

ERGODIC PROPERTIES OF CONVOLUTION OPERATORS

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ABSTRACT. Let G be a locally compact group and μ be a measure on G . In this paper we find conditions for the convolution operators $\lambda_p(\mu)$, defined on $L^p(G)$ and given by convolution by μ to be mean ergodic and uniformly mean ergodic. The ergodic properties of the operators $\lambda_p(\mu)$ are related to the ergodic properties of the measure μ as well.

1. INTRODUCTION

Von Neumann's mean ergodic theorem proves that the sequence $(T_{[n]})_n$ of averages of the first n powers of a unitary operator T (its Cesàro means) always converges to the projection onto the subspace of fixed vectors of T . After the appearance of von Neumann's mean ergodic theorem, and its Birkhoff's contemporary, the pointwise ergodic theorem, the convergence of averages of measure preserving transforms in diverse settings and senses rapidly became an important part of Ergodic Theory, see [17, 32] for more details. In this development, an operator on a Banach space whose sequence of Cesàro means converges in the strong operator topology came to be known as *mean ergodic*. When convergence holds in the operator norm, the coined term was *uniformly mean ergodic*.

Our approach to ergodicity can be traced back to [46] where Yosida used weak clustering of orbits to characterize mean ergodicity of operators on Banach spaces. Our contemporary antecedent is to be found in the work of Bonet and Domański [7, 8] that launched a systematic research on ergodicity from an operator theoretic point of view. Composition and multiplication operators in spaces of holomorphic functions have since been studied [4, 5, 7, 8] with a recent attention to composition operators acting on spaces of smooth real functions [20, 30]. In this paper, we address the ergodic properties of those operators on the Banach space $L^p(G)$ of p -integrable functions on a locally compact group G that are given by convolution by a fixed measure on G . We will refer to these operators simply as convolution operators.

When G is sigma-compact and μ is a probability measure, the operator given by convolution with μ is a Markov operator, having the Haar measure m_G as sigma-finite invariant measure; the Markov chain generated by μ with state space G is a random walk on G . If G is compact or Abelian, conditions for ergodicity and mixing of the random walk are well known, in terms of

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the algebraic properties of μ (the Itô-Kawada and Choquet-Deny theorems). See Remark 5.5 for some more information on this approach.

Among the most recent papers dealing with convolution operators from a point of view related to ours, we can mention the monograph by Derighetti [13], with special emphasis on their restriction and extension to and from closed subgroups, and the papers by Neufang, Salmi, Skalski and Spronk [38], with focus on the wider frame of quantum groups, and by Mustafayev [36, 37], where ergodicity for multipliers on Banach algebras and convolution operators on locally compact Abelian groups are studied.

1.1. Outline and summary of results. Our approach leads to characterizations of mean ergodicity and uniform mean ergodicity for convolution operators defined by several classes of measures. These characterizations can also be used to compare the convergence of averages of convolution powers of measures in the vague topology (mean ergodicity of measures) with mean ergodicity of the corresponding convolution operators. As it turns out, while mean ergodicity of measures often implies ergodicity of their convolution operators, the converse does not hold even for weak versions of ergodicity.

We next outline our results, showcasing the case of Abelian groups. Our methods however are not intrinsically commutative and most of our results are stated for more general classes of groups. We also have also tried to determine the limits of what can be expected beyond our results through a series of examples, both commutative and noncommutative, that appear, mostly, in Section 7.

Theorem A (Theorem 4.3, Theorem 5.1, Theorem 5.4). *Let G be a locally compact group and $\mu \in M(G)$. Consider the following conditions on the convolution operator $\lambda_p(\mu)$ (see Section 2 for undefined notation):*

- (a) $\lambda_p(\mu)$ is mean ergodic.
- (b) $\lambda_p(\mu)$ is power bounded.
- (c) $\|\mu\| \leq 1$.

These conditions can be related as follows.

- (1) *If G is Abelian and $p = 2$, then (a) and (b) are equivalent.*
- (2) *If G is amenable and μ is positive, then all three conditions are equivalent, for any $1 < p < \infty$.*
- (3) *If G is compact, μ positive and $p = 1$, then all three conditions are equivalent as well.*

The last statement of Theorem A actually contains all what needs to be known about the case $p = 1$, for $\lambda_1(\mu)$ cannot be mean ergodic unless either μ is a probability measure and the support of μ is contained in a compact subgroup of G , see Theorem 5.4, or $\|\mu\| < 1$. In this last case we have even that $(\lambda(\mu^n))_n$ is norm convergent to 0.

Theorem B (Theorem 4.6, Theorem 6.1, Theorem 6.3). *Let G be a locally compact Abelian group and let $\mu \in M(G)$. Consider the following conditions on the convolution operator $\lambda_p(\mu)$:*

- (a) $\lambda_p(\mu)$ is uniformly mean ergodic.
- (b) $\|\lambda_p(\mu)\| \leq 1$ and 1 is not an accumulation point of the spectrum of $\lambda_p(\mu)$.

Conditions (a) and (b) are then equivalent when any of the following conditions hold:

- (1) $p = 2$.
- (2) μ is positive and $1 < p < \infty$.
- (3) G is compact, μ positive and $p = 1$ or $p = \infty$. In either case $\lambda_p(\mu)$ is uniformly mean ergodic if and only if $\lambda_\infty(\mu)$ is mean ergodic.

In Theorem B, commutativity is only needed to make sure the operator $\lambda_2(\mu)$ is normal. Hence the theorem is also true under this weaker assumption.

Many of our results on ergodicity or mean ergodicity of $\lambda_p(\mu)$ depend on conditions on the ambient group G . To make them depend, as it is naturally expected, on the subgroup H_μ generated by the support of μ , we have had to relate the ergodic behaviour of $\lambda_p(\mu)$ as an operator on $L^p(G)$ with its behaviour as an operator on $L^p(H_\mu)$. Since this is a technically intricate issue, we have decided to deal with it in an Appendix at the end of the paper.

2. PRELIMINARIES

In this section we gather the basic definitions and basic facts around our two main subjects: ergodicity of operators and convolution operators.

2.1. Ergodic Operators.

Notation. If $T \in \mathcal{L}(X)$ is a bounded linear operator on a Banach space X , $T_{[n]}$ will denote the Cesàro means:

$$T_{[n]} = \frac{1}{n} \sum_{k=1}^n T^k.$$

Definition 2.1. We say that a bounded linear operator $T \in \mathcal{L}(X)$ is:

- (1) weakly mean ergodic if there is $P \in \mathcal{L}(X)$ such that $\lim_{n \rightarrow \infty} T_{[n]} = P$ in the weak operator topology.
- (2) mean ergodic if there is $P \in \mathcal{L}(X)$ such that $\lim_{n \rightarrow \infty} T_{[n]} = P$ in the strong operator topology.
- (3) uniformly mean ergodic if there is $P \in \mathcal{L}(X)$ such that $\lim_{n \rightarrow \infty} T_{[n]} = P$ in the operator norm.

Notation. If $T \in \mathcal{L}(X)$ is a bounded linear operator on a Banach space X for which $\ker(T - I)$ is a complemented subspace we will denote by P_T the projection operator onto $\ker(T - I)$.

The following two basic facts on mean ergodicity of operators can be found in or deduced from Section 8.4 of [18].

Theorem 2.2. Let T be a bounded linear operator on a Banach space X such that $\|T^n\|/n$ converges to 0 in the strong operator topology. Then

- (1) T is mean ergodic if and only if $X = \ker(T - I) \oplus \overline{\text{Ran } I - T}$.
- (2) If T is mean ergodic, then $\lim_n T_{[n]} = P_T$ and $P_T T = P_T = T P_T$.

We next state a general version of Yosida's mean ergodic Theorem. We include a proof for the sake of completeness. It does not include new ideas but those of Yosida. See [46] for the original proof and [2, Theorem 2.2], [18, Theorem 8.22] or [39, Theorem 1.3] for similar statements.

Theorem 2.3. *Let X be a Banach space and let τ be a locally convex topology on X compatible with $\sigma(X, X^*)$. Then there is a bounded linear operator P on X such that $(T_{[n]}x)_n$ is τ -convergent to $P(x)$ for each $x \in X$ if and only if the following two conditions hold for each $x \in X$:*

- (1) $\tau - \lim \frac{T^n(x)}{n} = 0$ and
- (2) $\{T_{[n]}(x) : n \in \mathbb{N}\}$ is relatively weakly compact.

Proof. The necessity of condition (1) comes from

$$\frac{T^n(x)}{n} = T_{[n]}(x) - \frac{n-1}{n}T_{[n-1]}(x), \quad \text{for any } x \in X.$$

Since τ is stronger than the weak topology, Condition (2) is obviously also necessary.

We now check the sufficiency of (1) and (2). We first observe that $T_{[n]}(I - T) = \frac{1}{n}(T - T^{n+1})$, hence $T_{[n]}(I - T)(x)$ is τ -convergent to 0 for every $x \in X$. Since, by the Banach Steinhaus theorem, $(\|T_{[n]}\|)_n$ is a bounded sequence, we get

$$(1) \quad \tau - \lim_n T_{[n]}(v) = 0 \text{ for every } v \in \overline{\text{Ran}(I - T)}.$$

Let now $x \in X$. As $\{T_{[n]}(x) : n \in \mathbb{N}\}$ is relatively weakly compact, we get from Eberlein's theorem an increasing sequence (n_k) of natural numbers and $y_x \in X$ such that $(T_{[n_k]}(x))$ is weakly convergent to y_x . Then, by the preceding paragraph, and using that $TT_{[n_k]} = T_{[n_k]}T$,

$$(2) \quad 0 = \sigma(X, X^*) - \lim_k T_{[n_k]}((I - T)(x)) = y_x - T(y_x).$$

On the other hand

$$\begin{aligned} x - y_x &= \sigma(X, X^*) - \lim_k \frac{1}{n_k} \left(\sum_{n=1}^{n_k} (x - T^n(x)) \right) \\ &= \sigma(X, X^*) - \lim_k \frac{1}{n_k} \left(\sum_{n=1}^{n_k} (I - T) ((I + T + \cdots + T^{n-1})(x)) \right). \end{aligned}$$

From this and $\text{Ran}(I - T)$ being a vector subspace we deduce that $x - y_x$ is in the norm closure $\overline{\text{Ran}(I - T)}$. Applying (1) once more we see that $\tau - \lim_n T_{[n]}(x - y_x) = 0$. From this and (2) we conclude that, putting $Px = y_x$,

$$\lim_n T_{[n]}x = Px, \quad \text{for every } x \in X.$$

Since $\|y_x\| \leq \sup_k \|T_{[n_k]}\| \|x\|$, it is clear that $P \in \mathcal{L}(X)$. \square

Corollary 2.4. *(Mean Ergodic Theorem) If X is reflexive, the sequence $(T_{[n]})_n$ is bounded (in other words, T is Cesàro bounded) and $\lim_n \frac{T^n}{n} = 0$ in the strong operator topology, then T is mean ergodic.*

Proposition 2.5. *If T is Cesàro bounded then $r(T) \leq 1$.*

Proof. If $r(T) > 1$ then there is $M > 1$ such that, eventually, $\|T^n\| \geq M^n$, which makes the sequence $(\frac{T^n}{n})_n$ unbounded. Since $\frac{T^n}{n} = T_{[n]} - \frac{n-1}{n}T_{[n-1]}$, this goes against Cesàro-boundedness of T . \square

Corollary 2.6. *Let X be a Banach space. If T is weakly mean ergodic, then $r(T) \leq 1$.*

And, next, we collect some basic facts on uniform mean ergodicity. (1) is an immediate consequence of the spectral radius formula, (2) comes again from $T_{[n]} - \frac{n-1}{n}T_{[n-1]} = \frac{T^n}{n}$ and (3) is the classical Yosida Kakutani theorem [47, Theorem 4, Corollary (i)].

