LIOUVILLE’S EQUATIONS FOR RANDOM SYSTEMS

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Abstract. Given a random system, a Liouville’s equation is an exact partial differential equation that describes the evolution of the probability density function of the solution. In this paper, we derive Liouville’s equations for the first-order homogeneous semilinear random partial differential equation. This is done for all finite-dimensional distributions of the random field solution, starting with dimension one, then dimension two, and finally generalizing to any dimension. Several examples, including the linear advection equation with random coefficients, are treated. As a corollary, we deduce Liouville’s equations for path-wise stochastic integrals and nonlinear random ordinary differential equations.

Keywords: probability density function; random partial, ordinary and fractional differential equation; Liouville’s equation

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1. Introduction

Motivated by the inherent uncertainties of many physical phenomena, a random ordinary or partial differential equation problem considers the inputs of the corresponding deterministic problem (equation parameters, initial states, boundary values, forcing terms, etc.) as random variables, stochastic processes and/or random fields [1–5]. Contrary to the case of stochastic differential equations of Itô type, which are driven by irregular processes, the inputs of a random differential equation problem are smooth, with any type of probability distribution. The path-wise solution is a differentiable stochastic process or random field. For the difference between random and stochastic differential equations, the reader is referred to [6, pp. 96–98], [7].

The solution to a random differential equation problem may have a probability density function with respect to the standard Lebesgue measure for each finite-dimensional distribution. A Liouville’s equation is an exact partial differential equation that dictates the evolution of the probability density function associated with the solution. In the framework of Itô stochastic differential equations, which might be more familiar to the reader, Liouville’s equations are usually termed Fokker-Planck equations or forward Kolmogorov equations [8].

In this paper, we derive some Liouville’s equations in a concise manner. Section 2 deals with general first-order and homogeneous semilinear random partial differential equations. All finite-dimensional distributions of the solution are tackled, beginning with dimension one, then dimension two, and eventually generalizing to any dimension. Several examples, including the linear advection equation with random coefficients, are treated. With the goal
of obtaining Liouville’s formulas, some studies on the transport random equation are found in the literature: [9], [10, Section 3.1], [11, Prop. 3.1], [12], [13, Th. 4.1].

As a corollary of the exposition on semilinear random partial differential equations, in Section 3 we obtain Liouville’s equations for path-wise stochastic integrals and random ordinary differential equations. The former case provides an alternative proof of the transformation of random variables formula, as well as the temporal development of the probability density function for the primitives of some important Gaussian processes, such as fractional Brownian motion, Brownian bridge, white noise and Ornstein-Uhlenbeck. The latter case has already been documented in the literature by means of characteristic functions [14], [2, Th. 6.2.2], dynamical systems theory [15, Th. 8.4], and the principle of preservation of probability [16], with some applications in various fields [17–21]. But our derivation, based on semilinear random partial differential equations, seems to be novel. Also original is the application in the context of controlled shrimp growth [7].

Finally, in Section 4, the main results of the paper are summarized. The case of random fractional differential equations [22,23] is briefly discussed. Future research lines are indicated.

2. First-order and homogeneous semilinear random partial differential equations

We work with the generic first-order homogeneous semilinear random partial differential equation

$$
\sum_{i=1}^{n} g_i(x) u_{x_i}(x) = 0,
$$

(2.1)

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the independent variable, and $g_i : \mathbb{R}^n \to \mathbb{R}$ and $u : \mathbb{R}^n \to \mathbb{R}^m$ are random fields. Here the subscript $x_i$ denotes the partial derivative with respect to $x_i$, also denoted via $\partial/\partial x_i = \partial_{x_i} = \partial_i$. Let $g = (g_1, \ldots, g_n)$. It is assumed that $g$ is sufficiently smooth, hence non-white noise and non-Brownian. Bold letters always denote vectors of any length. Non-bold letters always denote scalars.

Let $f(u; x)$ be the probability density function of the state vector $u(x)$ with respect to the standard Lebesgue measure, evaluated at $u \in \mathbb{R}^m$. In this section, first, we aim at deriving a partial differential equation for $f(u; x)$. This is referred to as Liouville’s equation.

**Theorem 2.1.** Given (2.1), it holds

$$
\nabla_x \cdot (\mathbb{E}[g(x)|u(x) = q] f(q; x)) = \mathbb{E}[\nabla_x \cdot g(x)|u(x) = q] f(q; x),
$$

(2.2)

where $q \in \mathbb{R}^m$, $\nabla_x$ is the gradient vector, the dot $\cdot$ denotes the dot product, $\nabla_x \cdot$ denotes the divergence operator, $\mathbb{E}[\ast]$ is the expectation operator, and $\mathbb{E}[\ast|\ast]$ is the conditional expectation.

In particular, when $g$ is deterministic,

$$
\nabla_x \cdot (g(x) f(q; x)) = (\nabla_x \cdot g(x)) f(q; x) \equiv g(x) \cdot \nabla_x f(q; x) = 0.
$$

(2.3)

**Proof.** Let $h : \mathbb{R}^m \to \mathbb{R}$ be any smooth function with compact support. We have

$$
\sum_{i=1}^{n} g_i(x)(h(u))_{x_i}(x) = \nabla h(u(x)) \cdot \sum_{i=1}^{n} g_i(x) u_{x_i}(x) = 0.
$$

(2.4)
That is, \( h(u) \) is a scalar solution to (2.1). We apply the expectation operator:

\[
\sum_{i=1}^{n} \mathbb{E} [g_i(x)(h(u))_{x_i}(x)] = 0.
\]

We use the product rule for differentiation and commute derivative and expectation:

\[
\mathbb{E} [g_i(x)(h(u))_{x_i}(x)] = \mathbb{E} [(g_i h(u))_{x_i}(x)] - \mathbb{E} [(g_i)_{x_i}(x)h(u(x))]
\]

\[
= \mathbb{E} [g_i(x)h(u(x))]_{x_i} - \mathbb{E} [(g_i)_{x_i}(x)h(u(x))].
\]

We compute these last two expectations. First,

\[
\mathbb{E} [g_i(x)h(u(x))] = \int_{\mathbb{R}^m} \mathbb{E} [g_i(x)h(u(x))|u(x) = q] f(q; x) \, dq
\]

\[
= \int_{\mathbb{R}^m} h(q) \mathbb{E} [g_i(x)|u(x) = q] f(q; x) \, dq,
\]

so

\[
\mathbb{E} [g_i(x)h(u(x))]_{x_i} = \int_{\mathbb{R}^m} h(q) \frac{\partial}{\partial x_i} \{ \mathbb{E} [g_i(x)|u(x) = q] f(q; x) \} \, dq.
\]

Second,

\[
\mathbb{E} [(g_i)_{x_i}(x)h(u(x))] = \int_{\mathbb{R}^m} h(q) \mathbb{E} [(g_i)_{x_i}(x)|u(x) = q] f(q; x) \, dq.
\]

As a consequence,

\[
\int_{\mathbb{R}^m} h(q) \left[ \sum_{i=1}^{n} \left( \frac{\partial}{\partial x_i} \{ \mathbb{E} [g_i(x)|u(x) = q] f(q; x) \} - \mathbb{E} [(g_i)_{x_i}(x)|u(x) = q] f(q; x) \right) \right] \, dq = 0.
\]

Since this holds for any \( h \), as a consequence of the fundamental lemma of calculus of variations [24, Lemma 1.1.1, p. 6] we obtain (2.2), using the divergence notation. \( \square \)

In the following two examples, we show the Liouville’s equation for the one-dimensional linear advection equation with uncertainties, by using Theorem 2.1. In the third example, we work with a particular partial differential equation problem with uncertainties and obtain the Liouville’s equation as an application of Theorem 2.1.

**Example 2.2.** Let us consider the one-dimensional linear advection equation with random initial condition: \( u_t(t, x) + au_x(t, x) = 0 \), \( u(0, x) = u_0(x) \), \( t \geq 0, x \in \mathbb{R} \), where \( a \in \mathbb{R} \) is a constant and \( u_0 \) is a stochastic process. In this case, \( g = (1, a) \). The partial differential equation (2.3) becomes \( \partial_t f(q; t, x) + a \partial_x f(q; t, x) = 0 \). This formula may be checked by employing the exact solution \( u(t, x) = u_0(x - at) \), which gives \( f(q; t, x) = f_{u_0}(q; x - at) \).

