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The spectrum of the Laplacian and volume growth of proper minimal submanifolds

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Abstract
We give upper bounds for the bottom of the essential spectrum of properly immersed minimal submanifolds of $\mathbb{R}^n$ in terms of their volume growth. Our result can be viewed as an extrinsic version of Brooks's essential spectrum estimate (Brooks, Math Z 178(4): 501–508, 1981, Thm. 1) and it gives a fairly general answer to a question of Yau (Asian J Math 4(1): 235–278, 2000) about upper bounds for the first eigenvalue (bottom of the spectrum) of immersed minimal surfaces of $\mathbb{R}^3$.

Keywords Essential spectrum · Minimal submanifolds · Volume growth

Mathematics Subject Classification 58C40 · 53C42

1 Introduction

Let M be a complete Riemannian n-manifold and let $\triangle = \operatorname{div} \circ \operatorname{grad}$ be the Laplace-Beltrami operator (Laplacian) acting on $C_0^\infty(M)$ the space of smooth functions with compact support. The Laplacian has a unique self-adjoint extension to an operator $\triangle\colon \mathcal{D}(\triangle) \to L^2(M)$ whose domain is $\mathcal{D}(\triangle) = \{f \in L^2(M)\colon \triangle f \in L^2(M)\}$. The spectrum $\triangle$ is the set of $\lambda \in \mathbb{R}$ for which $\operatorname{Ker}(\triangle + \lambda I) \neq \{0\}$ or $(\triangle + \lambda I)^{-1}$ is unbounded. We will refer to $\sigma(\triangle)$ as the spectrum

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of M and denote it by $\sigma(M)$. Those λ's for which $\mathrm{Ker}(\triangle + \lambda I) \neq \{0\}$ are the eigenvalues of M and the elements of $\mathrm{Ker}(\triangle + \lambda I)$ are the eigenfunctions associated to λ. The set of all eigenvalues of M is the point spectrum $\sigma_p(M)$ and the subset of the point spectrum formed by the isolated eigenvalues with finite multiplicity ($\dim \mathrm{Ker}(\triangle + \lambda I) < \infty$) is called the discrete spectrum, $\sigma_d(M)$. The essential spectrum of M is $\sigma_{ess}(M) = \sigma(M)\backslash\sigma_d(M)$, [8]. When M is compact the spectrum of $\triangle$ is discrete while when M is non-compact the spectrum may be purely continuous, meaning $\sigma_p(M) = \emptyset$, like the Euclidean space $\mathbb{R}^n$, purely discrete ($\sigma_{ess}(M) = \emptyset$), as the simply connected Riemannian manifolds with highly negative curvature, [11] or a mixture of both types, see [9,10].

The very basic question [25] is: *for what geometries* $\inf \sigma(M) > 0$? It was shown by McKean [23] that if M is a simply connected Riemannian manifold with curvature $K_M \leq -\delta^2 < 0$ then $\inf \sigma(M) \geq (n-1)^2\delta^2/4$. Cheng has shown in [7] that if M is complete with non-negative Ricci curvature $\mathrm{Ric}_M \geq 0$ then $\inf \sigma(M) = 0$. On the other hand, a curvature free estimate for the bottom of the spectrum was obtained by R. Brooks in [6]. Precisely, let M be a complete Riemannian manifold of infinite volume and $v(r) = \mathrm{vol}(B_p(r))$ be the volume of the geodesic ball $B_p(r)$ of radius r centred at $p \in M$. Set

$$\mu = \limsup_{r\to\infty} \frac{\log(v(r))}{r}\cdot$$

Brooks proved that $\inf \sigma_{ess}(M) \leq \mu^2/4$.