Theorem 2.7. *Let T be a bounded linear operator on a Banach space X :*

- (1) *If $r(T) < 1$ then T is uniformly mean ergodic (further, T^n converges to 0 in the norm operator topology).*
- (2) *If T is uniformly mean ergodic, then $\lim_{n \rightarrow \infty} \frac{\|T^n\|}{n} = 0$.*
- (3) *If T is power bounded, compact and mean ergodic then it is uniformly mean ergodic.*

Uniform mean ergodicity can be characterized in a functional analytic way.

Theorem 2.8 (Dunford, Theorem 8 of [16], Lin, Theorem of [33]). *Let T be a bounded linear operator on the Banach space X . The following assertions are equivalent:*

- (1) *T is uniformly mean ergodic.*
- (2) *$\text{Ran}(I - T)^2$ is closed and $\lim_n \frac{\|T^n\|}{n} = 0$.*
- (3) *Either $1 \in \rho(T)$ or 1 is a pole of order 1 of the resolvent mapping $R(z, T)$ and $\lim_n \frac{\|T^n\|}{n} = 0$.*
- (4) *$\text{Ran}(I - T)$ is closed and $\lim_n \frac{\|T^n\|}{n} = 0$.*
- (5) *$\text{Ran}(I - T)$ is closed, $X = \text{Ran}(I - T) \oplus \ker(I - T)$ and $\lim_n \frac{\|T^n\|}{n} = 0$.*

If in addition we have $\ker(I - T) = \{0\}$ then all the statements are equivalent to

- (6) *$1 \notin \sigma(T)$ and $\lim_n \frac{\|T^n\|}{n} = 0$.*

The equivalence of the first 3 conditions and that they imply (4) and (5) were proved by Dunford [16, Theorem 8]. Dunford also proved that (4) implies (1) if T is assumed to be mean ergodic. Lin proved in [33] that (4) implies (1) with no extra assumptions, and then also (5), which certainly implies (4). Condition (6) is simply a particular case of the theorem which is specially relevant in our work. The decomposition in direct sum of (5) when there are not fixed points is equivalent to $I - T$ being an injective operator and $\text{Ran}(I - T) = X$, i.e. to $I - T$ being an isomorphism.

We finish this subsection stating a classic result of Lotz, in the form that we need it

Theorem 2.9 (Theorem 5 of [34]). *Let (Ω, Σ, μ) be a positive measure space. Then every bounded linear operator T on $L^\infty(\Omega)$ satisfying $\lim_n \|T^n\|/n = 0$ which is mean ergodic is also uniformly mean ergodic.*

2.2. Convolution operators. Our convolution operators will be defined on Banach spaces of functions on a locally compact group. For any such G , we will denote by m_G its Haar measure, i.e., its (essentially unique) left invariant measure. The Banach space of (equivalence classes of) Haar p -integrable functions will be simply denoted as $L^p(G)$.

The measure m_G needs not be right invariant. Its behaviour under right translations is gauged by the modular function Δ_G . Δ_G is a homomorphism of G into the multiplicative group of positive real numbers such that

$$\int f(xy^{-1}) dm_G(x) = \Delta_G(y) \int f(x) dm_G(x).$$

The Banach space of bounded regular measures on G will be denoted by $M(G)$. Hence $M(G) = C_0(G)^*$. The weak* topology $\sigma(M(G), C_0(G))$ will be referred to as the *vague topology*.

We will always regard $L^1(G)$ as a subalgebra (an ideal, actually) of $M(G)$ through the embedding $f \mapsto f \cdot m_G$.

Definitions 2.10. Let $\mu, \mu_1, \mu_2 \in M(G)$ be bounded regular measures. We consider:

- (1) The convolution of measures:

$$\langle \mu_1 * \mu_2, f \rangle = \iint f(xy) d\mu_1(x) d\mu_2(y), \text{ for every } f \in C_{00}(G).$$

- (2) The left convolution operator $1 \leq p \leq \infty$:

$$\lambda_p(\mu): L^p(G) \rightarrow L^p(G)$$

given by

$$\lambda_p(\mu)(f)(s) = (\mu * f)(s) := \int f(x^{-1}s) d\mu(x), \quad f \in L^p(G), s \in G.$$

- (3) The right convolution operator for $1 \leq p \leq \infty$:

$$\rho_p(\mu): L^p(G) \rightarrow L^p(G)$$

given by

$$\rho_p(\mu)(f)(s) = \int \Delta_G^{1/p}(x) f(sx) d\mu(x), \quad f \in L^p(G), s \in G.$$

The ergodic behaviour of the operators $\lambda_p(\mu)$ and $\rho_p(\mu)$ is the same when $1 < p < \infty$. This is due to the following easily verifiable fact.

Fact 2.1. Let $1 < p < \infty$. The operators $\rho_p(\mu)$ and $\lambda_p(\mu)$ are intertwined by the linear isometry $U_p: L^p(G) \rightarrow L^p(G)$, given by $U_p(f)(s) = \Delta_G^{1/p}(s^{-1})f(s^{-1})$, i.e. $\lambda_p(\mu)U_p = U_p\rho_p(\mu)$.

Proofs for items (1) and (2) of the following theorem can be found in Theorem 20.12 of [27], for a proof of item (3), see e.g., page 47 of [21].

Theorem 2.11. Let G be a locally compact group and let $\mu \in M(G)$. Then:

- (1) $\mu * f \in L^p(G)$ for every $f \in L^p(G)$ and $\|\mu * f\|_p \leq \|\mu\| \cdot \|f\|_p$. As a consequence,
- (2) $\lambda_p(\mu) \in \mathcal{L}(L^p(G))$ and $\|\lambda_p(\mu)\| \leq \|\mu\|$.
- (3) $\|\lambda_1(\mu)\| = \|\mu\|$.

$M(G)$ is a $*$ -Banach algebra under convolution with involution defined by $\langle \mu^*, h \rangle = \overline{\langle \mu, \tilde{h} \rangle}$ for every $h \in C_0(G)$, where $\tilde{h}(s) = \overline{h(s^{-1})}$, for every $s \in G$. If $\mu = \mu^*$, the measure is called *Hermitian*. Since $\lambda_2(\mu)^* = \lambda_2(\mu^*)$, the operator $\lambda_2(\mu)$ is self-adjoint precisely when μ is Hermitian. We will say that the measure μ is *operator-normal* when $\mu * \mu^* = \mu^* * \mu$. The operator $\lambda_2(\mu)$ will be normal precisely when μ is operator-normal, in particular when G is commutative, [27, Theorem 20.23].

We will often deal with $\langle L^p(G), L^{p'}(G) \rangle$ dualities. For $p \neq 2$ and $1/p + 1/p' = 1$, this duality is given by:

$$\langle f, g \rangle = \int f(x)g(x) \, dm_G(x) \quad f \in L^p(G), g \in L^{p'}(G).$$

When $p = 2$, the duality is derived from the inner product in $L^2(G)$ and is therefore given by

$$\langle f, g \rangle = \int f(x)\overline{g(x)} \, dm_G(x) \quad f, g \in L^2(G).$$

2.3. Abelian groups: Fourier-Stieltjes transforms. If G is Abelian, the Fourier-Stieltjes transform establishes a unitary equivalence between convolution operators and multiplication operators. We use this subsection to recall this fact and some of its consequences.

Let G be a locally compact and let \mathbb{T} denote the multiplicative group of complex numbers of modulus 1. By a *character* of G we understand a continuous homomorphism of G into \mathbb{T} . With the topology of uniform convergence on compact sets and pointwise multiplication, the set \widehat{G} of all continuous characters of G acquires the structure of a locally compact Abelian group.

If $\mu \in M(G)$, the function $\widehat{\mu}: \widehat{G} \rightarrow \mathbb{C}$ given by $\widehat{\mu}(\chi) = \int \chi(t) d\mu(t)$ is a bounded uniformly continuous function known as the *Fourier-Stieltjes transform* of μ . The Riemann-Lebesgue theorem shows that, $\widehat{f} \in C_0(\widehat{G})$ for every $f \in L^1(G)$ (recall that we simply write f for the measure $f \cdot dm_G$). The Fourier-Stieltjes transform restricted to $L^1(G)$ is usually known as the *Fourier transform*.

The symbol $M_{\widehat{\mu}}$ will denote the multiplication operator $M_{\widehat{\mu}}(f): L^2(\widehat{G}) \rightarrow L^2(\widehat{G})$ given by $M_{\widehat{\mu}}(f)(\chi) = \widehat{\mu}(\chi) \cdot f(\chi)$.

Theorem 2.12. *Let G be a locally compact Abelian group and let $\mu \in M(G)$.*

- (1) *The convolution operator $\lambda_2(\mu) \in \mathcal{L}(L^2(G))$ is unitarily equivalent to the multiplication operator $M_{\widehat{\mu}} \in \mathcal{L}(L^2(\widehat{G}))$. Hence, $\|\lambda_2(\mu)\| = \|\widehat{\mu}\|_\infty$.*
- (2) *Let $K_\mu := \{\widehat{f}: f \in \ker(I - \lambda_2(\mu))\}$ and let $\widehat{P_\lambda(\mu)} \in \mathcal{L}(L^2(\widehat{G}))$ be the projection operator onto K_μ . Then $\widehat{P_\lambda(\mu)} = M_{\mathbf{1}_{A_\mu}}$, where $A_\mu = \widehat{\mu}^{-1}(\{1\})$.*

2.4. Amenable groups. Spectrum of convolution operators. Amenability is a far-reaching property of topological groups that is characterized by the existence of invariant means on their space of uniformly continuous bounded functions. All compact groups and all locally compact Abelian groups are amenable. Free groups and locally compact groups containing

them, such as semisimple Lie groups, are among the most prominent non-amenable groups. We will refrain from defining what an amenable group is. For our purposes, we only need to recall the following properties.

Theorem 2.13. *Let G be a locally compact group, let μ be a positive measure on G and let $1 \leq p < \infty$. Then:*

- (1) (Theorem 3.2.2 of [21], for instance) *If G is amenable, then $\|\mu\| \in \sigma(\lambda_p(\mu))$.*
- (2) ([6, Théorème 5]) *H_μ is amenable if and only if $r(\lambda_p(\mu)) = \|\mu\|$.*

Proposition 2.14 below is well known when G is Abelian. It is also well-known that $\sigma(\lambda_p(\mu)) \subseteq \sigma(\lambda_1(\mu)) = \sigma(\mu)$ for every locally compact group, not necessarily amenable, and every $\mu \in M(G)$. The inclusion $\sigma(\lambda_2(\mu)) \subseteq \sigma(\lambda_p(\mu))$ for amenable groups is not, as far as we know, explicitly stated in the literature.

Proposition 2.14. *Let G be an amenable locally compact and let $\mu \in M(G)$. Then, for any $1 \leq p \leq q \leq 2$ or $2 \leq q \leq p$, $\sigma(\lambda_q(\mu)) \subseteq \sigma(\lambda_p(\mu))$.*

Proof. Let $z \in \mathbb{C}$, $z \notin \sigma(\lambda_p(\mu))$. Then there is an operator $S \in \mathcal{L}(L^p(G))$, such that $(\lambda_p(\mu) - zI) \circ S = S \circ (\lambda_p(\mu) - zI) = I$. Since $(\lambda_p(\mu) - zI)$ commutes with right translations, so will do its inverse S . In the terminology of [24] this means that $S \in \text{Conv}_p(G)$. By [24, Theorem C] (proved in Corollary of [26], see also Section 8.3 of [13]) we have then that $S \in \mathcal{L}(L^q(G))$, what means that $z \notin \sigma(\lambda_q(G))$. \square

2.5. Fixed points of convolution operators. The following theorem is part of Corollary 6.6 of [38]. For $\mu \geq 0$, it can be deduced from the results of Derriennic [14, Théorème 8] (also obtained by Mukherjea [35, Theorem 2] in the second countable case) and Dérriennic and Lin, [15, Proposition 2.1].

