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CMMSE: Study of a new symmetric anomaly in the elliptic, hyperbolic, and parabolic Keplerian motion

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In the present work, we define a new anomaly, Ψ, termed semifocal anomaly. It is determined by the mean between the true anomaly, f, and the antifocal anomaly, f'; Fukushima defined f' as the angle between the periapsis and the secondary around the empty focus. In this first part of the paper, we take an approach to the study of the semifocal anomaly in the hyperbolic motion and in the limit case corresponding to the parabolic movement. From here, we find a relation between the semifocal anomaly and the true anomaly that holds independently of the movement type. We focus on the study of the two-body problem when this new anomaly is used as the temporal variable. In the second part, we show the use of this anomaly—combined with numerical integration methods—to improve integration errors in one revolution. Finally, we analyze the errors committed in the integration process—depending on several values of the eccentricity—for the elliptic, parabolic, and hyperbolic cases in the apsidal region.

KEYWORDS

Celestial Mechanics, computational algebra, orbital motion, ordinary differential equations

MSC CLASSIFICATION

70F15, 70F05, 70H09, 74H10, 74H15

1 | INTRODUCTION

The study of the motion in the Solar System is one of the strengths of Celestial Mechanics. This issue involves the development of planetary theories and the motion of artificial satellites around the Earth. In this paper, we deal with both topics. To tackle these problems, first, we will consider the elliptic movement as the leading case; we will then study the hyperbolic movement due to its great importance in astronautics, and we will finally deal with the limit case of the parabolic movement. The elliptic movement case is the most studied and is of enormous importance because of its adequacy to the study of most of the problems in the Solar System, such as planetary movements or artificial satellites.

The study of the movement in the Solar System can be accomplished by numerical or analytical ways. Analytical methods are appropriate when the eccentricities of the bodies are small. In this case, it may be feasible to describe their movements by use of series developments; this is the case, for instance, of the planetary theories. Analytical methods are complicated, but they present a significant advantage: Once the method has been designed, the position of the bodies included in the model can be easily obtained; the process is as simple as replacing the time in a function, hence the high interest of these methods. Numerical methods allow high precision results in any case, but the analytical methods are only suitable for small eccentricities. In the elliptic case and when the eccentricities are high, given that there will always exist perturbative masses, it is necessary the use of numerical integrators. Among these integrators, we mention the symplectic integrators and the variable step size integrators.

In the process of constructing a planetary theory, two major approaches can be considered: the use of numerical integrators[1,2] or the use of analytical methods to integrate the problem.[3–6]

Analytical methods are based on the solution of the two-body problem (Sun–planet). This solution is given through a set of six orbital elements. In this paper, we have chosen the third set of Brouwer and Clemence[7] $(a, e, i, \Omega, \omega, M)$. In this set, a stands for the major semi-axis, e is the eccentricity, i is the inclination with respect to the espatial reference system, Ω represents the argument of the ascendent node, ω is the argument of the periapsis, and M is the mean anomaly for the epoch. M is related to the mean anomaly in the initial epoch by means of $M = M_0 + n(t - t_0)$, where n is the mean motion, t_0 is the initial epoch, and M_0 is the mean anomaly in the initial epoch. This solution can be considered as a first approximation of the perturbed problem, and we can use the Lagrange method of variation of constants to replace the first elements by the osculating ones given by the Lagrange planetary equations[8]

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \sigma}, \\
\frac{de}{dt} &= -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega} + \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial \sigma}, \\
\frac{di}{dt} &= -\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \Omega} + \frac{\operatorname{ctg} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \omega}, \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}, \\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}, \\
\frac{d\sigma}{dt} &= -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e}.
\end{aligned} \tag{1}$$

Notice that σ is a new variable defined by the equation

$$M = \sigma + \int_{t_0}^{t} n\,dt, \tag{2}$$

and it coincides with M_0 in the case of the unperturbed motion. R is the disturbing potential, $R = \sum_{k=1}^{N} R_i$, due to the disturbing bodies $i = 1, \dots, N$ and is defined as[8]

$$R = \sum_{k=1}^{N} G m_k \left[\left(\frac{1}{\Delta_k} \right) - \frac{x \cdot x_k + y \cdot y_k + z \cdot z_k}{r_k^3} \right], \tag{3}$$

where G is the gavitational constant, $\vec{r} = (x, y, z)$ and $\vec{r}_k = (x_k, y_k, z_k)$ are the heliocentric vector position of the secondary and the kth disturbing body, respectively, Δ_k is the distance between the secondary body and the disturbing body, and m_k the mass of the disturbing body.

In order to integrate the Lagrange planetary equations through analytical methods, it is necessary to develop the second member of the Lagrange planetary equations as truncated Fourier series; this is a classical problem in Celestial Mechanics.[6,7,9–11] The analytical methods provide long-term series as solutions, but it would be more convenient to obtain more compact developments; this can be accomplished by using an appropriate anomaly as temporal variable.

To obtain the expansions according to an anomaly Ψ_i, it is necessary to obtain the developments of the coordinates for each planet, i, and the inverse of the radius in Fourier series of Ψ_i. Then, the integration of the Lagrange planetary equations with respect to the Ψ_i anomalies requires to compute the corresponding Kepler equation $M_i = M_i(\Psi_i)$.[12,13]

When using numerical integration methods, it is more appropriate to consider the equation of the motion in the form of the Newton second law. The efficiency of the numerical integrators can improve with an appropriate change in the temporal variable. In this paper, we will study the performance of the previous family of anomalies. To this aim, we have chosen the problem of the motion of an artificial satellite around the Earth. The relative motion of the secondary with respect to the Earth is defined by the second-order differential equations:

$$\frac{d^2\vec{r}}{dt^2} = -GM\frac{\vec{r}}{r^3} - \vec{\nabla}U - \vec{F}, \tag{4}$$

where $\vec{r}$ is the radius vector of the satellite, U the potential from which the perturbative conservative forces are derived, and $\vec{F}$ combines the non-conservative forces, such as the atmospheric friction forces or the solar radiation pressure. To integrate the system (4), it is necessary to know the initial value of the radius vector $\vec{r}_0$ and velocity $\vec{v}_0$.

