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A note on b_f -spaces and on the distribution of the functor of the Dieudonné completion

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Abstract: A subset B of a space X is said to be bounded (in X) if the restriction to B of every real-valued continuous function on X is bounded. A real-valued function on X is called b_f -continuous if its restriction to each bounded subset of X has a continuous extension to the whole space X . b_f -spaces are spaces such that b_f -continuous functions are continuous. We take advantage to the exponential map in the realm of b_f -spaces in order to study b_f -extensions of b_f -continuous functions. This allows us to improve several results concerning the distribution of the functor of the Dieudonné completion. We also prove that a relative version of the classical Glicksberg's theorem characterizing the product of two pseudocompact spaces is valid for k_r -spaces. In the last section we show that b_f -hemibounded groups are Moscow spaces and, consequently, they are strong- PT -groups.

Keywords: b_f -space, b_f -continuous function, exponential map, Dieudonné completion, k_r -space, Moscow space, strong- PT -group

MSC: 54C10, 54C20, 54C35, 54C45, 54D60, 54G99, 54H11

1 Introduction

Glicksberg in [16] showed an elegant theorem on the distribution of the functor β of the Stone-Čech compactification reproved later by another method in [13] by Z. Frolík: For infinite spaces X and Y , the equality $\beta(X \times Y) = \beta(X) \times \beta(Y)$ holds if and only if the product space $X \times Y$ is pseudocompact. The interest and importance of the Glicksberg-Frolík theorem lies in the fact that the distribution of the functor β is characterized by means of a topological property of the product space $X \times Y$.

The attempt of obtaining a similar result for the functors ν and μ of the Hewitt realcompactification and of the Dieudonné completion, respectively, have been unsuccessful. Usually the assumptions on X and Y are *asymmetric* (see, for example, the classic papers [9, 22, 23] by Comfort and Ohta). Moreover, Husěk proved in [19] that there is no topological property characterizing the equality $\nu(X \times Y) = \nu(X) \times \nu(Y)$ (and, consequently, the equality $\mu(X \times Y) = \mu(X) \times \mu(Y)$). Thus, the problem of establishing implications, one of whose terms is roughly of the form $\nu(X \times Y) = \nu(X) \times \nu(Y)$ (respectively, $\mu(X \times Y) = \mu(X) \times \mu(Y)$), has attracted very considerable attention.

It is worth noting that the equality $\mu(X \times Y) = \mu(X) \times \mu(Y)$ has a strong relationship with a Pestov-Tkachenko question ([24, 34]): Let G be a topological group, and μG the Dieudonné completion of the space G . Can the operations in G be extended to $\mu(G)$ in such a way that $\mu(G)$ becomes a topological group containing G as a topological subgroup?

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Arhangel'skii answered this question in the negative in [2] and he found a relationship between a positive answer to this problem and the distribution of the functor μ (see also [3, 4, 29, 30]). Arhangel'skii called PT -groups the groups which provide a positive answer to the previous question.

The aim of this paper is to study the relationship between the exponential map and b_f -continuous extensions of b_f -continuous functions to $\mu(X) \times Y$. The results applies to the equality $\mu(X \times Y) = \mu(X) \times \mu(Y)$. We generalize several results on the theory by Arhangel'skii [3], Comfort [9], Noble [21] and Ohta [23]. We also obtain a relative version of the classical Glicksberg's theorem characterizing the product of two pseudocompact spaces in the realm of k_r -spaces. In the last section we present several results on PT -groups.

