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Conjugacy classes and union of cosets of normal subgroups

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Abstract
Let G be a finite group, N a normal subgroup of G and K a conjugacy class of G. We prove that if $K \cup K^{-1}$ is union of cosets of N, then N is soluble, K is a real-imaginary class, that is, every irreducible character of G takes real or purely imaginary values on K, and if, in addition, the elements of K are p-elements for some prime p, then N has normal p-complement. We also prove that if $K \cup K^{-1}$ is a single coset of N, then $\langle K \rangle$ has a normal 2-complement.

Keywords Conjugacy classes · Cosets of normal subgroups · Finite simple groups · Characters

Mathematics Subject Classification 20C15 · 20D20 · 20E45

1 Introduction

Let G be a finite group and N a proper normal subgroup of G. The set $G - N$ is union of conjugacy classes of G, and at the same time, is the union of all non-trivial cosets of N. It makes sense then to consider certain relations between unions of some conjugacy classes of G out of N and unions of some cosets of N. A particularly studied case is *Camina groups* (also *generalized Camina groups*), in which every conjugacy class of G outside the derived subgroup G' is a single coset of G'. It turns out that Camina groups are soluble and, in fact, were completely classified in [8, 15]: A Camina group is either a p-group or a Frobenius group whose complement is cyclic or quaternion. More generally, A.R. Camina introduced in [5] the subsequently called *Camina pairs*, that is, those groups G having a normal subgroup N such that each non-trivial coset of N is contained in a conjugacy class of G. He proved that if G is not a Frobenius group, then one of N or G/N is a p-group. In the latter case, moreover, Isaacs proved

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that N has normal p-complement [14]. Several decades later, instead of pondering all conjugacy classes of G out of N, Guralnick and Navarro focused on a single conjugacy class K of G that is union of cosets of N [10,Theorem B]. It was necessary, however, to appeal to the Classification of Finite Simple Groups so as to demonstrate the solubility of N.

Inspired by the above results, in this note we address the case in which the union $K \cup K^{-1}$, where K is a conjugacy class of G, is union of some cosets of N. This is easily seen to be equivalent to the fact that $xN \subseteq K \cup K^{-1}$ where K is the conjugacy class of x. Actually, the solubility of N under these assumptions remained an unsolved problem in [3]. We give an affirmative answer by following a different approach to that of [3], which requires the Classification of Finite Simple Groups as well as several extensibility properties of irreducible characters of normal subgroups and simple groups.

Theorem A *Suppose that G is a finite group, N is a normal subgroup of G, and K a conjugacy class of G such that $K \cup K^{-1}$ is union of cosets of N. Then*

(a) N is soluble.
(b) K is a real-imaginary class, that is, every $\chi \in \mathrm{Irr}(G)$ takes real or purely imaginary values on K.
(c) if, in addition, the elements of K are p-elements, then N has normal p-complement.

We will show that under the hypotheses of Theorem A, the normal subgroup $\langle K\rangle$ need not be soluble. However, in the particular case in which $K \cup K^{-1}$ is exactly one coset, we do have the solubility of $\langle K\rangle$, in fact, we get a slight improvement of Theorem A of [3].

Corollary B *Suppose that G is a finite group, N is a normal subgroup of G and $K = x^G$ is a conjugacy class of G such that $xN = K \cup K^{-1}$. Then $\langle K\rangle$ has a normal 2-complement and x is a 2-element. Furthermore, if x has order 2, then the normal 2-complement of $\langle K\rangle$ is nilpotent.*

All groups considered are finite, the notation and terminology are usual and we will employ [12] and [13] to refer to standard results on Character Theory.

2 Preliminaries

We include some results that were previously developed and are necessary for our purposes, the first of which helps to solve the particular case $KC = K^{-1}$ for some conjugacy class C of G contained in N under the assumptions of Theorem A.