In order to uniformize the truncation errors when a numerical integrator is used, three main techniques can be followed:

1. The use of a very small step size.
2. The use of an adaptative step size method.
3. A change in the temporal variable to arrange an appropriate distribution of the points on the orbit so that the points are mostly concentrated in the regions where the speed and curvature are maxima.

This paper deploys the third technique. Several authors have already studied this question; see, for instance, Sundman,[14] who introduced a new temporal variable, τ, related to the time, t, through $dt = Crd\tau$; Nacozy[15] proposed a new temporal variable $dt = Cr^{3/3}d\tau$; Brumberg[16] proposed the usage of the regularized length of arc, and Brumberg and Fukushima[17] introduced the elliptic anomaly as temporal variable. Janin,[18] Janin and Bond,[19] and Velez and Hilinski[20] extended this technique defining a new one-parameter family of transformations α called generalized Sundman transformations $dt = Q(r,\alpha)d\tau_\alpha$, where $Q(r,\alpha) = C_\alpha r^\alpha$. The function $Q(r)$ is usually known as the partition function. A more complicated family of transformations was introduced by Ferrándiz et al.[21] $Q(r) = r^{2/3}(a_0 + a_1 r)^{-1/2}$. López et al[22] introduce an new family of anomalies, called natural anomalies, as $d\Psi_\alpha = (1-\alpha)f' + \alpha f$; $\alpha \in [0,1]$, where f, f' are the true and secondary anomalies. f is the angle between the periapsis and the secondary position taking as origin the primary focus F of the ellipse; f' is the angle between the periapsis and the position of the secondary taking as origin the empty focus F'. Analytical and numerical properties of the generalized Sundman anomalies and the natural anomaly have been studied by López et al.[23–25]

The generalized Sundman family and the natural anomalies may have several inconveniences: the main quantities of the two-body problem, such as the orbital coordinates (ξ, η), the radius vector, and the generalized Kepler equation cannot be written by means of a closed formula, except for a small set of values of the parameter α. Besides, in general terms, the coefficients of the necessary developments for the construction of analytical theories of the planetary motion cannot be written using closed formulas, either. Finally, these anomalies have not an easy geometrical interpretation.

In 2016, a new family of anomalies was introduced by López et al.[26] This family includes the eccentric anomaly, g, the true anomaly, f, the antifocal anomaly, f', submitted by Fukushima,[27] by means of a simple set of geometric transformations. López demonstrated that the main magnitudes involved in the two-body problem can be obtained in closed form for all the anomalies in the family. Besides, he also determined that the coefficients of the series developments can be obtained in closed form too.

In the year 2017, López et al[28] extended the mentioned family to the hyperbolic movement in both cases: the attractive branch, linked to a two-body problem, and the repulsive branch, connected to the movent of two magnetic charges of the same sign. Moreover, López et al[29] defined a new biparametric family of anomalies comprising Sundman's generalized family, the elliptic anomaly, the regularized length of arch anomaly, and the antifocal anomaly.

In this paper, we define, in the first place, a new anomaly Ψ as the mean between the true and antifocal anomaly, $\Psi = \frac{f'+f}{2}$. Then, we extend this anomaly to the hyperbolic case of the two-body problem, and finally, we study this anomaly in the limit case of the parabolic motion. This anomaly is by definition contained in the natural family of anomalies[22] and we will show that this anomaly is included in the biparametric family too.

In this introductory section, the general background has been settled. The rest of this article is organized as follows. In Section 2, we introduce the semifocal anomaly in the elliptic case. There is a subsection that describes the series developments of the most important magnitudes that appear in the two-body problem according to the semifocal anomaly obtaining the exact analytical expressions for the coefficients of the mentioned developments. Section 3 extends the usage of the semifocal anomaly to the hyperbolic case and the results are akin to the ones obtained in the elliptic case. Section 4 covers the study of the parabolic movement; it is shown that it is possible to extend the use of the semifocal anomaly to this limit case. In Section 5, we analyze, from a numerical point of view, the study of the integration errors in the elliptic, hyperbolic, and parabolic movements. Finally, in Section 6, we show the main conclusions that have been drawn from the study.

2 | THE SEMIFOCAL ANOMALY IN THE ELLIPTIC MOTION

In this section, we are studying the analytical properties of the semifocal anomaly in the elliptic motion. In this sense, López et al[30] defined a family of symmetric anomalies depending on one parameter, α. This family includes, as a particular case, the semifocal anomaly, Ψ, defined as the mean between the true anomaly, f, and the antifocal anomaly, f'; see Figure 1.

In order to make the article more reader-friendly, first, we will show a set of closed-form formulas—included in[30]—that connect the fundamental magnitudes in the two-body problem when using the new anomaly.

Let us define (ξ, η) as the orbital coordinates referred to the primary focus, F, and let r and r' be the distance between the secondary, P, and the primary focus, F, and the secondary focus F', respectively. The angle g is called eccentric anomaly, the angle f is called true anomaly, and the angle f' is called antifocal anomaly.

Let Ψ be the new anomaly defined as

$$\Psi = \frac{f + f'}{2}. \tag{5}$$

In order to link g to Ψ, we consider the classical relations described below; the main quantities of the two-body problem can be described through the eccentric g, true f, and antifocal f' anomalies using the relationships

$$r + r' = 2a, \tag{6}$$

$$\xi = a(e - \cos g),\ \ \eta = a\sqrt{1 - e^2}\sin g, \tag{7}$$

$$r = a(1 - e\cos g),\ \ \ r' = a(1 + e\cos g), \tag{8}$$

$$\xi = r\cos f,\ \ \eta = r\sin f,\ \ r = \frac{a(1 - e^2)}{1 + e\cos f}, \tag{9}$$

$$\xi = r'\cos f' - 2ae,\ \ \eta = r'\sin f',\ \ r' = \frac{a(1 - e^2)}{1 - e\cos f'}. \tag{10}$$