Throughout, all spaces are by default Tychonoff and all topological groups are Hausdorff. A subset B of a space X is said to be bounded (in X) if each real-valued continuous function on X is bounded on B . A space is pseudocompact if it is bounded in itself. Locally pseudocompact spaces are spaces X such that each point of X has a pseudocompact (equivalently, a bounded) neighborhood. We will denote by b the family of all bounded subsets of a space X . A function f from a space X into a space Y is said to be b_f -continuous if the restriction of f to each member of b can be extended to a continuous function on the whole space X . A space X is called a b_f -space if every real-valued b_f -continuous function on X (equivalently, if every b_f -continuous function from X into a Tychonoff space Y) is continuous. Notice that locally pseudocompact spaces and k_r -spaces (spaces X where a real-valued function is continuous when its restriction to each compact subset of X is continuous) are examples of b_f -spaces. Thus, k -spaces (spaces X where a subset A is closed if and only if $A \cap K$ is closed in K for every compact subset K of X), first countable (so metrizable) spaces are b_f -spaces. It is worth noting that a Σ -product of an arbitrary product of locally pseudocompact groups and an arbitrary product of locally pseudocompact groups are also examples of b_f -spaces which are not (in general) k_r -spaces (for details, see [25, Corollary 4.1, Theorem 4.3]). The image of a b_f -space under an onto quotient map is a b_f -space [25, Lemma 4.2]. Thus, quotients of b_f -groups are b_f -spaces [25, Lemma 4.2]. More examples are given in the last section.

Our terminology and notation are standard. For instance, \mathbb{N} means the set of natural numbers, \mathbb{R} the real numbers and $f|_A$ the restriction of a function f to a subset A . We say that a space X is topologically complete (o Dieudonné complete) if X is homeomorphic to a closed subspace of a product of metrizable spaces. It is a well-known fact that for every space X there exists a unique topologically complete space $\mu(X)$, up to homeomorphisms which leave X pointwise fixed, in which X is dense and every continuous function f from X into a topologically complete space Z can be extended to a continuous function on $\mu(X)$. This space is called the Dieudonné completion of X . It is clear that a space X is topologically complete if $X = \mu(X)$. It is worth mentioning that, by an outstanding result of Shirota [31], $\nu X = \mu X$ if the cardinality of every closed discrete subspace of $\mu(X)$ is a non-measurable cardinal number. Thus, our results apply to the Hewitt realcompactification $\nu(X)$ of X under the assumption that the cardinal of X is non-measurable. For notions and terminology which are not explicitly defined here, the reader might consult [5, 12].

2 Preliminaries

Let $F(M, Z)$ denote the set of all real-valued functions from a set M into a set Z . Our basic tool will be the so-called *exponential map*, that is, the natural one-to-one correspondence γ between the set $F(X \times Y, Z)$ and $F(X, F(Y, Z))$ which is defined as

$$\gamma(f)(x)(y) = f(x, y), \quad f \in F(X \times Y, Z).$$

The restriction of this map to subspaces will also denoted by γ . Given two spaces X and Z , $b_f C(X, Z)$ stands for the set of all b_f -continuous functions from X into Z . We write $b_f C(X)$ when the space Z is the space \mathbb{R} of the real numbers endowed with its usual topology. The exponential map has been used by several authors for studying the relationship $\mu(X \times Y) = \mu(X) \times \mu(Y)$ in the realm of k -spaces (see, among others, [9, 19]). We will follow this way in a more general setting, To do this, we take advantage of the following theorem:

Theorem 2.1. [25, Theorem 3.2] *Let Y be a b_f -space. For each space X , the following assertions are equivalent:*

- (i) $\gamma(b_f C(X \times Y, Z)) = b_f C(X, C_b(Y, Z))$ for each topologically complete space Z .
- (ii) $\text{cl}_{\beta(X \times Y)}(A \times B) = \text{cl}_{\beta(X)}(A) \times \text{cl}_{\beta(Y)}(B)$ whenever A and B are bounded subsets of X and Y , respectively.

Note that we can consider condition (ii) as a relative version of the Glicksberg's theorem. So, taking as a stating point this fact, we say that a pair (X, Y) enjoys property (Gl) (in short, (X, Y) is a (Gl) pair) if $X \times Y$ satisfies condition (ii) of Theorem 2.1. By a result of Hernández, Sanchis and Tkachenko [18], the condition (ii) is satisfied if either X or Y is a topological group.

It is a well-known fact that the closure of a bounded subset of a topologically complete space is compact. Therefore, if either X or Y are topologically complete, then (X, Y) is a (Gl) pair. Actually, we have a more general result. We will denote by \mathfrak{B} the Frol'ik's class of all pseudocompact spaces X such that $X \times Y$ is pseudocompact for every pseudocompact space Y .

