Supersymmetric gaps of a numerical semigroup with two generators

Preprint · November 2020

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SUPERSYMMETRIC GAPS OF A NUMERICAL SEMIGROUP WITH TWO GENERATORS

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ABSTRACT. In this paper we introduce the new concepts of supersymmetric and self-symmetric gaps of a numerical semigroup with two generators. Those concepts are based on certain symmetries of the gaps of the semigroup with respect to their Wilf number. Finally, we prove that the set of supersymmetric and self-symmetric gaps completely determines the semigroup and we compare this set with the fundamental gaps of the semigroup.

1. INTRODUCTION

A numerical semigroup Γ is an additive sub-semigroup of the monoid $(\mathbb{N},+)$ such that the greatest common divisor of all its elements is equal to 1. The complement $\mathbb{N}\setminus\Gamma$ is therefore finite and the elements of that complement are called gaps of Γ. Moreover, Γ is finitely generated and it is not difficult to find a minimal system of generators of Γ. In 2004, Rosales et al. [11] defined the concept of fundamental gaps of a numerical semigroup as an alternative way to represent a numerical semigroup as the set

$$\mathscr{FG}=\mathscr{FG}(\Gamma):=\{g\in\mathbb{N}\setminus\Gamma:\ \{2g,3g\}\subset\Gamma\}.$$

From $\mathscr{FG}$ one can define the set $\mathscr{D}(\mathscr{FG}):=\{x\in\mathbb{N}:\ x|x_i \text{ for some } x_i\in\mathscr{FG}\}$ and see that $\Gamma=\mathbb{N}\setminus\mathscr{D}(\mathscr{FG})$ provides an alternative representation of Γ. Moreover, $\mathscr{FG}$ is the smallest subset of $\mathbb{N}\setminus\Gamma$ that H–determines the semigroup Γ: a subset X of $\mathbb{N}$ is said to H-determine Γ if Γ is the maximum (with respect to the set inclusion) numerical semigroup such that X is a subset of $\mathbb{N}\setminus\Gamma$ (see [11]; also Section 4.2).

In this paper, we are mainly concern about numerical semigroups of the form $\Gamma=\langle\alpha,\beta\rangle$. For those semigroups, we are going to introduce the concept of *supersymmetric gap* and *self-symmetric gap*. Let us denote SG resp. SSG the set of supersymmetric resp. self-symmetric gaps. We prove that the set $\mathsf{SG}\cup\mathsf{SSG}$ completely determines the semigroup Γ (see Theorem 4.5). Our construction lies on the application of certain affine linear transformations to the sets SG,SSG represented in the lattice $\mathbb{N}^2$; we call the process of representation of Γ via those transformations *polyomino game*. The set $\mathsf{SG}\cup\mathsf{SSG}$ presents advantages over the set $\mathscr{FG}$: it is not contained in $\mathscr{FG}$ and its cardinality is less than or equal to the cardinality of the set of fundamental gaps whenever $\alpha>2$ (and in the case $\Gamma=\langle 2,3\rangle$), see Section 4.2.

2010 *Mathematics Subject Classification.* Primary: 20M14; Secondary: 05A19,20M30.

Key words and phrases. Numerical semigroup, Frobenius problem, Γ-semimodule, syzygy.

The first author was partially supported by Spanish Goverment, Ministerios de Ciencia e Innovación y de Universidades MTM2016-76868-C2-1-P. The second author was partially supported by the Spanish Government, Ministerios de Ciencia e Innovación y de Universidades, grant PGC2018-096446-B-C22, as well as by Universitat Jaume I, grant UJI-B2018-10. Both authors contributed equally to this work.

Our new concept of supersymmetric and self-symmetric gaps is closely related to a long standing conjecture proposed by H. Wilf in 1978 that can be formulated as follows [14]:

Conjecture 1.1 (Wilf conjecture)**.** *Let* Γ *be a numerical semigroup minimally generated by* $x_1,\dots,x_n$. *Let us denote by* $c(\Gamma)$ *the conductor of* Γ. *Then,*

$$c(\Gamma) \geq \frac{n}{n-1}|\mathbb{N}\setminus\Gamma|.$$

Inspired by this inequality, we consider a Γ–semimodule Δ minimally generated by $\mathrm{ed}(\Delta)$ elements and we define the Wilf number of Δ as

$$W(\Delta) = c(\Delta) - \mathrm{ed}(\Delta)\cdot\delta(\Delta),$$

where $c(\Delta)$ denotes the conductor of Δ and $\delta(\Delta) = \{x \in \Delta : x < c(\Delta)\}$. Observe that for $\Delta = \Gamma$ this number was already considered in [3, p. 45] (both numbers coincide up to sign).

In the case $\Gamma = \langle\alpha,\beta\rangle$, the second author together with Uliczka [7] proved that every Γ–semimodule corresponds to a certain lattice path of the lattice $\mathbb{N}^2$. The lattice path representation has already led to a formula for the conductor $c(\Delta)$ of those Γ–semimodules, see [1]. In particular, this allows us to explicitly compute the Wilf number associated to a Γ–semimodule. Moreover, in the case that Δ is generated by $[0,g]$ for $g \in \mathbb{N}\setminus\Gamma$ we see that $W(\Delta)$ only depends on g (see Proposition 3.3) so in this case it will be referred to as the Wilf number associated to g. Finally, we observe that the Wilf number provides a beautiful symmetry on the set of gaps (see Sect. 4) which motivates our definition of supersymmetric gaps. Moreover, if some of the generators of Γ is even, then there exist some gaps whose Wilf number vanishes and that are invariant under several operations; this motivates the name self-symmetric gaps (see Sections 3 and 4).

To conclude, we discuss some issues regarding the possible extensions of the concepts of supersymmetric and self-symmetric gaps to the general case where Γ is a numerical semigroup with an arbitrary number of generators (see Subsection 4.3). More concretely, we propose a general definition for symmetric gaps (see Definition 4.15) and we ask if this definition allows us to define the concepts of supersymmetric and self-symmetric gaps for a semigroup with any number of generators. We hope that—if this extension succeeds—these new concepts could be helpful to the solution of the Wilf conjecture.

Acknowledgments. The authors would like to thank Alfredo Granell Marqués for the computation of general examples.

2. Lattice paths and semimodules over a numerical semigroup

Let Γ be a numerical semigroup. The reader is referred to [12] or [9] for specific material about numerical semigroups. We are interested in subsets of $\mathbb{N}$ which have an additive structure over Γ (in analogy with the structure of module over a ring): a Γ-semimodule is a non-empty subset Δ of $\mathbb{N}$ with $\Delta+\Gamma \subseteq \Delta$. A system of generators of Δ is a subset $\mathscr{E}$ of Δ with $\Delta = \bigcup_{x\in\mathscr{E}}(x+\Gamma)$; it is called minimal if no proper subset of $\mathscr{E}$ generates Δ. Notice that, since $\Delta\setminus\Gamma$ is finite, every Γ-semimodule is finitely generated and has a conductor

$$c(\Delta) = \max(\mathbb{N}\setminus\Delta)+1.$$

To a semimodule Δ we also associate the *δ-invariant* of Δ, namely

$$\delta(\Delta) = \{x \in \Delta : x < c(\Delta)\}.$$

Every Γ-semimodule Δ has a unique minimal system of generators (see e.g. [7, Lemma 2.1]); its cardinality is said to be the *embedding dimension* of Δ, written $\mathrm{ed}(\Delta)$. Two Γ-semimodules Δ and Δ' are called isomorphic if there is an integer n such that $x \mapsto x+n$ is a bijection from Δ to Δ'; we write then $\Delta \cong \Delta'$. For every Γ-semimodule Δ there is a unique semimodule $\Delta' \cong \Delta$ containing 0; such a semimodule is called *normalized*. The Γ-semimodule

$$\Delta^\circ := \{x - \min \Delta : x \in \Delta\}$$

is called the normalization of Δ; this is the unique Γ-semimodule isomorphic to Δ and containing 0. Moreover, the minimal system of generators $\{x_0 = 0, \ldots, x_n\}$ of a normalized Γ-semimodule is a Γ-lean set, i.e. it satisfies that

$$|x_i - x_j| \notin \Gamma \text{ for any } 0 \leq i < j \leq n,$$

and conversely, every Γ-lean set of $\mathbb{N}$ minimally generates a normalized Γ-semimodule; we will write then $[x_0 = 0, \ldots, x_n]$. Hence there is a bijection between the set of isomorphism classes of Γ-semimodules and the set of Γ-lean sets of $\mathbb{N}$; see Sect. 2 in [7] for the proofs of those statements.