Theorem 2.15. *Let G be a locally compact group and let $\mu \in M(G)$ be a measure with $\|\mu\| \leq 1$ that satisfies one of the following conditions:*

- (1) *H_μ is not compact.*
- (2) *H_μ is compact and $\|\mu\| < 1$.*
- (3) *H_μ is compact, $\|\mu\| = 1$ and $\mu \neq \chi|\mu|$, for any $\chi \in \widehat{H}_\mu$.*

*If $f \in L_p(G)$, $1 \leq p < \infty$, and $\mu * f = f$ (a.e.), then $f = 0$ (a.e.)*

The impact of Theorem 2.15 in the ergodic behaviour of convolution operators is reflected in the following two consequences

Proposition 2.16. *Let G be a locally compact group and let $\mu \in M(G)$ be a measure with $\|\mu\| \leq 1$ such that H_μ is not compact. Then $(\lambda_p(\mu_{[n]})f)_n$ converges to 0 for each $f \in L_p(\mu)$.*

Proof. Since $L_p(\mu)$ is reflexive and $\|\lambda_p(\mu)\| \leq 1$, we get from Corollary 2.4 that $(\lambda_p(\mu))$ is mean ergodic, and $(\lambda_p(\mu_{[n]}))_n$ converges in the strong operator topology to the projection P on the fixed points of $\lambda_p(\mu)$. Theorem 2.15 yields $P = 0$. \square

Proposition 2.17. *Let G be a locally compact group and let $\mu \in M(G)$ be a measure with $\|\mu\| \leq 1$ such that H_μ is not compact. The following assertions are equivalent for $1 \leq p < \infty$:*

- (1) $\lambda_p(\mu)$ is uniformly mean ergodic.
- (2) $1 \notin \sigma(\lambda_p(\mu))$.

Proof. Follows from Theorem 2.15 and (6) of Theorem 2.8. \square

2.6. Vague ergodicity. Using convolution as multiplication, it also makes sense to consider the ergodic behaviour of a measure without reference to the operator it induces. The limiting process is then studied under the $\sigma(M(G), C_0(G))$ topology which is usually called the *vague* topology. Notice that by the Alaoglu-Bourbaki theorem, this topology agrees, on bounded subsets of $M(G)$, with the $\sigma(M(G), C_{00}(G))$ -topology

Definition 2.18. Let G be a locally compact group and let $\mu \in M(G)$. For $n \in \mathbb{N}$, we define $\mu_{[n]} = \frac{\mu + \mu^2 + \dots + \mu^n}{n}$. We say that μ is a vague-ergodic measure if there is a measure $\mu_c \in M(G)$ such that, in the vague topology,

$$\lim_{n \rightarrow \infty} \mu_{[n]} = \mu_c.$$

Probability measures are always vague-ergodic. We record here this important classical result that can be deduced from the Mean Ergodic Theorem. Although it is usually proved assuming the group is second countable, see for instance Theorem 3.0 of [22], this assumption is quite straightforward to avoid, see, e.g., Lemma 4.1 of [38]. For compact groups, a proof dating back to 1960 can be found in [42, Theorem 3]. See Theorem 3.4 for a more general statement.

Theorem 2.19. If G is a locally compact group, then every probability measure μ is vague-ergodic and if $\mu_c = \lim_{n \rightarrow \infty} \mu_{[n]}$, then μ_c is a convolution idempotent measure.

3. GENERAL RESULTS

In this section we develop tools that are not directly related with ergodicity but have a strong impact in our work. Some hints on that impact are also included in this section.

We first establish the continuity properties of the regular representation on $M(G)$. These properties, will permeate most sections of the paper.

Our second tool will be a well-known result on the vague convergence of powers of probability measures. This has a clear impact on the existence of fixed points for the corresponding convolution operator.

3.1. Continuity properties of regular representations on $M(G)$. We next establish the continuity properties of the left regular representation on $M(G)$. This is necessary to connect vague-ergodicity of a measure with the mean ergodicity of the corresponding convolution operator.

Proposition 3.1. Let G be a locally compact group and consider the mapping $\lambda_p : M(G) \rightarrow \mathcal{L}(L^p(G))$.

- (1) λ_p is vague-WOT continuous on norm bounded subsets of $M(G)$, for every $1 < p < \infty$.
- (2) Let $H \subseteq G$ be a compact subgroup and let $(\mu_n)_n$ be a sequence vaguely convergent to 0 such that $\text{supp}(\mu_n) \subseteq H$ for all $n \in \mathbb{N}$. Then $(\|\mu_n * f\|_p)_n$ is convergent to 0 for each $f \in L_p(G)$, $1 \leq p < \infty$.

Consequently, if G is compact then λ_p is vague-SOT sequentially continuous for every $1 \leq p < \infty$.

Proof. We start with (1). Let $(\mu_\alpha)_\alpha$ be a (norm) bounded net that converges vaguely to 0. Since $(\lambda_p(\mu_\alpha))_\alpha$ is also norm bounded, and $C_{00}(G)$ is norm dense in $L^p(G)$, we only need to show that $(\lambda^p(\mu_\alpha)f)_\alpha$ is weakly convergent to 0 in $L^p(G)$ for each $f \in C_{00}(G)$. As $(\lambda^p(\mu_\alpha)f)_\alpha$ is a bounded net in $L^p(G)$, and $C_{00}(G)$ is dense in $L^p(G)$, it will again suffice to prove that $\lim_\alpha \langle \lambda_p(\mu_\alpha)f, h \rangle = 0$ for every $h \in C_{00}(G)$.

So, let $f, h \in C_{00}(G)$. We first observe that, by Fubini's theorem,

$$\langle \lambda_p(\mu_\alpha)f, h \rangle = \langle \mu_\alpha, (h * \check{f}) \rangle,$$

where the first bracket refers to the $(L^p(G), L^p(G))$, the second to the $(M(G), C_{00}(G))$ -duality and, for any $u: G \rightarrow \mathbb{C}$, $\check{u}(t) = u(t^{-1})$.

Once this is clear, it immediately follows from the vague convergence of $(\mu_\alpha)_\alpha$ and the inclusion $C_{00}(G) * L^p(G) \subseteq C_0(G)$, that

$$\lim_\alpha \langle \lambda_p(\mu_\alpha)(f), h \rangle = \lim_\alpha \langle \mu_\alpha, (h * \check{f}) \rangle = 0.$$

For Statement (2) we start with a sequence $(\mu_n)_n$ made of measures supported in the compact subgroup H that is vaguely convergent to 0. As before, we only need to check that $\|\lambda_p(\mu_n)f\|_p$ converges to 0 for every $f \in C_{00}(G)$. Let $K := \text{supp}(f)$

Now,

$$\|\lambda_p(\mu_n)f\|_p^p = \int \left| \langle \mu_n, \check{f}_x \rangle \right|^p dx,$$

where f_x denotes the right translate of f by x , so that $\check{f}_x(t) = f(t^{-1}x)$. As before, vague convergence of $(\mu_n)_n$ implies that $\lim_n \left| \langle \mu_n, \check{f}_x \rangle \right|^p = 0$ for every $x \in G$. Taking into account

$$\left| \langle \check{f}_x, \mu_n \rangle \right|^p \leq \|f\|_\infty^p \cdot \sup_n \|\mu_n\|_{M(G)}^p \mathbf{1}_{HK}(x),$$

we can apply Lebesgue dominated convergence to conclude $\lim_n \|\lambda_p(\mu_n)f\|_p = 0$, as we wanted to prove. \square

Vague-WOT continuity of λ_1 and Vague-SOT convergence of λ_p ($1 < p < \infty$) on bounded sets, both fail when G is not compact, as the following example shows.

Example 3.2. *If G is an infinite discrete group, the mapping $\lambda_1: M(G) \rightarrow \mathcal{L}(\ell^1(G))$ is not vague-WOT continuous on bounded sets, and, for $1 < p < \infty$, $\lambda_p: M(G) \rightarrow \mathcal{L}(\ell^p(G))$ is not vague-SOT sequentially continuous.*

Proof. We prove that λ_p fails to be vague-SOT sequentially continuous for $1 \leq p < \infty$. Since $\ell^1(G)$ has the Schur property, this implies both statements. Let $(s_n)_n$ be a faithfully indexed sequence in G . The sequence $(\delta_{s_n})_n$ clearly converges vaguely to 0. Take any $f \in \ell^1(G)$, $f \neq 0$. Then, for any $1 \leq p < \infty$,

$$\|\lambda_p(\delta_{s_n})f\|_p = \|f\|_p.$$

Hence, $(\lambda_p(\delta_{s_n})f)_n$ cannot converge to 0 in $\ell^p(G)$. \square

Theorem 3.1 establishes an immediate relation between vague-ergodicity and mean ergodicity of convolution operators.

Proposition 3.3. *Let G be a locally compact group and $\mu \in M(G)$. The following statements are equivalent:*

- (1) μ is vague-ergodic.
- (2) μ is Cesàro bounded and $\lim_n \frac{\mu^n}{n} = 0$, vaguely.
- (3) μ is Cesàro bounded and $\lambda_p(\mu)$ is weakly mean ergodic for all $1 < p < \infty$.
- (4) μ is Cesàro bounded and $\lambda_p(\mu)$ is weakly mean ergodic for some $1 < p < \infty$.

If H_μ is compact we can replace (3) and (4) by

- (3)' μ is Cesàro bounded and $\lambda_p(\mu)$ is mean ergodic for all $1 \leq p < \infty$.
- (4)' μ is Cesàro bounded and $\lambda_p(\mu)$ is mean ergodic for some $1 \leq p < \infty$.

Proof. We prove first the necessity of the conditions. If μ is vague-ergodic it is clear that $(\|\mu_{[n]}\|)_n$ is bounded. The equality,

$$\frac{\mu^n}{n} = \mu_{[n]} - \frac{n-1}{n} \mu_{[n-1]},$$

shows that $\frac{\mu^n}{n}$ converges vaguely to 0. Therefore (1) implies (2)

We assume now that (2) holds and fix $1 < p < \infty$. We deduce from Proposition 3.1 that $(\lambda_p(\mu)_{[n]})_n$ is a bounded sequence in $\mathcal{L}(L^p(G))$ and that $\lim \lambda_p(\mu)^n/n = 0$ in the weak operator topology. Theorem 2.3 then implies that $\lambda_p(\mu)$ is weakly mean ergodic.

Statement (3) certainly implies (4). Assume now (4), i.e., that $(\mu_{[n]})_n$ is norm bounded and $(\lambda_p(\mu)_{[n]})_n$ converges in the weak operator topology to some $P \in \mathcal{L}(L^p(G))$. Since the sequence $(\mu_{[n]})_n$ is norm bounded, Theorem 3.1 (i) implies that for any accumulation point μ of $(\mu_{[n]})_n$, $\lambda_p(\mu) = P$. It follows that $(\mu_{[n]})_n$ has only precisely one accumulation point, it is therefore convergent.

If H_μ is compact, the equivalence with properties (3)' and (4)' is proved analogously using Proposition 3.1 (2). \square

Convolution operators $\lambda_p(\mu)$ can be mean ergodic and yet μ fail to be vague ergodic, even if G is compact and Abelian, see Remarks 4.4. For an example with $\mu \geq 0$, see Example 7.5.

The foregoing theorem and Corollary 3.3 lead to the following generalization of Theorem 2.19 that goes beyond probability measures.

Theorem 3.4. *Let G be a locally compact group and let $\mu \in M(G)$. If $(\|\mu^n\|)_n$ is a bounded sequence, then μ is vague-ergodic.*

As a further corollary we deduce [38, Corollary 6.4] from Corollary 6.6 loc. cit, as a further sample of how easily results on L^p -ergodicity can be turned into results on ergodicity of measures. Note that the authors of [38] proceed in the opposite direction and obtain Theorem 2.15 from Corollary 3.5.