The eccentric anomaly is connected to the mean anomaly, M, through the Kepler equation

$$M = g - e\sin g. \tag{11}$$

To describe Ψ in terms of g, we consider

$$\sin f = \frac{a}{r}\sqrt{1 - e^2}\sin g,\ \ \sin f' = \frac{a}{r'}\sqrt{1 - e^2}\sin g, \tag{12}$$

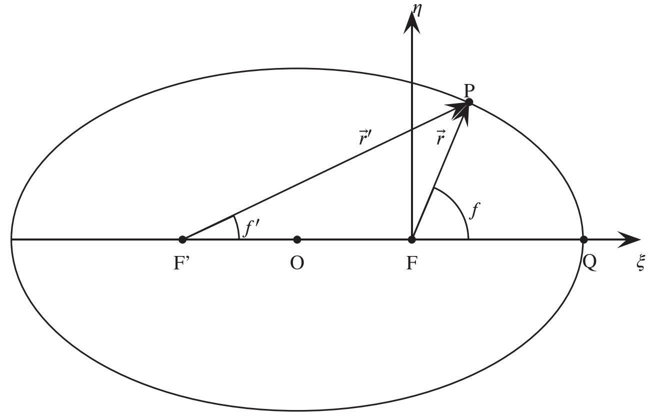

FIGURE 1 Elliptic motion: O is the center of the ellipse, F is the primary focus, F' is the secondary focus, Q is the periapsis, and P is the position of the secondary on the ellipse. The vector radii $\vec{r}$ and $\vec{r}'$ represent the secondary position with respect to F and F', respectively, f is the true anomaly, and f' the antifocal anomaly. Finally, (ξ, η) are the Cartesian coordinates in the orbital system

and

$$\cos f = \frac{a}{r}\left(\cos g - e\right), \ \cos f' = \frac{a}{r'}\left(\cos g + e\right). \tag{13}$$

From (12) and (13), it is easy to deduce

$$\sin 2\Psi = \frac{a^2}{rr'}\sqrt{1-e^2}\,\sin 2g, \tag{14}$$

and

$$\cos 2\Psi = \frac{a^2}{rr'}\left\{\cos 2g - \frac{e^2}{2}(1+\cos 2g)\right\}. \tag{15}$$

Thus,

$$1-\cos 2\Psi = 2\frac{a^2}{rr'}\sin^2 g, \ 1+\cos 2\Psi = 2\frac{a^2}{rr'}(1-e^2)\cos^2 g. \tag{16}$$

The values of $\sin\Psi$ and $\cos\Psi$ are given by

$$\sin\Psi = \frac{a}{\sqrt{rr'}}\sin g, \ \cos\Psi = \frac{a}{\sqrt{rr'}}\sqrt{1-e^2}\cos g, \tag{17}$$

and taking into account (8), we obtain

$$\sin\Psi = \frac{\sin g}{\sqrt{1-e^2\cos^2 g}}, \ \cos\Psi = \frac{\sqrt{1-e^2}\cos g}{\sqrt{1-e^2\cos^2 g}}. \tag{18}$$

It is important to mention that in this paper, in order to connect $d\Psi$ and dM, we will follow an alternate procedure as the one in López et al.[30] So, from (18), it is easy to get

$$\sin g = \frac{\sqrt{1-e^2}\sin\Psi}{\sqrt{1-e^2\sin^2\Psi}}, \ \cos g = \frac{\cos\Psi}{\sqrt{1-e^2\sin^2\Psi}}. \tag{19}$$

In order to deal with Ψ and M, we derive (19), and after operating, we get

$$dg = \frac{\sqrt{1-e^2}}{1-e^2\sin^2\Psi}d\Psi. \tag{20}$$

If we replace (19) in (8), we obtain

$$r = a\left(1-\frac{e\cos\Psi}{\sqrt{1-e^2\sin^2\Psi}}\right), \ r' = a\left(1+\frac{e\cos\Psi}{\sqrt{1-e^2\sin^2\Psi}}\right), \tag{21}$$

and from this equation,

$$rr' = \frac{a^2(1-e^2)}{1-e^2\sin^2\Psi}. \tag{22}$$

If we replace (22) in (20), we get

$$dg = \frac{rr'}{a^2\sqrt{1-e^2}}d\Psi, \tag{23}$$

and taking into account that $dM = \frac{r}{a}dg$, we have

$$dM = \frac{r^2 r'}{a^3\sqrt{1-e^2}}d\Psi. \tag{24}$$

From the last equation, we easily conclude that the semifocal anomaly is included in the biparametric family for $\alpha = 2$ and $\beta = 1$.

Besides, if we take into account that

$$dM = \frac{r^2}{a^2\sqrt{1-e^2}}df, \tag{25}$$

and we replace this value in (24), we obtain

$$df = \frac{r'}{a} d\Psi. \tag{26}$$

If we replace (21) in (26) and integrate the equality, we obtain

$$f = \Psi + \arcsin\left(e \sin \Psi\right), \tag{27}$$

and from here, we deduce the important equation

$$\sin\left(f - \Psi\right) = e \sin \Psi. \tag{28}$$

2.1 | Analytical developments in the elliptic case

In order to integrate the Lagrange planetary equations by using analytical or semianalytical methods, it is necessary to develop their second members as Fourier series according to the selected anomalies for each couple of planets. To this aim, it is necessary to obtain the expansions with respect to the selected anomaly of the two-body problem quantities g, $\sin g$, $\cos g$, r/a, a/r, and M.

The developments of the main magnitudes in the two-body problem can be straightforwardly deduced from the corresponding ones in the biparametric family. This can be easily achieved by replacing $\alpha = 2$ and $\beta = 1$. Likewise, the same developments can be derived from the family of natural anomalies when $\alpha = \frac{1}{2}$.

Another way to obtain the developments mentioned before is to resort to the classical technique of the inversion.[23–25] Starting from the development of Ψ as a function of g and by the use of Deprit inversion algorithm,[31] it is possible to obtain the developments of g, $\sin g$, $\cos g$, r/a, and a/r as a function of Ψ; then, with the help of those developments the Kepler equation can also be transformed.