The dual Δ^* of a Γ-semimodule Δ is defined to be

$$\Delta^* := \mathrm{Hom}_\Gamma(\Delta, \Gamma) = \{x \in \mathbb{N} : x + \Delta \subseteq \Gamma\},$$

cf. [8, p. 677]. A Γ-semimodule is said to be *selfdual* if $\Delta = \Delta^*$. In addition, we define the set of syzygies of a Γ-semimodule $\Delta = \Delta_I$ with minimal set of generators $I = [g_0, \ldots, g_n]$ as

$$\mathrm{Syz}(\Delta) := \bigcup_{i,j \in I, i \neq j} \Big((\Gamma + g_i) \cap (\Gamma + g_j) \Big).$$

In this paper we will consider numerical semigroups with two generators, say $\Gamma = \langle \alpha, \beta \rangle$, with $\alpha, \beta \in \mathbb{N}$ and $\alpha < \beta$. As mentioned above, the conductor of Γ can be expressed as $c = c(\Gamma) = (\alpha-1)(\beta-1)$. The gaps of $\langle \alpha, \beta \rangle$ are also easy to describe: they admit a representation $\alpha\beta - a\alpha - b\beta$, where $a \in \,]0, \beta-1] \cap \mathbb{N}$ and $b \in \,]0, \alpha-1] \cap \mathbb{N}$, see Rosales [10, Lemma 1]. This writing yields a map from the set of gaps of $\langle \alpha, \beta \rangle$ to $\mathbb{N}^2$ given by $\alpha\beta - a\alpha - b\beta \mapsto (a,b)$, which allows us to identify a gap with a point in the lattice $\mathscr{L} = \mathbb{N}^2$; since the gaps are positive numbers, the point lies inside the triangle with vertices $(0,0), (0,\alpha), (\beta,0)$. Let us denote by LG the image of the map $\alpha\beta - a\alpha - b\beta \mapsto (a,b)$, i.e. the points of $\mathscr{L}$ inside the triangle of vertices $(0,0), (0,\alpha), (\beta,0)$.

In the following we will use the notation

$$e = \alpha\beta - a(e)\alpha - b(e)\beta$$

for a gap e of the semigroup $\langle \alpha, \beta \rangle$; if the gap is subscripted as e_i then we write $a_i = a(e_i)$ and $b_i = b(e_i)$.

Let us denote by $\leq$ the total ordering in $\mathbb{N}$, if needed we will denote it by $\leq_{\mathbb{N}}$ to emphasize that it is the natural order. We also consider the following partial ordering $\preceq$ on the set of gaps:

Definition 2.1. Given two gaps e_1, e_2 of $\langle \alpha, \beta \rangle$, we define

$$e_1 \preceq e_2 \;:\Longleftrightarrow\; a_1 \leq a_2 \;\wedge\; b_1 \geq b_2$$

and

$$e_1 \prec e_2 \;:\Longleftrightarrow\; a_1 < a_2 \;\wedge\; b_1 > b_2.$$

Let $\mathscr{E} = \{0, e_1, \ldots, e_n\}$ be a subset of $\mathbb{N}$ with gaps $e_i = \alpha\beta - a_i\alpha - b_i\beta$ of $\langle \alpha, \beta \rangle$ for every $i = 1, \ldots, n$ such that $a_1 < a_2 < \cdots < a_n$. Corollary 3.3 in [7] ensures that $\mathscr{E}$ is $\langle \alpha, \beta \rangle$-lean if and only if $b_1 > b_2 > \cdots > b_n$. This simple fact leads to an identification (cf. [7, Lemma 3.4]) between an $\langle \alpha, \beta \rangle$-lean set and a lattice path with steps downwards and to the right from $(0, \alpha)$ to $(\beta, 0)$ not crossing the line joining these two points, where the lattice points identified with the gaps in $\mathscr{E}$ mark the turns from the x-direction to the y-direction; these turns will be called ES-turns for abbreviation. Figure 2.1 shows the lattice path corresponding to the $\langle 5, 7 \rangle$-lean set $[0, 9, 11, 8]$.

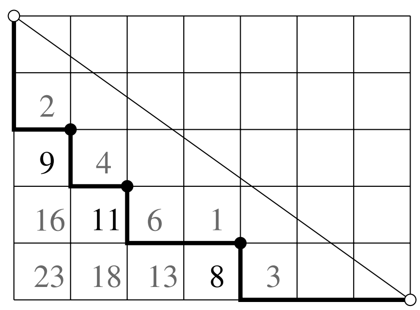

FIGURE 2.1. Lattice path for the $\langle 5, 7 \rangle$-lean set $[0, 9, 11, 8]$.

Let $g_0 = 0, g_1, \ldots, g_n$ be the minimal system of generators of a $\langle \alpha, \beta \rangle$-semimodule Δ. From now on, we will assume that the indexing in the minimal set of generators of Δ is such that $g_0 = 0 \preceq g_1 \preceq \cdots \preceq g_n$. Under this assumption, we can give an explicit formula for the minimal generators of Δ^* in terms of those of Δ:

$$\Delta^* = (\Gamma + a_1\alpha) \cup \bigcup_{k=1}^{n-1} (\Gamma + a_{k+1}\alpha + b_k\beta) \cup (\Gamma + b_n\beta). \tag{2.1}$$

Moreover, the semimodule $\mathrm{Syz}(\Delta)$ of syzygies of Δ can be characterized as follows (see [7, Theorem 4.2]):

Proposition 2.2.

$$\mathrm{Syz}(\Delta) = \bigcup_{0 \leq k < j \leq n} \Big((\Gamma + g_k) \cap (\Gamma + g_j) \Big) = \bigcup_{k=0}^{n} (\Gamma + h_k),$$

where $h_1, \ldots, h_{n-1}$ are gaps of Γ, $h_0, h_n \leq \alpha\beta$, and

$$h_k \equiv g_k \bmod \alpha, \; h_k > g_k \textit{ for } k = 0, \ldots, n$$
$$h_k \equiv g_{k+1} \bmod \beta, \; h_k > g_{k+1} \textit{ for } k = 0, \ldots, n-1$$
$$h_n \equiv 0 \bmod \beta, \text{ and } h_n \geq 0$$

In particular, $J=[h_0,\dots,h_n]$ is a minimal system of generators of the semimodule $\Delta_J=\mathrm{Syz}(\Delta)$, hence $h_0 \preceq h_1 \preceq \cdots \preceq h_n$. Therefore it is easily seen that the SE-turns of the lattice path associated to Δ can be identified with the minimal set of generators of the syzygy module (we call SE-turns to the turns from the y–direction to the x–direction). After that, we can associate to any Γ-semimodule Δ a lean couple (I,J) where I is a minimal set of generators of Δ and J a minimal set of generators of $\mathrm{Syz}(\Delta)$; or equivalently a lattice path. The syzygies allowed us to give a formula for the conductor $c(\Delta)$ of Δ:

Theorem 2.3. [1, Theorem 1] *Let Δ be a Γ-semimodule with Γ-lean couple (I,J), and let $M:=\max_{\leq_{\mathbb{N}}}\{h\in J\}$ denote the biggest, with respect to the order of the natural numbers, minimal generator of the syzygy module. Then*

$$c(\Delta)=M-\alpha-\beta+1.$$

In particular, if we denote by (m_1,m_2) the point in the lattice $\mathscr{L}$ representing M, then we have

$$c(\Delta)=c(\Gamma)-m_1\alpha-m_2\beta.$$

The syzygies lead also to the concept of *fixed point* for a semimodule:

Definition 2.4. An $\langle\alpha,\beta\rangle$-semimodule Δ_I with associated Γ-lean couple (I,J) is said to be a $\langle\alpha,\beta\rangle$-fixed point (or simply a fixed point if the semigroup is clear from the context) if the semimodule $(\Delta_J)^\circ$ admits I again as a minimal system of generators.