Corollary 3.5. *Let G be a locally compact group and let $\mu \in M(G)$ with $\|\mu\| \leq 1$. Then $(\mu_{[n]})_n$ is vague convergent to 0 if and only if one of the following conditions hold:*

- (1) H_μ is not compact.
- (2) H_μ is compact and $\|\mu\| < 1$.
- (3) H_μ is compact, $\|\mu\| = 1$ and $\mu \neq \chi|\mu|$, for any $\chi \in \widehat{H}_\mu$.

Proof. By Proposition 3.4 μ is vague ergodic, i.e. there is $\mu_c \in M(G)$ such that $(\mu_{[n]})_n$ is convergent to μ_c in the vague topology. Also by Proposition 3.3 $(\lambda_2(\mu_{[n]}))_n$ is convergent to $\lambda_2(\mu_c)$, which must be the projection on the fixed points of $\lambda_2(\mu)$.

If any of the statements (1)–(3) hold, Theorem 2.15 shows that $\lambda_2(\mu)$ has no fixed points, and, therefore $\mu_c = 0$.

If, conversely, none of them holds then $\mu = \chi|\mu|$, with H_μ being compact. One can easily check that, in this situation, the function $\chi \cdot \mathbf{1}_{H_\mu}$ is a nontrivial fixed point of $\lambda_2(\mu)$, showing that $\lambda_2(\mu_c)$ is a non null projection. \square

Remark 3.6. The condition $\|\mu\| \leq 1$ in Corollary 3.5 is imposed by its dependence on Theorem 2.15. The simple proof of Corollary 3.5 will remain valid under any condition on μ that keeps $(\|\mu^n\|)_n$ bounded and makes sure that $\mu_c = 0$. Since μ_c is an idempotent measure, this latter condition is freely obtained in groups that do not admit nontrivial idempotent measures. Any locally compact Abelian group with no nontrivial compact subgroup satisfies this property, [44, Theorem 3.3.2].

4. OPERATOR-NORMAL MEASURES

In this section we restrict our study to measures that give rise to convolution operators that are normal, i.e., to operator-normal measures according to our definitions in Section 2. This will automatically involve all measures when G is Abelian.

Normal operators on Hilbert spaces satisfy the identity $r(T) = \|T\|$, and that greatly simplifies the analysis of mean ergodicity. The following is an easy consequence of Theorems 2.2 and 2.8.

Theorem 4.1. *Let \mathbb{H} be a Hilbert space and let $T \in \mathcal{L}(\mathbb{H})$ be a normal operator. Then*

- (1) *The operator T is mean ergodic if and only if it is weakly mean ergodic if and only if $\|T\| \leq 1$.*
- (2) *The operator T is uniformly mean ergodic if and only if $\|T\| \leq 1$ and 1 is not an accumulation point of $\sigma(T)$.*

Proof. Since $\|T\| = r(T)$, corollaries 2.4 and 2.6 imply that T is (weakly) mean ergodic if and only if $\|T\| \leq 1$.

If T is uniformly mean ergodic it follows from (1) implies (3) on Theorem 2.8 that 1 cannot be an accumulation point of $\sigma(T)$.

For the converse we only have to recall that for a normal operator T and $\lambda \in \mathbb{C}$, $\text{Ran}(T - \lambda I)$ is closed if and only if λ is not an accumulation point of $\sigma(T)$, see, e.g., [10, Proposition 4.5 of Chapter XI]. Hence T is uniformly mean ergodic if 1 is not an accumulation point of $\sigma(T)$, by (4) implies (1) of Theorem 2.8. \square

Corollary 4.2. *Let \mathbb{H} be a Hilbert space and let $T \in \mathcal{L}(\mathbb{H})$ be a normal operator. Then T is uniformly mean ergodic whenever T^2 is.*

Proof. If T is normal also T^2 is. Assume that T is not uniformly mean ergodic. Then 1 is an accumulation point of $\sigma(T)$, and therefore also is an accumulation point of $\sigma(T^2) = \sigma(T)^2$. Therefore T^2 is not uniformly mean ergodic. \square

We will see in Example 7.1 that the converse of Corollary 4.2 is not true.

4.1. Mean ergodicity of normal convolution operators. We can now completely characterize the mean ergodicity of $\lambda_2(\mu)$ when μ is operator-normal. This provides a complete characterization of the mean ergodicity of $\lambda_2(\mu)$ when G is Abelian and $\mu \in M(G)$.

Theorem 4.3. *Let G be a locally compact group G and let μ be an operator-normal measure on G . Then the following assertions are equivalent:*

- (1) *The operator $\lambda_2(\mu)$ is mean ergodic.*
- (2) *The operator $\lambda_2(\mu)$ is weakly mean ergodic.*
- (3) *$r(\lambda_2(\mu)) = \|\lambda_2(\mu)\| \leq 1$.*
- (4) *The operator $\lambda_2(\mu)$ is power bounded.*
- (5) *The operator $\lambda_2(\mu)$ is Cesàro bounded.*

Proof. The equivalence of Statements (1) to (3) follows at once from Theorem 4.1. It is obvious that (3) implies (4) and (4) implies (5). Proposition 2.5 and the normality of $\lambda_2(\mu)$ show that Statement (5) implies Statement (3). \square

Remarks 4.4. The assumptions of Theorem 4.3 cannot completely removed and its conclusions cannot be easily strengthened.

- (1) Example 6.24 of [45] shows that every nondiscrete locally compact Abelian group contains a measure $\mu \in M(G)$ with $\|\mu^n\| \geq 2^n$ and $r(\lambda_2(\mu)) = \|\lambda_2(\mu)\| = \|\hat{\mu}\|_\infty < 1$. The operator $\lambda_2(\mu)$ is mean ergodic but the measure μ is not Cesàro bounded, let alone vague-ergodic. After Theorem 3.3, one deduces that, at least for operator-normal measures, vague-ergodicity is strictly stronger than weak mean ergodicity of the convolution operator.
- (2) Both the normality condition $\mu^* * \mu = \mu * \mu^*$ and the restriction to the Hilbert case $p = 2$ can be removed if $\mu \geq 0$ and H_μ is amenable, see Theorem 5.1 *infra*. Example 7.5 shows that Theorem 4.3 is no longer true when μ is not operator-normal and H_μ is not amenable, even if μ is positive.
- (3) If we keep the condition $\mu^* * \mu = \mu * \mu^*$ but consider $p \neq 2$, the result also fails, as witnessed by the following example.

Example 4.5. *Let $p > 2$. On $G = \mathbb{Z}_3$, the cyclic group of order 3, there is a measure $\mu \in M(G)$ such that $\|\lambda_p(\mu)\| > 1$ but $\lambda_p(\mu)$ is uniformly mean ergodic.*

Proof. If $G = \{e, x, x^2\}$, let $\mu = (\delta_x - \delta_{x^2})$. An elementary computation yields that, for any p . $r(\lambda_p(\mu)) = \|\lambda_2(\mu)\| = \sqrt{3}$. Computing $\|\lambda_p(\mu)f\|_p$ with $f = 6^{-1/p}\delta_e + 6^{-1/p}\delta_x - (3/2)^{-1/p}\delta_{x^2}$, one sees that, for every p ,

$$\|\lambda_p(\mu)\|^p \geq \frac{1}{3} \left(1 + 4^{\frac{1}{p}}\right)^p.$$

It is easy to check then that for every $p > 2$, $\|\lambda_p(\mu)\| > \sqrt{3}$. Pick now t with $\|\lambda_p(\mu)\| > t > \sqrt{3}$. The measure $\mu_t = \frac{1}{t}\mu$ is then the desired measure with $\|\lambda_p(\mu_t)\| > 1$ but $r(\lambda_p(\mu_t)) < 1$. \square

4.2. Uniform mean ergodicity of normal convolution operators. Turning to uniform mean ergodicity the direct consequence of Theorem 4.1 is the following.

Theorem 4.6. *Let G be a locally compact group G and let μ be an operator-normal measure on G . Then the following assertions are equivalent:*

- (1) *The operator $\lambda_2(\mu)$ is uniformly mean ergodic.*
- (2) *$\|\lambda_2(\mu)\| = r(\lambda_2(\mu)) \leq 1$ and 1 is not an accumulation point of $\sigma(\lambda_2(\mu))$.*

We next extend the preceding Theorem to the case $p \neq 2$. This will follow from Proposition 4.9 which ensures that if uniform mean ergodicity of $\lambda_q(\mu)$ is assumed for some q , then condition (3) in Theorem 2.8 can be relaxed as in Theorem 4.1 to characterize uniform mean ergodicity for arbitrary p . We first need a known result on extension of vector-valued holomorphic functions.

Definition 4.7. *Let X be a Banach space. A subset $H \subseteq X^*$ is said to be separating in X when $x^*(x) = 0$ for all $x^* \in H$ implies $x = 0$.*

Proposition 4.8. [19, Theorem 1],[9, Corollary 10, Remark 11] *Let X be a Banach space, H a separating subspace of X^* , $\Omega \subseteq \mathbb{C}$ a domain, $a \in \Omega$. Let $f: \Omega \setminus \{a\} \rightarrow X$ be a holomorphic function such that $x^* \circ f$ admits holomorphic extension to Ω for each $x^* \in H$. Then f admits a (unique) holomorphic extension to Ω .*

Proposition 4.9. *Let G be a locally compact group and let $\mu \in M(G)$. Assume there is $q > 1$ such that $\lambda_q(\mu)$ is uniformly mean ergodic. Then, for each $1 \leq p \leq \infty$, $\lambda_p(\mu)$ is uniformly mean ergodic if and only if $\lim \|\lambda_p(\mu^n)\|/n = 0$ and 1 is not an accumulation point of $\sigma(\lambda_p(\mu))$.*

Proof. The necessity follows from (1) implies (3) in Theorem 2.8. We prove the converse. Let $r > 0$ such that $(B(1, r) \setminus \{1\}) \cap \sigma(\lambda_p(\mu)) = \emptyset$ and $(B(1, r) \setminus \{1\}) \cap \sigma(\lambda_q(\mu)) = \emptyset$. The resolvent mapping restricted to $B(1, r) \setminus \{1\}$

$$R(\cdot, \lambda_p(\mu)): B(1, r) \setminus \{1\} \rightarrow \mathcal{L}(L^p(G))$$

is then a holomorphic function. From uniform mean ergodicity in $L_q(G)$ and Theorem 2.8 we get that, for each $f, g \in C_{00}(G)$, the function

$$z \mapsto \langle (z - 1)R(z, \lambda_q(\mu))(f), g \rangle,$$

$z \in B(1, r) \setminus \{1\}$, is holomorphic and admits a holomorphic extension to 1. For $f, g \in C_{00}(G)$ define $I_{f,g} \in \mathcal{L}(L_q(G))^*$ by $I_{f,g}(T) = \langle T(f), g \rangle$. Observe that $\{I_{f,g} : f, g \in C_{00}(G)\}$ is a separating subset of $\mathcal{L}(L_p(G))^*$. We conclude from $R(z, \lambda_q(\mu))(f) = R(z, \lambda_p(\mu))(f)$ for each $f \in C_{00}(G)$ that the function $B(1, r) \setminus \{1\} \rightarrow \mathbb{C}$, $z \mapsto \langle (z - 1)R(z, \lambda_p(\mu))(f), g \rangle$ admits a holomorphic extension to 1 for each $f, g \in C_{00}(G)$. The conclusion follows from Proposition 4.8 and (3) implies (1) in Theorem 2.8. \square

Theorem 4.10. *Let G be a locally compact group and let μ be an operator-normal measure. Assume also that H_μ is amenable. The following assertions are equivalent for $1 \leq p \leq \infty$.*

- (1) $\lambda_p(\mu)$ is uniformly mean ergodic.
- (2) 1 is not an accumulation point of $\sigma(\lambda_p(\mu))$, and $\lim_n \|\lambda_p(\mu^n)\|/n = 0$.