It is a classical result due to Efimov-Hilbert that complete surfaces with strictly negative curvature $K_M \leq -\delta^2 < 0$ can not be isometrically immersed in $\mathbb{R}^3$, [12,15]. Naturally, one is lead to ask if a complete surface with positive $\inf \sigma(M) > 0$ can be isometrically immersed in $\mathbb{R}^3$. It turns out that the examples, constructed by Nadirashvili [24] and the Spanish School of Geometry [1,19–22], of bounded complete minimal surfaces of $\mathbb{R}^3$ all have $\inf \sigma(M) > 0$, see [2–4]. However, the question whether a manifold with $\inf \sigma(M) > 0$ can be minimally and properly immersed in the Euclidean space is still valid. In some sense, it complements the question raised by S. T. Yau in [26, p. 240] when he asked what upper bounds can one give for the bottom of the spectrum of complete immersed minimal surfaces in $\mathbb{R}^3$.

Ilias et al. [16, Cor.3] gave an answer to Yau's question establishing a Brook's type upper estimate for the bottom of the essential spectrum of any properly immersed submanifold of $\mathbb{R}^n$ with infinite volume. They observed that $v(r) = \mathrm{vol}(B_p(r)) \leq \mathrm{vol}(\Omega_r)$ where $\Omega_r = \varphi^{-1}(B_p(r))$ is the extrinsic ball of radius r of a properly immersed m-submanifold $\varphi\colon M \to \mathbb{R}^n$. Thus applying Brooks spectral estimate they obtain that

$$\inf \sigma_{ess}(M) \leq \left[\liminf_{r\to\infty} r^{-1}\log(\mathrm{vol}(\Omega_r))\right]^2/4.$$

In this note we prove a stronger Brook's type upper estimate for the bottom of the essential spectrum of properly immersed minimal m-submanifolds of the Euclidean n-space. Indeed, letting $\Omega_r \subset M$ be the extrinsic geodesic ball of radius $r > 0$ of a properly minimal immersion $\xi : M \hookrightarrow \mathbb{R}^n$ of a complete Riemannian m-manifold M into $\mathbb{R}^n$ with $\xi(p) = o$, i.e. $\Omega_r = \xi^{-1}(B_o(r))$, we prove the following result.

Theorem 1.1 *Let $\xi\colon M \to \mathbb{R}^n$ be a proper isometric minimal immersion of a complete m-submanifold M of $\mathbb{R}^n$ with $\xi(p) = o$. The bottom of the essential spectrum is bounded above by*

$$\inf \sigma_{ess}(M) \leq m \cdot \liminf_{r\to\infty} \left[r^{-2}\log(\mathrm{vol}(\Omega_r))\right].$$

Theorem 1.1 has a number of corollaries. Let $\Theta(r) = \mathrm{vol}(\Omega_r)/\mathrm{vol}(B^m(r)$, where $B^m(r)$ is a geodesic ball of radius r in the Euclidean space $\mathbb{R}^m$. In [18], Lima et al., proved that if $\liminf_{s\to\infty}(\log(\Theta(s))/\log(s)) = 0$ then $\sigma(M) = [0, \infty)$. Regarding the bottom of the essential spectrum of a properly immersed minimal submanifold our next result extends greatly Lima et al.'s.

Corollary 1.2 *Let $\xi\colon M \to \mathbb{R}^n$ be a complete properly immersed minimal m-submanifold M of $\mathbb{R}^n$ with $\xi(p) = o$. If*

$$\liminf_{r\to\infty} \frac{\log(\Theta(r))}{r^2} = 0$$

then $\inf \sigma_{ess}(M) = 0$.

Corollary 1.3 *Let $\xi\colon M \to \mathbb{R}^3$ be a complete properly embedded minimal surface M of $\mathbb{R}^3$ with $\xi(p) = o$ and let $\kappa(r) = \inf_{x\in\Omega_r} \{K_M(x)\}$, where $K_M(x)$ is the Gaussian curvature of M at x. If*

$$\liminf_{r\to\infty} \frac{\log(|\kappa(r)|)}{r^2} = 0$$

then $\inf \sigma_{ess}(M) = 0$.