The chosen name *fixed point* has a reason: it refers to the orbits of period 1 of the Picard sequence associated to the map $f=h\circ\mathrm{Syz}$, where Syz is the map $\Delta_I\mapsto\Delta_J$ and h is the normalization map for $\Delta_J=\mathrm{Syz}(\Delta_I)$; this is further explained in [7, Sect. 5].

3. Wilf number of a gap of a numerical semigroup

In this section, we are going to make use of the conductor formula of a Γ–semimodule (Theorem 2.3) to associate to a gap a number which will be invariant under certain symmetries of the lattice. This number is motivated by Wilf's conjecture. First, we introduce the *Wilf number* of a Γ-semimodule:

Definition 3.1. Let Δ be a Γ-semimodule, then the Wilf number of Δ is defined to be

$$W(\Delta):=c(\Delta)-\mathrm{ed}(\Delta)\cdot\delta(\Delta).$$

In addition, we define the Wilf number of a gap $g\in\mathbb{N}\setminus\Gamma$ by assigning the Wilf number of the Γ-semimodule $\Delta=\Delta_I$ minimally generated by $I=[0,g]$:

$$W(g):=W(\Delta_I)=c(\Delta_I)-2\delta(\Delta_I).$$

If $g=\alpha\beta-a\alpha-b\beta$ then we will also denote by $W(a,b):=W(g)$ its Wilf number.

Observe that for $\Delta=\Gamma$, Wilf's conjecture claims that $W(\Gamma)\leq 0$, see [14].

Now, we restrict our attention to the case $\Gamma=\langle\alpha,\beta\rangle$. Here $W(g)$ only depends on the gap g as a consequence of the formula for the conductor of a Γ–semimodule given in Theorem 2.3.

Proposition 3.2. *Let $I=[g_0=0,g_1,\dots,g_n]$ be the minimal system of generators of a Γ-semimodule Δ ordered as $0\prec g_1\prec\dots\prec g_n$; recall the writing $g_i=\alpha\beta-a_i\alpha-b_i\beta$, and set $a_0=b_0=0$. Then*

$$\begin{aligned}\delta(\Delta)=&c(\Delta)-\delta(\Gamma)+\sum_{i=0}^{n}(a_{i+1}-a_i)b_{i+1}\\ =&c(\Delta)-\delta(\Gamma)+\sum_{i=0}^{n}(b_i-b_{i+1})a_{i+1}.\end{aligned}$$

Furthermore, for any $I=[0,g_i]$ with $i=1,\dots,n$, we have

$$\delta(\Delta_I)=c(\Delta_I)-\delta(\Gamma)+a_ib_i.$$

Proof. Since $\delta(\Gamma)=|\mathbb{N}\setminus\Gamma|$, it is easily deduced from the lattice path representation of Δ_I that

$$|\mathbb{N}\setminus\Delta_I|=\delta(\Gamma)-\sum_{i=0}^{n}(a_{i+1}-a_i)b_{i+1}=\sum_{i=0}^{n}(b_i-b_{i+1})a_{i+1}.$$

By the definition of $\delta(\Delta_I)$ we have the claim. □

Therefore, we can compute explicitly the Wilf number of a gap of $\langle\alpha,\beta\rangle$:

Proposition 3.3. *Let $g=\alpha\beta-a\alpha-b\beta$ be a gap of Γ. Let us denote by $[h_0,h_1]$ the minimal system of generators of the Γ-semimodule $\mathrm{Syz}(\Delta_{[0,g]})$. Then*

$$W(g)=\begin{cases} a\alpha-2ab & \text{if } \min\{h_0,h_1\}=\alpha\beta-b\beta,\\ b\beta-2ab & \text{if } \min\{h_0,h_1\}=\alpha\beta-a\alpha.\end{cases}$$

Proof. Consider the Γ–semimodule Δ_I generated by $I=[0,g]$. From the representation of the lattice path we know that $h_0=\alpha\beta-b\beta$ and $h_1=\alpha\beta-a\alpha$. Let us first assume $\min\{h_0,h_1\}=h_0$, thus $\max\{h_0,h_1\}=h_1=\alpha\beta-a\alpha$. Therefore, by Theorem 2.3 we have

$$c(\Delta_I)=\alpha\beta-a\alpha-\alpha-\beta+1.$$

Lemma 3.2 yields

$$c(\Delta_I)-2\delta(\Delta_I)=-c(\Delta_I)+2\delta(\Gamma)-2ab=c(\Gamma)-c(\Delta_I)-2ab=a\alpha-2ab.$$

The same reasoning applies after exchanging the roles of a and b to obtain the second case of the formula. □

We are interested in the case in which the Wilf number of a gap is zero:

Theorem 3.4. *Let $\Gamma=\langle\alpha,\beta\rangle$ and let $g=\alpha\beta-a\alpha-b\beta$ be a gap of Γ. Consider the Γ-semimodule $\Delta=\Delta_I$ minimally generated by $I=[0,g]$. The following statements are equivalent:*

(1) $W(g)=0$;
(2) either $\alpha=2b$ or $\beta=2a$;
(3) Δ is a fixed point;
(4) Δ is selfdual;
(5) Δ is symmetric, i.e. for every $x\in\Delta$ we have that $c(\Delta)-1-x\notin\Delta$ if and only if $x\in\Delta$.

Proof. (1) $\iff$ (2) is obvious by Proposition 3.3.
(2) $\iff$ (3): If $\alpha = 2b$, then $g = \alpha\beta - a\alpha - b\beta = b\beta - a\alpha = h_1 - h_0$, and this is positive since g is a gap, hence $\mathrm{Syz}(\Delta_{[0,g]}) = \Delta_{[h_0,h_1]} = \Delta_{[0,h_1-h_0]} = \Delta_{[0,g]}$. *Mutatis mutandis*, if $\beta = 2a$, then Δ is a fixed point. Conversely, assume without loss of generality that $h_0 < h_1$; since $\alpha\beta - a\alpha - b\beta = g = h_1 - h_0 = b\beta - a\alpha$, it is easily seen that $\alpha = 2b$.
(3) $\iff$ (4): First we observe that the number of semimodules Δ_I with $I = [0,g]$ which are fixed points coincide with the number of selfdual modules of that form, as a direct application of Theorem 5.5 in [7, Theorem 5.5], as well as Proposition 4.1 and Theorem 4.4 in [8]. Moreover, every selfdual semimodule is a fixed point: For $\Delta = \Delta_I$ with $I = [0,g]$, eq. (2.1) implies that Δ^* is minimally generated by $[a_1\alpha, b_1\beta]$; the selfduality implies that $a_1 = 0$ and $b_1\beta = \alpha\beta - b_1\beta$, hence $I = [0, \alpha\beta - b_1\beta]$. On the other hand, $\mathrm{Syz}(\Delta_I)$ is minimally generated by $[a_1\alpha, b_1\beta] = [0, \alpha\beta - b_1\beta]$, that equals its own normalization. Therefore Δ_I coincides with the normalization of $\mathrm{Syz}(\Delta_I)$ and Δ_I is a fixed point.
(4) $\iff$ (5): this is a consequence of Proposition 4 in [13], and Theorem 2.11 together with Proposition 3.8 in [6]. □

For $\langle \alpha, \beta \rangle$-semimodules Δ with $\mathrm{ed}(\Delta) > 2$, Theorem 3.4 is no longer true: for instance the $\langle 5, 8 \rangle$-semimodule minimally generated by the lean set $I = [0,4,6,7]$ has Wilf number $W(\Delta_I) = c(\Delta_I) - 4\delta(\Delta_I) = 4 - 4 \cdot 1 = 0$ and it is not a fixed point. Moreover, let us consider the numerical semigroup $\Gamma = \langle 10, 14, 27 \rangle$. This semigroup is both symmetric and complete intersection, however if we consider the Γ–semimodule generated by $I = [0,9]$ then $W(\Delta_I) = 0$ and $I = [0,9]$ is neither a fixed point nor symmetric. On the other hand, if we consider $\Gamma = \langle 10, 14, 29 \rangle$ every Γ–semimodule with Wilf number equal to zero is a fixed point.
Therefore, we cannot expect a generalization of Theorem 3.4 for numerical semigroup Γ with more than two minimal generators just by imposing the condition of symmetric or complete intersection. This encourages us to propose the following question.