Proof. From $\lim_n \|\lambda_p(\mu^n)\|/n = 0$ and the spectral radius formula it follows that $r(\lambda_p(\mu)) \leq 1$. Combining Theorem 4.6 with Proposition 2.14 we get that (2) implies that $\lambda_2(\mu)$ is uniformly mean ergodic. Proposition 4.9 then proves that Statement (2) implies Statement (1). The other direction comes directly from Theorem 2.8. \square

4.3. Abelian groups. When G is Abelian, Corollaries 4.3 and 4.6 can be rephrased, using Theorem 2.12, in terms of Fourier-Stieltjes transforms.

Corollary 4.11. *Let G be a locally compact Abelian group G and let $\mu \in M(G)$. Then:*

- (1) *The operator $\lambda_2(\mu)$ is mean ergodic if and only if $\|\widehat{\mu}\|_\infty \leq 1$.*
- (2) *The operator $\lambda_2(\mu)$ is uniformly mean ergodic if and only if $\|\widehat{\mu}\|_\infty \leq 1$ and $1 \notin \overline{\{\widehat{\mu}(\chi) : \chi \in \widehat{G}, \widehat{\mu}(\chi) \neq 1\}}$.*

There is an obvious relation between mean ergodicity of $\lambda_2(\mu)$ and how A_μ embeds topologically in \widehat{G} . Recall, Section 2.3, that $A_\mu = \widehat{\mu}^{-1}(\{1\})$ and hence that A_μ is always a closed set. We now clarify this.

Corollary 4.12. *Let G be a locally compact Abelian group and let $\mu \in M(G)$. If $\lambda_2(\mu)$ is uniformly mean ergodic, then A_μ is a closed and open subset of \widehat{G} .*

Proof. If $\lambda_2(\mu)$ is uniformly mean ergodic, we deduce from (2) of Corollary 4.11 that there is $\delta > 0$ such that:

$$|\widehat{\mu}(\chi) - 1| > \delta \quad \text{for all } \chi \in \widehat{G} \text{ with } \chi \notin A_\mu.$$

We see then that $\widehat{\mu}^{-1}(1 - \delta, 1 + \delta) = A_\mu$ and therefore that A_μ is open. \square

Remark 4.13. It is easy to find measures with A_μ open that do not produce uniformly mean ergodic operators $\lambda_2(\mu)$. Take for instance $G = \mathbb{T}$ and $\mu = \delta_s$ for any $s = e^{2\pi i x}$ with $x \notin \mathbb{Q}$. Identifying $\widehat{\mathbb{T}}$ with the discrete group \mathbb{Z} we have that $\widehat{\mu}: \mathbb{Z} \rightarrow \mathbb{C}$ is defined by $\widehat{\mu}(k) = s^k$ for $k \in \mathbb{Z}$. The range of $\widehat{\mu}$ is well-known to be dense in \mathbb{T} , whence we see that μ does not satisfy condition (2) of Corollary 4.11. $\widehat{\mathbb{T}}$ being discrete, A_μ is sure open.

The situation in this respect is quite different if $\mu \in L^1(\mathbb{T})$ or, more generally, if $\widehat{\mu} \in c_0(\mathbb{Z})$, as we next see.

Definition 4.14. *If G is a locally compact Abelian group, we denote by $M_0(G)$ the set of measures $\mu \in M(G)$ such that $\widehat{\mu} \in C_0(\widehat{G})$.*

Since the Riemann-Lebesgue theorem proves that $L^1(G) \subset M_0(G)$, the following Corollary applies in particular to all $\lambda_2(f)$ with $f \in L^1(G)$.

Corollary 4.15. *Let G be an locally compact Abelian group and let $\mu \in M_0(G)$. Then $\lambda_2(\mu)$ is uniformly mean ergodic if and only if $\|\widehat{\mu}\|_\infty \leq 1$ and A_μ is a closed and open subset of \widehat{G} .*

Proof. After Theorem 4.11 and Corollary 4.12, we only have to prove the sufficiency part.

Suppose therefore that A_μ is an open (and closed) subset of \widehat{G} but there is a sequence $(\chi_n)_n$ in \widehat{G} with $\chi_n \notin A_\mu$ for every $n \in \mathbb{N}$ such that $1 = \lim_n \widehat{\mu}(\chi_n)$. Since $\widehat{\mu} \in C_0(\widehat{G})$, we deduce that there is a compact subset K of \widehat{G} such that $\chi_n \in K$ for every n . There is then $\chi_0 \in \widehat{G}$ and a subnet $(\chi_\alpha)_\alpha$ of $(\chi_n)_n$ such that $\lim_\alpha \chi_\alpha = \chi_0$. Since $\lim_n \widehat{\mu}(\chi_n) = 1$, we have that $\chi_0 \in A_\mu$; but A_μ being open, this would imply that the net $(\chi_\alpha)_\alpha$ is eventually in A_μ . \square

If G is compact, then \widehat{G} is discrete, we thus have:

Corollary 4.16. *Let G be a compact Abelian group and let $\mu \in M_0(G)$. The following assertions are then equivalent:*

- (1) $\lambda_2(\mu)$ is power bounded.
- (2) $\lambda_2(\mu)$ is mean ergodic.
- (3) $\|\widehat{\mu}\|_\infty \leq 1$.
- (4) $\lambda_2(\mu)$ is uniformly mean ergodic.

Theorem 4.10 suggests that, for a given $\mu \in M(G)$, the uniform mean ergodicity of $\lambda_p(\mu)$ may depend on p . To confirm this we will need the following result due to S. Igari.

Theorem 4.17 (Particular case of Theorem 1 of [28]). *Let G be a nondiscrete locally compact Abelian group, let $1 \leq p < 2$ and let $\Phi: [-1, 1] \rightarrow \mathbb{C}$. If Φ does not extend to an entire function, there are $\mu \in M(G)$ with $\widehat{\mu}(\widehat{G}) \subseteq [-1, 1]$, and $h \in L^p(G)$ such that $(\Phi \circ \widehat{\mu}) \cdot \widehat{h}$ is not the Fourier transform of any function in $L^p(G)$.*

Proposition 4.18. *For any nondiscrete locally compact Abelian group G and for every $1 \leq p < 2$ there is a measure $\mu \in M(G)$ such that $\lambda_2(\mu)$ is uniformly mean ergodic but $\lambda_p(\mu)$ is not.*

Proof. Define $\Phi: [-1, 1] \rightarrow \mathbb{C}$ by $\Phi(t) = 1/(t - z)$ where $z \in \mathbb{T} \setminus [-1, 1]$. Since Φ cannot be extended to an entire function we can find $\mu_z \in M(G)$ and $h_z \in L^p(G)$ with the properties of Theorem 4.17.

Suppose that $z \notin \sigma(\lambda_p(\mu_z))$. In that case, there would be an operator $T \in \mathcal{L}(L^p(G))$ such that $\mu_z * Tf - zTf = f$ for every $f \in L^p(G)$. Taking Fourier-Stieltjes transforms we see that, for every $f \in L^p(G)$,

$$\widehat{Tf}(\widehat{\mu}_z - z) = \widehat{f}.$$

The preceding equality applied to $f = h_z$ implies then that

$$\widehat{Th}_z = (\Phi \circ \widehat{\mu}_z) \cdot \widehat{h}_z,$$

which goes against the choice of μ_z and h_z from Theorem 4.17. We conclude that $z \in \sigma(\lambda_p(\mu_z))$.

Once we have found $\mu_z \in M(G)$ with $\widehat{\mu}_z(\widehat{G}) \subseteq [-1, 1]$ and $z \in \sigma(\lambda_p(\mu_z))$ we consider $\mu = \overline{z}\mu_z$. Then

$$\sigma(\lambda_2(\mu)) = \overline{\widehat{\mu}_z(\widehat{G})}$$

is contained in a diameter of the unit circle not passing through 1. According to Theorem 4.6, the operator $\lambda_2(\mu)$ is then uniformly ergodic. On the other

hand, $1 \in \sigma(\lambda_p(\mu))$. Since isolated points of the spectrum are necessarily in the range of the Fourier-Stieltjes transform (see [48, Lemma 2.2]) and $1 \notin \widehat{\mu}(\widehat{G})$, we deduce, now from Theorem 4.10, that $\lambda_p(\mu)$ is not uniformly mean ergodic. \square

5. POSITIVE MEASURES: MEAN ERGODICITY

This section is devoted to make clear that the ergodic behaviour of the operators $\lambda_p(\mu)$ is simpler when μ is positive.

5.1. Positive measures supported in an amenable subgroup: reflexive case. In this section we analyze mean ergodicity of convolution operators induced by positive measures whose support is contained in an amenable subgroup. The set of techniques at reach for this case is much richer and leads to conclusive results.

Theorem 5.1. *Let G be a locally compact group and let $\mu \in M(G)$ be a positive measure with H_μ amenable. If $1 < p < \infty$, the following assertions are equivalent:*

- (1) $\lambda_p(\mu)$ is power bounded.
- (2) $\lambda_p(\mu)$ is mean ergodic.
- (3) $\lambda_p(\mu)$ is weakly mean ergodic.
- (4) $\lambda_p(\mu)$ is Cesàro bounded.
- (5) $\|\mu\| \leq 1$.
- (6) μ is vague-ergodic.

Proof. By Corollary A.4 we can restrict ourselves to the case when G itself is amenable. The Mean Ergodic Theorem (Corollary 2.4) proves that (1) implies (2). This one obviously implies (3) and, by Banach-Steinhaus, (3) implies (4). Since $\|\mu\| = r(\lambda_p(\mu))$, by Theorem 2.13, Corollary 2.5 shows that (4) implies (5). Since $r(\lambda_p(\mu)) = \|\mu\|$ implies that $\|\lambda_p(\mu)\| = \|\mu\|$, Statement (5) certainly implies Statement (1). Hence, statements (1)–(5) are all equivalent.

Finally, (5) implies (6), by Theorem 3.4, and (6) implies (3) by Theorem 3.3. \square

Remark 5.2. Theorem 5.1 does not hold if H_μ is not amenable, see Example 7.5.

5.2. Positive measures: mean ergodicity in $L^1(G)$. Mean ergodicity of $\lambda_1(\mu)$ is a much more restrictive condition, as we now see. Here we are not assuming conditions a priori on H_μ . First of all, we observe that we can reduce our study to probability measures.

Proposition 5.3. *Let G be a locally compact group and let $\mu \in M(G)$ be positive.*

- (1) *If $\|\mu\| < 1$ then $(\lambda_1(\mu^n))_n$ is norm convergent to 0, and then $\lambda_1(\mu)$ is uniformly mean ergodic.*
- (2) *If $\|\mu\| > 1$ then $\lambda_1(\mu)$ is not mean ergodic.*

Proof. Observe that, for each $n \in \mathbb{N}$, $\|\lambda_1(\mu^n)\| = \|\mu^n\| = \mu(G)^n$. Thus (1) is immediate and (2) follows from the unboundedness of the sequence $(\|\lambda_1(\mu^n)\|/n)_n$ when $\mu(G) > 1$. \square

Theorem 5.4. *Let μ be a probability measure on G . Then $\lambda_1(\mu)$ is mean ergodic if and only if H_μ is compact.*

Proof. If H_μ is compact, we only have to apply (2) implies (3)' of Theorem 3.3.

Assume now that H_μ is not compact. By Corollaries 2.4 and 2.15, $\lambda_2(\mu)$ is a mean ergodic operator without nonzero fixed points. As a consequence, $\|\lambda_2(\mu)_{[n]}f\|_2$ must converge to 0 for every $f \in L^2(G)$.

Let $f \in C_{00}(G)$, $f \geq 0$. By the above $(\langle \lambda_1(\mu)_{[n]}f, g \rangle)_n$ converges to 0 for each $g \in C_{00}(G)$, i.e. $(\lambda_1(\mu)_{[n]}f)_n$ converges to 0 in the $\sigma(L^1(G), C_{00}(G))$ -topology. Since we are assuming that $\lambda_1(\mu)$ is mean ergodic, $\lambda_1(\mu)_{[n]}f$ must converge in $L^1(G)$. But, as the $\sigma(L^1, C_{00}(G))$ -topology is Hausdorff and is weaker than the norm topology, we conclude that, in fact,

$$(3) \quad \lim_n \|\lambda_1(\mu)_{[n]}f\|_1 = 0.$$

This actually holds for every $f \in L^1(G)$ but we will only need that fact for some $f \in C_{00}(G)$, $f \geq 0$, $f \neq 0$.