To that aim, from (18), we deduce

$$\tan \Psi = \sqrt{1 - e^2} \tan g, \tag{29}$$

and taking into account the classical development[32] of

$$\tan y = \left(\frac{1+m}{1-m}\right) \tan x, \tag{30}$$

as

$$y = x + \sum_{k=1}^{\infty} \frac{m^k}{k} \sin(kx), \tag{31}$$

we obtain

$$\Psi = g + \sum_{k=1}^{\infty} \frac{\beta^k}{k} \sin(kg), \tag{32}$$

where

$$\beta = \frac{2 - e^2 - 2\sqrt{1 - e^2}}{e^2}. \tag{33}$$

With this, we are provided with the fundamental development of Ψ depending on g as a power series of the parameter β. To make it dependable on e, it is enough to consider that

$$\beta = \sum_{k=1}^{\infty} \frac{(2k-1)!}{2^{2k-1}(k+1)!(k-1)!} e^{2k}. \tag{34}$$

The first terms of this series (34) are given by

$$\beta = \frac{1}{4}e^2 + \frac{1}{8}e^4 + \frac{5}{64}e^6 + \frac{7}{128}e^8 + \frac{21}{512}e^{10} + \dots ,$$

and if we adopt Deprit's series inversion method, we obtain the required developments.

On the other hand, in this particular case, it is preferable to follow another way in order to obtain the exact values of the coefficients of the series. To do so, we proceed by developing in Fourier series the function

$$\frac{1}{\sqrt{1-e^2\sin^2\Psi}} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2n\Psi, \tag{35}$$

where

$$a_0 = \frac{d\Psi}{\pi}\int_0^{2\pi} \frac{1}{\sqrt{1-e^2\sin^2\Psi}} = \frac{4}{\pi}K(e), \tag{36}$$

and $K(e)$ is the complete elliptic integral of the first kind. Also

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos 2\Psi d\Psi}{\sqrt{1-e^2\sin^2\Psi}} = \frac{8E(e) - 4\left(2-e^2\right)K(e)}{\pi e^2}, \tag{37}$$

where $E(e)$ is the complete elliptic integral of second kind.

To obtain the rest of the coefficients, we derive Equation (35) with respect to Ψ

$$\frac{e^2 \sin 2\Psi}{2\left(1-e^2\sin^2\Psi\right)^{3/2}} = -2\sum_{n=1}^{\infty} n\, a_n \sin 2n\Psi, \tag{38}$$

and from them,

$$e^2 \sin 2\Psi \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2n\Psi\right) = -\left(\left(2-e^2\right) + e^2\cos 2\Psi\right)\sum_{n=1}^{\infty} n\, a_n \sin 2n\Psi. \tag{39}$$

After operating, we obtain the recurrence formula

$$(2n+1)\, e^2 a_n + 4\,(n+1)\left(2-e^2\right) a_{n+1} + (2n+3)\, e^2 a_{n+2} = 0. \tag{40}$$

To obtain the developments of $\sin g$ and $\cos g$, we replace (35) in (19), and so we have

$$\cos g = \sum_{n=0}^{\infty}\left(\frac{a_n + a_{n+1}}{2}\right)\cos(2n+1)\Psi, \tag{41}$$

and

$$\sin g = \sum_{n=0}^{\infty}\sqrt{1-e^2}\left(\frac{a_n - a_{n+1}}{2}\right)\sin(2n+1)\Psi. \tag{42}$$

The development of $\frac{r}{a}$ can be easily obtained from (8).

Finally, to get $\frac{a}{r}$, we consider

$$\frac{a}{r} = \frac{\sqrt{1-e^2\sin^2\Psi}}{\sqrt{1-e^2\sin^2\Psi} - e\cos^{\Psi}} = \frac{1-e^2\sin^2\Psi + e\cos\Psi\sqrt{1-e^2\sin^2\Psi}}{1-e^2}. \tag{43}$$

Given this expression, it will be enough to obtain the development of $\sqrt{1-e^2\sin^2\Psi}$. To this aim, we have

$$\sqrt{1-e^2\sin^2\Psi} = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos(2n\Psi), \tag{44}$$

and if we calculate the derivative, we have

$$\frac{\sin(2\Psi)}{2\sqrt{1-e^2\sin^2\Psi}} = 2\sum_{n=1}^{\infty} n\,b_n \sin(2n\Psi). \tag{45}$$

From here,

$$\sin(2\Psi)\sqrt{1-e^2\sin^2\Psi} = 4\left(1-e^2\sin^2\Psi\right)\sum_{n=1}^{\infty} n\,b_n \sin(2n\Psi), \tag{46}$$

and after some algebraic manipulations, we finally obtain the recurrence

$$(2n-1)\,e^2 b_n + 4\,(n+1)\left(2-e^2\right) b_{n+1} + (2n+5)\,e^2 b_{n+2} = 0. \tag{47}$$

The initial values b_0 and b_1 are given by

$$b_0 = \frac{4}{\pi}E(e),\ b_1 = \frac{4}{3e^2\pi}\left[\left(2-e^2\right)E(e) + \left(1-e^2\right)K(e)\right], \tag{48}$$

where $K(e)$ and $E(e)$ are the elliptic integrals of first and second kinds, respectively.

3 | THE SEMIFOCAL ANOMALY IN THE HYPERBOLIC MOTION

In this section, we will study the hyperbolic case of the two-body problem when the new anomaly, Ψ, is considered. Ψ has been similarly defined as in the elliptic case, but considering the peculiarity that in the elliptic case, both focuses lie at the same side of the periapsis, while in the hyperbolic case, the primary focus is placed to the left of the periapsis and the secondary focus to the right. According to this particularity, in order to be 0 in the apoapsis, the anomaly Ψ must be defined as shown in Figure 2.

$$\Psi = \frac{f+f'}{2} - \frac{\pi}{2}. \tag{49}$$

In the hyperbolic case, we have the relations

$$r\cos f - r'\cos f' = 2ae,\ r\cos f = ae - a\cosh H,\ r\sin f = a\sqrt{e^2-1}\,\sinh H, \tag{50}$$