Question 3.5. Given a numerical semigroup Γ with $\mathrm{ed}(\Gamma) > 2$, does there exist a family of numerical semigroups for which any of the equivalences of Theorem 3.4 remain true?

4. Supersymmetry of the gap set with respect to the Wilf number

This section is devoted to introduce and to develop the concept of supersymmetric gaps. This notion is based on certain symmetries encoded in the Wilf number of a gap. Before introducing those concepts, let us see some properties of the image LG of the map $\alpha\beta - a\alpha - b\beta \mapsto (a,b)$, cf. Sect. 2.

4.1. Determinacy of the semigroup.

A first observation is that a Γ–semimodule generated by $[0,g]$ has its syzygy module with two minimal generators. Moreover, by the formulas for the minimal set of generators of the syzygy module the following lemma is a straightforward computation.

Lemma 4.1. *Let $g = \alpha\beta - a\alpha - b\beta$ be a gap of $\Gamma = \langle \alpha, \beta \rangle$. Let $[h_0, h_1]$ be the minimal set of generators of $\mathrm{Syz}(\Delta_{[0,g]})$. Then,*

(1) If $b > \lfloor \frac{\alpha}{2} \rfloor$ and $a \le \lfloor \frac{\beta}{2} \rfloor$ then $\min\{h_0, h_1\} = \alpha\beta - b\beta$.
(2) If $b \le \lfloor \frac{\alpha}{2} \rfloor$ and $a > \lfloor \frac{\beta}{2} \rfloor$ then $\min\{h_0, h_1\} = \alpha\beta - a\alpha$.

This lemma allows us to describe the behavior of the set of gaps with respect to the Wilf number. First, observe that any integral point inside the triangle delimited by the y–axis, the line $y = \lfloor \frac{\alpha}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$ represents a gap g of Γ with expression $g = \alpha\beta - a\alpha - b\beta$ and $b > \lfloor \frac{\alpha}{2} \rfloor$. Hence this gap has Wilf number $W(g) = a\alpha - 2ab$. Now, let us consider the symmetric point to g with respect to the reflection along the line $y = \lfloor \frac{\alpha}{2} \rfloor$. This reflection is given by the map $(a,b) \mapsto (a, \alpha - b)$. Therefore, we have

Lemma 4.2. *If (a,b) is an integral point inside the triangle delimited by the y–axis, the line $y = \lfloor \frac{\alpha}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$ then*

$$W(a,b) = -W(a, \alpha - b).$$

Proof. By Lemma 4.1 and Proposition 3.3 we have that $W(a,b) = a\alpha - 2ab$. Now, let us denote $g_{\text{sym}} = \alpha\beta - a\alpha - (\alpha - b)\beta$ the symmetric gap with respect to the reflection $(a,b) \mapsto (a, \alpha - b)$. Let us consider the minimal set of generators $[h'_0, h'_1]$ of $\text{Syz}(\Delta_{[0, g_{\text{sym}}]})$. It is thus clear that $\min\{h'_0, h'_1\} = \alpha\beta - b\beta$, since

$$h'_1 - h'_0 = \alpha\beta - a\alpha - \alpha\beta + (\alpha - b)\beta = \alpha\beta - a\alpha - b\beta = g > 0,$$

and the proof follows. □

An analogous situation occurs when considering the triangle delimited by the x–axis, the line $x = \lfloor \frac{\beta}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$. In this case, the map $(a,b) \mapsto (\beta - a, b)$ yields the following result:

Lemma 4.3. *If (a,b) is an integral point inside the triangle delimited by the x–axis, the line $x = \lfloor \frac{\beta}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$, then*

$$W(a,b) = -W(\beta - a, b).$$

In particular, the set of fixed points of each of the previous symmetries is exactly the set of points with $W(a,b) = 0$. As we have seen in Theorem 3.4 those are exactly fixed points of the orbits of the associated lattice path, i.e. of the associated semimodule.

The previous discussion leads to the following definition:

Definition 4.4. Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup. Let us denote by $\mathscr{T}_r$ the set of points of $\mathscr{L}$ inside the triangle delimited by the x–axis, the line $x = \lfloor \frac{\beta}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$ and $\mathscr{T}_u$ the set of points of $\mathscr{L}$ inside the triangle delimited by the y–axis, the line $y = \lfloor \frac{\alpha}{2} \rfloor$ and the diagonal $\alpha\beta = x\alpha + y\beta$. The set of *supersymmetric gaps* is defined to be

$$\mathsf{SG} := \begin{cases} \mathscr{T}_u & \text{if } |\mathscr{T}_u| < |\mathscr{T}_r| \\ \mathscr{T}_r & \text{if } |\mathscr{T}_r| < |\mathscr{T}_u|. \end{cases}$$

We also define the set of *self-symmetric gaps*

$$\mathsf{SSG} := \{g \in \mathbb{N} \setminus \Gamma : \, W(g) = 0\}.$$

At this point, we are able to prove the main result of the paper.

Theorem 4.5. *Let $\Gamma = \langle \alpha, \beta \rangle$ be a numerical semigroup. Then the set $\mathsf{SG} \cup \mathsf{SSG}$ of supersymmetric and self-symmetric gaps completely determines the set of gaps of Γ. In particular, it determines Γ itself.*

Proof. With the notation of Definition 4.4, consider the symmetry $s_\alpha : \mathscr{T}_u \to LG$ along the line $y = \lfloor \frac{\alpha}{2} \rfloor$ defined by $(a,b) \mapsto (a, \alpha - b)$, as well as the symmetry $s_\beta : \mathscr{T}_r \to LG$ along the line $x = \lfloor \frac{\beta}{2} \rfloor$ defined by $(a,b) \mapsto (\beta - a, b)$. First we are going to show that $s_\alpha(\mathscr{T}_u) \cap s_\beta(\mathscr{T}_r) = \emptyset$. Consider $(a,b) \in \mathscr{T}_u$ then

$$(a,b) \mapsto (a, \alpha - b) \mapsto (\beta - a, \alpha - b),$$

where $\alpha\beta - (\beta - a)\alpha - (\alpha - b)\beta = a\alpha + b\beta - \alpha\beta < 0$, since $\alpha\beta - a\alpha - b\beta$ is the representation of a gap. Therefore, $s_\beta^{-1}(s_\alpha(\mathscr{T}_u)) = \emptyset$. Analogously, it can be shown that $s_\alpha^{-1}(s_\beta(\mathscr{T}_u)) = \emptyset$. Now, let $B(\mathscr{T}_u)$ resp. $B(\mathscr{T}_r)$ the border points of sets $\mathscr{T}_u$ resp. $\mathscr{T}_r$, i.e. those points such that $(a,b) \in \mathscr{T}_u$ resp. $\mathscr{T}_r$ and $(a+1,b) \notin LG$ or $(a,b+1) \notin LG$. Moreover, let $RB(\mathscr{T}_u)$ resp. $RB(\mathscr{T}_r)$ denote the set of border points of the type $(a+1,b) \notin LG$. Observe that those points determine ES–turns, hence the borders $B(\mathscr{T}_u)$ and $B(\mathscr{T}_r)$ are determined by $RB(\mathscr{T}_u)$ and $RB(\mathscr{T}_r)$.