**Example 2.3.** Let us work in the same setting of Example 2.2, but now let us assume that \( a \) is a random variable, independent of the stochastic initial condition \( u_0 \). The partial differential equation (2.2) then becomes

\[
\partial_t f(q; t, x) + \partial_x \left( \mathbb{E}[a u_0(x - at) = q] f(q; t, x) \right) = 0.
\]

The conditional expectation can be written in closed form. Indeed, by Bayes’ formula and the independence,

\[
f_{a|u_0(x-at)=q}(a_0) = \frac{f_{u_0}(q; x - at)f_a(a_0)}{\int_{\mathbb{R}} f_{u_0}(q; x - at)f_a(a_0) \, da_0},
\]
so
\[ \mathbb{E}[a|u_0(x - at) = q] = \frac{\int_\mathbb{R} a_0 f_{w_0}(q; x - a_0 t) f_a(a_0) \, da_0}{\int_\mathbb{R} f_{w_0}(q; x - a_0 t) f_a(a_0) \, da_0}. \]

Then the partial differential equation (2.5) becomes
\[ \partial_t f(q; t, x) + \partial_x \left( \frac{\int_\mathbb{R} a_0 f_{w_0}(q; x - a_0 t) f_a(a_0) \, da_0}{\int_\mathbb{R} f_{w_0}(q; x - a_0 t) f_a(a_0) \, da_0} f(q; t, x) \right) = 0. \quad (2.6) \]

This equation (2.6) can be checked alternatively, since the probability density function of the composition \( u(t, x) = u_0(x - at) \) is computable via the law of total probability:
\[
\mathbb{P}[u_0(x - at) \in B] = \int_\mathbb{R} \mathbb{P}[u_0(x - at) \in B|a = a_0] f_a(a_0) \, da_0
= \int_\mathbb{R} \mathbb{P}[u_0(x - a_0 t) \in B] f_a(a_0) \, da_0
= \int_\mathbb{R} \int_B f_{w_0}(q; x - a_0 t) \, dq f_a(a_0) \, da_0
= \int_B \int_\mathbb{R} f_{w_0}(q; x - a_0 t) f_a(a_0) \, da_0 \, dq.
\]

Here \( \mathbb{P} \) denotes the underlying probability measure and \( B \) is any Borel set in \( \mathbb{R} \). Then we derive that
\[
f_{w_0(x-at)}(q) = \int_\mathbb{R} f_{w_0}(q; x - a_0 t) f_a(a_0) \, da_0.
\]

It is trivial to verify that this density function satisfies (2.6), which is in agreement with Theorem 2.1

**Example 2.4.** Let \( xu_x - ayu_y = 0 \), where \( a \) is a random variable, \( x > 1 \) and \( y \in \mathbb{R} \). We have that \( u(x, y) = x^a y \) is the solution to the problem with non-random initial state \( u(1, y) = y \).

Let \( f(q; x, y) \) be the probability density function of \( u(x, y) \) evaluated at \( q \in \mathbb{R} \). Notice that \( f(q; x, y) = 0 \) when \( q \) and \( y \) have opposite signs. We then assume that \( qy > 0 \). By (2.2) with \( g(x, y) = (x, -ay) \),
\[
\frac{\partial}{\partial x} (xf(q; x, y)) + \frac{\partial}{\partial y} (-y \mathbb{E}[a|x^a y = q] f(q; x, y)) = \mathbb{E}[1 - a|x^a y = q] f(q; x, y).
\]

We have \( \mathbb{E}[a|x^a y = q] = \log(q/y) / \log x \). Then, after simple computations,
\[
x \frac{\partial}{\partial x} f(q; x, y) - y \log(q/y) \frac{\partial}{\partial y} f(q; x, y) = -\frac{1}{\log x} f(q; x, y)
\]
is a Liouville’s equation for \( f(q; x, y) \).

Given \( x, y \in \mathbb{R}^n \), let \( f(q_1, q_2; x, y) \) be the probability density function of the joint vector \( \langle u(x), u(y) \rangle \) with respect to the standard Lebesgue measure, evaluated at \( (q_1, q_2) \in \mathbb{R}^m \times \mathbb{R}^m \); this corresponds to the second finite-dimensional distributions of \( u \). We derive a partial differential equation for \( f(q_1, q_2; x, y) \). The ideas are similar to Theorem 2.1, but the expressions are more cumbersome because the dimensions are increasing.
Theorem 2.5. Given (2.1), it holds

\[
(\nabla_x \otimes \nabla_y) \cdot (\mathbb{E} [g(x) \otimes g(y)] u(x) = q_1, u(y) = q_2] f(q_1, q_2; x, y))
\]

- \nabla_x \cdot (\mathbb{E} [(\nabla_y \cdot g(y)) g(x)] u(x) = q_1, u(y) = q_2] f(q_1, q_2; x, y))

- \nabla_y \cdot (\mathbb{E} [(\nabla_x \cdot g(x)) g(y)] u(x) = q_1, u(y) = q_2] f(q_1, q_2; x, y))

+ \mathbb{E} [(\nabla_x \cdot g(x)) (\nabla_y \cdot g(y)) | u(x) = q_1, u(y) = q_2] f(q_1, q_2; x, y) = 0,

where \((q_1, q_2) \in \mathbb{R}^m \times \mathbb{R}^m\) and \(\otimes\) denotes the Kronecker product \(((w_i), \otimes (z_j)) = (w_i z_j)_{i,j}\), all of them in vector form.

In particular, when \(g\) is deterministic,

\[
(\nabla_x \otimes \nabla_y) \cdot ((g(x) \otimes g(y)) f(q_1, q_2; x, y))
\]

- \(\nabla_y \cdot (g(y)) \nabla_x \cdot (g(x) f(q_1, q_2; x, y))\)

- \(\nabla_x \cdot (g(x)) \nabla_y \cdot (g(y) f(q_1, q_2; x, y))\)

+ \(\nabla_x \cdot (g(x)) (\nabla_y \cdot g(y)) f(q_1, q_2; x, y)\) = \([g(x) \otimes g(y)] \cdot [(\nabla_x \otimes \nabla_y) f(q_1, q_2; x, y)] = 0.\]

Proof. By (2.4), we have

\[
\sum_{i=1}^{n} g_i(x) (h_1(u))_{x_i}(x) = 0, \quad \sum_{j=1}^{n} g_j(y) (h_2(u))_{y_j}(y) = 0,
\]

for any two smooth functions \(h_1, h_2 : \mathbb{R}^m \to \mathbb{R}\) with compact support. Here \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) and \(y = (y_1, \ldots, y_n) \in \mathbb{R}^n\). Multiplying both expressions and applying expectation,

\[
\sum_{i,j=1}^{n} \mathbb{E} \left[ g_i(x) (h_1(u))_{x_i}(x) g_j(y) (h_2(u))_{y_j}(y) \right] = 0. \quad (2.7)
\]

By the product rule for differentiation,

\[
g_i(x) (h_1(u))_{x_i}(x) g_j(y) (h_2(u))_{y_j}(y)
\]

\[
= \left( (g_i h_1(u))_{x_i}(x) - (g_i)_{x_i}(x) h_1(u(x)) \right) \left( (g_j h_2(u))_{y_j}(y) - (g_j)_{y_j}(y) h_2(u(y)) \right).
\]

Expand this last expression and apply expectation: from (2.7),

\[
0 = \sum_{i,j=1}^{n} \left\{ \mathbb{E} \left[ (g_i h_1(u))_{x_i}(x) (g_j h_2(u))_{y_j}(y) \right] - \mathbb{E} \left[ (g_i)_{x_i}(x) h_1(u(x)) (g_j h_2(u))_{y_j}(y) \right] \right. 
\]

\[
- \mathbb{E} \left[ (g_i h_1(u))_{x_i}(x) (g_j)_{y_j}(y) h_2(u(y)) \right] + \mathbb{E} \left[ (g_i)_{x_i}(x) h_1(u(x)) (g_j)_{y_j}(y) h_2(u(y)) \right] \right\}. \quad (2.8)
\]