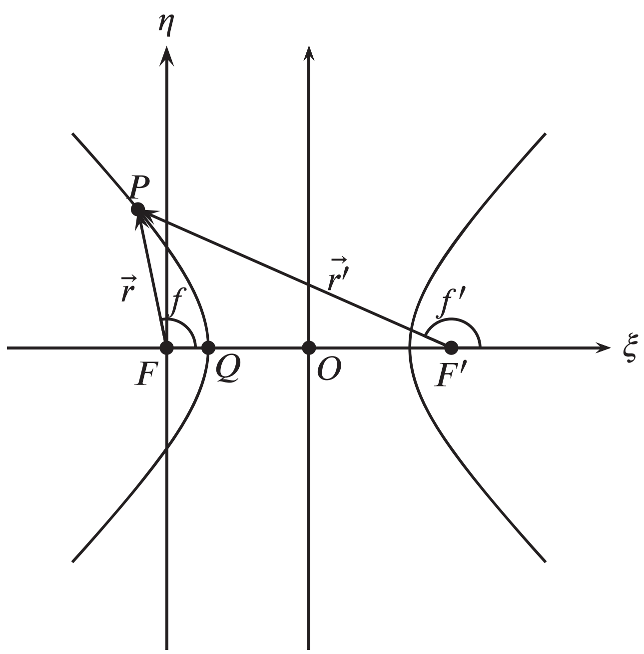

FIGURE 2 Hyperbolic motion: O is the center of the hyperbola, F is the primary focus, F' is the secondary focus, Q is the periapsis, and P is the position of the secondary on the hyperbola. The vector radii $\vec{r}$ and $\vec{r'}$ represent the secondary position with respect to F and F', respectively, f is the true anomaly, and f' is the antifocal anomaly. Finally, (ξ, η) are the Cartesian coordinates in the orbital system

where H is the equivalent variable to the eccentric anomaly in the hyperbolic motion,[28]

$$e \sinh(H) - H = M, \tag{51}$$

where $M = \nu(t - t_0)$, $\nu = \sqrt{\mu/a^3}$, $\mu = GM$, and t_0 is the transit epoch at the periapsis,

$$r = a\left(e \cosh H - 1\right), \; r' = a\left(e \cosh H + 1\right), \tag{52}$$

and obviously, $r' - r = 2a$. From these equations, it is easy to obtain

$$\begin{aligned} \cos f &= \frac{a}{r}\left(e - \cosh H\right), \; \sin f = \frac{a}{r}\sqrt{e^2-1}\sinh H, \\ \cos f' &= -\frac{a}{r'}\left(e + \cosh H\right), \; \sin f' = \frac{a}{r'}\sqrt{e^2-1}\sinh H. \end{aligned} \tag{53}$$

On the other hand, $\cosh^2 H - \sinh^2 H = 1$. It is a necessary condition for Ψ to be an anomaly that $\Psi = 0$ when $f = 0$, and for this reason, in the hyperbolic case, it is necessary to define Ψ as $\Psi = \frac{f+f'}{2} - \frac{\pi}{2}$, as mentioned before.

We obtain the relationships

$$\cos\left(f + f'\right) = -\frac{a^2}{rr'}\left(e^2 - \cosh^2 H + (e^2-1)\sinh^2 H\right), \tag{54}$$

and from them,

$$\begin{aligned} \sin\left(\frac{f+f'}{2}\right) &= \frac{1-\cos(f+f')}{2} = \frac{a}{\sqrt{rr'}}\sqrt{e^2-1}\cosh H, \\ \cos\left(\frac{f+f'}{2}\right) &= \frac{1+\cos(f+f')}{2} = -\frac{a}{\sqrt{rr'}}\sinh H. \end{aligned} \tag{55}$$

We have

$$\begin{aligned} \cos\Psi = \sin\left(\frac{f+f'}{2}\right) &= \frac{a}{\sqrt{rr'}}\sqrt{e^2-1}\cosh H, \\ \sin\Psi = -\cos\left(\frac{f+f'}{2}\right) &= \frac{a}{\sqrt{rr'}}\sinh H, \end{aligned} \tag{56}$$

$$\cos\Psi = \frac{\sqrt{e^2-1}\cosh H}{\sqrt{e^2\cosh^2 H - 1}}, \; \sin\Psi = \frac{\sinh H}{\sqrt{e^2\cosh^2 H - 1}}, \tag{57}$$

and so

$$\cosh H = \frac{\cos\Psi}{\sqrt{1 - e^2\sin^2\Psi}}, \; \sinh H = \frac{\sqrt{e^2-1}\sin\Psi}{\sqrt{1 - e^2\sin^2\Psi}}. \tag{58}$$

Replacing (58) in (52), we obtain

$$r = a\left(\frac{e\cos\Psi}{\sqrt{1-e^2\sin^2\Psi}} - 1\right), \; r' = a\left(\frac{e\cos\Psi}{\sqrt{1-e^2\sin^2\Psi}} + 1\right). \tag{59}$$

On the other hand,

$$dH = \frac{e^2-1}{1-e^2\sin^2\Psi}d\Psi, \tag{60}$$

$$\frac{rr'}{a^2(e^2-1)} = \frac{1}{1-e^2\sin^2\Psi}, \tag{61}$$

and we have

$$dH = \frac{r\,r'}{a\sqrt{e^2-1}}d\Psi. \tag{62}$$

Finally, after some algebraic manipulations, the change of anomaly can be written as

$$dM = \frac{r^2\,r'}{a^3\sqrt{e^2-1}}d\Psi = Q(r,r')d\Psi, \tag{63}$$

where $Q(r,r') = \frac{r^2\,r'}{a^3\sqrt{e^2-1}}$ is the partition function.