Let us denote by $\tau : \mathscr{L} \to \mathscr{L}$ the translation defined by $(a,b) \mapsto (a+1,b)$. We claim that

$$s_\beta^{-1}(\tau(s_\alpha(RB(\mathscr{T}_u)))) = RB(\mathscr{T}_r).$$

Indeed, consider the point $(a,b) \in \mathscr{T}_u$, then $s_\beta^{-1}(\tau(s_\alpha((a,b)))) = (\beta - a - 1, \alpha - a) \in RB(\mathscr{T}_r)$ due to the fact that $\alpha\beta - (\beta - a - 1)\alpha - b\beta > 0$ and $\alpha\beta - (\beta - a)\alpha - b\beta < 0$. A similar reasoning allows us to prove

$$s_\alpha^{-1}(\tau^{-1}(s_\beta(RB(\mathscr{T}_r)))) = RB(\mathscr{T}_u).$$

The proof will finish by distinguishing three cases concerning the parity of α and β. Let us start with easiest one and assume that α, β are both odd. By Theorem 3.4 there are no gaps with $W(g) = 0$. Thus, we have a configuration as in Figure 4.1.

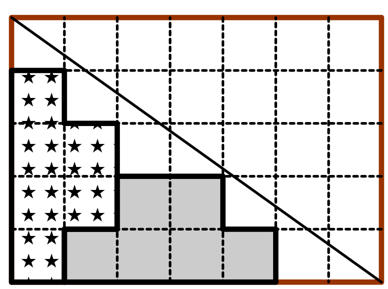

FIGURE 4.1. The sets $\mathscr{T}_u \cup s_\alpha(\mathscr{T}_u)$ (starred) and $\mathscr{T}_r \cup s_\beta(\mathscr{T}_r)$ (shaded).

In fact, it is easily checked that

$$B(\mathscr{T}_u) \cup B(\mathscr{T}_r) \cup s_\beta(B(\mathscr{T}_r)) \supseteq B(LG),$$

and the sets fit as shown in Figure 4.1.

Next we assume that α is even (the case β even follows analogously). By Theorem 3.4 the set SSG consists of exactly those gaps given by the lattice points $(a, \alpha/2)$ with $1 \le a \le \lfloor \beta/2 \rfloor$. Let $B(\mathsf{SSG})$ denote the set of border points in SSG, then we have a configuration as in Figure 4.2.

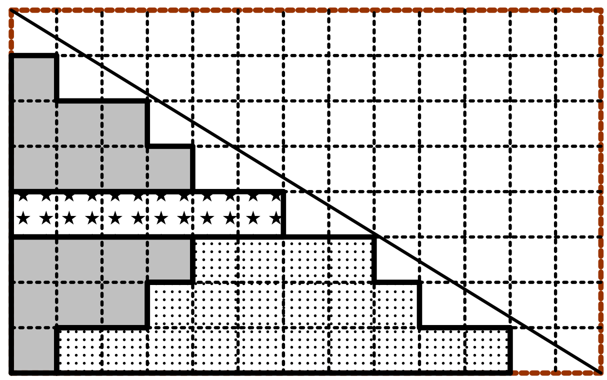

FIGURE 4.2. Sets $\mathscr{T}_u \cup s_\alpha(\mathscr{T}_u)$ (shaded), SSG (starred), and $\mathscr{T}_r \cup s_\beta(\mathscr{T}_r)$ (dotted).

So it is easily seen that

$$B(\mathscr{T}_u) \cup B(\mathsf{SSG}) \cup B(\mathscr{T}_r) \cup s_\beta(B(\mathscr{T}_r)) \supseteq B(LG).$$

All this together shows that the union of the triangles $\mathscr{T}_u, \mathscr{T}_r$, their images and the set of self-symmetric gaps build a partition of the set of gaps into disjoint sets

$$\mathbb{N} \setminus \Gamma = \mathscr{T}_u \bigsqcup s_\alpha(\mathscr{T}_u) \bigsqcup \mathsf{SSG} \bigsqcup \mathscr{T}_r \bigsqcup s_\beta(\mathscr{T}_r).$$

We are finished as soon as the procedure to recover $\mathbb{N} \setminus \Gamma$ —hence Γ— from the set $\mathsf{SG} \cup \mathsf{SSG}$ will be given.
Let us assume that $\mathsf{SG} = \mathscr{T}_u$ (similarly for $\mathscr{T}_r$). Thus, we have

$$s_\beta(B(\mathscr{T}_r)) = \tau(s_\alpha(B(\mathsf{SG}))).$$

We distinguish two cases:

(1) If α, β are both odd, then we can recover $s_\beta(\mathscr{T}_r)$ as the polyomino corresponding to the complement of $s_\alpha(\mathsf{SG})$ in the lattice square with vertices $(0,0), (\lfloor \frac{\beta}{2} \rfloor, 0), (\lfloor \frac{\beta}{2} \rfloor, \lfloor \frac{\alpha}{2} \rfloor)$, and $(0, \lfloor \frac{\alpha}{2} \rfloor)$.

(2) If α or β is even, then consider the polyomino $\mathsf{SSG} \cup s_\alpha(\mathsf{SG})$. Thus, we can recover $s_\beta(\mathscr{T}_r)$ as the polyomino corresponding to the complement of $s_\alpha(\mathsf{SG}) \cup \mathsf{SSG}$ in the lattice square with vertices $(0,0), (\lfloor \frac{\beta}{2} \rfloor, 0), (\lfloor \frac{\beta}{2} \rfloor, \lfloor \frac{\alpha}{2} \rfloor), and (0, \lfloor \frac{\alpha}{2} \rfloor)$.

Observe that, if $\mathsf{SG} = \mathscr{T}_r$, then the roles of α, β in (2) and (3) need to be exchange.

In short, we have checked that in all cases we can obtain $s_\beta(\mathscr{T}_r)$ resp. $s_\alpha(\mathscr{T}_u)$, hence $\mathscr{T}_r$ resp. $\mathscr{T}_u$ from certain linear transformations of the set $\mathsf{SG} \cup \mathsf{SSG}$ in the lattice. Therefore, by the previous partition of the set of gaps we can reconstruct completely the set of gaps from the set $\mathsf{SG} \cup \mathsf{SSG}$. □

The proof of Theorem 4.5 shows in particular that SG and SSG are polyominoes, and that we can obtain the whole set LG making operations with them. These necessary operations which allow us to obtain the set LG from SG and SSG will be called *polyomino game*. We illustrate both the polyomino game and the proof of Theorem 4.5 with an example.

Example 4.6. Let $\Gamma = \langle 7, 8\rangle$. We start with SG which in this case is $\mathscr{T}_r$. Then the set of gaps represented in $\mathscr{T}_r$ is $\{5, 6, 13\}$ (see Figure 4.3). Now, we consider the polyomino $s_\beta(\mathsf{SG}) \cup \mathsf{SSG}$ which represents $\{4, 12, 19, 20, 27, 34\}$ inside the square of vertices $(0,0), (4,0), (4,3), (0,3)$ (see Figure 4.4).

After that, we consider the complement of the polyomino $s_\beta(\mathsf{SG}) \cup \mathsf{SSG}$ which represents the set of gaps $\{41, 33, 26, 25, 18, 11\}$ and we apply the map s_α as we can see in Figure 4.5. Finally, we put all polyominoes together to give rise $\mathbb{N} \setminus \Gamma$ as shown in Figure 4.6.