Lemma 2.1 [2,Lemma 2] *Let G be a finite group and K, L and D non-trivial conjugacy classes of G such that $KL = D$ with $|D| = |K|$. Then $\langle L\rangle$ is soluble.*

Lemma 2.2 [3,Lemma 2.3] *Let G be a finite group and let N be a normal subgroup of G. Let $K = x^G$ be the conjugacy class of an element $x \in G$. Suppose that $xN \subseteq K \cup K^{-1}$. If $\chi \in \mathrm{Irr}(G)$ does not contain N in its kernel, then $\chi(x)$ is purely imaginary.*

The proof of Theorem A(a) reduces to the case when N is product of non-abelian simple groups and then we make use of the existence of irreducible characters of coprime degree in certain non-abelian simple groups. Likewise, we need some results concerning extensions (sometimes real or rational extensions) of irreducible characters of simple groups. Regarding the alternating and sporadic groups we will use the following.

Theorem 2.3 [4,Theorem 3] *If $n \geq 6$, then $\mathrm{Alt}(n)$ has irreducible characters of degree $(n-1)(n-2)/2$ and $n(n-3)/2$ that extend to $\mathrm{Sym}(n)$.*

Observe that $(n-1)(n-2)/2$ and $n(n-3)/2$ are coprime numbers.

Theorem 2.4 [4,Theorem 4] *Let S be a sporadic simple group or the Tits group and let A be the automorphism group of S. Then there exist nonlinear characters $\chi_m, \chi_n \in \mathrm{Irr}(S)$ such that $(\chi_m(1), \chi_n(1)) = 1$ and that both χ_m and χ_n extend to A.*

Moreover, Table 1 of [4] lists, according to the Atlas [7], the specific characters of Theorem 2.4 with their respective degrees. Furthermore, by using tensor inductions of characters, the authors of [4] attained the following result.

Theorem 2.5 [4,Lemma 5] *Let N be a minimal normal subgroup of a group G so that $N = S_1 \times \ldots \times S_t$, where $S_i \cong S$ is a non-abelian simple group. Let A be the automorphism group of S. If $\sigma \in \mathrm{Irr}(S)$ extends to A, then $\sigma \times \ldots \times \sigma \in \mathrm{Irr}(N)$ extends to G.*

With regard to simple groups of Lie type, we also require some well-known properties of the Steinberg character. We refer the reader to [6] for its standard properties. The Steinberg character of a simple group of Lie type S is a (rational) character that extends to $\mathrm{Aut}(S)$. This was first proved by Schmid in [17, 18]. Furthermore, according to the main result of [9], the Steinberg character extends to the character of a rational representation of $\mathrm{Aut}(S)$. This is not enough for our purposes, however, and we still need to be more accurated so as to find real extensions of real characters of normal subgroups that are direct product of simple groups of Lie type. For that purpose, instead of using tensor inductions like in the proof of Theorem 2.5, we will employ a particular case of a result in [16].

Theorem 2.6 *Let G be a finite group, $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$ be a G-invariant real character. Suppose that $\theta(1)$ is odd and that N is perfect. Then θ extends to a real character of G.*

Proof This is an immediate consequence of [16,Theorem 2.3]. $\square$

3 Proofs

We begin by giving a characterization, in terms of irreducible characters, of an equality involving conjugacy classes. This equation arises when considering the hypotheses of Theorem A and will play a relevant role in its proof.

Lemma 3.1 *Let G be a finite group and $x, c \in G$. Let $K = x^G$ and $C = c^G$ be conjugacy classes of G. The following conditions are equivalent:*

(1) $KC = K \cup K^{-1}$.
(2) For every $\chi \in \mathrm{Irr}(G)$

$$|C|\chi(x)\chi(c) = \chi(1)(a\chi(x) + b\chi(x^{-1})).$$

for some integers $a, b \geq 0$ *such that* $a + b = |C|$.