In addition, taking into account that

$$dM = \frac{r^2}{a^2\sqrt{e^2-1}}df, \tag{64}$$

if we substitute (63) in (64), we get

$$df = \frac{r'}{a}d\Psi. \tag{65}$$

Now, considering (59) and integrating, we obtain

$$f = \arcsin(e\sin\Psi) + \Psi, \tag{66}$$

which leads to the noteworthy relation

$$\sin(f - \Psi) = e\sin\Psi. \tag{67}$$

4 | THE SEMIFOCAL ANOMALY IN THE PARABOLIC MOTION

The parabolic movement can be considered as a limit case, both in the elliptic and in the hyperbolic problems. In the parabolic movement, the apsidal distance, q, and the parameter, p, satisfy the relation $p = 2q$, and this fact holds both in the elliptic and in the hyperbolic cases when $e = 1$. The parabola can be considered a boundary or separatrix between conics; in the ellipse, both focuses are placed on the left of the periapsis, while in the hyperbola, one focus is on the left and the other on the right of the periapsis. In the parabolic case, the secondary focus does not exist. If we approximate to the parabolic case from the elliptic case, the secondary focus tends to $(-\infty, 0)$; if we do so from the hyperbolic case, the secondary focus tends to $(0, +\infty)$. This is the reason why we can state that in the parabolic case, if we approximate from the elliptic case, the f' anomaly tends to 0; on the contrary, if our approximation is from the hyperbolic movement, f' tends to π and in both cases $\Psi = \frac{f}{2}$. This intuitive fact suggests that the semifocal anomaly, f', exists in the parabolic case and is continuous in $e = 1$.

After this introduction, we show two analytical proofs of this fact. In the first proof, if we take into account (26) and knowing that $r + r' = 2a$, we have

$$df = \frac{r'}{a}d\Psi = \frac{2a - r}{a}d\Psi = \left(2 - \frac{r}{a}\right)d\Psi. \tag{68}$$

First, if we approximate the parabolic case from the elliptic case, a tends to $-\infty$ and we have $df = 2d\Psi$; consequently, we arrive to the conclusion that $\Psi = f/2$. Similarly, if we approach from the hyperbolic case, taking into account (65) and the fact that $r' - r = 2a$, we have

$$df = \frac{r'}{a}d\Psi = \frac{2a + r}{a}d\Psi = \left(2 + \frac{r}{a}\right)d\Psi. \tag{69}$$

Now, considering that we approach the parabola from the hyperbolic case, we have that a tends to $+\infty$ and then $df = 2d\Psi$; therefore, $\Psi = \frac{f}{2}$.

A second proof—also conclusive for both elliptic and hyperbolic cases—is based on the Equations (28) and (67); considering that in the periapsis $f = \Psi = 0$ and the parabolic limits corresponds to $e = 1$, we have that $f - \Psi = \Psi$, hence $\Psi = \frac{f}{2}$.

This fact is of paramount importance, since the semifocal anomaly can be considered as a universal anomaly, valid for elliptic, parabolic, and hyperbolic movements.

5 | NUMERICAL EXAMPLES

In this section, we have designed some numerical experiments in order to test the new variable. The first of these experiments applies to the case of the elliptical motion and consists in the integration of a fictitious artificial satellite orbiting the Earth. Its semi-axis, $a = 118{,}363.47$ km, is the same as the satellite Heos II and the eccentricity ranges from 0 to 0.95 with a fixed step of 0.025. The experiment consists of the integration of the problem. We consider (x, y) the coordinates in the orbital plane originating in the primary, and we take the axis OX in the direction of the periapsis. Then, we obtain the position and speed in the periapsis, and with the help of a fourth-order classical Runge–Kutta method, we integrate one revolution using the semifocal anomaly and the mean anomaly as temporal variables; in both cases, we have taken 1000 steps. After a revolution, the error modulus is calculated in position and speed for the two variables. Table 1 shows the numerical results of the integration; in this table, we appreciate that for low eccentricities the difference between the results obtained using the semifocal and the mean anomalies are similar. On the contrary, for high values of the eccentricity, the use of the semifocal anomaly significantly improves the results obtained when using the mean anomaly, that is, the natural time. In all the calculations the value of $GM = 3.986004415 \cdot 10^5\ \text{km}^3/\text{s}^2$ corresponding to the Earth has been taken.

In the numerical integration, it is preferable to start from the equations of the movement expressed in Cartesian coordinates

$$\frac{dx}{dt} = v_x,\ \frac{dv_x}{dt} = -GM\frac{x}{r^3},$$
$$\frac{dy}{dt} = v_y,\ \frac{dv_y}{dt} = -GM\frac{y}{r^3}, \tag{70}$$

where $r = \sqrt{x^2 + y^2}$.

Next, we proceed by studying the errors generated in the general case integration, where it is no longer possible to ensure the periodicity. To that aim, we start from a point in the coordinates $(-p, 0)$, where p is the parameter of the conic. After 1000 integration steps, the point becomes $(p, 0)$, and since symmetry has to be preserved, it can be taken as a measure of the error $(x_i - x_f, y_i + y_f)$, where the subscripts i and f denote the values of the starting point and the ending point obtained in the integration. In order to measure the error in speed, we have taken $(v_i + v_f, v_i - v_f)$. The calculation will be made by taking the apsidal distance q of the artificial satellite Heos II, for which the value of the parameter p is given by $p = q(1 + e)$. Table 2 shows the errors for a set of eccentricities ranging from 0 to 2 with a step of 0.05.

TABLE 1 Integration errors in position (km) and velocity (km/s) in a revolution for several values of e using the semifocal and mean anomalies