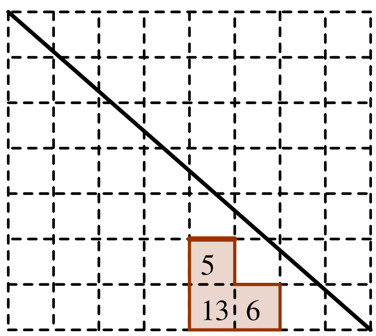

FIGURE 4.3. The set SG (shaded).

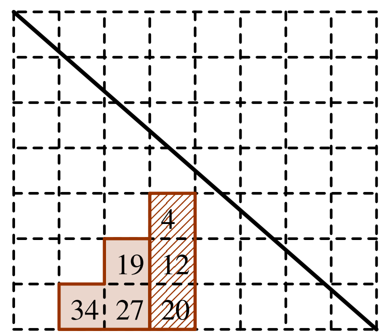

FIGURE 4.4. The sets $s_\beta(\mathsf{SG})$ (shaded) and SSG (striped).

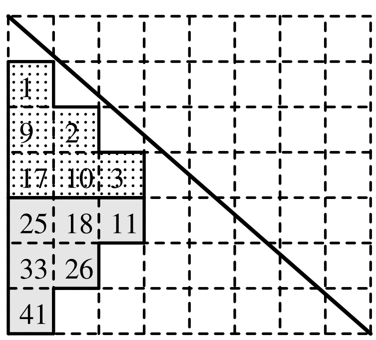

FIGURE 4.5. The set $\mathscr{T}_r \cup s_\alpha(\mathscr{T}_r)$.

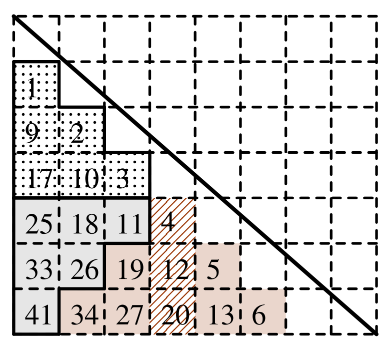

FIGURE 4.6. Lattice representation of the gap set $\mathbb{N} \setminus \Gamma$.

Let us present formulas for the cardinal of the sets of supersymmetric and self-symmetric gaps.

Proposition 4.7. *Let $\Gamma = \langle \alpha, \beta\rangle$ be a numerical semigroup. Then*

$$|\mathsf{SSG}| = \begin{cases} 0 & \text{if } \alpha, \beta \text{ are odd} \\ (\beta - 1)/2 & \text{if } \alpha \text{ even} \\ (\alpha - 1)/2 & \text{if } \beta \text{ even} \end{cases}$$

Moreover, if α is even,

$$|\mathsf{SG}| = \begin{cases} \displaystyle\sum_{j=1}^{\lfloor \frac{\alpha}{2} \rfloor - 1} \left\lfloor \frac{j\beta}{\alpha} \right\rfloor & \text{if } \mathsf{SG} = \mathscr{T}_u \\ \displaystyle\sum_{j=h}^{\alpha-1} \left(\left\lfloor \frac{j\beta}{\alpha} \right\rfloor - \left\lfloor \frac{\beta}{2} \right\rfloor \right) & \text{if } \mathsf{SG} = \mathscr{T}_r \end{cases}$$

where $h = \left\lfloor \frac{\alpha}{2} \right\rfloor + 1$ if α is even, and $h = \left\lfloor \frac{\alpha}{2} \right\rfloor$ if α is odd.

Proof. The formula for $|\mathsf{SSG}|$ is a direct consequence of Theorem 3.4. So let us prove the formula for $|\mathsf{SG}|$. We start by showing that $(a,b) \in RB(\mathscr{T}_u)$ resp. $(a,b) \in RB(\mathscr{T}_r)$ if it is of the form $(\lfloor j\beta/\alpha \rfloor, \alpha - j)$ with $j = 1, \ldots, \lfloor \alpha/2 \rfloor - 1$, resp. $j = h, \ldots, \alpha - 1$, where where $h = \lfloor \alpha/2 \rfloor + 1$ if α is even, and $h = \lfloor \alpha/2 \rfloor$ if α is odd. Obviously, $(\lfloor j\beta/\alpha \rfloor, \alpha - j)$ lies always on the right-hand sided border, since

$$\alpha\beta - \lfloor j\beta/\alpha \rfloor \alpha - (\alpha - j)\beta \geq 0 \quad \text{and} \quad \alpha\beta - (\lfloor j\beta/\alpha \rfloor + 1)\alpha - (\alpha - j)\beta \leq 0.$$

Now, observe that by definition the points on $RB(\mathscr{T}_u)$ have second coordinate varying from $\alpha - 1$ to $\alpha - \lfloor \alpha/2 \rfloor + 1$. For the points in $RB(\mathscr{T}_r)$ we need to distinguish two cases: if α is even, then the points with second coordinate $\alpha - \alpha/2$ are self-symmetric gaps so they do not belong to $\mathscr{T}_r$ and we need to start the summation running from $\lfloor \alpha/2 \rfloor + 1$ on. If α is odd, then there are no self-symmetric gaps of the previous form. The unique self-symmetric gaps may be those with coordinates $(\beta/2, \lfloor \alpha/2 \rfloor)$, but if one of them is actually a border point, then it adds zero in the summation. □

4.2. **Fundamental gaps vs supersymmetric gaps and self-symmetric gaps.** The fundamental gaps for semigroups of the form $\Gamma = \langle \alpha, \beta \rangle$ are explicitly described by Rosales in [10, Theorem 9]. As part of the proof, he characterized the elements $x \in \mathbb{N} \setminus \Gamma$ such that $2x \in \Gamma$. From this characterization we are able to prove the following.

Proposition 4.8. *Let $\Gamma = \langle \alpha, \beta \rangle$, and let $x \in \mathbb{N} \setminus \Gamma$ be a gap of Γ. Then the following are equivalent:*

(1) $2x \in \Gamma$;
(2) $x = \alpha\beta - a\alpha - b\beta$ with $1 \leq a \leq \beta/2$ and $1 \leq b \leq \alpha/2$;
(3) $W(x) \geq 0$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is [10, Proposition 4], and (2) $\Leftrightarrow$ (3) is a straightforward computation from the formula given in Proposition 3.3. □

In particular, nonnegative Wilf number is a necessary condition for a gap to be a fundamental gap.

Corollary 4.9. *Let $\Gamma = \langle \alpha, \beta \rangle$, and let $x \in \mathbb{N} \setminus \Gamma$ be a gap of Γ. If $x \in \mathscr{FG}(\Gamma)$, then $W(x) \geq 0$.*

Notice that the converse is not true: consider the semigroup $\Gamma = \langle 8, 13 \rangle$ and take the gap 25, then $W(25) = 9 > 0$ but $25 \notin \mathscr{FG}(\Gamma)$.

We recall that a subset X of the set of nonnegative integers H–determines a numerical semigroup Γ if Γ is the maximal numerical semigroup with respect to set inclusion such that $X \subset \mathbb{N} \setminus \Gamma$.

Under this description of Γ, the set of fundamental gaps is the smallest subset H–determining Γ. Moreover, Rosales et al. [11] proved the following important result about minimality of the fundamental gaps with respect the H–determinacy.

Proposition 4.10. [11, Corollary 7] *Let Γ be a numerical semigroup and let be $X \subset \mathbb{N}\setminus\Gamma$. The set X H–determines Γ if and only if $\mathscr{FG}(\Gamma) \subset X$.*

On the other hand, we have proven in Theorem 4.5 that $\mathsf{SG}\cup\mathsf{SSG}$ completely determines Γ. In this way, it is natural to compare $\mathsf{SG}\cup\mathsf{SSG}$ with $\mathscr{FG}(\Gamma)$. However, this comparison is not set-theoretically possible since we do not have inclusion relations, i.e. $\mathsf{SG}\cap\mathscr{FG}(\Gamma)=\emptyset$ and $\mathsf{SSG}\subset\{x\in\mathbb{N}\setminus\Gamma:\ 2x\in\Gamma\}$ but in general $\mathsf{SSG}\nsubseteq\mathscr{FG}(\Gamma)$ as Example 4.13 shows; this example also shows that if $\mathsf{SG}=\mathscr{T}_u$ resp. $\mathscr{T}_r$ then $s_\alpha(\mathsf{SG})$ resp. $s_\beta(\mathsf{SG})$ does not need to be contained in $\mathscr{FG}(\Gamma)$.