Proof Suppose first that $KC = K \cup K^{-1}$. We remark that the possibility $K = K^{-1}$ is included. Let us denote by $\widehat{K}$, $\widehat{K^{-1}}$ and $\widehat{C}$ the sum of all elements in K, K^{-1} and C respectively, when considering them as elements of the complex group algebra $\mathbb{C}[G]$. Write $\widehat{K}\widehat{C} = a\widehat{K} + b\widehat{K^{-1}}$ for some integers $a, b \geq 0$. Notice that, by counting elements, $|K||C| = a|K| + b|K^{-1}|$, which implies that $a + b = |C|$. On the other hand, by applying [12,Lemma 3.8 and Theorem 3.9], we have

$$\frac{|K|\chi(x)}{\chi(1)}\frac{|C|\chi(c)}{\chi(1)} = a\frac{|K|\chi(x)}{\chi(1)} + b\frac{|K^{-1}|\chi(x^{-1})}{\chi(1)}$$

for every $\chi \in \mathrm{Irr}(G)$, and hence, the equality stated in (2) follows.

Conversely, suppose that (2) holds. Let $\{C_1, \ldots, C_t\}$ be the set of conjugacy classes of G. We know that (for instance by [13, Exercise 3.9]) for every pair of conjugacy class sums $\widehat{C}_m$ and $\widehat{C}_n$ with representatives c_m and c_n

$$\widehat{C}_m\widehat{C}_n = \sum_i \alpha_i \widehat{C}_i$$

where

$$\alpha_i = \frac{|C_m||C_n|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(c_m)\chi(c_n)\overline{\chi(c_i)}}{\chi(1)}$$

and $c_i \in C_i$. In particular,

$$\widehat{K}\widehat{C} = \sum_i \alpha_i \widehat{C}_i \quad \text{with} \quad \alpha_i = \frac{|K||C|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(x)\chi(c)\chi(c_i^{-1})}{\chi(1)}. \qquad (3.1)$$

On the other hand, if we pour out $\chi(x)\chi(c)$ from the equation in (2)

$$\chi(x)\chi(c) = \frac{\chi(1)(a\chi(x) + b\chi(x^{-1}))}{|C|},$$

and by replacing it in Eq. (3.1), we have

$$\alpha_k = \frac{|K||C|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{(a\chi(x) + b\chi(x^{-1}))\chi(c_i^{-1})}{|C|} =$$
$$= \frac{|K|}{|G|}(a \sum_{\chi \in \mathrm{Irr}(G)} \chi(x)\chi(c_i^{-1}) + b \sum_{\chi \in \mathrm{Irr}(G)} \chi(x^{-1})\chi(c_i^{-1})).$$

We use now the second orthogonality relation. If we assume $K \neq K^{-1}$ we easily deduce that: $\alpha_i = a$ when $C_i = K$; $\alpha_i = b$ when $C_i = K^{-1}$; and $\alpha_i = 0$ otherwise. This implies that $KC = K \cup K^{-1}$. If, on the contrary, we assume $K = K^{-1}$, it follows that $\alpha_i = a + b = |C|$ when $C_i = K$, and $\alpha_i = 0$ when $C_i \neq K$, so we get $\widehat{K}\widehat{C} = |C|\widehat{K}$. Therefore, the equality in (1) also holds. □

Before giving the proof of Theorem A, we prove an easy observation made in the Introduction, which allows us to restate Theorem A in a somewhat different manner for our convenience.