Theorem 2.2. *Let X, Y two spaces. If $A \subseteq X$ belongs to \mathfrak{B} , then $\text{cl}_{\beta(X \times Y)}(A \times B) = \text{cl}_{\beta(X)}(A) \times \text{cl}_{\beta(Y)}(B)$ for every bounded subset B of Y .*

Proof. By Corollary 6 in [7], $A \times B$ is bounded in $A \times Y$ and, a posteriori, it is bounded in $X \times Y$. Apply now Theorem 2.8 of [15]. \square

Corollary 2.3. *A space X belongs to the class \mathfrak{B} if and only if (X, Y) is a (Gl) pair for every space Y .*

In the next section we prove that, if X is a k_r -space, then (X, Y) enjoys property (Gl) for each space Y . It seems interesting to remark that the equality $\text{cl}_{\beta(X \times Y)}(A \times B) = \text{cl}_{\beta(X)}(A) \times \text{cl}_{\beta(Y)}(B)$ implies that $A \times B$ is bounded in $X \times Y$. The question of whether the converse is true remains open (see [26, Chapter 4]).

Throughout what follows, we shall freely use without explicit mention the previous facts. More detailed information about the equality $\text{cl}_{\beta(X \times Y)}(A \times B) = \text{cl}_{\beta(X)}(A) \times \text{cl}_{\beta(Y)}(B)$ for bounded subset is available in [14], [27] and in the survey [26, Chapter 4].

3 The exponential map and b_f -continuous extensions

Given two spaces M and Z , $C(M, Z)$ (respectively $C(M)$) stands for the family of all continuous functions from M to Z (respectively, from M to the reals). Let $C_b(M, Z)$ denote the space $C(M, Z)$ equipped with the topology of the uniform convergence on bounded subsets. The first result provides an extension theorem for b_f -continuous functions.

Theorem 3.1. *Let X, Y be two b_f -spaces. If (X, Y) is a (Gl) pair and Z is a topologically complete space, then the following hold:*

- (i) *Every b_f -continuous function from $X \times Y$ into Z has a b_f -continuous extension to $\mu(X) \times Y$.*
- (ii) *Every b_f -continuous function from $X \times Y$ into Z has a b_f -continuous extension to $X \times \mu(Y)$.*

Proof. (i) Let f be a b_f -continuous function on $X \times Y$. By Theorem 2.1, $\gamma(f)$ belongs to $b_f C(X, C_b(Y, Z))$. Being X a b_f -space, $\gamma(f)$ is a continuous function. Since Y is a b_f -space, the function space $C_b(Y, Z)$ is Dieudonné complete (see [25, Lemma 3.1]) so that there exists a continuous extension, say g , of $\gamma(f)$ to $\mu(X)$.

Consider now the function $\gamma^{-1}(g)$ on $\mu(X) \times Y$. Since $(\mu(X), Y)$ is a (Gl) pair, by Theorem 2.1 again, we have

$$\gamma(b_f C(\mu(X) \times Y, Z)) = b_f C(\mu(X), C_b(Y, Z))$$

which implies that $\gamma^{-1}(g)$ belongs to $b_f C(\mu(X) \times Y, Z)$. Note that the restriction to $X \times Y$ of $\gamma^{-1}(g)$ coincides with f . This completes the proof of (i). The claim (ii) follows in a similar way. \square

For a b_f -group G we understand a topological group which is a b_f -space.

Corollary 3.2. *Let G be a b_f -group and let Y be a b_f -space. If Z is a topologically complete space, then the following hold:*

- (i) *Every b_f -continuous function from $G \times Y$ into Z has a b_f -continuous extension to $\mu(G) \times Y$.*
- (ii) *Every b_f -continuous function from $G \times Y$ into Z has a b_f -continuous extension to $G \times \mu(Y)$.*

Observe that the previous results are valid when Z is a metric space (in particular the real line \mathbb{R}). In fact, Dickinson showed that paracompact spaces are topologically complete [11] and metric spaces are paracompact [33]. As in the previous corollary, being every continuous function b_f -continuous, Theorem 3.1 enables us to obtain extension theorems for continuous functions when either the space $\mu(G) \times X$ or the space $G \times \mu(X)$ is a b_f -space. Moreover, our approach allows us to generalize several results obtained in the literature. As a flavor of example, we present several applications.