We compute each expectation from (2.8). The first one is given by

\[
\mathbb{E} \left[ (g_i h_1(u))_{x_i}(x) (g_j h_2(u))_{y_j}(y) \right] = \frac{\partial^2}{\partial x_i \partial y_j} \mathbb{E} [g_i(x) h_1(u(x)) g_j(y) h_2(u(y))]
\]

\[
= \frac{\partial^2}{\partial x_i \partial y_j} \int_{\mathbb{R}^{2m}} h_1(q_1) h_2(q_2) \mathbb{E} [g_i(x) g_j(y)] u(x) = q_1, u(y) = q_2] f(q_1, q_2; x, y) \, dq_1 \, dq_2
\]

\[
= \int_{\mathbb{R}^{2m}} h_1(q_1) h_2(q_2) \frac{\partial^2}{\partial x_i \partial y_j} \left\{ \mathbb{E} [g_i(x) g_j(y)] u(x) = q_1, u(y) = q_2] f(q_1, q_2; x, y) \right\} \, dq_1 \, dq_2.
\]
Analogously, the second expectation in (2.8) is given by

\[
\mathbb{E} \left[ (g_i)_x(x) h_1(u(x)) (g_j)_y(y) \right] = \int_{\mathbb{R}^2} h_1(q_1) h_2(q_2) \frac{\partial}{\partial y_j} \{ \mathbb{E} \left[ (g_i)_x(x) g_j(y) \right] u(x) = q_1, u(y) = q_2 \} \, dq_1 \, dq_2.
\]

And analogously for the remaining two expectations in (2.8). Hence, expression (2.8) becomes

\[
0 = \int_{\mathbb{R}^2} h_1(q_1) h_2(q_2) \sum_{i,j=1}^{n} \left( \frac{\partial^2}{\partial x_i \partial y_j} \{ \mathbb{E} \left[ (g_i)_x(x) g_j(y) \right] u(x) = q_1, u(y) = q_2 \} \right) dq_1 dq_2.
\]

Since this equality holds for any \( h_1 \) and \( h_2 \), we obtain the desired Liouville’s equation for \( f(q_1, q_2; x, y) \).

In the following two examples, we show Liouville’s equations for joint density functions of one-dimensional linear advection equations with uncertainties. In the third example, we inspect a particular equation. We apply Theorem 2.5.

Example 2.6. We deal with the one-dimensional linear advection equation with random initial condition: \( u_t(t, x) + a u_x(t, x) = 0, u(0, x) = u_0(x), t \geq 0, x \in \mathbb{R} \), where \( a \in \mathbb{R} \) is a constant and \( u_0 \) is a stochastic process. In this case, \( g = (1, a) \). Let \( f(q_1, q_2; t_1, t_2, y) \) be the joint probability density function of \((u(t_1, x), u(t_2, y))\) evaluated at \((q_1, q_2) \in \mathbb{R}^2\). By Theorem 2.5 and after some simple computations,

\[
\left( \frac{\partial}{\partial t_1} + a \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t_2} + a \frac{\partial}{\partial y} \right) f(q_1, q_2; t_1, t_2, y) = 0.
\]

This formula may be verified by employing the exact random field solution \( u(t, x) = u_0(x-\alpha t) \) and its exact density function \( f(q_1, q_2; t_1, t_2, y) = f_{u_0}(q_1, q_2; x-\alpha t_1, y-\alpha t_2) \).

Example 2.7. In the setting of Example 2.6 with a random and \( u_0 \) stochastic, both independent, Theorem 2.5 yields the following partial differential equation for the joint probability density function \( f(q_1, q_2; t_1, t_2, y) \):

\[
\frac{\partial^2}{\partial t_1 \partial t_2} f(q_1, q_2; t_1, t_2, y) + \left( \frac{\partial^2}{\partial t_2 \partial y} + \frac{\partial^2}{\partial x \partial y} \right) \{ \mathbb{E} [a | u(t_1, x) = q_1, u(t_2, y) = q_2] f(q_1, q_2; t_1, t_2, y) \} = 0.
\]

By Bayes’ formula and the independence,

\[
f_{a|u_0(x-\alpha t_1)=q_1,u_0(y-\alpha t_2)=q_2}(a_0) = \frac{f_{u_0}(q_1, q_2; x-\alpha_0 t_1, y-\alpha_0 t_2) f_a(a_0)}{\int_{\mathbb{R}} f_{u_0}(q_1, q_2; x-\alpha_0 t_1, y-\alpha_0 t_2) f_a(a_0) \, da_0}.
\]
Thus, the Liouville’s equation becomes

\[ \mathbb{E}[u|u_0(x-at)] = q = \frac{\int_{\mathbb{R}} a_0 f_{u_0}(q_1, q_2; x - a_0 t_1, y - a_0 t_2) f_a(a_0) \, da_0}{\int_{\mathbb{R}} f_{u_0}(q_1, q_2; x - a_0 t_1, y - a_0 t_2) f_a(a_0) \, da_0} \]

and

\[ \mathbb{E}[u^2|u_0(x-at)] = q = \frac{\int_{\mathbb{R}} a_0^2 f_{u_0}(q_1, q_2; x - a_0 t_1, y - a_0 t_2) f_a(a_0) \, da_0}{\int_{\mathbb{R}} f_{u_0}(q_1, q_2; x - a_0 t_1, y - a_0 t_2) f_a(a_0) \, da_0}. \]

Thus, the Liouville’s equation becomes

\[
\frac{\partial^2}{\partial t_1 \partial t_2} f(q_1, q_2; t_1, x, t_2, y) + \left( \frac{\partial^2}{\partial t_1 \partial y} + \frac{\partial^2}{\partial t_2 \partial x} \right) \left\{ \int_{\mathbb{R}} a_0 f_{u_0}(q_1, q_2; x - a_0 t_1, y - a_0 t_2) f_a(a_0) \, da_0 \right\} f(q_1, q_2; t_1, x, t_2, y) \]

\[
+ \frac{\partial^2}{\partial x \partial y} \left\{ \int_{\mathbb{R}} a_0^2 f_{u_0}(q_1, q_2; x - a_0 t_1, y - a_0 t_2) f_a(a_0) \, da_0 \right\} f(q_1, q_2; t_1, x, t_2, y) \] = 0.

This formula may be checked by considering the exact density function

\[ f(q_1, q_2; t_1, x, t_2, y) = \int_{\mathbb{R}} f_{u_0}(q_1, q_2; x - a_0 t_1, y - a_0 t_2) f_a(a_0) \, da_0. \]

This exact density is obtained by applying the law of total probability, as in Example 2.3.

**Example 2.8.** Let \( x_2 - ayu_0 = 0 \), where \( a \) is a random variable, \( x > 1 \) and \( y \in \mathbb{R} \). Consider the random initial condition \( u(1, y) = by \), where \( b \) is a random quantity independent of \( a \). This is the same equation as in Example 2.4 but now it contains a random initial condition. The solution is \( u(x, y) = b x^a y \). Let \( f(q_1, q_2; x_1, y_1, x_2, y_2) \) be the joint probability density function of \((u(x_1, y_1), u(x_2, y_2))\) evaluated at \((q_1, q_2) \in \mathbb{R}^2\). Notice that \( f(q_1, q_2; x_1, y_1, x_2, y_2) = 0 \) when \( q_2 y_2/(q_1 y_2) \leq 0 \). We then assume that \( q_2 y_2/(q_1 y_2) > 0 \). Theorem 2.5 is applied with \( g(x, y) = (x, -ay) \). An important part of the Liouville’s equation is the conditional expectation

\[ \alpha = \mathbb{E}[a|bx_1 y_1 = q_1, bx_2 y_2 = q_2] = \frac{\log(q_2 y_2/(q_1 y_2))}{\log(x_2/x_1)}. \]

Then Theorem 2.5 gives

\[
\frac{\partial^2}{\partial x_1 \partial x_2} (x_1 x_2 f) - \frac{\partial^2}{\partial x_1 \partial y_2} (x_1 y_2 \alpha f) - \frac{\partial^2}{\partial y_1 \partial x_2} (x_2 y_1 \alpha f) + \frac{\partial^2}{\partial y_1 \partial y_2} (y_1 y_2 \alpha f) - \frac{\partial}{\partial x_1} ((1-\alpha)x_1 f) + \frac{\partial}{\partial y_1} ((1-\alpha)\alpha y_1 f) - \frac{\partial}{\partial x_2} ((1-\alpha)x_2 f) + \frac{\partial}{\partial y_2} ((1-\alpha)\alpha y_2 f) + (1-\alpha)^2 f = 0.
\]