Lemma 3.2 *Let G be a finite group, N a normal subgroup of G and $K = x^G$ the conjugacy class of $x \in G$. Then $xN \subseteq K \cup K^{-1}$ if and only if $K \cup K^{-1}$ is union of cosets of N.*

Proof Suppose first that $xN \subseteq K \cup K^{-1}$. As $K \cup K^{-1}$ is a normal subset, it is clear that $KN = (xN)^G \subseteq K \cup K^{-1}$. By taking inverses, $K^{-1}N = (KN)^{-1} \subseteq K \cup K^{-1}$. Therefore, $(K \cup K^{-1})N = K \cup K^{-1}$. This equality shows that $K \cup K^{-1}$ is union of cosets of N. Conversely, suppose now that $K \cup K^{-1}$ is union of cosets of N and let $yN \subseteq K \cup K^{-1}$ for some $y \in G$. As $y \in K \cup K^{-1}$, there exists $g \in G$ such that either $y^g = x$ or $y^g = x^{-1}$, and then, either $xN = y^gN = (yN)^g \subseteq K \cup K^{-1}$ or analogously $x^{-1}N \subseteq K \cup K^{-1}$. In the latter case, we also obtain $xN \subseteq K \cup K^{-1}$, as wanted. □

Theorem 3.3 *Suppose that G is a finite group, N is a normal subgroup of G and $K = x^G$ is a conjugacy class of G such that $xN \subseteq K \cup K^{-1}$. Then*

(a) N is soluble.
(b) K is a real-imaginary class, that is, $\chi(x)$ is real or purely imaginary for every $\chi \in \mathrm{Irr}(G)$.
(c) if x is a p-element, then N has normal p-complement.

Proof (a) We argue by induction on $|G|$ and take M to be a minimal normal subgroup of G contained in N. Let us denote with bars the factor group G/M. Since $\overline{xN} \subseteq \overline{K} \cup \overline{K^{-1}}$, the inductive hypothesis implies that $\overline{N}$ is soluble, so we only have to prove the solubility of M. Since $xM \subseteq K \cup K^{-1}$ we can assume then that N is minimal normal in G. We will assume henceforth that N is non-soluble, that is, is the direct product of isomorphic non-abelian simple groups, say $N \cong S \times \ldots \times S$, and seek a contradiction.

First, note that $xN \subseteq K \cup K^{-1}$ includes the possibility $xN \subseteq K$, but this case is solved in [10,Theorem B(a)]. Hence, it is clear that we can assume $K \neq K^{-1}$ and that $KN = K \cup K^{-1}$. So, for every conjugacy class C of G contained in N we have $KC \subseteq K \cup K^{-1}$. If $KC = K$ or $KC = K^{-1}$ for some class $C \neq 1$, then $\langle C \rangle = N$ would be soluble by Lemma 2.1, against our assumption. Thus we assume throughout the proof that every class $C \neq 1$ contained in N satisfies $KC = K \cup K^{-1}$.

Suppose that $\chi \in \mathrm{Irr}(G)$ extends some $\theta \in \mathrm{Irr}(N)$ with $\theta \neq 1_N$, and we calculate the possible values that χ may take on x. By Lemma 2.2, we know that $\chi(x)$ is purely imaginary (possibly zero), and write $\chi(x) = \alpha i$, with $\alpha \in \mathbb{R}$. Furthermore, by applying [13,Lemma 8.14(c)] we have

$$|N| = \sum_{y \in xN} \chi(y)\overline{\chi(y)} = \sum_{y \in xN \subseteq K \cup K^{-1}} |\alpha i|^2 = |N|\alpha^2.$$

This forces $\alpha = \pm 1$, so $\chi(x) = \pm i$. The rest of the proof consists of proving that this property leads to a contradiction according to the distinct cases for S given by the Classification of the Finite Simple Groups.

Assume first that S is isomorphic to an alternating or a sporadic simple group. We postpone the case $\mathrm{Alt}(5) \cong \mathrm{PSL}(2,5)$, which will be treated as a group of Lie type. Then, by Theorems 2.3 and 2.4, there exist two (nonlinear) irreducible characters of S, say θ_1 and θ_2, having coprime degrees that extend to $\mathrm{Aut}(S)$. By Theorem 2.5, the characters $\theta_j \times \ldots \times \theta_j \in \mathrm{Irr}(N)$ for $j = 1, 2$ also extend to G. Let $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ denote two respective extensions of $\theta_j \times \ldots \times \theta_j$ for $j = 1, 2$. Since we have proved that $\chi_j(x) = \pm i$ for $j = 1, 2$, if we fix a non-trivial class C of G contained in N and take $c \in C$, by Lemma 3.1, there exist $a, b \geq 0$ with $a + b = |C|$ such that