e	$\lvert\Delta r_\Psi\rvert$	$\lvert\Delta v_\Psi\rvert$	$\lvert\Delta r_M\rvert$	$\lvert\Delta v_M\rvert$	e	$\lvert\Delta r_\Psi\rvert$	$\lvert\Delta v_\Psi\rvert$	$\lvert\Delta r_M\rvert$	$\lvert\Delta v_M\rvert$
0.000	9.66e-06	1.50e-10	9.66e-06	1.50e-10	0.500	7.36e-04	2.63e-08	3.73e-03	1.42e-07
0.025	3.18e-05	5.20e-10	2.64e-05	4.49e-10	0.525	8.57e-04	3.27e-08	5.79e-03	2.34e-07
0.050	3.75e-05	6.41e-10	2.64e-05	4.88e-10	0.550	9.96e-04	4.09e-08	9.16e-03	3.95e-07
0.075	4.45e-05	7.89e-10	2.74e-05	5.50e-10	0.575	1.16e-03	5.14e-08	1.48e-02	6.86e-07
0.100	5.29e-05	9.69e-10	2.98e-05	6.42e-10	0.600	1.35e-03	6.49e-08	2.46e-02	1.23e-06
0.125	6.29e-05	1.19e-09	3.39e-05	7.72e-10	0.625	1.57e-03	8.25e-08	4.22e-02	2.29e-06
0.150	7.49e-05	1.46e-09	4.01e-05	9.56e-10	0.650	1.82e-03	1.06e-07	7.47e-02	4.45e-06
0.175	8.92e-05	1.79e-09	4.92e-05	1.21e-09	0.675	2.12e-03	1.37e-07	1.38e-01	9.05e-06
0.200	1.06e-04	2.19e-09	6.22e-05	1.58e-09	0.700	2.48e-03	1.79e-07	2.66e-01	1.95e-05
0.225	1.26e-04	2.69e-09	8.05e-05	2.10e-09	0.725	2.91e-03	2.38e-07	5.41e-01	4.47e-05
0.250	1.49e-04	3.29e-09	1.06e-04	2.83e-09	0.750	3.43e-03	3.21e-07	1.18e+00	1.11e-04
0.275	1.77e-04	4.03e-09	1.43e-04	3.89e-09	0.775	4.06e-03	4.43e-07	2.77e+00	3.04e-04
0.300	2.09e-04	4.94e-09	1.95e-04	5.43e-09	0.800	4.88e-03	6.31e-07	7.22e+00	9.37e-04
0.325	2.46e-04	6.06e-09	2.69e-04	7.70e-09	0.825	5.96e-03	9.37e-07	2.14e+01	3.36e-03
0.350	2.89e-04	7.44e-09	3.77e-04	1.11e-08	0.850	7.53e-03	1.48e-06	7.50e+01	1.47e-02
0.375	3.39e-04	9.14e-09	5.34e-04	1.62e-08	0.875	1.01e-02	2.59e-06	3.33e+02	8.54e-02
0.400	3.97e-04	1.12e-08	7.66e-04	2.41e-08	0.900	1.50e-02	5.34e-06	2.09e+03	7.41e-01
0.425	4.64e-04	1.38e-08	1.11e-03	3.65e-08	0.925	2.73e-02	1.49e-05	1.92e+04	6.83e+00
0.450	5.42e-04	1.71e-08	1.64e-03	5.63e-08	0.950	7.30e-02	7.26e-05	1.41e+05	1.17e+01
0.475	6.32e-04	2.12e-08	2.45e-03	8.86e-08	0.975	4.34e-01	1.21e-03	8.30e+06	2.96e+01

TABLE 2 Integration errors in position (km) and velocity (km/s) for several values of e using the semifocal anomaly

e	$\lvert\Delta r_\Psi\rvert$	$\lvert\Delta v_\Psi\rvert$	e	$\lvert\Delta r_\Psi\rvert$	$\lvert\Delta v_\Psi\rvert$	e	$\lvert\Delta r_\Psi\rvert$	$\lvert\Delta v_\Psi\rvert$	e	$\lvert\Delta r_\Psi\rvert$	$\lvert\Delta v_\Psi\rvert$
0.025	1.4e-07	4.5e-07	0.525	1.3e-07	2.9e-06	1.025	3.8e-07	9.2e-11	1.525	2.4e-06	6.8e-08
0.050	1.4e-07	5.0e-07	0.550	1.3e-07	3.2e-06	1.050	4.1e-07	5.2e-08	1.550	2.7e-06	7.0e-08
0.075	1.4e-07	5.5e-07	0.575	1.4e-07	3.4e-06	1.075	4.5e-07	5.1e-08	1.575	2.9e-06	7.1e-08
0.100	1.4e-07	6.1e-07	0.600	1.4e-07	3.7e-06	1.100	5.0e-07	5.1e-08	1.600	3.2e-06	7.3e-08
0.125	1.4e-07	6.6e-07	0.625	1.4e-07	4.1e-06	1.125	5.5e-07	5.1e-08	1.625	3.4e-06	7.5e-08
0.150	1.4e-07	7.3e-07	0.650	1.4e-07	4.4e-06	1.150	6.1e-07	5.0e-08	1.650	3.7e-06	7.7e-08
0.175	1.4e-07	8.1e-07	0.675	1.5e-07	4.8e-06	1.175	6.6e-07	5.1e-08	1.675	4.1e-06	7.9e-08
0.200	1.4e-07	8.8e-07	0.700	1.5e-07	5.2e-06	1.200	7.3e-07	5.1e-08	1.700	4.4e-06	8.1e-08
0.225	1.3e-07	9.8e-07	0.725	1.6e-07	5.7e-06	1.225	8.1e-07	5.2e-08	1.725	4.8e-06	8.3e-08
0.250	1.3e-07	1.1e-06	0.750	1.7e-07	6.1e-06	1.250	8.8e-07	5.3e-08	1.750	5.2e-06	8.5e-08
0.275	1.3e-07	1.2e-06	0.775	1.8e-07	6.6e-06	1.275	9.8e-07	5.4e-08	1.775	5.7e-06	8.7e-08
0.300	1.3e-07	1.3e-06	0.800	1.9e-07	7.2e-06	1.300	1.1e-06	5.5e-08	1.800	6.1e-06	9.0e-08
0.325	1.3e-07	1.4e-06	0.825	2.0e-07	7.7e-06	1.325	1.2e-06	5.6e-08	1.825	6.6e-06	9.2e-08
0.350	1.3e-07	1.6e-06	0.850	2.1e-07	8.3e-06	1.350	1.3e-06	5.7e-08	1.850	7.2e-06	9.4e-08
0.375	1.3e-07	1.7e-06	0.875	2.3e-07	9.0e-06	1.375	1.4e-06	5.8e-08	1.875	7.7e-06	9.6e-08
0.400	1.3e-07	1.9e-06	0.900	2.5e-07	9.7e-06	1.400	1.6e-06	6.0e-08	1.900	8.3e-06	9.9e-08
0.425	1.3e-07	2.0e-06	0.925	2.7e-07	1.0e-05	1.425	1.7e-06	6.1e-08	1.925	9.0e-06	1.0e-07
0.450	1.3e-07	2.2e-06	0.950	2.9e-07	1.1e-05	1.450	1.9e-06	6.3e-08	1.950	9.7e-06	1.0e-07
0.475	1.3e-07	2.4e-06	0.975	3.2e-07	8.1e-11	1.475	2.0e-06	6.4e-08	1.975	1.0e-05	1.1e-07
0.500	1.3e-07	2.7e-06	1.000	3.5e-07	8.6e-11	1.500	2.2e-06	6.6e-08	2.000	1.1e-05	1.1e-07