A simple application of Fubini's theorem shows that, for every $f \in L^1(G)$,

$$\int (\mu * f)(x) \, dm_G(x) = \int f(x) \, dm_G(x)$$

Then, for any $f \in C_{00}(G)$, $f \geq 0$,

$$\|f\|_1 = \|\mu_{[n]} * f\|_1 = \|\lambda_1(\mu)_{[n]}f\|_1,$$

If we let n go to infinity and apply (3) we reach a contradiction as soon as $f \neq 0$. \square

Remark 5.5. As can be remarked in the proof of Theorem 5.4, the reason for the failure of $\lambda_1(\mu)$ to be mean ergodic lies in its action against positive functions. A way to avoid these functions is to consider the subspace $L_0^1(G) = \{f \in L^1(G) : \int f \, dm_G = 0\}$. Since this subspace is invariant under the action of λ_1 , one can consider the operator $\lambda_1^0(\mu)|_{L_0^1(G)}$ and study its mean ergodicity. This was done by Rosenblatt [43] who defined a measure $\mu \in M(G)$ to be *ergodic by convolutions* if $\lambda_1^0(\mu)_{[n]}$ converges to 0. In that same paper, Rosenblatt proves that a locally compact group contains a probability measure that is ergodic by convolutions if and only if the group is σ -compact and amenable. When G is compact, the Itô-Kawada Theorem proves that a probability measure $\mu \in M(G)$ is ergodic by convolutions if and only if $H_\mu = G$, and the Choquet-Deny theorem implies that the same assertion is true when G is Abelian [43, Theorems 1.4 and 1.5], see also [11] and [29] and the references therein for more on this property. It follows from the facts just collected that every adapted (that is, with $H_\mu = G$) probability measure supported in a noncompact Abelian group satisfies that $\lambda_1^0(\mu)$ is mean ergodic, while $\lambda_1(\mu)$ is not. On the other hand, if G is compact and Abelian, and $\mu \in M(G)$ is a probability measure which is not adapted, then $\lambda_1(\mu)$ is mean ergodic but $\lambda_1^0(\mu)$ is not. Neither concept is therefore stronger than the other.

We can now complete the panorama of Proposition 3.3, Theorem 5.1 and Theorem 5.4.

Theorem 5.6. *Let G be a locally compact group and let $\mu \in M(G)$ be a positive measure with H_μ compact. For $1 \leq p < \infty$, the following assertions are equivalent:*

- (1) μ is vague-ergodic.
- (2) μ is Cesàro bounded.
- (3) $\|\mu\| \leq 1$.
- (4) $\lambda_p(\mu)$ is mean ergodic.

Proof. That Statement (1) implies Statement (2) is a simple consequence of the Banach-Steinhaus theorem. Next, (2) is the same as Cesàro boundedness of $\lambda_1(\mu)$ (by (3) of Theorem 2.11) and the latter implies, by Proposition 2.5, that $r(\lambda_1(\mu)) \leq 1$. Since, by Proposition 2.13, this implies $\|\mu\| \leq 1$, we see that Statement (2) implies Statement (3). (3) implies (4) by Theorem 5.1 if $p > 1$ and by Theorem 5.4 for $p = 1$. Finally (4) implies (1) follows from Proposition 3.3. \square

6. POSITIVE MEASURES: UNIFORM MEAN ERGODICITY

If μ is positive, the equality $\|\lambda_p(\mu)\| = \|\mu\|$ makes Theorem 4.10 into the following somewhat cleaner characterization.

Theorem 6.1. *Let G be a locally compact group and let μ be a positive operator-normal measure with H_μ amenable. The following assertions are equivalent for $1 \leq p \leq \infty$.*

- (1) $\lambda_p(\mu)$ is uniformly mean ergodic.
- (2) 1 is not an accumulation point of $\sigma(\lambda_p(\mu))$, and $\|\mu\| \leq 1$.

Taking Theorem 5.4 into account, Theorem 6.1 yields the following Corollary. It applies to every probability measure in an Abelian group, although in that case a simpler proof using duality theory can be applied to every measure of norm 1.

Corollary 6.2. *Let G be a locally compact group and let μ be an operator-normal probability measure such that H_μ is amenable and not compact. Then 1 is an accumulation point of $\sigma(\lambda_1(\mu))$ ($= \sigma(\mu)$).*

Theorem 6.1 leads to a complete characterization in the non reflexive case that is valid for, at least, all Abelian G .

Theorem 6.3. *Let G be a locally compact group and let $\mu \in M(G)$ be operator-normal and positive. The following are equivalent*

- (1) $\lambda_1(\mu)$ is uniformly mean ergodic.
- (2) $\lambda_\infty(\mu)$ is uniformly mean ergodic.
- (3) Either $\|\mu\| < 1$, or $\|\mu\| = 1$, H_μ is compact and 1 is not an accumulation point of $\sigma(\mu)$.
- (4) $\lambda_\infty(\mu)$ is mean ergodic.

Proof. Statement (2) is equivalent to Statement (4) by Theorem 2.9. The remaining equivalences follow from Theorem 4.10 and Theorem 5.4, observing that $\lambda_1(\mu)^* = \lambda_\infty(\mu^*)$, $H_\mu = H_{\mu^*}$ and $\sigma(\mu) = \sigma(\mu^*)$ and taking into account that, for a bounded linear operator on a Banach space X , the uniform ergodicity of T is equivalent to that of T^* . \square

Remark 6.4. When G is compact, Theorem 5.6 of [3] can be used to find examples of positive measures with $\lambda_2(\mu)$ uniformly mean ergodic for which $\lambda_1(\mu)$ is not. The measures obtained in [3, Theorem 5.6] are positive, belong to the space $M_0(G)$ formed by the measures whose Fourier Stieltjes transform vanishes at infinity on \widehat{G} , and have the property that $\sigma(\lambda_1(\mu))$ is the whole unit disk (see Lemma 4.1 of [3] for this). For any such μ , $\lambda_2(\mu)$ is uniformly mean ergodic, Corollary 4.16, yet $\lambda_1(\mu)$ is not, Theorem 6.1. Note that $\lambda_1(\mu)$ is mean ergodic by Theorem 5.4.

6.1. Positive measures with noncompact support. When the support of a probability measure μ is not contained in a compact subgroup of G , $\lambda_p(\mu)$ does not have fixed points and its dynamic behaviour is much simpler.

Theorem 6.5. *Let G be a locally compact group and let $\mu \in M(G)$ be a probability measure with H_μ noncompact. If $1 < p < \infty$, the following assertions are equivalent.*

- (1) $\lambda_p(\mu)$ is uniformly mean ergodic.
- (2) $r(\lambda_p(\mu)) < 1$.
- (3) $(\lambda_p(\mu^n))_n$ is norm convergent to 0.
- (4) H_μ is not amenable.

Proof. According to Corollary A.4, we can assume along this proof that $G = H_\mu$. By Theorem 2.13, this immediately shows that Statements (2) and (4) are equivalent.

As for the remaining equivalences, only that (1) implies (4) needs proof. Assume to that end that $\lambda_p(\mu)$ is uniformly mean ergodic. Taking into account that, by Proposition 2.17, $1 \notin \sigma(\lambda_p(\mu))$ Theorem 2.13, applies then to show that (4) holds. \square

For convolution operators of norm at most one induced by measures $\mu = f \, dm_G$ for some positive $f \in L^1(G)$, uniform mean ergodicity can be neatly characterized for all p , $1 \leq p < \infty$.

Theorem 6.6. *Let G be a locally compact group and let $f \geq 0$, $f \in L^1(G)$, with $\|f\| \leq 1$. Then $\lambda_p(f)$, $1 \leq p < \infty$, is uniformly mean ergodic if and only if either $\text{supp}(f)$ is contained in a compact subgroup of G or $r(\lambda_p(f)) < 1$.*

Proof. After Theorems 6.5 and 5.4, it only remains to prove that $\lambda_p(f)$ is uniformly mean ergodic when the support of f is contained in a compact subgroup of G .

The operator $\lambda_p^{H_f}(f)$ obtained by regarding $\lambda_p(f)$ as an operator on $L^p(H_f)$, is mean ergodic. This follows from Theorem 5.4 when $p = 1$ and from the Mean Ergodic Theorem, Corollary 2.4 in the reflexive case $p > 1$.

Since convolution with a function in $L^1(H_f)$ defines a compact operator on $L^p(H_f)$ (see, e.g., [12, Exercise 10.4.2]) and, for power bounded compact operators, mean ergodicity is equivalent to uniform mean ergodicity, (3) of Theorem 2.7, we have that $\lambda_p^{H_f}(f)$ is a uniformly mean ergodic operator. Corollary A.4 then shows that $\lambda_p^G(f)$ is uniformly mean ergodic as well. \square

The simple example in Remark 4.13 shows that when $\mu \notin L^1(G)$, H_μ compact does not necessarily imply that $\lambda_p(\mu)$ is a uniformly mean ergodic operator.

The characterization of Theorem 6.5 leaves room for a convolution operator $\lambda_p(\mu)$, with μ positive and H_μ noncompact, to be uniformly mean ergodic and yet $r(\lambda_p(\mu)) = 1$. We see below that this cannot happen when H_μ is amenable.

Corollary 6.7. *Let G be a locally compact group and let $\mu \in M(G)$ be a positive measure with H_μ amenable and noncompact. If $1 < p < \infty$, the following assertions are equivalent:*

- (1) $\lambda_p(\mu)$ is uniformly mean ergodic.
- (2) $\|\mu\| < 1$.
- (3) $\lambda_p(\mu^n)$ is norm convergent to 0.

Proof. Only that (1) implies (2) requires proof. The equality $\|\lambda_p(\mu)\| = \|\mu\|$ (Theorem 2.13) permits us to proceed as in the proof of Proposition 5.3 to get $\|\mu\| < 1$. \square

Remark 6.8. The above Corollary can be used to prove the converse of Corollary 4.2 for positive measures on amenable groups that do not contain nontrivial compact subgroups (as, e.g., $G = \mathbb{R}$ or $G = \mathbb{Z}$). For Abelian G , more is true. In that case, it is not difficult to prove (relying on the positive-definiteness of $\hat{\mu}$) that -1 is an accumulation point of $\hat{\mu}(\hat{G})$ if and only if 1 is. It follows that for a probability measure $\mu \in M(G)$, $\lambda_2(\mu)$ is uniformly mean ergodic if and only if $\lambda_2(\mu^2)$ is. Positivity of μ is essential here, see Remark 7.2.

7. TRACING LIMITS

7.1. Counterexamples for nonpositive measures. When G is Abelian and μ positive, Theorem 5.1 and Theorem 5.4 in Section 5 characterize the mean ergodicity of $\lambda_p(\mu)$, for $1 \leq p < \infty$. Also under the same assumptions, Theorem 6.1 and Theorem 6.3 in Section 6 give a complete characterization of the uniform mean ergodicity of $\lambda_p(\mu)$, $1 \leq p \leq \infty$. In this section we give examples showing that these characterizations are not longer true when μ is not required to be positive.

Since our counterexamples will be Abelian, we recall from subsection 2.3 that, for Abelian G , $\lambda_2(\mu)$ is unitarily equivalent to the multiplication operator by $\hat{\mu}$ on $L^2(\hat{G})$. As a consequence, the spectrum of $\lambda_2(\mu)$ is exactly $\overline{\hat{\mu}(\hat{G})}$.

The example below shows that Theorem 6.5 fails if μ is not assumed to be positive, even in the Hilbert case, i.e., that $\lambda_2(\mu)$ can be uniformly mean ergodic even if $\|\mu\| = 1$.