Corollary 3.3. *Let X be a b_f -space. If $\mu(X) \times Y$ is a b_f -space and (X, Y) is a (Gl)-pair, then each continuous function on $X \times Y$ into a topologically complete space Z has a continuous extension to $\mu(X) \times Y$.*

We need the following result.

Theorem 3.4. [9, Corollary 2.2] *If X is a locally compact topological complete space, then $\mu(X \times Y) = X \times \mu(Y)$ for each space Y .*

Theorem 3.5. *If μX is locally compact and Y is a b_f -space such that (X, Y) is a (Gl)-pair, then the equality $\mu(X \times Y) = \mu(X) \times \mu(Y)$ holds.*

Proof. Since $\mu(X)$ is locally compact, X is locally pseudocompact and, consequently, a b_f -space. Moreover, $\mu(X)$ is locally in the class \mathfrak{B} so that, by Corollary 6 in [7], the space $\mu(X) \times Y$ is a b_f -space. Then, if Z is a topologically complete space, Corollary 3.3 says us that every continuous function on $X \times Y$ into Z has a continuous extension to $\mu(X) \times Y$. Now, by means of Theorem 3.4, it is routine to prove that $\mu(X \times Y) = \mu(X) \times \mu(Y)$ holds. \square

It is well known that the Dieudonné completion of a locally pseudocompact group G is a locally compact group: in fact, it coincides with the Weil completion of G . Then we obtain

Corollary 3.6. *If G is a locally pseudocompact group and X is a b_f -space (in particular, a b_f -group), then the equality $\mu(G \times X) = \mu(G) \times \mu(X)$ holds.*

Comfort and Ross show that the product of two pseudocompact groups is a pseudocompact group [10]. Tkachenko [35] improves this result by proving that every pseudocompact group belongs to \mathfrak{B} . According to Glicksberg's theorem, Tkachenko's result is equivalent to the fact that $\beta(G \times X) = \beta(X) \times \beta(Y)$ when G is a pseudocompact group and X is a pseudocompact space. Thus, Corollary 3.6 generalizes both outcomes.

Obviously, Corollary 3.6 holds if *locally pseudocompact group* is replaced by *pseudocompact group*. Thus, as a consequence of the previous result, we have

Corollary 3.7. [3, Theorem 5.16] *If G is a pseudocompact group and H is a k -group, then the equality $\mu(G \times H) = \mu(G) \times \mu(H)$ holds.*

Recall that a subset A is bounded in X if every countable locally finite family of pairwise disjoint open sets meeting A is finite. Let p be a free ultrafilter on \mathbb{N} (in symbols, $p \in \omega^*$). A subset A of a space X is said to be p -bounded (in X) if for every sequence $\{U_n\}_{n \in \mathbb{N}}$ of pairwise disjoint open subsets of X meeting A , then $\{n \in \mathbb{N} : A \cap U_n \neq \emptyset\} \in p$. Obviously, p -bounded subsets are bounded. It is apparent that a compact subset of X is p -bounded for every $p \in \omega^*$. For the origins and applications of these concepts in bounded sets theory see [26, Chpater 4].

It was showed by Noble [20] that every k_r -pseudocompact space belongs to the Frolík's class \mathfrak{B} . This result was generalizes in [7] where the authors show that if X is a k_r -space, then, for each space Y , $A \times B$ is bounded in $X \times Y$ for each bounded subset A of X and each bounded subset B of Y . By Corollary 4 and Corollary 6 of [7], each locally pseudocompact k_r -space is locally in the class \mathfrak{B} . The following theorem improves these results.

Theorem 3.8. *If X is a k_r -space, then (X, Y) is a (Gl) pair for every space Y .*

Proof. Let A be a bounded subset of X . Consider a sequence $\{U_n\}_{n \in \mathbb{N}}$ of pairwise disjoint open sets meeting A . We prove that there is a compact subset of X meeting infinitely many elements of $\{U_n\}_{n \in \mathbb{N}}$. For this, suppose, contrary we claim, that every compact subset of X meets finitely many elements of $\{U_n\}_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$, choose a point $x_n \in U_n$.