After elementary but tedious calculations, if \( \beta = \log(x_2/x_1) \), the Liouville’s equation becomes

\[
x_1 x_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} - x_1 y_2 \alpha \frac{\partial^2 f}{\partial x_1 \partial y_2} - x_2 y_1 \alpha \frac{\partial^2 f}{\partial y_1 \partial x_2} + y_1 y_2 \alpha \frac{\partial^2 f}{\partial y_1 \partial y_2} + x_1 \frac{1}{\beta} \frac{\partial f}{\partial x_1} - x_2 \frac{1}{\beta} \frac{\partial f}{\partial x_2} - y_1 \frac{\alpha}{\beta} \frac{\partial f}{\partial y_1} + y_2 \frac{\alpha}{\beta} \frac{\partial f}{\partial y_2} = 0,
\]
The Liouville’s equations for the first and second finite-dimensional distributions of $u$ have been obtained in Theorems 2.1 and 2.5. The next result solves the general case. The ideas are the same as those used in Theorem 2.5 but the expressions and indices become difficult to handle. The proof is sketched. Given $x_1, \ldots, x_s \in \mathbb{R}^n$, $s \geq 1$, let $f(q_1, \ldots, q_s; x_1, \ldots, x_s)$ be the probability density function of the joint vector $(u(x_1), \ldots, u(x_s))$ with respect to the standard Lebesgue measure, evaluated at $(q_1, \ldots, q_s) \in \mathbb{R}^m \times \cdots \times \mathbb{R}^m$.

**Theorem 2.9.** Given (2.1), it holds

$$\sum_{I \subseteq [s]} (-1)^{s-|I|} \left( \bigotimes_{i \in I} \frac{\partial}{\partial x_i} \right) \cdot \left( \mathbb{E} \left[ \left( \prod_{i \in I^c} (\nabla x_i \cdot g(x_i)) \right) \left( \bigotimes_{i \in I} g(x_i) \right) \bigg| u(x_k) = q_k, \forall k = 1, \ldots, s \right] f \right) = 0,$$

where $f \equiv f(q_1, \ldots, q_s; x_1, \ldots, x_s)$, $[s] = \{1, \ldots, s\}$, $|I|$ is the cardinality of $I$, and $I^c$ is the complement of $I$.

In particular, when $g$ is deterministic,

$$\left[ \bigotimes_{i=1}^{s} g(x_i) \right] \cdot \left[ \left( \bigotimes_{i=1}^{s} \frac{\partial}{\partial x_i} \right) f \right] = 0.$$

**Proof.** Let $h_1, \ldots, h_s : \mathbb{R}^m \to \mathbb{R}$ be smooth functions with compact support. Denote $x_i = (x_{i1}, \ldots, x_{in})$. By (2.4),

$$\sum_{j=1}^{n} g_j(x_i) (h_i(u))_{x_{ij}}(x_i) = 0, \quad i = 1, \ldots, s.$$

Multiply all these equalities for $i = 1, \ldots, s$ and apply expectation:

$$0 = \sum_{j_1=1}^{n} \cdots \sum_{j_s=1}^{n} \mathbb{E} \left[ \prod_{i=1}^{s} g_{j_i}(x_i) (h_i(u))_{x_{ij_i}}(x_i) \right]$$

$$= \sum_{j_1=1}^{n} \cdots \sum_{j_s=1}^{n} \mathbb{E} \left[ \prod_{j=1}^{s} \left\{ (g_{j_i} h_i(u))_{x_{ij_i}}(x_i) - (g_{j_i})_{x_{ij_i}}(x_i) h_i(u(x_i)) \right\} \right]. \quad (2.9)$$

Those products of the form $\prod_{i=1}^{s}(\alpha_i + \beta_i)$, where $\alpha_i, \beta_i \in \mathbb{R}$, can be written as

$$\prod_{i=1}^{s} (\alpha_i + \beta_i) = \sum_{I \subseteq [s]} \alpha_I \beta_{I^c}, \quad (2.10)$$

where, by definition,

$$\alpha_J = \prod_{j \in J} \alpha_j$$

for $J \subseteq [s]$, and $\alpha_\emptyset = 1$. By applying (2.10), equation (2.9) can be rewritten. The linearity of expectation is used. Then conditional expectations are employed, as in the proofs of Theorems 2.1 and 2.5. The final expression is written using the divergence notation. It is understood that $\bigotimes_{i \in \emptyset}$ is equal to the vector (1) and that $\prod_{i \in \emptyset}$ is 1. \qed
3. PATH-WISE STOCHASTIC INTEGRALS AND FIRST-ORDER RANDOM ORDINARY
DIFFERENTIAL EQUATIONS

As a consequence of Section 2 in this section we derive partial differential equations for the
probability density function of the path-wise stochastic integral and the solution to random
ordinary differential equations. These are referred to as Liouville’s equations. A key fact
is the relation between first-order homogeneous semilinear partial differential equations
and ordinary differential equations through first integrals.

**Proposition 3.1.** Let \( g = (g_1, \ldots, g_n) \), where \( g_i(t), t \geq 0 \), is a stochastic process. Let \( v_0 \) be
a random vector of length \( n \). Let \( f(q; t) \) be the probability density function of \( v_0 + \int_0^t g(s) \, ds \)
evaluated at \( q \in \mathbb{R}^n \). Then

\[
\frac{\partial}{\partial t} f(q; t) + \nabla_q \cdot \left( \mathbb{E} \left[ g(t) \bigg| v_0 + \int_0^t g(s) \, ds = q \right] f(q; t) \right) = 0.
\]

**Proof.** Consider the partial differential equation

\[
u_i(t, x) + \sum_{i=1}^n g_i(t) u_i(t, x) = 0, \quad u(0, x) = u_0(x),
\]

(3.1)

where \( u_0 \) is a random field to be specified later, \( x \in \mathbb{R}^n \). Let \( u(t, x) = u_0 \left( x - \int_0^t g(s) \, ds \right) \).

Take \( u_0(z) = -z + v_0 \), so that

\[
u(t, x) = v_0 + \int_0^t g(s) \, ds - x.
\]

(3.2)

Let \( f(q; t, x) \) be the probability density function of \( u(t, x) \) at \( q \in \mathbb{R}^n \). Notice that \( f(q; t, x) = f(x + q; t) \). By Theorem 2.1,

\[
\frac{\partial}{\partial t} f(q; t, x) + \nabla_x \cdot (\mathbb{E}[g(t)|u(t, x) = q] f(q; t, x)) = 0.
\]

In particular, at \( x = 0 \),

\[
0 = \frac{\partial}{\partial t} f(q; t, 0) + \nabla_{x=0} \cdot (\mathbb{E}[g(t)|u(t, x) = q] f(q; t, x))
\]

\[
= \frac{\partial}{\partial t} f(q; t) + \nabla_{x=0} \cdot \left( \mathbb{E} \left[ g(t) \bigg| v_0 + \int_0^t g(s) \, ds = x + q \right] f(x + q; t) \right)
\]

\[
= \frac{\partial}{\partial t} f(q; t) + \nabla_q \cdot \left( \mathbb{E} \left[ g(t) \bigg| v_0 + \int_0^t g(s) \, ds = q \right] f(q; t) \right).
\]

Some simple examples are made hereafter. They reflect the natural connection between
Liouville’s equation, which is a consequence of the properties of semilinear partial differential
equations, and the method of transformation of random variables, which arises from the
change of variables formula for integration.
Example 3.2. Let \( g(t) = a \) be a time-independent random quantity, with probability density function \( f_a \). Let \( v(t) = \int_0^t g(s) \, ds = at, \ t \geq 0 \). We have \( \mathbb{E}[a|v(t) = q] = \mathbb{E}[a|at = q] = \mathbb{E}[a|a = q/t] = q/t \). Let \( f(q; t) \) be the probability density function of \( v(t) \). Proposition 3.1 renders

\[
\frac{\partial}{\partial t} f(q; t) + \frac{1}{t} \frac{\partial}{\partial q} \{qf(q; t)\} = 0.
\]

This equation is clear, by just differentiating the relation \( f(q; t) = f_a(q/t)/t \).