Example 7.1. *A measure $\mu \in M(\mathbb{Z})$ whose support generates \mathbb{Z} , $r(\lambda_2(\mu)) = 1$ and yet $\lambda_2(\mu)$ is uniformly mean ergodic. Moreover $(\lambda_2(\mu^n))_n$ is not norm convergent.*

Proof. Take $\mu = (1/2)(\delta_1 - \delta_2)$. By Corollary 4.6, $\lambda_2(\mu)$ will be uniformly mean ergodic if and only if its spectrum is contained in the unit disc and does not contain 1 as an accumulation point.

In this case, for every $0 \leq t < 2\pi$,

$$\hat{\mu}(e^{it}) = \frac{1}{2}(e^{it} - e^{2it}) = \frac{1}{2}e^{it}(1 - e^{it}).$$

For $\widehat{\mu}(e^{it}) = 1$ one needs that $|1 - e^{it}| = 2$ and this only happens when $t = \pi$, but in that case $\widehat{\mu}(e^{it}) = -1$. Hence $r(\lambda_2(\mu)) = 1$ but $1 \notin \sigma(\lambda_2(\mu))$.

Since $\widehat{\mu}^n(e^{it}) = (-1)^n$ for each $n \in \mathbb{N}$ and $\|\lambda_2(\mu^n)\| = r(\lambda_2(\mu^n)) = r(\lambda_2(\mu))^n = 1$, $(\lambda_2(\mu^n))_n$ cannot be norm convergent to 0. Observe that $\lambda_2(\mu)$ does not have non null fixed points by Theorem 2.15. We conclude that $(\lambda_2(\mu^n))$ cannot converge in norm. \square

Remark 7.2. In the above example $1 \in \sigma(\lambda_2(\mu^2))$, hence $\lambda_2(\mu^2)$ is not uniformly mean ergodic. This shows that, unlike the positive case, the converse to Corollary 4.2 is not true when μ is not positive.

The following example of a measure μ with $\|\mu\| > 1$, $H_\mu = \mathbb{Z}$ and $\lambda_p(\mu)$ uniformly mean ergodic for every p , reveals that positivity of μ is not a disposable condition in Proposition 5.3, Theorem 5.6, Theorem 6.3 or Corollary 6.7.

Example 7.3. Let $\mu = \frac{1}{3}(\delta_1 + \delta_0 - \delta_{-1}) \in M(\mathbb{Z})(= l_1(\mathbb{Z}))$. For each $1 < t < 3/\sqrt{7}$, $\|t\mu\| > 1$ and $\|\lambda_p(t\mu^n)\|$ is convergent to 0 and, hence, $\lambda_p(t\mu)$ is uniformly mean ergodic, for $1 \leq p \leq \infty$.

Proof. We only have to observe that $\|\mu^2\| = 7/9$. As a consequence, as long as $t < 3/\sqrt{7}$, $\lim_n \|t\mu^n\| = 0$. \square

7.2. Uniformly mean ergodic convolution operators induced by positive measures with large support. We have seen in Theorem 4.3 that for an operator-normal measure $\mu \in M(G)$, $\lambda_2(\mu)$ is mean ergodic if and only if $\|\lambda_2(\mu)\| \leq 1$. The same equivalence is shown to hold when μ is positive and supported in an amenable subgroup of G , Theorem 5.1, this time for $\lambda_p(\mu)$ and $1 < p < \infty$. In this latter case mean ergodicity of $\lambda_p(\mu)$ is, in addition, equivalent to vague-ergodicity of μ .

Here we show that, when the measure is positive but not operator-normal and H_μ is not amenable, none of these equivalences remains true. Our examples will consist on convolution operators defined by finitely supported measures in discrete groups. We first collect some facts on $\|\lambda_2(\mu)\|$ for $\mu = \frac{1}{|S|} \sum_{s \in S} \delta_s$ with $S \subseteq G$.

Recall that elements of the free group $F(X)$ on the set of generators X can be described uniquely as words of the form $w = x_1^{\epsilon_1} \cdots x_2^{\epsilon_2} \cdots \cdots x_n^{\epsilon_n}$ with $x_i \in X$, $\epsilon_i = \pm 1$, with no redundant pairs included, i.e. with $\epsilon_i = \epsilon_{i+1}$ whenever $x_i = x_{i+1}$. The length of w is then the minimum number of terms in such an expression of w . When all the exponents are positive, then w belongs to the semigroup generated by X .

Theorem 7.4. Let G be a discrete group and let $S \subseteq G$ with $|S| = n$. Consider $\mu = \sum_{s \in S} \delta_s \in M(G)$. Then:

- (1) (Particular case of [40, Theorem 18.3]) $r(\lambda_2(\mu)) = n$ if and only if $\langle S \rangle$ is amenable. If S contains the identity, $\|\lambda_2(\mu)\| = n$ if and only if $\langle S \rangle$ is amenable.
- (2) ([1, Theorem IV.K]) If S generates a free group: $\|\lambda_2(\mu)\| = 2\sqrt{n-1}$.
- (3) (Haagerup inequality, [23]) If $G = F(X)$ is a free group and S consists of words of length N on X , then: $\|\lambda_2(\mu)\|_{\mathcal{L}(L^2(G))} \leq (N+1)\|\mu\|_{l_2}$.

- (4) (*strong Haagerup inequality*, [31]) *If G is a free group and S consists of words of length N that are in the semigroup generated by X , then*
- $$\|\lambda_2(\mu)\|_{\mathcal{L}(L^2(G))} \leq e\sqrt{N+1}\|\mu\|_{\ell^2}.$$

This is the promised counterexample to Theorem 5.1 for positive measures which are not supported in amenable subgroups. This example also shows that the hypothesis of normality of the measure cannot be dropped to get the necessity of $\|\lambda_2(\mu)\| \leq 1$ when $\lambda_2(\mu)$ is uniformly mean ergodic in Theorem 4.3, even if μ is positive.

Example 7.5. *Let G be the free group on three generators $G = F(x_1, x_2, x_3)$. There is a finitely supported positive measure $\nu \in M(G)$ such that:*

- (1) $\|\nu\| > \|\lambda_2(\nu)\| > 1$.
- (2) $\lambda_2(\nu)$ is uniformly mean ergodic.
- (3) ν is not vague-ergodic.

Proof. Let $\mu = (\delta_{x_1} + \delta_{x_2} + \delta_{x_3})$ and define, and for $\frac{1}{\sqrt{8}} < r < \frac{1}{\sqrt{3}}$, $\nu = r\mu$.

Observe to begin with that, for each $n \in \mathbb{N}$, the measures μ^n is precisely the characteristic function of the set of all words of length n of the free semigroup generated by $\{x_1, x_2, x_3\}$.

Since, clearly, $\|\nu^n\|_{\ell^2} = (\sqrt{3}r)^n$ we deduce from the strong Haagerup inequality ((4) of Theorem 7.4) that

$$\|\lambda_2(\nu^n)\| \leq (\sqrt{3}r)^n e\sqrt{n+1}.$$

The spectral radius formula together with the choice of r yields

$$r(\lambda_2(\nu)) \leq \sqrt{3}r < 1$$

Therefore, by (1) of Theorem 2.7, $\lambda_2(\nu)$ is uniformly mean ergodic ($(\lambda_2(\nu^n))_n$ is even convergent to 0)

On the other hand, (2) of Theorem 7.4 shows that

$$\|\lambda_2(\nu)\| = \sqrt{8}r > 1.$$

Finally, Since $\|\nu\| = 3r > \sqrt{8}r > 1$ we get

$$\lim_n \frac{\|\nu^n\|}{n} = \lim_n \frac{(3r)^n}{n} = \infty,$$

and we see that μ cannot be vague-ergodic. □

8. OPEN QUESTIONS

We remark that all our examples of mean ergodic operators are power bounded. It is natural to conjecture a positive answer to the following problem.

Problem 1. Let G be a locally compact group. Let $\mu \in M(G)$ and $1 < p < \infty$. Is it true that $\lambda_p(\mu)$ is power bounded whenever it is mean ergodic? What if $\mu \geq 0$? What if H_μ is amenable?

Examples 7.3 and 7.5 both introduce *big* measures with $(\lambda_2(\mu^n))$ convergent in norm to 0, it seems also natural to ask if an example in the spirit of 7.1 can be obtained for positive measures (necessarily on non-amenable groups).

Problem 2. Let G be a free group (or any other nonamenable locally compact group). Is there any positive measure $\mu \in M(G)$ such that $\mu(G) > 1$, $\lambda_2(\mu)$ is mean ergodic and $r(\lambda_2(\mu)) = 1$? If the answer is positive, could μ be taken in such a way that also $\lambda_2(\mu)$ is uniformly mean ergodic?

Our last question refers to possible generalizations of Theorem 5.1 to nonpositive measures or nonamenable groups.

Problem 3. Is there a locally compact group G supporting a measure which is vague ergodic but $\lambda_p(\mu)$ is not mean ergodic? A positive answer would provide a convolution operator which is weakly mean ergodic but not mean ergodic.

APPENDIX. CONVOLUTION OPERATORS $\lambda_p(\mu)$ AS OPERATORS ON $L^p(H_\mu)$:
REDUCING TO THE SUPPORT

If H is a closed subgroup of G with $H_\mu \subseteq H$, convolving by μ can be seen both as an operator on $L^p(H)$ and as an operator on $L^p(G)$. Our aim is to show that the ergodic behaviour of both operators is, as expected, the same. The proofs of these results are rather technical and rely on several involved results of abstract harmonic analysis. We have therefore preferred to defer their proof to this Appendix.

These facts are best described when convolution operators by measures are seen in the wider frame of algebras of p -pseudomeasures. The algebra $PM_p(G)$ of p -pseudomeasures is defined as the weak-operator closure of $\{\rho_p(f) : f \in L^1(G)\}$ in $\mathcal{L}(L^p(G))$. $PM_p(G)$ is a Banach subalgebra of $\mathcal{L}(L^p(G))$ that contains $\rho_p(\mu)$ for every $\mu \in M(G)$. It is easy to see that operators in $PM_p(G)$ commute with left translations.

As a Banach space, the algebra $PM_p(G)$ can be seen as the dual space of a function algebra $A_p(G)$ known as the Figà-Talamanca Herz algebra. We will not need to provide a precise definition of this algebra here. It suffices to say that for any $f \in L^p(G)$ and $g \in \overline{L^{p'}(G)}$, with $1/p + 1/p' = 1$, the convolution $\bar{f} * \check{g} \in A_p(G)$, where $\bar{f}(s) = \overline{f(s)}$ and $\check{f}(s) = f(s^{-1})$, and that, for each $T \in PM_p(G)$, $\langle T, \bar{f} * \check{g} \rangle = \langle Tf, g \rangle$, where the first bracket corresponds to the $(PM_p(G), A_p(G))$ -duality and the second to the $(L^p(G), L^{p'}(G))$ -duality.

In our setting, it would have been more natural to introduce the algebra of pseudomeasures as the weak-operator closure of $\{\lambda_p(f) : f \in L^1(G)\}$, as it is often done in the literature. This would have produced a different but linearly isometric algebra. By technical reasons related to Theorem A.3 we find it preferable to use the right-handed version here.

Since in this section we are going to see the operator $\rho_p(\mu)$, $\mu \in M(H)$, both as an operator on $L^p(G)$ and as an operator on $L^p(H)$, where H is a subgroup of G , it will be convenient to use the notation $\rho_p^G(\mu)$ and $\rho_p^H(\mu)$ (or $\lambda_p^G(\mu)$ and $\lambda_p^H(\mu)$ for the left-handed versions) to distinguish both cases.

The basic tool to explore the relation between operators on $L^p(G)$ and $L^p(H)$ is the Mackey-Bruhat integration formula described in the next lemma.