Next, for each $n \in \mathbb{N}$, let f_n be a continuous function on X with $f(x_n) = n$ and $f_n(X \setminus U_n) = 0$. Define a function f on X as follows:

$$f(x) = \sum_{n \in \mathbb{N}} f_n(x), \quad x \in X.$$

Notice that, being the elements of the sequence $\{U_n\}_{n \in \mathbb{N}}$ pairwise disjoint, f is well defined. Moreover, by our assumption, f is k_r -continuous. But $f|_A$ is unbounded which contradicts that A is bounded in X . Thus, there exists a compact subset $K \subseteq X$ and a subsequence $\{n_t\}_{t \in \mathbb{N}}$ such that $K \cap U_{n_t} \neq \emptyset$ for every $t \in \mathbb{N}$. Then, since K is p -bounded for every $p \in \omega^*$, so also is A . The results now follows from [29, Corollary 13]. \square

In the proof of the previous result, the following is implicit.

Corollary 3.9. *Each point of a locally pseudocompact k_r -space has a neighborhood that is p -bounded for every $p \in \omega^*$.*

Theorem 3.8, Theorem 3.1 and Theorem 3.5 permit us to generalize several results of the literature. The next results improve Theorem 2.7 and Theorem 3.11 of [9] by Comfort and (1) \implies (2) of [23] by Ohta.

Corollary 3.10. *Assume that both $\mu(X) \times Y$ and $\mu(X) \times \mu(Y)$ are k_r -space. Then $\mu(X \times Y) = \mu(X) \times \mu(Y)$.*

Proof. Let Z be a topologically complete space. Being Y a k_r -space, (X, Y) is a (Gl) pair by Theorem 3.8. Now Theorem 3.1 says us that each continuous function on $X \times Y$ into Z can be continuously extended to $\mu(X) \times Y$. Let f be a continuous function from $\mu(X) \times Y$ into Z . Being $\mu(X)$ topologically complete, Theorem 3.1 applies in order to obtain that $\gamma(f) \in b_f C(Y, C_b(\mu(X), Z))$. Since $\mu(X)$ and Y are k_r -spaces, an argument similar to the one used in the proof of Theorem 3.1 shows that f has a continuous extension to $\mu(X) \times \mu(Y)$. Thus, $\mu(X \times Y) = \mu(X) \times \mu(Y)$. \square

Corollary 3.11. *If $\mu(X)$ is locally compact and Y is a k_r -space, then $\mu(X \times Y) = \mu(X) \times \mu(Y)$.*

Corollary 3.12. [Compare [23, Theorem 2]] *If X is locally pseudocompact and Y is a k_r -space, then every continuous function f on $X \times Y$ into a topologically complete space Z has a continuous extension to $X \times \mu(Y)$.*

Proof. Being X locally pseudocompat, it suffices to prove that f has a continuous extension to $P \times Y$ for each pseudocompact subset P of X . By Teorema 3.8 and [7, Theorem 9], $P \times Y$ is a b_f -space. The result now follows from Theorem 3.8 and Theorem 3.1. \square

Remark 3.13. In Corollaries 3.10, 3.11 and 3.12 we can replaced k_r -space by *topological group*. In fact, as we have commented above, if G is a topological group, then (G, Y) enjoys the property (Gl) for every space Y .

Remark 3.14. For a family \mathfrak{a} of bounded subset of a space X , the definitions of \mathfrak{a}_f -space and \mathfrak{a}_f -continuous function are self-explanatory. Then, if \mathfrak{b} denotes the family of all subsets of a space X that belong to the Frolík's class \mathfrak{B} , a natural question is if each \mathfrak{b}_f -space satisfies Corollaries 3.10, 3.11 and 3.12.