Corollary 1.4 *Let $\xi : M \to \mathbb{R}^n$ be an isometric minimal immersion of a complete m-dimensional Riemannian manifold into $\mathbb{R}^n$ with $\xi(p) = o$. Suppose that for some $\sigma > 0$*

$$\int_M e^{-\sigma\|\xi(x)\|^2} d\mu(x) < \infty.$$

Then the immersion ξ is proper, see [13, Thm.1.1], *and*

$$\inf \sigma_{ess}(M) \le m\sigma.$$

2 Proof of the results

A *model n-manifold* $\mathbb{M}^n_h$, with radial sectional curvature $-G(r)$ along the geodesics issuing from the origin, where $G\colon \mathbb{R} \to \mathbb{R}$ is a smooth even function, is the quotient space

$$\mathbb{M}^n_h = [0, R_h) \times \mathbb{S}^{n-1}/\sim$$

with $(\rho, \theta) \sim (\tilde{\rho}, \beta) \Leftrightarrow \rho = \tilde{\rho} = 0$ or $\rho = \tilde{\rho}$ and $\theta = \beta$, endowed with the metric $ds_h^2 = d\rho^2 + h^2(\rho)d\theta^2$ where $h\colon [0, \infty) \to \mathbb{R}$ is the unique solution of the Cauchy problem

$$\begin{cases} h'' - Gh = 0, \\ h'(0) = 1, \\ h^{(2k)}(0) = 0, \ k = 0, 1, \ldots, \end{cases}$$

and R_h is the largest positive real number such that $h|_{(0,R_h)} > 0$. The model $\mathbb{M}^n_h$ is non-compact with pole at the origin $o = \{0\} \times \mathbb{S}^{n-1}/\sim$ if $R_h = \infty$. Observe that $\mathbb{M}^n_t = \mathbb{R}^n$, $\mathbb{M}^n_{\sinh(t)} = \mathbb{H}^n(-1)$ and if $h(t) = \sin(t)$ and $R_h = \pi$ then $\mathbb{M}^n_{\sin(t)} = \mathbb{S}^n$.

If G satisfies

$$G_- \in L^1(\mathbb{R}^+) \text{ and } \int_t^\infty G_-(s)ds \le \frac{1}{4t}$$

then $h' > 0$ in $\mathbb{R}^+$ and $\mathbb{M}_h^n$ is geodesically complete, [5, Proposition 1.21]. The geodesic ball centered at the origin with radius $r < R_h$ is the set $B_h(r) = [0, r) \times \mathbb{S}^{n-1}/\sim$ whose volume and the volume of its boundary are given respectively, by

$$V(r) = \omega_n \int_0^r h^{n-1}(s)ds \text{ and } S(r) = \omega_n h^{n-1}(r),$$

where $\omega_n = \mathrm{vol}(\mathbb{S}^{n-1})$. The Laplace operator on $B_h(r)$, expressed in polar coordinates (ρ, θ), is given by

$$\triangle = \frac{\partial^2}{\partial\rho^2} + (n-1)\frac{h'}{h}\frac{\partial}{\partial\rho} + \frac{1}{h^2}\triangle_\theta.$$

Let $\rho(x) = \mathrm{dist}_{\mathbb{M}_h^n}(o, x)$ be the distance function to the origin o on $\mathbb{M}_h^n$. The hessian of ρ is given by the following expression

$$\mathrm{Hess}\,\rho(x)(e_i, e_i) = \frac{h'}{h}(\rho(x))\left\{\langle e_i, e_i\rangle - d\rho \otimes d\rho(e_i, e_i)\right\}, \tag{1}$$

where $\{e_1, \ldots, e_m\}$ is an orthormal basis of $T_x\mathbb{M}_h^n$. Let $\varphi\colon M \hookrightarrow \mathbb{M}_h^n$ be an isometric immersion of a complete m-manifold into $\mathbb{M}_h^n$. Suppose that $\varphi(p) = o$ for some $p \in M$. The function $t\colon M \to \mathbb{R}$ given by $t(y) = \rho \circ \varphi(y)$ is smooth in $M\backslash\varphi^{-1}(o)$. The hessian of t is given by

$$\mathrm{Hess}_M\, t(q)(e_i, e_i) = \mathrm{Hess}_{\mathbb{M}_h^n}\,\rho(e_i, e_i) + \langle \mathrm{grad}\,\rho, \alpha(e_i, e_i)\rangle.$$