$$|C|(\pm i)\chi_j(c) = \chi_j(1)(a(\pm i) + b(\mp i)).$$

for $j = 1, 2$. This yields

$$\chi_j(c) = \pm \frac{(a-b)\chi_j(1)}{|C|} \quad \text{for} \quad j = 1, 2.$$

Now, since $\chi_1(1)$ and $\chi_2(1)$ are coprime, there exist $m, n \in \mathbb{Z}$ such that $m\chi_1(1) + n\chi_2(1) = 1$. It follows that

$$m\chi_1(c) + n\chi_2(c) = m\frac{(a-b)\chi_1(1)}{|C|} + n\frac{(a-b)\chi_2(1)}{|C|} = \frac{a-b}{|C|}$$

is algebraic integer and rational, so it must be integer. However, as we are assuming $K \neq K^{-1}$, then $0 < a, b < |C|$, and this leads to a contradiction unless $a = b$. But if $a = b$ for every class $C \neq 1$ of G contained in N, then $(\theta_j \times \ldots \times \theta_j)(n) = \chi_j(n) = 0$ for $j = 1, 2$ and for every $1 \neq n \in N$, which is not possible. This contradiction finishes this case.

Assume now that S is a simple group of Lie type and let St denote the Steinberg character of S. As St extends to $\mathrm{Aut}(S)$, by Theorem 2.5 we know that $\theta = \mathsf{St} \times \ldots \times \mathsf{St} \in \mathrm{Irr}(N)$ also extends to G. We take then $\chi \in \mathrm{Irr}(G)$ to be an extension of θ. Suppose first that the defining characteristic of S is 2. It is well-known that St vanishes on 2-singular elements of S, and this means that $\chi(c) = 0$ for every $c \in N$ of order divisible by 2. Say, for instance, $c \in C$ for some conjugacy class C of G. Moreover, we have proved above that $\chi(x) = \pm i$. Then, since $K \neq K^{-1}$, by applying Lemma 3.1, there exist integers $a, b > 0$ with $a + b = |C|$ such that

$$0 = \chi(1)(a(\pm i) + b(\mp(i)).$$

This forces $a = b = |C|/2$, so every conjugacy class of G of any element of even order of N must have even cardinality. This is certainly not true for if we choose P a Sylow 2-subgroup of G, then $N \cap P \trianglelefteq P$, and also $\mathbf{Z}(P) \cap N \neq 1$. Then, if $1 \neq c \in \mathbf{Z}(P) \cap N$, we certainly have that $|c^G|$ is odd.

Assume finally that the defining characteristic of S is $p > 2$. Since θ is G-invariant, rational and has degree a power of p, and hence is an odd number (St is G-invariant and $\mathsf{St}(1) = |S|_p$), we can apply Theorem 2.6 to affirm that θ has a real extension to G. This contradicts the fact that every extension to G of a non-principal irreducible character of N must take values $\pm i$ on x. This contradiction completes the proof of (a).

(b) To prove that K is a real-imaginary class we will make use of a property [1,Theorem A] that characterizes real-imaginary classes within the complex group algebra $\mathbb{C}[G]$. We employ the following notation in $\mathbb{C}[G]$. Let $C_1, \ldots, C_t$ be the conjugacy classes of G and let $\widehat{S} = \sum_{i=1}^{t} n_i \widehat{C_i} \in \mathbb{C}[G]$ with $n_i \in \mathbb{N}$ for $1 \leq i \leq t$. Then we write $(\widehat{S}, \widehat{C_i}) = n_i$. We just recall an elementary property of these multiplicities (for instance, as a result of Theorem 4.6 of [12]):

$$(\widehat{C_i}\widehat{C_j}, \widehat{C_k}) = \frac{|C_j|}{|C_k|}(\widehat{C_i}\widehat{C_k^{-1}}, \widehat{C_j^{-1}})$$

for every $1 \leq i, j, k \leq t$.