This means that, in general, the set $\mathsf{SG}\cup\mathsf{SSG}$ does not H–determines Γ, but it determines Γ in the sense that Γ can be recovered from $\mathsf{SG}\cup\mathsf{SSG}$. Moreover, the polyomino game cannot be recovered from the set of fundamental gaps. Then, the most we can do is to compare the cardinality of both sets:

Proposition 4.11. *Let $\Gamma=\langle\alpha,\beta\rangle$ be a numerical semigroup with $\alpha>2$, then*

$$|\mathsf{SG}\cup\mathsf{SSG}|\leq|\mathscr{FG}(\Gamma)|.$$

Proof. We will distinguish three cases depending on the parity of the semigroup generators.

Case (A): If α and β are both of them odd numbers, then $\mathsf{SSG}=\emptyset$ and

$$\min\{|\tau_u|,|\tau_r|\}\leq\frac{1}{8}(\alpha-1)(\beta-1). \tag{4.1}$$

In order to prove Equation (4.1) we observe that

$$|\mathbb{N}\setminus\Gamma|=\sum_{j=1}^{\alpha-1}\left\lfloor\frac{j\beta}{\alpha}\right\rfloor=\frac{1}{2}(\alpha-1)(\beta-1)=2\left\lfloor\frac{\alpha}{2}\right\rfloor\left\lfloor\frac{\beta}{2}\right\rfloor=2\cdot|\{x\in\mathbb{N}\setminus\Gamma:2x\in\Gamma\}|.$$

Therefore, $|\tau_u|+|\tau_r|=|\mathbb{N}\setminus\Gamma|-\frac{1}{4}(\alpha-1)(\beta-1)=\frac{1}{4}(\alpha-1)(\beta-1)$, cf. Proposition 4.7. This give us directly Equation (4.1). Now, in view of [10, Corollary 11] it is enough to show that

$$\frac{1}{4}(\alpha-1)(\beta-1)-\left\lceil\frac{\alpha-3}{6}\right\rceil\left\lceil\frac{\beta-3}{6}\right\rceil\geq\frac{1}{8}(\alpha-1)(\beta-1),$$

which is equivalent to the inequality

$$\frac{1}{8}(\alpha-1)(\beta-1)\geq\left\lceil\frac{\alpha-3}{6}\right\rceil\left\lceil\frac{\beta-3}{6}\right\rceil.$$

This is true: since

$$\left\lceil\frac{\alpha-3}{6}\right\rceil\left\lceil\frac{\beta-3}{6}\right\rceil=\left(\left\lfloor\frac{\alpha-3}{6}\right\rfloor+1\right)\left(\left\lfloor\frac{\beta-3}{6}\right\rfloor+1\right)\leq\frac{1}{36}(\alpha\beta+3\alpha+3\beta+9),$$

we just need to realize that

$$\frac{1}{36}(\alpha\beta+3\alpha+3\beta+9)\leq\frac{1}{8}(\alpha\beta-\alpha-\beta+1),$$

which leads to the inequality

$$7\alpha\beta-15\alpha-15\beta-9\geq 0.$$

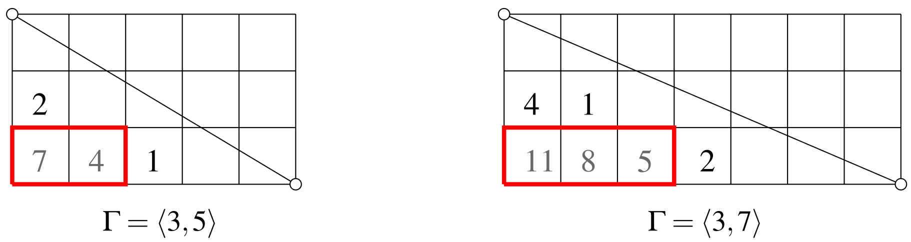

FIGURE 4.7. Special cases in (A) of Proposition 4.11.

This holds for $\alpha = 3$ and $\beta \geq 11$ odd as well as for any $\alpha \geq 5$ odd and $\beta > \alpha$ odd. The cases $\alpha = 3, \beta = 5$ and $\alpha = 3, \beta = 7$ must be treated separately, see Figure 4.7. In the first case, an easy computation shows that $|\mathscr{FG}(\Gamma)| = 2$ and $|\tau_u| = 1$, $|\tau_r| = 1$, and the result follows; in the second case, we have that $|\mathscr{FG}(\Gamma)| = 3$, and $|\tau_u| = 2$, $|\tau_r| = 1$, so the result remains also true.
Case (B): If $\alpha > 2$ is even and $\beta > \alpha$ is odd, then $|\mathsf{SSG}| = \left\lfloor \frac{\beta}{2} \right\rfloor$ and also

$$|\tau_u| + |\tau_r| = \frac{1}{4}(\alpha - 1)(\beta - 1).$$

By reasoning as in Case (A), it suffices to prove that

$$\frac{1}{4}\alpha(\beta - 1) - \left\lceil \frac{\alpha - 3}{6} \right\rceil \left\lceil \frac{\beta - 3}{6} \right\rceil \geq \frac{1}{8}(\alpha - 1)(\beta - 1) + \frac{1}{2}\left\lfloor \frac{\beta}{2} \right\rfloor,$$

again by Proposition 4.7 and [10, Corollary 11]. But this leads us to Case (A) since

$$\frac{1}{4}\alpha(\beta - 1) - \frac{1}{8}(\alpha - 1)(\beta - 1) = \frac{1}{8}(\alpha - 1)(\beta - 1) + \frac{\beta - 1}{4} \quad \text{and} \quad \frac{\beta - 1}{4} = \frac{1}{2}\left\lfloor \frac{\beta}{2} \right\rfloor.$$

Case (C): If $\alpha \geq 3$ is odd and $\beta > \alpha$ is even, we may repeat *mutatis mutandis* the argument in Case (B), and the result follows.

□

Remark 4.12. Observe that for $\alpha = 2, \beta = 3$ the statement of Proposition 4.11 holds; but this is no longer true for $\alpha = 2$ and any $\beta \geq 3$ odd, since in that case $|\tau_u| = |\tau_r| = 0$ and $|\mathscr{FG}(\Gamma)| = \frac{\beta - 1}{2} - \left\lceil \frac{\beta - 3}{6} \right\rceil > 0$.

Finally, let us show in an example how all the important sets presented in this paper look like.

Example 4.13. Consider the numerical semigroup $\Gamma = \langle 8, 13\rangle$. In this case $\mathsf{SG} = \mathscr{T}_u$ as we can see in Figure 4.8. This figure is labelled in the following manner: as usual, every lattice cell represents the gap of Γ given by (a, b), where these are the coordinates of the upper-right corner of the cell. Every cell is endowed with two numbers: the one lying on the bottom of the cell is just the corresponding gap, while the number on the top of the cell is the Wilf number of the gap.

The figure also presents a filling code: we have shadowed the set of fundamental gaps and dotted the set of self-symmetric gaps. This makes it clear that self-symmetric gaps are not fully contained in the set of fundamental gaps neither the images by s_α, s_β of the triangles $\mathscr{T}_u, \mathscr{T}_r$. The red rectangle contains the gaps x such that $2x \in \Gamma$, cf. Proposition 4.8. The polyominoes corresponding to $\mathscr{T}_r \cup s_\alpha(\mathscr{T}_r)$ and $\mathscr{T}_u \cup s_\beta(\mathscr{T}_u)$ are also distinguishable.