Example 3.3. Let \( v(t) = e^{at} \), where \( a \) is a random variable with probability density function \( f_a \). Given \( g(t) = v'(t) = ae^{at} \) and \( v_0 = 1 \), the relation \( v(t) = v_0 + \int_0^t g(s) \, ds \) holds. Let \( f(q; t) \) be the probability density function of \( v(t) \). We have \( \mathbb{E}[g(t)|v(t) = q] = \mathbb{E}[ae^{at}|e^{at} = q] = q \log(q)/t \). Proposition 3.1 gives

\[
\frac{\partial}{\partial t} f(q; t) + \frac{1}{t} \frac{\partial}{\partial q} \{q \log(q)f(q; t)\} = 0.
\]

This equation can also be derived by differentiating the identity \( f(q; t) = f_a(\log(q)/t)/(qt) \), obtained from transformation of random variables.

Example 3.4 (Proof of the transformation of random variables formula). Let \( v(t) = G(a, t) \), where \( G \) is a deterministic function and \( a \) is a random variable with probability density function \( f_a \). It is assumed that \( G(\ast, t) \) is invertible for each \( t \) (strictly monotone, for instance), with inverse \( H(\ast, t) \). Let \( g(t) = v'(t) = \partial_t G(a, t) \) and \( v_0 = G(a, 0) \), so that the relation \( v(t) = v_0 + \int_0^t g(s) \, ds \) holds. Let \( f(q; t) \) be the probability density function of \( v(t) \). We have \( \mathbb{E}[g(t)|v(t) = q] = \mathbb{E}[\partial_t G(a, t)|G(a, t) = q] = (\partial_t G)(H(q, t), t) \). This value can be rewritten. From the identity \( G(H(q, t), t) = q \), one has, by differentiating with respect to \( q \) and \( t \), that \( (\partial_t G)(H(q, t), t) = -(\partial_t H/\partial_q H)(q, t) \). By Proposition 3.1,

\[
\frac{\partial}{\partial t} f(q; t) = \frac{\partial}{\partial q} \left\{ \frac{\partial_t H(q, t)}{\partial_q H(q, t)} f(q; t) \right\}.
\]

This partial differential equation is satisfied by

\[
\frac{\partial}{\partial t} f(q; t) = f_a(H(q, t))\partial_q H(q, t),
\]

which is the probability density function of a transformation of random variables under a strictly monotone mapping \( G(\ast, t) \). Let us prove (3.4) from (3.3). The homogeneous part of the partial differential equation is

\[
\frac{\partial}{\partial t} f(q; t) - \frac{\partial_t H(q, t)}{\partial_q H(q, t)} \frac{\partial}{\partial q} f(q; t) = 0,
\]

whose general solution is \( \varphi(H(q, t)), \varphi \in C^1(\mathbb{R}) \). For (3.3), variables are changed as \( \xi(q, t) = H(q, t) \) (a solution of the homogeneous part) and \( \lambda(q, t) = t \). That is, \( q = G(\xi, \lambda) \) and \( t = \lambda \).
Therefore (3.3) becomes, in shorthand notation,
\[
\frac{\partial f}{\partial \lambda} = \frac{\partial}{\partial \xi} \left( \frac{\partial H(G(\xi, \lambda))}{\partial H(G(\xi, \lambda))} \right) \frac{\partial f}{\partial q} = (\partial^2_{12} H)(\partial_2 G)(\partial_1 H) \frac{\partial f}{\partial q} = \frac{\partial H}{\partial q} + (\partial^2_{12} H)(\partial_2 G) f = \frac{\partial}{\partial \lambda} \log(\partial H(G(\xi), \lambda)) f.
\]

Then \( f(\xi, \lambda) = k(\xi)\partial H(G(\xi, \lambda), \lambda) \), where \( k \) is a function. Undoing the change of variables, \( f(q; t) = k(H(q, t))\partial q H(q, t) \). This formula captures the set of solutions to (3.3). The function \( k \) is determined through a known state. If \( G(a, t_1) = a \) for some \( t_1 \), then \( f_a(q) = f(q; t_1) = k(q) \), so that (3.4) holds, as wanted.

In the following example, we deal with renowned Gaussian processes and determine Liouville’s equations for their path-wise integrals, as an application of Proposition 3.1.

**Example 3.5.** In dimension one, let \( g \) be a Gaussian stochastic process. Let \( u(t) = \int_0^t g(s) \, ds \). We find a Liouville’s equation for the probability density function of \( u(t) \), \( f(q; t) \).

By Proposition 3.1, \( \partial_t f(q; t) + \partial_q (\mathbb{E}[g(t)u(t) = q] f(q; t)) = 0 \). It is easy to see that the random vector \((g(t), u(t))\) is Gaussian, for each \( t \) (the integral is a limit of Riemann sums, each Riemann sum is Gaussian, and the limit of Gaussian vectors is Gaussian). By Example 4.51, the conditional law \( g(t)|u(t) = q \) is Gaussian and
\[
\mathbb{E}[g(t)|u(t) = q] = \mathbb{E}[g(t)] + \frac{\text{Cov}[g(t), u(t)]}{\mathbb{V}[u(t)]} (q - \mathbb{E}[u(t)]),
\]
where \( \mathbb{E}[u(t)] = \int_0^t \mathbb{E}[g(s)] \, ds \), \( \mathbb{V}[u(t)] = \int_0^t \int_0^t \text{Cov}[g(s), g(\tau)] \, ds \, d\tau \) and \( \text{Cov}[g(t), u(t)] = \int_0^t \text{Cov}[g(t), g(s)] \, ds \). The Liouville’s equation becomes
\[
\frac{\partial}{\partial t} f(q; t) + \frac{\partial}{\partial q} \left\{ \left( \mathbb{E}[g(t)] + \frac{\text{Cov}[g(t), u(t)]}{\mathbb{V}[u(t)]} (q - \mathbb{E}[u(t)]) \right) f(q; t) \right\} = 0. \tag{3.5}
\]

Other Liouville’s equations may be found in this context. For example, by differentiating the generic density function of a Normal distribution directly, it is easy to check that
\[

\mathbb{V}[u(t)] \frac{\partial^2}{\partial q^2} f(q; t) + \frac{\partial}{\partial q} \{ f(q; t) (q - \mathbb{E}[u(t)]) \} = 0.
\]

If we combine the two Liouville’s formulas, we arrive at the following convection-diffusion equation:
\[
\frac{\partial}{\partial t} f(q; t) + \mathbb{E}[g(t)] \frac{\partial}{\partial q} f(q; t) = \text{Cov}[g(t), u(t)] \frac{\partial^2}{\partial q^2} f(q; t). \tag{3.6}
\]

We can apply (3.5) and (3.6) to derive, after elementary calculations, explicit Liouville’s equations for important Gaussian processes. When \( g \) is the standard Brownian motion (zero-mean and covariance function equal to \( \min\{t, s\} \) at times \( t, s \)) [25 Ch. 5], the probability density function of its primitive \( u(t) \) evolves as
\[
\frac{\partial}{\partial t} f(q; t) + \frac{3}{2t} \frac{\partial}{\partial q} (q f(q; t)) = 0,
\]
\[
\frac{\partial}{\partial t} f(q; t) = \frac{t^2}{2} \frac{\partial^2}{\partial q^2} f(q; t).
\]

When \( g \) is the standard Brownian bridge on \([0, 1]\) (zero-mean and covariance function equal to \( \min\{t, s\} - ts \) at times \((t, s)\)) \cite[p. 193]{25}, then
\[
\frac{\partial}{\partial t} f(q; t) + \frac{6(t - 1)}{t(3t - 4)} \frac{\partial}{\partial q} (q f(q; t)) = 0,
\]
\[
\frac{\partial}{\partial t} f(q; t) = \frac{(1 - t)^2}{2} \frac{\partial^2}{\partial q^2} f(q; t).
\]

If \( g \) is a Gaussian white noise (zero-mean and delta-correlated, given by the formal derivative of the standard Brownian motion) \cite[p. 196]{25}, then
\[
\frac{\partial}{\partial t} f(q; t) + \frac{1}{2t} \frac{\partial}{\partial q} (q f(q; t)) = 0,
\]
\[
\frac{\partial}{\partial t} f(q; t) = \frac{1}{2} \frac{\partial^2}{\partial q^2} f(q; t).
\]