We will assume that K is non-real, otherwise the result is trivial. Now, the hypotheses imply that $KN \subseteq K \cup K^{-1}$, so if we take any conjugacy class C of G contained in N we can write

$$\widehat{K}\widehat{C} = a\widehat{K} + b\widehat{K^{-1}} \quad \text{and} \quad \widehat{K}\widehat{C^{-1}} = c\widehat{K} + d\widehat{K^{-1}}$$

for some integers $a, b, c, d \geq 0$. Then $\widehat{K^{-1}}\widehat{C} = c\widehat{K^{-1}} + d\widehat{K}$ and

$$\begin{aligned}\widehat{K^{-1}}(\widehat{K}\widehat{C}) &= a\widehat{K^{-1}}\widehat{K} + b\widehat{K}^{-2}\\ \widehat{K}(\widehat{K^{-1}}\widehat{C}) &= c\widehat{K}\widehat{K^{-1}} + d\widehat{K}^{2}.\end{aligned}$$

Since K is assumed to be non-real, then $(\widehat{K}^{-2}, \widehat{1}) = (\widehat{K}^2, \widehat{1}) = 0$, and then, by applying the above property on multiplicities

$$\begin{aligned}(\widehat{K^{-1}}(\widehat{K}\widehat{C}), \widehat{1}) &= (a\widehat{K^{-1}}\widehat{K} + b\widehat{K}^{-2}, \widehat{1}) = a|K|(\widehat{K}, \widehat{K}) = a|K|\\ (\widehat{K}(\widehat{K^{-1}}\widehat{C}), \widehat{1}) &= (c\widehat{K}\widehat{K^{-1}} + d\widehat{K}^{2}, \widehat{1}) = c|K|(\widehat{K}, \widehat{K}) = c|K|.\end{aligned}$$

As class sums belong to $\mathbf{Z}(\mathbb{C}[G])$ then $\widehat{K^{-1}}(\widehat{K}\widehat{C}) = \widehat{K}(\widehat{K^{-1}}\widehat{C})$, and hence $a = c$. It follows that $b\widehat{K}^{-2} = d\widehat{K}^2$, and by taking cardinalities we get $b = d$. Consequently, $\widehat{K}^{-2} = \widehat{K}^2$, and this is exactly the aforementioned property that characterizes real-imaginary classes in $\mathbb{C}[G]$.

(c) The proof is practically identical to the proof of [10,Theorem B(b)], but we include a sketch for the reader's convenience. Let H be a p-complement of N. By the Frattini argument we have $G = \mathbf{N}_G(H)N$, so we can write $x = mn$ with $m \in \mathbf{N}_G(H)$ and $n \in N$. Hence $xn^{-1} \in K \cup K^{-1}$ normalizes H. Since $xn^{-1} = x^g$ or $(x^{-1})^g$ for some $g \in G$, it follows that x normalizes some G-conjugate of H, which by the Frattini argument, is some N-conjugate of H. We deduce that every element in xN fixes at least one p-complement of N. The same argument that appears in the last paragraph of the proof of [10,Theorem 3.2(c)], based on an extension of Burnside's Lemma, works to show that N has just one p-complement, and thus, it is normal in N. □

Proof of Corollary B. Note that by hypothesis $xN = K \cup K^{-1} = (xN)^{-1} = x^{-1}N$, and then $x^2N = N$. Since $x^2 \in N$, then the $2'$-part of x belongs to N. Therefore, as all elements of xN have the same order, it follows that every element of xN is a 2-element, in particular, x is a 2-element too. By Theorem A(c), N has a normal 2-complement. Since

$$\langle K \rangle = \langle xN \rangle = \langle x, N \rangle = \langle x \rangle N,$$

then $\langle K \rangle / N$ is a 2-group (cyclic of order 2, indeed), and consequently, $\langle K \rangle$ has a normal 2-complement too.