Perhaps it is appropriate at this point to note that there exists pseudocompact spaces in \mathfrak{B} which are not p -pseudocompact (that is, p -bounded in itself) for any $p \in \omega^*$ ([28, Example 2.9]). Moreover, a b_f -space need not be pseudocompact: indeed, if $X \in \mathfrak{B}$, consider the disjoint union of countable many copies of X equipped with its natural topology.

However the answer to the previous question is affirmative. To see this, it suffices to use the techniques presented in this paper by applying Theorem 2.2 and the following result. It is possible to obtain it by simply mimicking the proof of Theorem 2.1. All the details are left to the interesting reader.

Theorem 3.15. *Let Y be a b_f -space. If $\mathfrak{b} \times \mathfrak{b}$ stands for the family*

$$\mathfrak{b} \times \mathfrak{b} = \{A \times B : A \subset X, B \subset Y \text{ with } A \in \mathfrak{B}\},$$

then, for each space X , the following holds:

(i) $\gamma((\mathfrak{b} \times \mathfrak{b})_f C(X \times Y, Z)) = b_f C(X, C_{\mathfrak{b}}(Y, Z))$ for each topologically complete space Z .

Moreover, if X is a b_f -space, we have

(ii) $\gamma((\mathfrak{b} \times \mathfrak{b})_f C(X \times Y, Z)) = C(Y, C_{\mathfrak{b}}(X, Z))$ for each topologically complete space Z .

4 A note on PT -groups

In this section we take up a result on PT -groups. For this, we turn now a brief discussion of b_f -hemibounded groups (that is, groups that are b_f -hemibounded spaces) and Moscow spaces.

Recall that a space X is called hemibounded if there exists a countable family $\{B_n\}_{n \in \mathbb{N}}$ of bounded subsets such that, for every bounded subset $B \subseteq X$, there is a B_n such that $B \subseteq B_n$. An interesting example of b_f -hemibounded group is the free (abelian) topological group $G(X)$ over a pseudocompact space X (see [17]). b_f -hemibounded spaces appear in a natural way in the theory of function spaces: $C_b(X)$ is a Fréchet space if, and only if, X is a b_f -hemibounded space (see [6]). The definition of a hemicompact space is self-explanatory.

A stimulant result on b_f -hemibounded spaces is

Theorem 4.1. [8] *If X is a b_f -hemibounded space, then $\mu(X)$ is a k_r -hemicompact space. Moreover, every compact subset of $\mu(X)$ is contained in the closure (in $\mu(X)$) of a bounded subset of X .*

The notion of a Moscow space was introduced by Arhangel'skii in [1]. A space X is called a Moscow space if $\text{cl}_X U$ is the union of G_δ -subsets of X for each open subset U of X . Each Moscow group G is a PT -group. Actually, G is a strong- PT -group, that is, every real-valued continuous function on G has a continuous extension to the G_δ -closure $\rho_\omega G$ of G in its Rajkov completion (see Theorem 1.5, Corollary 1.6 and Theorem 1.7 in [2]).

Our aim is to show that b_f -hemibounded groups are strong- PT groups. First we have the following general result.

Theorem 4.2. *Let G be a topological group. If $\mu(G \times G) = \mu(G) \times \mu(G)$, then G is a PT -group.*

Proof. Denote by \circ the operation on G and consider the function $\alpha: G \times G \rightarrow \mu(G)$ defined as

$$\alpha(g_1, g_2) = g_1 \circ g_2, \quad g_1, g_2 \in G.$$

Being $\mu(G)$ topologically complete, there is a continuous extension, say $\tilde{\alpha}$, of the function α to $\mu(G \times G)$. By our hypothesis, $\mu(G \times G) = \mu(G) \times \mu(G)$.