The Ornstein-Uhlenbeck process is defined in \cite[Example 3.9]{26} as
\[
g(t) = \frac{B(e^t)}{e^{t/2}},
\]
where \( B \) is the standard Brownian motion. Such process is Gaussian, has zero mean and covariance function equal to \( e^{-(1/2)|t-s|} \) at times \((t, s)\). The law of its primitive is governed by the equations
\[
\frac{\partial}{\partial t} f(q; t) + \frac{1 - e^{-t/2}}{4(e^{-t/2} - 1) + 2t} \frac{\partial}{\partial q} (q f(q; t)) = 0,
\]
\[
\frac{\partial}{\partial t} f(q; t) = 2(1 - e^{-t/2}) \frac{\partial^2}{\partial q^2} f(q; t).
\]

The fractional Brownian motion \cite[p. 196]{25}, with Hurst parameter \( H \in (0, 1) \), is defined as a Gaussian process \( g \) with zero mean and covariance function equal to \( \frac{1}{2}([t]^{2H} + [s]^{2H} - |t-s|^{2H}) \) at times \((t, s)\). The Liouville’s equations for its primitive are
\[
\frac{\partial}{\partial t} f(q; t) + \frac{1 + H}{t} \frac{\partial}{\partial q} (q f(q; t)) = 0,
\]
\[
\frac{\partial}{\partial t} f(q; t) = \frac{1}{2} t^{1+2H} \frac{\partial^2}{\partial q^2} f(q; t).
\]

As a consequence of \cite{3.5}, the density function of the Itô integral of a square-integrable deterministic function \( \varphi(t) \), given by \( u(t) = \int_0^t \varphi(s) \eta(s) ds \), where \( \eta(s) \) is a Gaussian white noise, satisfies
\[
\left( \int_0^t \varphi(s)^2 ds \right) \frac{\partial}{\partial t} f(q; t) + \frac{1}{2} \varphi(t)^2 \frac{\partial}{\partial q} (q f(q; t)) = 0.
\]

The example is finished.

In what follows, Proposition 3.1 is extended to higher dimensions: two instants of time \( t_1, t_2 \), are considered.
Proposition 3.6. Let $g = (g_1, \ldots, g_n)$, where $g_i(t)$, $t \geq 0$, is a stochastic process. Let $v_0$ be a random vector of length $n$. Let $f(q_1, q_2; t_1, t_2)$ be the probability density function of $(v_0 + \int_0^{t_1} g(s) \, ds, v_0 + \int_0^{t_2} g(s) \, ds)$ evaluated at $(q_1, q_2) \in \mathbb{R}^n \times \mathbb{R}^n$. Then

\[
0 = \frac{\partial^2}{\partial t_1 \partial t_2} f(q_1, q_2; t_1, t_2) + \frac{\partial}{\partial t_1} \nabla_{q_2} \cdot \left( \mathbb{E} \left[ g(t_1) \mid q_1, q_2 \right] \right)v_0 + \int_0^{t_1} g(s) \, ds = q_1, v_0 + \int_0^{t_2} g(s) \, ds = q_2 \right) f(q_1, q_2; t_1, t_2) \\
+ \frac{\partial}{\partial t_2} \nabla_{q_1} \cdot \left( \mathbb{E} \left[ g(t_1) \mid q_1, q_2 \right] \right)v_0 + \int_0^{t_1} g(s) \, ds = q_1, v_0 + \int_0^{t_2} g(s) \, ds = q_2 \right) f(q_1, q_2; t_1, t_2) \\
+ \left( \nabla_{q_1} \otimes \nabla_{q_2} \right) \cdot \left( \mathbb{E} \left[ g(t_1) \otimes g(t_2) \mid q_1, q_2 \right] \right)v_0 + \int_0^{t_1} g(s) \, ds = q_1, v_0 + \int_0^{t_2} g(s) \, ds = q_2 \right) f(q_1, q_2; t_1, t_2).
\]

Proof. Consider the partial differential equation (3.1), with initial condition $u_0(z) = -z + v_0$ and solution given by (3.2). Let $f(u_1, u_2; t_1, t_2, y)$ be the probability density function of $(u(t_1, x), u(t_2, y))$ at $(u_1, u_2) \in \mathbb{R}^n \times \mathbb{R}^n$. Notice that $f(q_1, q_2; t_1, x, t_2, y) = f(x + q_1, y + q_2; t_1, t_2)$. By Theorem 2.5

\[
0 = \frac{\partial^2}{\partial t_1 \partial t_2} f(q_1, q_2; t_1, t_2, y) + \frac{\partial}{\partial t_1} \nabla_y \cdot \left( \mathbb{E} \left[ g(t_1) \mid u(t_1, x) = q_1, u(t_2, y) = q_2 \right] \right)v_0 + \int_0^{t_1} g(s) \, ds = q_1, v_0 + \int_0^{t_2} g(s) \, ds = q_2 \right) f(q_1, q_2; t_1, t_2, y)) \\
+ \frac{\partial}{\partial t_2} \nabla_x \cdot \left( \mathbb{E} \left[ g(t_1) \mid u(t_1, x) = q_1, u(t_2, y) = q_2 \right] \right)v_0 + \int_0^{t_1} g(s) \, ds = q_1, v_0 + \int_0^{t_2} g(s) \, ds = q_2 \right) f(q_1, q_2; t_1, t_2, y)) \\
+ \left( \nabla_x \otimes \nabla_y \right) \cdot \left( \mathbb{E} \left[ g(t_1) \otimes g(t_2) \mid u(t_1, x) = q_1, u(t_2, y) = q_2 \right] \right)v_0 + \int_0^{t_1} g(s) \, ds = q_1, v_0 + \int_0^{t_2} g(s) \, ds = q_2 \right) f(q_1, q_2; t_1, t_2, y)).
\]

Evaluating at $x = 0$ and $y = 0$, we obtain the desired Liouville’s equation for $f(q_1, q_2; t_1, t_2)$. \qed

Example 3.7. Let $v(t) = be^{at}$, where $a$ and $b$ are independent random variables. Given $g(t) = v'(t) = abe^{at}$ and $v_0 = b$, the relation $v(t) = v_0 + \int_0^t g(s) \, ds$ holds. Let $f(q_1, q_2; t_1, t_2)$ be the joint probability density function of $(v(t_1), v(t_2))$. Note that this density is defined when $t_1 \neq t_2$ (by the independence), otherwise $v(t_1) = v(t_2)$ and therefore $(v(t_1), v(t_2))$ does not possess a density. On the other hand, $v(t_1)$ and $v(t_2)$ always have the same sign, therefore $f$ is zero whenever $q_1q_2 < 0$. Let us suppose that $q_1q_2 > 0$ and $t_1 < t_2$. We have

\[
\mathbb{E}[g(t_2)|v(t_1) = q_1, v(t_2) = q_2] = \frac{q_2 \log(q_2/q_1)}{t_2 - t_1};
\]
\[
\mathbb{E}[g(t_1)|v(t_1) = q_1, v(t_2) = q_2] = \frac{q_1 \log(q_2/q_1)}{t_2 - t_1};
\]
\[
\mathbb{E}[g(t_1)g(t_2)|v(t_1) = q_1, v(t_2) = q_2] = \frac{q_1q_2 \log^2(q_2/q_1)}{(t_2 - t_1)^2}.
\]

By Proposition 3.6,

\[
\frac{\partial^2 f}{\partial t_1 \partial t_2} + \frac{\partial^2}{\partial t_1 \partial q_2} \left( \frac{q_2 \log(q_2/q_1)}{t_2 - t_1} f \right) + \frac{\partial^2}{\partial t_2 \partial q_1} \left( \frac{q_1 \log(q_2/q_1)}{t_2 - t_1} f \right) + \frac{\partial^2}{\partial q_1 \partial q_2} \left( \frac{q_1q_2 \log^2(q_2/q_1)}{(t_2 - t_1)^2} f \right) = 0
\]

is a Liouville’s equation.
Proposition 3.8 is extended to general dimensions: several instants of time $t_1, \ldots, t_s$, $s \geq 1$, are considered.