Here we are identifying $e_i = d\varphi \cdot e_i$, see [17]. In particular,

$$\triangle_M t(q) = \sum_{i=1}^m \mathrm{Hess}_{\mathbb{M}_h^n}\,\rho(d\varphi \cdot e_i, d\varphi \cdot e_i) + \langle \mathrm{grad}\,\rho, \vec{H}\rangle. \tag{2}$$

Let $\varphi\colon M \to \mathbb{M}_h^n$ be a complete properly and minimally immersed m-submanifold of $\mathbb{M}_h^n$ with radial sectional curvature $-G(\rho) \le 0$, $\varphi(p) = o$ and let Ω_r be the pre-image $\varphi^{-1}(B_o(r))$.

Lemma 2.1 *For almost any $r > 0$ we have that*

$$\int_{\partial\Omega_r} |\mathrm{grad}\, t|\, d\nu \le m\,\frac{h'}{h}(r)\,\mathrm{vol}(\Omega_r).$$

Proof: Let $\phi\colon M \to \mathbb{R}$ be given by

$$\phi(y) = \left(\int_0^t h(s)ds\right) \circ \rho \circ \varphi(y) = \int_0^{\rho(\varphi(y))} h(s)ds. \tag{3}$$

At a point $q \in M$ and an orthonormal basis $\{e_1, \ldots, e_m\}$ of T_qM that, using (1) and (2), we have

$$\begin{aligned}\triangle_M \phi &= \sum_{i=1}^m \left[\phi''(\rho)\langle \mathrm{grad}\,\rho, e_i\rangle^2 + \phi'(\rho)\frac{h'}{h}(\rho)\left\{\langle e_i, e_i\rangle - \langle \mathrm{grad}\,\rho, e_i\rangle^2\right\}\right]\\ &= m\,h'(\rho).\end{aligned}$$

Since $G \ge 0$, we have that $h''(s) = G(s)h(s) \ge 0$ for $s > 0$, which implies that h' is non-decreasing. In view of Sard's theorem, Ω_r is smoothly bounded for almost any $r > 0$. For any such r, we compute

$$\begin{aligned} m\, h'(r)\mathrm{vol}(\Omega_r) &\geq \int_{\Omega_r} \triangle_M \phi \, d\mu \\ &= \int_{\partial\Omega_r} \left\langle \mathrm{grad}\, \phi, \frac{\mathrm{grad}\, t}{|\mathrm{grad}\, t|} \right\rangle d\nu \\ &= h(r) \int_{\partial\Omega_r} |\mathrm{grad}\, t| d\nu. \end{aligned}$$

Thus

$$\int_{\partial\Omega_r} |\mathrm{grad}\, t| d\nu \leq m \frac{h'(r)}{h(r)} \mathrm{vol}(\Omega_r).$$

This proves Lemma 2.1.

Let $\mathcal{H}_0^1(M)$ be the space of square-integrable functions with square-integrable gradient. Given a non-zero $u \in \mathcal{H}_0^1(M)$, set

$$\mathcal{R}(u) = \frac{\int_M \|\mathrm{grad}\, u\|^2}{\int_M u^2}.$$

For $r > 0$, define $u_r : M \to [0, +\infty)$ by

$$u_r(x) = \begin{cases} \phi(r) - \phi(t(x)) & \text{for } x \in \Omega_r, \\ 0 & \text{else.} \end{cases}$$

Here ϕ is defined in (3). It should be noticed that $u_r \in \mathcal{H}_0^1(M)$, being compactly supported and Lipschitz. Consider also the function

$$v_r = \frac{u_r}{\left(\int_M u_r^2\right)^{1/2}}.$$

This renormalization gives rise to sequences of functions which converge weakly to zero as the next lemma indicates.