Next we prove that the operation $\tilde{\circ}$ defined on $\mu(G)$ as

$$\begin{aligned} \tilde{\circ}: \mu(G) \times \mu(G) &\rightarrow \mu(G) \\ (h_1, h_1) &\rightarrow \tilde{\alpha}(h_1, h_1) \end{aligned}$$

makes $\mu(G)$ a topological group containing G as a dense subgroup. It is apparent that $\tilde{\circ}$ is continuous. First we show that $(\mu(G), \tilde{\circ})$ satisfies the associative property. Define the continuous functions $f_1, f_2: \mu(G) \times \mu(G) \times \mu(G) \rightarrow \mu(G)$ by the rule

$$f_1(h_1, h_2, h_3) = \tilde{\alpha}(h_1, \tilde{\alpha}(h_2, h_3)) \text{ and } f_2(h_1, h_2, h_3) = \tilde{\alpha}(\tilde{\alpha}(h_1, h_2), h_3).$$

As G is a topological group, $f_1|_{G \times G \times G} = f_2|_{G \times G \times G}$. Since $G \times G \times G$ is dense in $\mu(G) \times \mu(G) \times \mu(G)$, we conclude that $f_1 = f_2$, which means exactly that the operation $\tilde{\circ}$ is associative.

Next we show that $(\mu(G), \tilde{\circ})$ has a neutral element. Let e be the neutral element of G . If $h \in \mu(G) \setminus G$, by density, there exists a net $\{g_\beta\}_{\beta \in I}$ in G converging to h . Since $\tilde{\alpha}$ is continuous, the net $\{g_\beta, e\}_{\beta \in I}$ converges to (h, e) . But, for each $\beta \in I$, we have $g_\beta = g_\beta \circ e = \tilde{\alpha}(g_\beta, e)$. Then,

$$h = \lim_{\beta \in I} g_\beta = \lim_{\beta \in I} \tilde{\alpha}(g_\beta, e) = \tilde{\alpha}(h, e) = h \tilde{\circ} e.$$

In a similar way we can prove that $h = e \tilde{\circ} h$. Thus, e is a neutral element of $(G, \tilde{\circ})$. Now, consider the function $\eta: G \rightarrow \mu(G)$ defined as $\eta(g) = g^{-1}$ for each $g \in G$ with g^{-1} the inverse of g . Let $\tilde{\eta}$ be the continuous extension of η to $\mu(G)$. Given $h \in \mu(G) \setminus G$ and a net $\{g_\beta\}_{\beta \in I}$ in G with $\lim_{\beta \in I} g_\beta = h$, let h^* the limit of $\{\tilde{\eta}(g_\beta)\}_{\beta \in I}$. We show that $e = h \tilde{\circ} h^*$. Indeed, we have

$$e = \lim_{\beta \in I} (g_\beta, g_\beta^{-1}) = \lim_{\beta \in I} \tilde{\alpha}(g_\beta, g_\beta^{-1}) = \tilde{\alpha}(h, h^*) = h \tilde{\circ} h^*.$$

An argument similar to the previous one shows that $e = h^* \tilde{\circ} h$. Thus, h^* is the inverse of h . This completes the proof. \square

Remark 4.3. Topological groups containing a dense abelian group are abelian. So, under the hypothesis of the previous theorem, if G is abelian, so is $\mu(G)$.

If G is a b_f -hemibounded group and X is a b_f -hemibounded space, then $\mu(G \times X) = \mu(G) \times \mu(X)$ ([17, Theorem 2.6]). Moreover, k_r -hemicompact spaces are (normal) k -spaces (see Lemma 5.1 of [32]).

Theorem 4.4. *b_f -hemibounded groups are Moscou spaces.*

Proof. Let G be a b_f -hemibounded group. By the previous theorem, $\mu(G)$ is a k -group (that is, a topological group which is a k -space). According to [3, Theorem 2.16], $\mu(G)$ is a Moscou space. Since dense subsets of Moscou spaces are Moscou ([3, Proposition 1.1]), G is Moscou. \square

Applying Theorem 1.7 of [2], we obtain

Corollary 4.5. *If G is a b_f -hemibounded group, then G is a strong-PT-group.*

As a consequence of the previous corollary, we have

Corollary 4.6. *The (abelian) free topological group $G(X)$ over a pseudocompact space is a strong-PT group.*

The following result is a consequence of Theorem 2.7 in [17] and Theorem 5.6 in [3].

Theorem 4.7. *The product of two b_f -hemibounded groups is a PT-group. In particular, if X is a pseudocompact space, $G(X) \times G$ is a PT-group for every b_f -hemibounded group G .*

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