Finally, suppose that x has order 2, so every element of xN has order 2. Then

$$\langle K \rangle = N \cup Nx = N \cup K \cup K^{-1},$$

so every element of $\langle K \rangle - N$ has order 2 as well. It is clear that the 2-normal complement of $\langle K \rangle$ lies in the subgroup generated by all elements of $\langle K \rangle$ of order distinct from 2, and this subgroup is nilpotent by a renowned theorem of Hughes, Thompson and Kegel [11,V.8]). This proves the last assertion of the corollary. □

Corollary 3.4 *Suppose that G is a finite group, N is a normal subgroup of G such that every non-trivial coset of N is the union of a conjugacy class of G and its inverse. Then G has normal 2-complement and G/N is an elementary abelian 2-group. In particular, G is soluble.*

Proof For every $x \in G - N$, we apply Corollary B to conclude that $\langle x^G \rangle$ has a normal 2-complement. Since $G = \prod_{x \in G-N} \langle x^G \rangle$, we obtain the first assertion. The fact that G/N is an elementary abelian 2-group follows from the fact that $x^2 \in N$ for every $x \in G - N$. □

4 Some examples and questions

1) As we pointed out in the Introduction, the fact that $xN \subseteq K \cup K^{-1}$, with the notation in Theorem A, does not imply that $\langle K \rangle$ is soluble. The double cover $G = 2.M_{12}$ of the sporadic simple group M_{12} has four non-real conjugacy classes of elements of order 8, say $K = x^G$, K^{-1}, $L = y^G$ and L^{-1}, which satisfy $xZ \subseteq K \cup K^{-1}$ and $yZ \subseteq L \cup L^{-1}$, were $Z = \mathbf{Z}(G)$. Also, $KZ = K \cup K^{-1}$ and $LZ = L \cup L^{-1}$. These equalities can be easily checked by using the Atlas [7] and Lemma 3.1. Of course, $\langle K \rangle / Z$ is non-abelian simple and $\langle K \rangle = G$ is non-soluble. Likewise, the fact that $xN \subseteq K$ does not either lead to the solubillity of $\langle K \rangle$. The easiest example is the unique class K of elements of order 4 in SL(2,5), which satisfies $KZ = K$, where $Z = \mathbf{Z}(SL(2,5))$.

2) An easy example of a group satisfying the hypotheses of Theorem A, with N being non-central is $G = \langle a, b \mid a^8 = b^2 = 1, a^b = a^3 \rangle$, the semidihedral group of order 16. If we take $K = a^G$ and $N = \langle a^2 \rangle$, then we have $aN \subseteq K \cup K^{-1}$.

3) This example illustrates Corollary B and Corollary 3.4. Let G be the semidirect product of C_3^2 and C_4 acting faithfully,

$$G = \langle a, b, x \mid a^3 = b^3 = x^4 = 1, ab = ba, xax^{-1} = a^{-1}b, xbx^{-1} = ab \rangle,$$

and take $N = \langle a, b, x^2 \rangle \cong C_3 \rtimes S_3$, $K = x^G$ and $C = (x^2)^G$, with $|K| = |C| = 9$. Then $KC = K^{-1}$ and $xN = K \cup K^{-1}$. Notice that the 2-complement of N, which is also of $\langle K \rangle = G$, is nilpotent. Bearing in mind the results on Camina pairs quoted in the Introduction as well as Corollary 3.4, we are led to the following question.

Question. Is it possible to determine solubility conditions or classify those finite groups in which every coset of a given normal subgroup lies in the union of a conjugacy class and its inverse class?