Lemma 2.2 *For any sequence $(r_n)_{n\in\mathbb{N}} \subset (0, +\infty)$ with $r_n \to +\infty$ we have that $v_{r_n} \rightharpoonup 0$ in $L^2(M)$.*

Proof: From now on our model manifold is $\mathbb{R}^n$ and $h(s) = s$. For any $c > 0$ we compute,

$$\int_{\Omega_c} v_{r_n}^2 = \frac{\int_{\Omega_c} (r_n^2 - t^2)^2}{\int_{\Omega_{r_n}} (r_n^2 - t^2)^2} \leq \frac{r_n^4 \mathrm{vol}(\Omega_c)}{\int_{\Omega_{r_n/2}} (r_n^2 - t^2)^2} \leq \frac{16 \mathrm{vol}(\Omega_c)}{9 \mathrm{vol}(\Omega_{r_n/2})} \to 0.$$

Keeping in mind that $(v_{r_n})_{n\in\mathbb{N}}$ is bounded in $L^2(M)$, this shows that $v_{r_n} \rightharpoonup 0$. This completes the proof Lemma 2.2.

The significance of considering sequences that converge weakly to zero in order to estimate the bottom of the essential spectrum is illustrated in the following.

Proposition 2.3 *Consider $(v_n)_{n\in\mathbb{N}} \subset \mathcal{H}_0^1(M)$ with $\|v_n\|_{L^2(M)} = 1$ and $v_n \rightharpoonup 0$ in $L^2(M)$. Then the minimum of the essential spectrum of M is bounded by*

$$\inf \sigma_{ess}(M) \leq \liminf_n \mathcal{R}(v_n).$$

Proof: If the right hand side is infinite, there is nothing to prove. If it is finite, we denote it by λ and after passing to a subsequence if necessary, we may suppose that $\mathcal{R}(v_n) \to \lambda$. Assume by contradiction that $\inf \sigma_{ess}(M) > \lambda$. Then

$$\sigma(\Delta) \cap (-\infty, \lambda] = \sigma_d(\Delta) \cap [0, \lambda] = \{\lambda_1, \ldots, \lambda_k\},$$

where λ_i's are eigenvalues of the unique self-adjoint extension $-\triangle$ of minus the Laplacian of finite multiplicity, for some $k \in \mathbb{N}$. Let E_i be the eigenspace corresponding to λ_i and denote by E their sum. Then the spectrum of the restriction $\triangle|_{E^\perp}$ of $\triangle$ to the L^2-orthogonal complement of E is given by $\sigma(\triangle|_{E^\perp}) = \sigma(\triangle) \setminus \{\lambda_1, \ldots, \lambda_k\}$ and in particular, the minimum of its spectrum is greater than λ.

Writing $v_n = u_n + w_n$ with $u_n \in E$ and $w_n \perp E$, we readily see that $u_n \to 0$ and $\triangle u_n \to 0$ in $L^2(M)$, since $v_n \rightharpoonup 0$ and E is finite dimensional. This implies that

$$\int_M |\mathrm{grad}\, u_n|^2 = -\int_M u_n \triangle u_n \to 0.$$

Moreover, we obtain that $\|w_n\|_{L^2(M)} \to 1$ and

$$\int_M |\mathrm{grad}\, w_n|^2 = -\int_M |\mathrm{grad}\,(v_n - u_n)|^2 \to \lambda.$$

We conclude that $\mathcal{R}(w_n) \to \lambda$, which yields that the minimum of the spectrum of $\triangle|_{E^\perp}$ is less or equal to λ, which is a contradiction, that establishes Proposition 2.3.