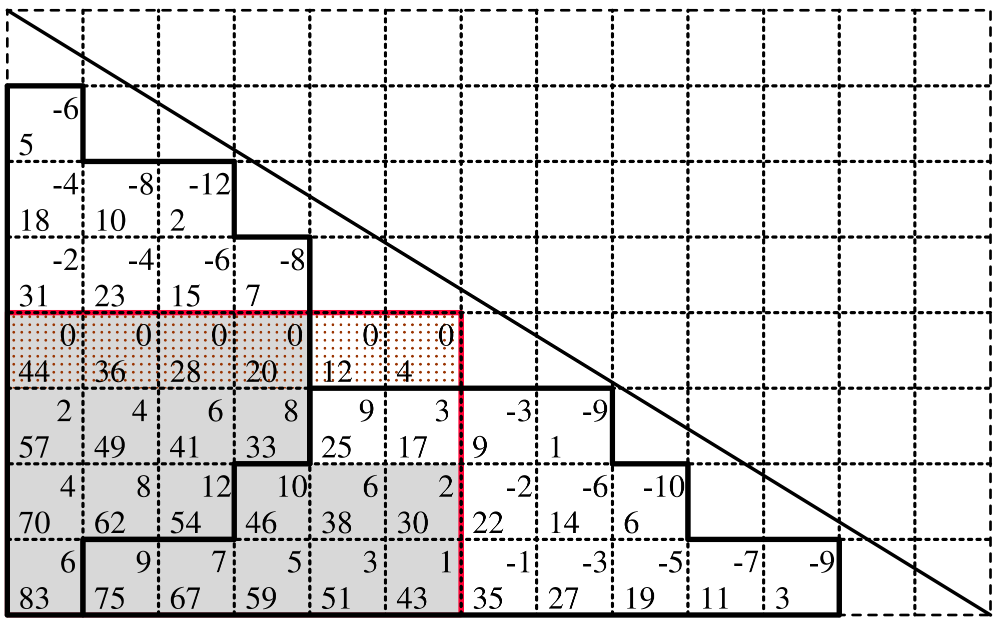

FIGURE 4.8. Polyomino game for the semigroup $\langle 8, 13\rangle$.

4.3. **Remarks on the concepts of supersymmetric and self-symmetric gaps.** We would like to finish the paper with a brief discussion about the possible extension of the concepts of supersymmetric and self-symmetric gaps to the case of a numerical semigroup Γ with arbitrary embedding dimension.

We remark first that our definition of supersymmetry does not coincide with the one given by Fröberg, Gottlieb and Häggkvist in [4]. Their notion lies on a lattice representation of each of the Apéry sets with respect to all minimal generators of the semigroup in such a way that supersymmetry in the sense of [4, Lemma 15] means symmetry plus uniqueness of the concerned lattice representation; recall that the Apéry set of Γ with respect to a nonzero element $s \in \Gamma$ is defined to be $\{w \in \Gamma : w - s \notin \Gamma\}$. On the other hand, our notion is defined from the lattice representation of the set of gaps of the semigroup together with its properties with respect to the Wilf numbers.

The extension of the notions of self-symmetric and supersymmetric gaps to higher embedding dimensions is trickier. The above mentioned example $\Gamma = \langle 4, 6, 13\rangle$ shows us that we cannot expect a definition through the sign of the Wilf number of the concerned gap, since for this example all gaps have positive Wilf number; this means, the sign here is not the important issue. In addition, the example $\Gamma = \langle 10, 14, 27\rangle$ after Theorem 3.4 shows that if we want to extend the concept of self-symmetric gap we cannot only focus on Wilf number zero; it seems that the notion of supersymmetry is deeper.

Moreover, observe that the symmetries under consideration imply the following property:

Proposition 4.14. *Let* $\Gamma = \langle \alpha, \beta\rangle$ *be a numerical semigroup. With the previous notation,*

(1) If $\alpha\beta - a\alpha - b\beta = g \in \mathscr{T}_u$*, then* $c(\Delta_{[0,g]}) = c(\Delta_{[0,s_\alpha(g)]}) = c(\Gamma) - a\alpha$.
(2) If $\alpha\beta - a\alpha - b\beta = g \in \mathscr{T}_r$*, then* $c(\Delta_{[0,g]}) = c(\Delta_{[0,s_\beta(g)]}) = c(\Gamma) - b\beta$.

Proof. We will prove only (1), and (2) follows *mutatis mutandis*.

Consider the gap $\alpha\beta - a\alpha - b\beta = g \in \mathscr{T}_u$. Let $[h_0, h_1]$ be the minimal set of generators of $\mathrm{Syz}(\Delta_{[0,g]})$. Then by Lemma 4.1

$$\min\{h_0, h_1\} = \alpha\beta - b\beta,$$

and Theorem 2.3 implies that $c(\Delta_{[0,g]}) = c(\Gamma) - a\alpha$.

Consider now $s_\alpha(g) = \alpha\beta - a\alpha - (\alpha - b)\beta$, and denote by

$$[h'_0 = \alpha\beta - a\alpha, h'_1 = \alpha\beta - (\alpha - b)\beta]$$

the minimal set of generators of $\mathrm{Syz}(\Delta_{[0,s_\alpha(g)]})$. In order to finish it would be enough to prove that $\min\{h'_0, h'_1\} = h'_1$; but this is immediate, as

$$h'_0 - h'_1 = \alpha\beta - a\alpha - \alpha\beta + (\alpha - b)\beta = \alpha\beta - a\alpha - b\beta = g > 0.$$

□

Proposition 4.14 the following definition for a numerical semigroup of arbitrary embedding dimension.

Definition 4.15. Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ be a numerical semigroup minimally generated by $x_1, \ldots, x_n$. We define on the set of gaps $G := \mathbb{N} \setminus \Gamma$ the relation

$$g_1 \sim_c g_2 \Longleftrightarrow c(g_1) = c(g_2) \text{ for any } g_1, g_2 \in G.$$

This is in fact an equivalence relation which thus provides a partition of the set gaps into equivalence classes. This partition will be called the *gap conductor partition* of G. We say that two gaps g_1, g_2 are *candidates to be symmetric* if $g_1 \sim_c g_2$. In addition, we say that two gaps g_1, g_2 are *symmetric* if $g_1 \sim_c g_2$ and $|W(g_1)| = |W(g_2)|$.

Remark 4.16. Observe that we are giving a definition of two gaps to be symmetric. This definition has no relation and has not to be mixed with the symmetry properties of the semigroup.

In the special case $\Gamma = \langle \alpha, \beta \rangle$, two gaps g_1, g_2 are symmetric if and only if $s_\alpha(g_1) = g_2$ or $s_\beta(g_1) = g_2$. Moreover, there is no three different symmetric gaps; i.e. either g_1 is symmetric to a unique $g_2 \neq g_1$ or g_1 is self-symmetric and then it is its own symmetric point. Therefore, Definition 4.15 allows us to extent the properties of the lattice symmetries to purely algebraic properties of the gaps. This discussion leads to pose the following closing questions:

Question 4.17. Given a symmetric numerical semigroup $\Gamma = \langle x_1, \ldots, x_n \rangle$, we ask:

(1) For $n = 2$, can supersymmetric and self-symmetric gaps be characterized from Definition 4.15 without the use of the lattice representation?
(2) For $n > 2$, does there exist an extension of the self-symmetric and supersymmetric gaps i.e. does there exist a subset of the set of gaps such that the symmetry property defined in Definition 4.15 allows to recover the whole semigroup from this set?

(3) If an extension of the concepts of self-symmetric and supersymmetric gaps to embedding dimension greater than 2 is possible, does there exist a lattice representation in $\mathbb{Z}^n$ of the set of gaps such that —in analogy with the case $n = 2$— it can be made up from the sets of self-symmetric and supersymmetric gaps together with some affine transformation of them?

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