**Proposition 3.8.** Let $g = (g_1, \ldots, g_n)$, where $g_i(t)$, $t \geq 0$, is a stochastic process. Let $v_0$ be a random vector of length $n$. Let $f(q_1, \ldots, q_s; t_1, \ldots, t_s)$, $s \geq 1$, be the probability density function of $\left(v_0 + \int_{0}^{t_1} g(\tau) \, d\tau, \ldots, v_0 + \int_{0}^{t_s} g(\tau) \, d\tau\right)$ evaluated at $(q_1, \ldots, q_s) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$. Then

$$
\sum_{I \subseteq [s]} \frac{\partial |I|}{\partial t_I} \left( \otimes_{i \in I^c} \nabla q_i \right) \cdot \left( \mathbb{E} \left[ \otimes_{i \in I} g(t_i) \right] \left[ v_0 + \int_{0}^{t_k} g(\tau) \, d\tau = q_k, \forall k = 1, \ldots, s \right] \mathbb{E} \right) = 0,
$$

where $f \equiv f(q_1, \ldots, q_s; t_1, \ldots, t_s)$, $|s| = \{1, \ldots, s\}$, $|I|$ is the cardinality of $I$, $I^c$ is the complement of $I$, and $\partial t_I = \prod_{i \in I} \partial t_i$.

**Proof.** Consider the partial differential equation (3.1), with initial state $u_0(z) = -z + v_0$ and solution given by (3.2). Let $\tilde{f}(u_1, \ldots, u_s; t_1, x_1, \ldots, t_s, x_s)$ be the probability density function of $(u(t_1, x_1), \ldots, u(t_s, x_s))$ at $(u_1, \ldots, u_s) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$. Notice that

$$
\tilde{f}(q_1, \ldots, q_s; t_1, x_1, \ldots, t_s, x_s) = f(x_1 + q_1, \ldots, x_s + q_s; t_1, \ldots, t_s).
$$

Theorem 2.9 is applied. In the sum of its Liouville’s equation, consider $I \subseteq [s]$ with $I \neq [s]$. If $i \in I^c$, then the term $\nabla (t_i, x_i) \cdot (1, g(t_i))$ is clearly 0. Thus, the sum has one term, corresponding to $I = [s]$. The equation is

$$
\left( \sum_{i = 1}^{s} \nabla (t_i, x_i) \right) \cdot \left( \mathbb{E} \left[ \sum_{i = 1}^{s} (1, g(t_i)) \left[ v_0 + \int_{0}^{t_k} g(\tau) \, d\tau = q_k, \forall k = 1, \ldots, s \right] \tilde{f} \right) = 0,
$$

where $\tilde{f} \equiv \tilde{f}(q_1, \ldots, q_s; t_1, x_1, \ldots, t_s, x_s)$. It remains rewriting this equation. After a careful inspection of indices, one arrives at the following form:

$$
\frac{\partial^{s} \tilde{f}}{\partial t_{[s]}^s} + \sum_{I \subseteq [s]} \frac{\partial |I|}{\partial t_I} \sum_{j = 1}^{n} \frac{\partial^{s-|I|}}{\partial \tau_{i,j}^{I^c}} \left( \mathbb{E} \left[ \prod_{i \in I^c} g_j(t_i) \left[ v_0 + \int_{0}^{t_k} g(\tau) \, d\tau = q_k, \forall k = 1, \ldots, s \right] \tilde{f} \right) = 0.
$$

Evaluating at $x_1 = 0, \ldots, x_s = 0$,

$$
\frac{\partial^{s} f}{\partial t_{[s]}^s} + \sum_{I \subseteq [s]} \frac{\partial |I|}{\partial t_I} \sum_{j = 1}^{n} \frac{\partial^{s-|I|}}{\partial q_{i,j}^{I^c}} \left( \mathbb{E} \left[ \prod_{i \in I^c} g_j(t_i) [ v_0 + \int_{0}^{t_k} g(\tau) \, d\tau = q_k, \forall k = 1, \ldots, s ] f \right) = 0.
$$

This is exactly the Liouville’s equation stated in the proposition. \qed

The Liouville’s equation for random ordinary differential equations is deduced, thus complementing the various proofs from the literature \cite{2,14,16}.

**Proposition 3.9.** Consider the general random ordinary differential equation problem

$$
v'(t) = g(t, v(t)), \quad v(0) = v_0,
$$

where $g : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a deterministic function, $v : \mathbb{R} \rightarrow \mathbb{R}^n$ is a stochastic process, and $v_0 \in \mathbb{R}^n$ is a random vector. Let $f(q; t)$ be the probability density function of $v(t)$ evaluated at $q \in \mathbb{R}^n$. Then

$$
\frac{\partial}{\partial t} f(q; t) + \nabla_q \cdot (g(t, q) f(q; t)) = 0.
$$
Proof. We have $v(t) = v_0 + \int_0^t g(s) \, ds$, where $g(s) = g(s, v(s))$. By Proposition [3.1] with $g$,

$$
0 = \frac{\partial}{\partial t} f(q; t) + \nabla_q \cdot \left( \mathbb{E} [ g(t, v(t)) | v(t) = q ] f(q; t) \right)
$$

(3.7)

$$
= \frac{\partial}{\partial t} f(q; t) + \nabla_q \cdot (g(t, q) f(q; t)).
$$

Notice that, in the setting of Proposition [3.9], it does not make sense to consider the probability density function of $(v(t_1), v(t_2))$; since randomness is only due to the initial condition $v_0 \in \mathbb{R}^n$, the vector $(v(t_1), v(t_2)) \in \mathbb{R}^{2n}$ cannot be absolutely continuous.

Note that $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ was considered as a non-random function in Proposition [3.9]. This was necessary to compute the conditional expectation. When $g$ depends on a finite number of random variables, $\xi = (\xi_1, \ldots, \xi_n)$, and the initial position $v_0 \in \mathbb{R}^n$ is random, then a Liouville’s equation is obtained for the joint density $f(q, \xi_0; t)$ of $(v(t), \xi)$ evaluated at $(q, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$. By changing the underlying probability measure $\mathbb{P}$ to $\mathbb{P}[\star \xi = \xi_0]$, Proposition [3.9] gives

$$
\frac{\partial}{\partial t} f(q; t) + \nabla_q \cdot (g(t, q, \xi_0) f(q; t; \xi_0)) = 0.
$$

By multiplying by the density of $\xi$ at $\xi_0$, $f(\xi; \xi_0)$, and taking into account that $f(q; t) f(\xi; \xi_0) = f(q, \xi_0; t)$, we arrive at

$$
\frac{\partial}{\partial t} f(q, \xi_0; t) + \nabla_q \cdot (g(t, q, \xi_0) f(q, \xi_0; t)) = 0.
$$

The density of the model output, $v(t)$, is obtained by marginalizing:

$$
f(q; t) = \int_{\mathbb{R}^n} f(q, \xi_0; t) d\xi_0.
$$

The situation in which the initial state $v_0$ is non-random does not seem to have been much explored in the literature. We investigate an interesting example in this regard.

Example 3.10. In the contribution [7], the authors considered mathematical models for the growth of shrimp cultured at the Shrimp Mariculture Research Facility, Texas Agricultural Experiment Station in Corpus Christi, TX. At different times $t_k$, fifty shrimp were randomly selected and individually weighed. By fitting the recorded data, it was found reasonable that the average weight (in grams) evolved in time exponentially. If $v(t)$ denotes the weight at time $t$, considered as a random variable, then the following model for the average weight was suggested:

$$
\frac{d\mathbb{E}[v(t)]}{dt} = b_0 \left( \mathbb{E}[v(t)] + c_0 \right) \implies \mathbb{E}[v(t)] = -c_0 + \left( \mathbb{E}[v_0] + c_0 \right) e^{b_0 t}.
$$

(3.8)

Here $b_0$ and $c_0$ are positive constants, which denote the intrinsic growth rate and the affine growth rate, respectively. The term $v_0 = v(0)$ denotes the initial weight, which shall be assumed non-random in this example. Taking into account (3.8), mathematical models based on random and Itô stochastic differential equations were proposed for $v(t)$. When $b$ follows a Normal distribution, with mean value $b_0$ and variance $\sigma_0^2$, the following random ordinary differential equation was considered:

$$
v'(t) = (b - \sigma_0^2 t) (v(t) + c_0).
$$

(3.9)

Variability of the intrinsic growth rate may be attributed to the effect of genetic differences or some disease. On the other hand, when the derivative $v'(t)$ is perturbed by a Gaussian white noise process $\eta(t)$ (formal derivative of the standard Brownian motion), the stochastic differential equation model was defined as follows:

$$
v'(t) = b_0 (v(t) + c_0) + \sqrt{2t} \sigma_0 (v(t) + c_0) \eta(t).
$$

(3.10)
It was proved that the solutions \( v(t) \) to (3.9) and (3.10) possess the same one-dimensional probability density function, \( f(q; t) \), by using properties of the log-Normal distribution:

\[
 f(q; t) = \frac{1}{(q + c_0)\sqrt{2\pi}\sigma_0 t} \exp \left( -\frac{1}{2\sigma_0^2 t^2} \left[ \log((q + c_0)/(v_0 + c_0)) - (b_0 t - (1/2)\sigma_0^2 t^2) \right]^2 \right). 
\]  