2.1 Proof of Theorem 1.1

To proceed with the proof of Theorem 1.1 we need to estimate

$$\mathcal{R}(v_r) = \mathcal{R}(u_r)$$

from above. To this end, using the co-area formula and Lemma 2.1, we compute

$$\begin{aligned}
\int_{\Omega_r} |\mathrm{grad}\, u|^2\, d\mu &= \int_{\Omega_r} h^2(t(x)) |\mathrm{grad}\, t|^2 d\mu \\
&= \int_0^r h^2(s) \int_{\partial\Omega_s} |\mathrm{grad}\, t|\, d\nu ds \\
&\leq m \int_0^r h(s) h'(s) \mathrm{vol}(\Omega_s) ds \\
&\leq m\, h'(r) \int_0^r h(s) \mathrm{vol}(\Omega_s) ds.
\end{aligned}$$

It follows from [18, Lemma 2, Propositions 2 and 3] that $\mathrm{vol}(\Omega_s)$ is locally absolutely continuous with

$$\frac{d\, \mathrm{vol}(\Omega_s)}{ds} = \int_{\partial\Omega_s} \frac{1}{|\mathrm{grad}\, t|} d\nu,$$

and

$$\int_{\Omega_r} u^2 d\mu = \int_0^r (\phi(r)-\phi(s))^2 \int_{\partial\Omega_s} \frac{1}{|\mathrm{grad}\, t|} d\nu ds$$
$$= \int_0^r (\phi(r)-\phi(s))^2 \frac{d\,\mathrm{vol}(\Omega_s)}{ds} ds$$
$$= 2\int_0^r (\phi(r)-\phi(s))\phi'(s)\,\mathrm{vol}(\Omega_s) ds.$$

Thus,

$$\mathcal{R}(u_r) \le \frac{m\, h'(r) \displaystyle\int_0^r h(s)\,\mathrm{vol}(\Omega_s) ds}{2\displaystyle\int_0^r (\phi(r)-\phi(s))h(s)\,\mathrm{vol}(\Omega_s) ds}.$$

Letting $F(r) = \int_0^r (\phi(r)-\phi(s))h(s)\,\mathrm{vol}(\Omega_s) ds$ we have that

$$F'(r) = h(r)\int_0^r h(s)\,\mathrm{vol}(\Omega_s) ds.$$

Therefore,

$$\mathcal{R}(v_r) = \mathcal{R}(u_r) \le \frac{m}{2}\frac{h'(r)}{h(r)}\frac{F'(r)}{F(r)}.$$

When $h(s) = s$, i.e. the model $\mathbb{M}_h^m = \mathbb{R}^m$, this inequality reads as

$$\mathcal{R}(v_r) \le m \frac{(\log F(r))'}{2r}$$

for any $r > 0$. We deduce from Lemma 2.2 and Proposition 2.3 that

$$\inf \sigma_{ess}(M) \le m \liminf_{r\to+\infty} \frac{(\log F(r))'}{2r}. \tag{4}$$

Consider any $c \in \mathbb{R}$ with

$$c \le \liminf_{r\to+\infty} \frac{(\log F(r))'}{2r}.$$

Then for any $\varepsilon > 0$ there exists $r_0 > 0$ such that

$$(\log F(r))' \ge 2(c-\varepsilon)r$$

for any $r \ge r_0$. Integrating gives that

$$\log F(r) - \log F(r_0) \ge (c-\varepsilon)(r^2 - r_0^2)$$

for any $r \ge r_0$, which yields that

$$\liminf_{r\to+\infty} \frac{\log F(r)}{r^2} = \liminf_{r\to+\infty} \frac{\log F(r) - \log F(r_0)}{r^2 - r_0^2} \ge c - \varepsilon.$$