Lemma A.1 (Mackey-Bruhat integration formula. Remark 8.2.3 of [41]). *Let G be a locally compact group and let H be a closed subgroup of G . There is a quasi-invariant measure $m_{G/H}$ on the space of left cosets G/H and a*

continuous strictly positive function $q: G \rightarrow \mathbb{R}$ such that:

$$\frac{q(xh)}{q(x)} = \frac{\Delta_H(h)}{\Delta_G(h)}, \text{ and}$$

$$\int f(x) \, dm_G(x) = \int_{G/H} \left(\int_H \frac{f(xh)}{q(xh)} \, dm_H(h) \right) dm_{G/H}(\dot{x}),$$

where \dot{x} denotes the right coset xH . If H is a normal subgroup of G , $q(x) = 1$ for every $x \in G$.

Lemma A.2. *Let G be a locally compact group, let H be a closed subgroup of G and let $1 < p < \infty$. Then,*

- (1) *On bounded subsets of $PM_p(G)$ the $\sigma(PM_p(G), A_p(G))$ - and weak operator topologies coincide.*
- (2) *Restriction defines a linear surjective mapping $R_H: A_p(G) \rightarrow A_p(H)$ such that for each $f \in A_p(H)$ and $\varepsilon > 0$ there is $f_H \in A_p(G)$ with $\|f\| \leq \|f_H\| \leq \|f\| + \varepsilon$ and $R_H(f_H) = f$.*
- (3) *The adjoint $(R_H)^*: PM_p(H) \rightarrow PM_p(G)$ is a multiplicative linear isometry.*
- (4) *If $Q \in PM_p(H)$, $f, g \in C_{00}(G)$, $x \in G$ and $h \in H$,*

$$\langle (R_H)^*(Q)f, g \rangle = \int_{G/H} \int_H Q(f_{x,p})(h) \frac{g(xh)}{q^{1/p'}(xh)} \, dm_H(h) \, dm_{G/H}(\dot{x}),$$

$$\text{where } f_{x,p}(h) = \frac{f(xh)}{q(xh)^{1/p}}.$$

- (5) *If $\mu \in M(G)$ and $H_\mu \subseteq H$, then $(R_H)^*(\rho_p^H(\mu)) = \rho_p^G(\mu)$.*

Proof. The first item follows, e.g., from Theorem 6 of [13]. Items (2) and (3) can be deduced from Theorems A and 1 of [25] and also from [13, Proposition 7.3.5 and Theorem 7.8.4].

Item (4) is Proposition 12 of [13] after observing (see Theorem 7.8.4 of [13] for this) that R^* coincides on $PM_p(G)$ with the operator $i: PM_p(H) \rightarrow PM_p(G)$ introduced in Definition 7.1.7 *loc. cit.*

In order to prove (5) we let $f, g \in C_{00}(G)$. The Mackey-Bruhat formula and item (4) applied to $\rho_p^H(\mu)$ then yield:

$$\begin{aligned} \langle \rho_p^G(\mu)f, g \rangle &= \int_{G/H} \int_H \frac{\rho_p^G(\mu)f(xh)g(xh)}{q(xh)} \, dm_H(h) \, dm_{G/H}(\dot{x}) \\ &= \int_{G/H} \int_H \int \Delta_G(u)^{1/p} \frac{f(xhu)}{q^{1/p}(xh)} d\mu(u) \frac{g(xh)}{q^{1/p'}(xh)} \, dm_H(h) \, dm_{G/H}(\dot{x}) \\ &= \int_{G/H} \int_H \int \Delta_H(u)^{1/p} \frac{f(xhu)}{q^{1/p}(xhu)} d\mu(u) \frac{g(xh)}{q^{1/p'}(xh)} \, dm_H(h) \, dm_{G/H}(\dot{x}) \\ &= \int_{G/H} \int_H \rho_p^H(\mu)(f_{x,p})(h) \frac{g(xh)}{q^{1/p'}(xh)} \, dm_H(h) \, dm_{G/H}(\dot{x}) \\ &= \langle (R_H)^*(\rho_p^H(\mu))f, g \rangle. \end{aligned}$$

□

Theorem A.3. *Let G be a locally compact group, H a closed subgroup of G and $\mu \in M(G)$ with $\text{supp}(\mu) \subseteq H$. Let as well $1 < p < \infty$. Then $\rho_p^H(\mu)$ is*

(uniformly, weakly) mean ergodic if and only if $\rho_G^p(\mu)$ is (uniformly, weakly) mean ergodic.

Proof. 1. Uniform mean ergodicity.

Since $(R_H)^*: PM_p(H) \rightarrow PM_p(G)$ is a multiplicative linear isometry and $(R_H)^*(\rho_p^H(\mu)) = \rho_p^G(\mu)$, Lemma A.2, it is clear that $\rho_p^H(\mu)$ is uniformly mean ergodic if and only if $\rho_p^G(\mu)$ is uniformly mean ergodic.

2. Weak mean ergodicity.

Lemma A.2 and the Banach-Steinhaus theorem imply that Condition (2) of Theorem 2.3 holds for $\rho_H^p(\mu)$ if and only if it holds for $\rho_G^p(\mu)$. It also follows that as soon as either $\rho_H^p(\mu)$ or $\rho_G^p(\mu)$ is weakly mean ergodic then $\lambda_H^p(\mu)_{[n]}$, $\rho_G^p(\mu)_{[n]}$, $\frac{1}{n}\rho_G(\mu^n)$ and $\frac{1}{n}\rho_H(\mu^n)$ will all be bounded in the operator norm.

Suppose now that $\frac{1}{n}\rho_H^p(\mu)^n$ converges to 0 in the weak operator topology and let $f \in L^p(G)$ and $g \in L^{p'}(G)$, then

$$\begin{aligned} \frac{1}{n} \langle \rho_G^p(\mu)^n f, g \rangle &= \frac{1}{n} \langle \rho_G^p(\mu^n), \bar{f} * \check{g} \rangle \\ &= \frac{1}{n} \langle \rho_H^p(\mu^n), R_H(\bar{f} * \check{g}) \rangle. \end{aligned}$$

So, since weak operator topology and $\sigma(PM_p(G), A_p(G))$ coincide on bounded sets of $PM_p(G)$ ((1) of Lemma A.2), $\frac{1}{n}\rho_G^p(\mu^n)$ converges to 0 in the weak operator topology.

If, conversely, $\frac{1}{n}\rho_G^p(\mu)^n$ converges to 0 in the weak operator topology and $f \in L^p(H)$, $g \in L^{p'}(H)$ we can consider $u \in A_p(G)$ with $R_H(u) = \bar{f} * \check{g}$ and, noting again that $R^*(\rho_p^H(\mu)) = \rho_p^G(\mu)$,

$$\begin{aligned} \frac{1}{n} \langle \rho_H^p(\mu)^n f, g \rangle &= \frac{1}{n} \langle \rho_H^p(\mu^n), \bar{f} * \check{g} \rangle \\ &= \frac{1}{n} \langle \rho_H^p(\mu^n), R_H(u) \rangle \\ &= \frac{1}{n} \langle \rho_G^p(\mu^n), u \rangle \end{aligned}$$

and we see that $\frac{1}{n}\rho_H^p(\mu)^n$ converges to 0 in the weak operator topology.

As both conditions of Theorem 2.3 hold precisely for $\rho_p^G(\mu)$ when they hold for $\rho_p^H(\mu)$, we conclude that $\rho_p^G(\mu)$ is weakly mean ergodic if and only if $\rho_p^H(\mu)$ is.

3. Mean ergodicity.

We are now going to use the norm topology version of Theorem 2.3. As above, Condition (2) will be satisfied for $\rho_p^G(\mu)$ if and only if it is satisfied for $\rho_p^H(\mu)$.

Let $\mu_n = \frac{\mu^n}{n}$. We will show that $\rho_p^G(\mu_n)$ converges to 0 in the strong operator topology if and only if so does $\rho_p^H(\mu_n)$. Our approach will follow closely Chapter 7 of [13].

So, let us first assume that $\rho_p^H(\mu_n)$ converges to 0 in the strong operator topology. Using the Mackey-Bruhat formula in Lemma A.1, for $f \in L^p(G)$,

$$(4) \quad \|\rho_G(\mu_n)f\|_{L^p(G)}^p = \int_{G/H} \int_H \frac{|\rho_G(\mu_n)f(sh)|^p}{q(sh)} dm_H(h) dm_{G/H}(\dot{s}).$$

Recalling that $f_{s,p}(h) = \frac{f(sh)}{q(sh)^{1/p}}$ and applying the properties of the function q to the inner integral in (4)

$$\begin{aligned} \int_H \frac{|\rho_G(\mu_n)f)(sh)|^p}{q(sh)} dm_H(h) &= \int_H \left| \int_H \frac{\Delta_G(u)^{1/p} f(shu)}{q(sh)^{1/p}} d\mu_n(u) \right|^p dm_H(h) \\ &= \int_H \left| \int_H \frac{\Delta_G(u)^{1/p} f(shu) \Delta_H(u)^{1/p}}{q(shu)^{1/p} \Delta_G(u)^{1/p}} d\mu_n(u) \right|^p dm_H(h) \\ &= \int_H \left| \int_H \frac{\Delta_H(u)^{1/p} f(shu)}{q(shu)^{1/p}} d\mu_n(u) \right|^p dm_H(h) \\ &= \|(\rho_H(\mu_n)f_{s,p})\|_{L^p(H)}^p. \end{aligned}$$

Since $\rho_H(\mu_n)$ converges to 0 in the strong operator topology, the sequence $\|\rho_H(\mu_n)f_{s,p}\|_{L^p(H)}^p$ converges to 0 for every $\dot{s} \in G/H$. By Lebesgue's dominated convergence theorem, applied to the integral in (4), the sequence $\|\rho_G(\mu_n)f\|_{L^p(G)}$ will converge to 0 as long as we can see that the functions $\|\rho_H(\mu_n)f_{s,p}\|_{L^p(H)}^p$ are dominated by some integrable function. But, for each $\dot{s} \in G/H$, $\|\rho_H(\mu_n)f_{s,p}\|_{L^p(H)}^p \leq \|\rho_H(\mu_n)\|^p \cdot \|f_{s,p}\|_{L^p(H)}^p$, and the Bruhat-Mackey formula implies that $\dot{s} \mapsto \|f_{s,p}\|_{L^p(H)}^p$ is integrable with, precisely,

$$\int_{G/H} \|f_{s,p}\|_{L^p(H)}^p dm_{G/H}(\dot{s}) = \|f\|_{L^p(G)}^p.$$

We conclude so that $\rho_p^G(\mu_n)f$ converges to 0 in norm.

Assume now that $\rho_p^G(\mu_n)$ converges to 0 and let $f \in L^p(H)$, $g \in L^{p'}(H)$ with $\|g\|_{L^{p'}(H)} \leq 1$. If we follow the proof of Theorem 7.3.2 of [13], we can find two functions $v_f \in L^p(G)$, $v_g \in L^{p'}(G)$ such that (this is the top formula of page 115 *loc. cit.*)

$$\begin{aligned} \|v_g\|_{L^{p'}(G)} &\leq \|g\|_{L^{p'}(H)} \leq 1 \quad \text{and} \\ |\langle \rho_p^H(\mu_n)f, g \rangle| &\leq |\langle \rho_p^G(\mu_n)v_f, v_g \rangle|. \end{aligned}$$

It follows then that $\lim_{n \rightarrow \infty} \|(\rho_p^H(\mu_n)f)\|_{L^p(H)} = 0$. □

The equivalence between $\rho_p(\mu)$ and $\lambda_p(\mu)$ stated in Fact 2.1 leads to the following Corollary.

Corollary A.4. *Let G be a locally compact group, H a closed subgroup of G and $\mu \in M(G)$ with $\text{supp}(\mu) \subseteq H$. Let as well $1 < p < \infty$. Then:*

- (1) $\|\lambda_H^p(\mu)\| = \|\lambda_G^p(\mu)\|$. Hence,
- (2) $r(\lambda_H^p(\mu)) = r(\lambda_G^p(\mu))$.
- (3) $\lambda_H^p(\mu)$ is (uniformly, weakly) mean ergodic if and only if $\lambda_G^p(\mu)$ is.

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