4) The hypotheses of Theorem A cannot be extended by increasing from 2 to 3 the number of conjugacy classes in which a given coset of N is contained, because in that case the solubility of N is not guaranteed. For instance, the Mathieu group M_{10} has three conjugacy classes out of the normal subgroup N isomorphic to Alt(6): a real class of elements order 4, say L, and two (non-real) classes K and K^{-1} of elements of order 8. It is not difficult to check that every conjugacy class $C \neq 1$ of M_{10} contained in N satisfies $KC = K^{-1}C = LC = K \cup K^{-1} \cup L$, and $KN = K \cup K^{-1} \cup L$, whereas N is obviously non-soluble.

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References

1. Beltrán, A., Robati, S.M.: Erratum to: Real-imaginary conjugacy classes and real-imaginary irreducible characters in finite groups. Math. Notes **110**, 638–639 (2021). https://doi.org/10.1134/0001434621090352
2. Beltrán, A., Felipe, M.J., Camina, R.D., Melchor, C.: Powers of conjugacy classes in a finite group. Ann. Mat. Pura Appl. **4**(2), 409–424 (2020). https://doi.org/10.1007/s10231-019-00885-2
3. Beltrán, A., Felipe, M.J.: Cosets of normal subgroups and powers of conjugacy classes. Math. Nachr. **294**(9), 1652–1656 (2021). https://doi.org/10.1002/mana.201900554
4. Bianchi, M., Chillag, D., Lewis, M.L., Pacifici, E.: Character degree graphs that are complete graphs. Proc. Amer. Math. Soc. **135**(3), 671–676 (2007). https://doi.org/10.1090/S0002-9939-06-08651-5
5. Camina, A.R.: Some conditions which almost characterize Frobenius Groups. Isr. J. Math. **31**, 153–160 (1978)
6. Carter, R.W.: Finite Groups of Lie Type. Wiley, New York (1985)
7. Conway, J.H., Curtis, R.T., Norton, S.P., Parker, R.A., Wilson, R.A.: An ATLAS of Finite Groups. Clarendon Press, Oxford (1985)
8. Dark, R., Scoppola, C.M.: On Camina groups of prime power order. J. Algebra **181**(3), 787–802 (1996)
9. Feit. W.: Extending Steinberg characters, Linear algebric groups and their representations (Los Angeles, CA, 1992), Contemp. Math. **153**, Amer. Math. Sco. Providence, RI, 1-9 (1993)
10. Guralnick, R.M., Navarro, G.: Squaring a conjugacy class and cosets of normal subgroups. Proc. Amer. Math. Soc. **144**(5), 1939–1945 (2016). https://doi.org/10.1090/proc/12874
11. Huppert, B.: Endliche Gruppen. Springer-Verlag, Berlin-Heidelberg-New York (1967)
12. Huppert, B.: Character theory of finite groups. Walter de Gruyter, Berlin, New York (1998)
13. Isaacs, I.M.: Character theory of finite groups. AMS Chelsea Publishing, Providence, RI. Corrected reprint of the 1976 original (2006)
14. Isaacs, I.M.: Coprime group actions fixing all nonlinear irreducible characters. Can. J. Math. **41**(1), 68–82 (1989)
15. Lewis, M.L.: Classifying Camina groups: a theorem of Dark and Scoppola. Rocky Mountain J. Math. **44**, 591–597 (2014). https://doi.org/10.1216/RMJ-2014-44-2-591
16. Navarro, G., Tiep, P.H.: Rational irreducible characters and rational conjugacy classes in finite groups. Trans. Amer. Math. Soc. **360**(5), 2443–2465 (2007). https://doi.org/10.1090/S0002-9947-07-04375-9
17. Schmid, P.: Rational matrix groups of a special type. Linear Algebra Appl. **71**, 289–293 (1985)
18. Schmid, P.: Extending the Steinberg representation. J. Algebra **150**, 254–256 (1992)