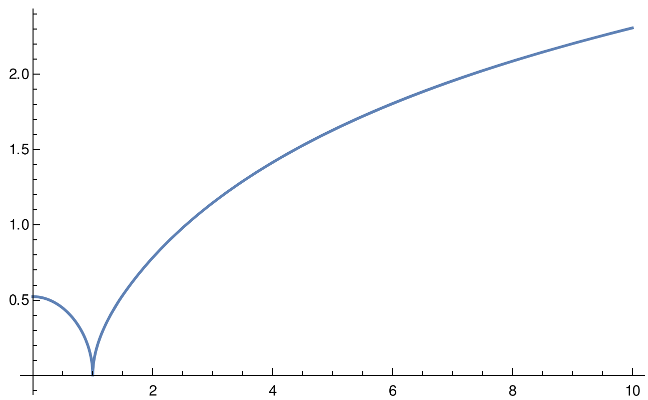

FIGURE 3 Anomalies E and H for $f = \pi/2$ depending on the eccentricity e. The horizontal axis is the value of the eccentricity, and the vertical axis represents the value of E for $e < 1$ and H for $e > 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

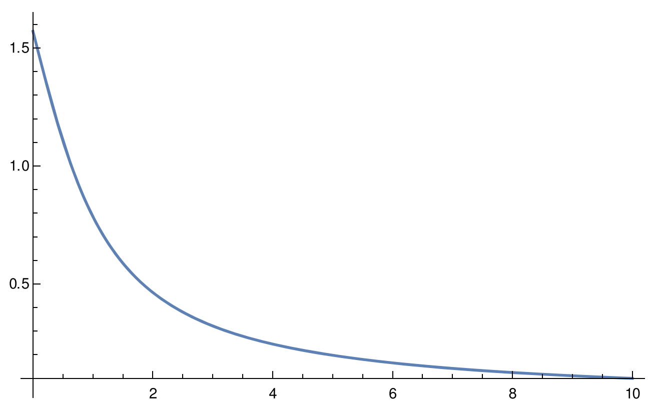

FIGURE 4 Semifocal anomaly Ψ for $f = \pi/2$ depending on the eccentricity e. The horizontal axis is the value of the eccentricity, and the vertical axis represents the value of the anomaly Ψ [Colour figure can be viewed at wileyonlinelibrary.com]

Finally, Figure 3 shows the values of E and H; both derived from the transformation $dt = Krd\tau$. Figure 4 shows the value of semifocal anomaly, Ψ, linked to the true anomaly for $f = \frac{\pi}{2}$ depending on the eccentricity.

In Figure 4, it can be clearly noticed that Ψ is a universal variable; we say it is universal because its usage is appropriate for the elliptic, parabolic, and also the hyperbolic motion. It is important to mention that the use of Ψ is not convenient in the hyperbolic case when the eccentricity is high. However, as shown in Figure 4, Ψ presents a remarkably good behavior in the parabolic case and its neighborhoods.

The semifocal anomaly connects Ψ to the true anomaly and the eccentricity in both elliptic (28) and hyperbolic case (67). Notice that the Equations (28) and (67) are the same. For this reason, it is an appropriate variable in the parabolic case and its surroundings.

However, in the case of the hyperbolic motion and when the eccentricities are very high (i.e. $e > 10$), it is not a very helpful variable because its variation range is very small, as it can be observed in Figure 4.

6 | CONCLUSIONS

In this paper, in order to study the elliptical motion, a new temporal variable has been defined. This variable is called semifocal anomaly, and it is defined as one half of the sum of true anomaly and the antifocal anomaly.

First of all, it is shown that the main magnitudes of the two-body problem can be expressed in closed form using this new anomaly as an independent variable. It is also proved that this anomaly is part of the family of symmetrical anomalies. Besides, an interesting relationship between the true anomaly and the semifocal anomaly is obtained.

Next, the Fourier series developments of the main magnitudes of the two-body problem depending on the semifocal anomaly are obtained. In the process, we demonstrate that these developments can be obtained from recurrence relations. Furthermore, we are able to obtain in both cases (35) and (44) their first two exact terms with the help of the complete elliptic integrals of the first and second kinds. These developments are of great interest when the values of the eccentricities are small, that is to say, when the analytical methods may be appropriate.

Then, the study is extended to the case of hyperbolic motion by defining the semifocal anomaly as the relationship $\Psi = \frac{f+f'}{2} - \frac{\pi}{2}$. In the hyperbolic Keplerian motion, it is also proved that the main magnitudes of the two-body problem can also be obtained in closed form depending on the new variable. Finally, it is obtained that the relationship between true anomaly and semifocal anomaly obtained in the elliptical movement case can be uncomplicatedly extended to the hyperbolic motion.

Next, the study of the limit case of the parabolic motion is addressed. We demonstrate by two analytical methods what the geometric intuition seems to indicate, that is to say, in the parabolic case the semifocal anomaly Ψ coincides with half of the true anomaly; in this sense, it has to be $\Psi = \frac{f}{2}$, maintaining continuity in the limit case and being, therefore, the semifocal anomaly a universal variable.

Finally, by way of example, a set of numerical results are presented. Those examples feature the efficiency of this new variable in the study of the Keplerian motion. The results obtained considerably improve those obtained when the ordinary time or, equivalently, the mean anomaly are used as the integration variable.

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest with regard to this work.

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