c-1)](\alpha+\beta+3)(1-\cos c)} \end{cases} \tag{53}$$

which leads to a non empty set of values when (see (52))

$$\arccos\left(\frac{\beta-\alpha}{(\alpha+\beta+4)} + 2\frac{\sqrt{(\alpha+2)(\beta+2)}}{\sqrt{(\alpha+\beta+3)}(\alpha+\beta+4)}\right) \leq c \leq \arccos\left(\frac{\alpha-\beta}{\alpha+\beta+2}\right).$$

In order to conclude the proof, we need to find $\widetilde{a}^c(\mathbb{M}^d)$ which is attained for

$$\psi_\gamma(\theta) := a_0^{\alpha,\beta} + a_1^{\alpha,\beta}\frac{P_1^{(\alpha,\beta)}(\cos\theta)}{P_1^{(\alpha,\beta)}(1)} + \gamma\frac{P_2^{(\alpha,\beta)}(\cos\theta)}{P_2^{(\alpha,\beta)}(1)}$$

where $a_0^{\alpha,\beta}, a_1^{\alpha,\beta}$ are given by (44) and (43) respectively and $\gamma = a_2^{\alpha,\beta}$.

To know the value for γ, note that by (35):

$$\begin{aligned} -\psi_\gamma''(0) &= \sum_{n=1}^{2} \frac{n(n+\alpha+\beta+1)}{2(\alpha+1)} a_n^{\alpha,\beta} \\ &= \underbrace{\frac{1}{(1-\cos c)}}_{\geq 0} + \gamma\left(\underbrace{\frac{-(\alpha+\beta+3)(\alpha+\beta+4)(\cos c-1)}{4(\alpha+1)(\alpha+2)}}_{\geq 0}\right). \end{aligned}$$

This implies that $\widetilde{a}^c(\mathbb{M}^d)$ is attained when γ assumes the lowest value on (53) that we call γ_0. After some algebraic manipulation, we obtain

$$-\psi''_{\gamma_0}(0) = \frac{\alpha+\beta+4}{(1-\cos c)2(\alpha+1)} \times \frac{(1-\cos c)^2(\alpha+\beta+3)(\alpha+\beta+2) - 4(\alpha+1)(\alpha+2)}{2(\alpha+2)(\alpha+\beta+2) - (\alpha+\beta+3)[4(\alpha+2) + (\alpha+\beta+4)(\cos c - 1)]},$$

that completes the proof. □

Declaration of competing interest

The authors have no competing interests to declare.

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