We conclude from this together with (4) that

$$\inf \sigma_{ess}(M) \le m \liminf_{r\to+\infty} \frac{\log F(r)}{r^2}.$$

This establishes Theorem 1.1 after noticing that

$$\begin{aligned} F(r) &= \int_0^r (\phi(r)-\phi(s))\, h(s)\mathrm{vol}(\Omega_s)ds \\ &\leq \mathrm{vol}(\Omega_r) \int_0^r \left(\frac{r^2}{2}-\frac{s^2}{2}\right) sds \\ &= \mathrm{vol}(\Omega_r)\, r^4/8 \end{aligned}$$

2.2 Proof of the colloraries

To prove the corollaries we proceed as follows. Observe that

$$\mathrm{vol}(\Omega_r) = \Theta(r)\mathrm{vol}(B^m(r))$$

then

$$\frac{\log(\mathrm{vol}(\Omega_r))}{r^2} = \frac{\log(\Theta(r))}{r^2} + \frac{\log(\mathrm{vol}(B^m(r))}{r^2}.$$

Thus by Theorem 1.1

$$\inf \sigma_{ess}(M) \leq m \liminf_{r\to\infty} \left(\frac{\log(\Theta(r))}{r^2}\right).$$

This proves Corollary 1.2.

Given a unit normal vector field $N : M \to \mathbb{R}^3$, consider the tubular neighbourhood of $\varphi(\Omega_r)$,

$$T_\epsilon(\Omega_r) = \{y \in \mathbb{R}^3 :\; y = q + xN(q), \quad -\epsilon < x < \epsilon, \quad q \in M\}.$$

By [14, p.9] the volume of $T_\epsilon(\Omega_r)$ for ϵ small enough is given by

$$\begin{aligned} \mathrm{vol}(T_\epsilon(\Omega_r)) &= 2\epsilon\, \mathrm{vol}(\Omega_r) + \frac{2\epsilon^2}{3}\int_{\Omega_r} K_M(x)d\mu(x) \\ &\leq \mathrm{vol}(B^3(\epsilon + r)) \\ &= \frac{4\pi}{3}\,(\epsilon + r)^3\,. \end{aligned}$$

On the other hand

$$2\epsilon\, \mathrm{vol}(\Omega_r) + \frac{2\epsilon^2}{3}\int_{\Omega_r} K_M(x)d\mu(x) \geq 2\epsilon\left(1 + \frac{\epsilon}{3}\kappa(r)\right)\mathrm{vol}(\Omega_r).$$

Observe that the immersion $\xi : M \to \mathbb{R}^3$ is minimal and assuming that $\xi(M)$ is not a plane we have that $\kappa(r) < 0$. Choosing $0 < \alpha < 1$ so that $\epsilon = -\frac{3\alpha}{\kappa(r)}$ we have that

$$\begin{aligned} \mathrm{vol}(\Omega_r) &\leq \frac{4\pi}{3}\frac{\left(r - \frac{3\alpha}{\kappa(r)}\right)^3}{-\frac{6\alpha}{\kappa(r)}(1-\alpha)} \\ &= \frac{4\pi}{18\alpha(1-\alpha)}\left(r - \frac{3\alpha}{\kappa(r)}\right)^3(-\kappa(r)). \end{aligned}$$

Thus

$$\liminf_{r\to\infty}\frac{\log(\mathrm{vol}(\Omega_r))}{r^2} \leq \liminf_{r\to\infty}\frac{\log(|\kappa(r)|)}{r^2}.$$

This proves Corollary 1.3.

Suppose that $C = \int_M e^{-\sigma \|\xi(x)\|^2} d\mu(x) < \infty$ for some $\sigma > 0$. Then

$$C \geq \int_{\Omega_r} e^{-\sigma \|\xi(x)\|^2} d\mu(x) \geq e^{-\sigma r^2} \mathrm{vol}(\Omega_r)$$

for any $r > 0$. We derive from [13, Thm. 1.1] that the immersion ξ is proper, and the proof of Corollary 1.4 is completed by Theorem 1.1.

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