(3.11)

Equation (3.8) for \( \mathbb{E}[v(t)] \) holds for both cases (3.9) and (3.10). Let us derive Liouville’s equations for the density (3.11). From step (3.7) within the proof of Proposition 3.9, a key issue is the computation of the conditional expectation \( \mathbb{E}[(b - \sigma_0^2 t)(v(t) + c_0)|v(t) = q] \). The explicit solution to (3.9) is

\[
 v(t) = -c_0 + (v_0 + c_0)e^{bt - \sigma_0^2 t^2/2}. 
\]  

(3.12)

Hence

\[
 b = \frac{1}{t}\log \frac{v(t) + c_0}{v_0 + c_0} + \frac{\sigma_0^2 t}{2}. 
\]

The conditional expectation then reads as follows:

\[
 \mathbb{E}[(b - \sigma_0^2 t)(v(t) + c_0)|v(t) = q] = (q + c_0) \left( \frac{1}{t}\log \frac{q + c_0}{v_0 + c_0} - \frac{1}{2}\sigma_0^2 t \right). 
\]

The Liouville’s equation is

\[
 \frac{\partial}{\partial t} f(q; t) + \frac{\partial}{\partial q} \left[ (q + c_0) \left( \frac{1}{t}\log \frac{q + c_0}{v_0 + c_0} - \frac{1}{2}\sigma_0^2 t \right) f(q; t) \right] = 0. 
\]

Note that this partial differential equation holds for any probability distribution of \( b \), not necessarily Gaussian of parameters \((b_0, \sigma_0^2)\). Check it by just differentiating the generic probability density function of (3.12):

\[
 f(q; t) = f_b \left( \frac{1}{t}\log \frac{q + c_0}{v_0 + c_0} + \frac{\sigma_0^2 t}{2} \right) \frac{1}{t(q + c_0)}, 
\]

where \( f_b \) is any density of \( b \) (this expression for \( f(q; t) \) is a consequence of the method of transformation of random variables). The Liouville’s partial differential equation is of first order, in contrast to the Fokker-Planck equation which is of second order. From the stochastic differential equation (3.10), the Fokker-Planck equation for \( f(q; t) \) is

\[
 \frac{\partial}{\partial t} f(q; t) = -\frac{\partial}{\partial q} (b_0(q + c_0)f(q; t)) + \frac{\partial^2}{\partial q^2} \left( t\sigma_0^2(q + c_0)^2 f(q; t) \right). 
\]

Differentiation of (3.11) reveals that the two partial differential equations are indeed true. By combining them, a third partial differential equation for (3.11) is derived (actually, a second-order ordinary differential equation for each fixed \( t \)):

\[
 2\sigma_0^2 t^2 (q + c_0)^2 \frac{\partial^2}{\partial q^2} f(q; t) + (q + c_0) \left( 7\sigma_0^2 t^2 - 2b_0 t + 2\log \frac{q + c_0}{v_0 + c_0} \right) \frac{\partial}{\partial q} f(q; t) 
\]

\[
 + \left( 3\sigma_0^2 t^2 - 2b_0 t + 2 + 2\log \frac{q + c_0}{v_0 + c_0} \right) f(q; t) = 0. 
\]

The example is finished.
Our context for random ordinary differential equations (Proposition 3.9 and below) assumes that the number of random terms is finite. When there is a stochastic process with infinite dimensionality, such as Brownian motion, a possible approach consists in truncating its Karhunen-Loève expansion [21]. This is legitimate. However, care must be exercised when smoothing out general Itô processes, since in the limit one may not obtain the same process but a variation of Stratonovich type [27]. Hence, when Itô processes are present in random ordinary differential equations, instead of smoothing them out by reducing dimensionality, it may be better to transform the random ordinary differential equation into an Itô stochastic differential equation, and then apply the Fokker-Planck equation for the probability density. For example, suppose the random ordinary differential equation \( v' = \mu(y) + \sigma(y) \eta(t) \) is an Itô process (\( \eta \) is the Gaussian white noise). Here \( v' = g(v,y) \) is a random ordinary differential equation because \( y \) is regular (it is Hölder continuous). The problem can be transformed into an Itô stochastic differential equation of dimension two: \( (v, y)' = (g(v,y), \mu(y)) + (0, \sigma(y)) \eta \). Then the Fokker-Planck equation for the joint density \( f(q_1, q_2; t) \) of \( (v(t), y(t)) \) is applicable:

\[
\frac{\partial f}{\partial t} = -\frac{\partial (gf)}{\partial q_1} - \frac{\partial (\mu f)}{\partial q_2} + \frac{\partial^2 (\sigma^2 f/2)}{\partial q_2^2}.
\]

The density of \( v(t) \) is finally obtained by marginalizing: \( \int_{\mathbb{R}} f(q_1, q_2; t) \, dq_2 \).

4. Conclusions and future work

4.1. Summary. In this paper, Liouville’s equations for random systems have been obtained. These are partial differential equations that describe the dynamics of the associated probability density function. We dealt with general first-order and homogeneous semilinear random partial differential equations and all finite-dimensional distributions. As a corollary, pathwise stochastic integrals and random ordinary differential equations were treated. A complete set of examples was included: linear random advection equation, transformation of random variables formula, primitives of important Gaussian processes, and controlled shrimp growth.

4.2. Random fractional differential equations. We include a brief discussion on random fractional differential equations and their associated Liouville’s equations. A fractional differential equation has the form

\[
C^D_{0^+,t} \mathbf{v}(t) = \mathbf{g}(t, \mathbf{v}(t)), \quad t \geq 0, \quad \mathbf{v}(0) = \mathbf{v}_0, \quad \text{where } 0 < \alpha < 1,
\]

and

\[
C^D_{0^+,t} \mathbf{v}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-u)^{-\alpha} \mathbf{v}'(u) \, du \text{ is the Caputo fractional derivative of order } \alpha \text{ at } t.
\]

We denote \( \mathbf{g} = (g_1, \ldots, g_n) \) and \( \mathbf{v} = (v_1, \ldots, v_n) \). In the random setting, we may consider the initial condition \( \mathbf{v}_0 \in \mathbb{R}^n \) as a random vector. When \( \mathbf{g} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is a deterministic function, there is a Liouville’s equation that governs the evolution of the probability density function \( f(\mathbf{v}; t) \) associated with the solution \( \mathbf{v}(t) \) [22]:

\[
\frac{\partial}{\partial t} f(\mathbf{v}; t) + \Gamma(2-\alpha) \sum_{k=1}^n v_k^{\alpha-1} C^D_{0^+,t} g_k(t, \mathbf{v}) f(\mathbf{v}; t) = 0. \tag{4.1}
\]

In [22], this formula (4.1) was obtained from the conservation of probability in a fractional volume element. It is an open question whether (4.1) can be deduced from Section 2 or similar results to ours but on fractional semilinear random partial differential equations. By [23], \( \mathbf{v}(t) = \mathbf{v}_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbf{g}(s, \mathbf{v}(s)) \, ds \) (random Volterra integral equation). Thus,
given $g(t, s) = \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}g(s, \nu(s))$, we would need to generalize Proposition 3.1 to $v_0 + \int_0^t g(t, s) \, ds$.

4.3. **Future work.** It is an open question whether Theorem 2.1 may be extended in some manner to other types of random partial differential equations (2.1), especially nonlinear ones, since a key fact of our reasoning was the closedness of solution to (2.1) under composition, see (2.4).

It is also an open problem whether (4.1) (Liouville’s equation for random fractional differential equations [22]) can be deduced from Section 2 or similar results to ours but on fractional semilinear random partial differential equations.

Finally, applications to uncertainty quantification shall be investigated, by numerically solving the Liouville’s equations. Comparison of performance between this hypothetical approach and other stochastic methods shall be conducted, such as kernel density estimation [28], method of transformation of random variables [29], polynomial chaos expansions combined with transformation of random variables [30], and finite difference schemes combined with transformation of random variables [31].

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