Research Article

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Foliations with isolated singularities on Hirzebruch surfaces

Abstract: We study foliations $\mathcal{F}$ on Hirzebruch surfaces S_δ and prove that, similarly to those on the projective plane, any $\mathcal{F}$ can be represented by a bi-homogeneous polynomial affine 1-form. In case $\mathcal{F}$ has isolated singularities, we show that, for $\delta = 1$, the singular scheme of $\mathcal{F}$ does determine the foliation, with some exceptions that we describe, as is the case of foliations in the projective plane. For $\delta \neq 1$, we prove that the singular scheme of $\mathcal{F}$ does not determine the foliation. However, we prove that, in most cases, two foliations $\mathcal{F}$ and $\mathcal{F}'$ given by sections s and s' have the same singular scheme if and only if $s' = \Phi(s)$, for some global endomorphism Φ of the tangent bundle of S_δ.

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1 Introduction

The study of complex planar polynomial differential systems goes back to the 19th century. Articles by Autonne [2, 3], Darboux [14], Painlevé [31] and Poincaré [33–37] can be considered as seminal references for this topic. Problems proposed more than a century ago, as to obtain conditions for the existence of first integrals for the above-mentioned systems, are still pending for resolution. Considering holomorphic foliations by curves with singularities (*foliations* in the sequel) on the complex projective plane have produced important advances in the knowledge of those systems [5, 9, 11, 15–20, 29, 32]. Foliations can be defined on another varieties extending the problems from the projective plane to those varieties [10, 12, 21, 26, 38, 39]. Focusing on foliations on surfaces, Hirzebruch surfaces S_δ with $\delta \neq 1$ (see Section 2 for our notation) constitute jointly with the projective plane the classical minimal rational surfaces, and the study of foliations on them is the first single step after that on the projective plane. Our aim is to deepen the study of foliations on Hirzebruch surfaces which have been treated within more general situations: as foliations on ruled surfaces (in the profound monograph [23]) or as foliations on toric varieties [12].

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Let M be a compact connected complex manifold. Recall that a foliation $\mathcal{F}$ on M may be defined by non-identically zero holomorphic vector fields X_i defined on a covering $\{V_i\}$ of M such that in each overlapping set $V_i \cap V_j$ we have

$$X_i = \xi_{ij} X_j, \tag{1.1}$$

where ξ_{ij} is a never vanishing holomorphic function. If L^* denotes the holomorphic line bundle constructed with the cocycle (ξ_{ij}), and $\mathcal{L}^*$ its corresponding invertible sheaf, then the X_i's give rise to a global section $s \in H^0(M, \Theta_M \otimes \mathcal{L}^*)$ or to a global section in $H^0(M, \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{L}, \Theta_M))$, where Θ_M is the tangent sheaf of M and $\mathcal{L}$ is the dual of $\mathcal{L}^*$. Two global sections (in the corresponding spaces) define the same foliation if and only if one is a non-zero scalar multiple of the other.

By following a somehow standard use (see [4], for instance), L^* will be called the *cotangent bundle of* $\mathcal{F}$, and its dual L will be called *its tangent bundle*. Hence, the space $\mathrm{Fol}(\mathcal{L}, M)$ of foliations $\mathcal{F}$ with tangent bundle L (or tangent sheaf $\mathcal{L}$) is $\mathbb{P}H^0(M, \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{L}, \Theta_M))$. Such an $\mathcal{F}$ corresponds to a foliation with cotangent bundle L^* (or cotangent sheaf $\mathcal{L}^*$) by regarding it as the class $[s] \in \mathbb{P}H^0(M, \Theta_M \otimes \mathcal{L}^*)$ of a global section $s \in H^0(M, \Theta_M \otimes \mathcal{L}^*)$.

Given a global section $s \in H^0(M, \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{L}, \Theta_M))$, the scheme $Z = Z_s$ of those points $p \in M$ where the induced morphism $\mathcal{L}_p \to \Theta_{M,p}$ becomes zero will be referred to as the *singular scheme* of s: its sheaf of ideals $I_Z \subset \mathcal{O}_M$ is the sheaf obtained by gluing the ideals $(a_i, b_i) \subset \mathcal{O}(V_i)$, where a_i and b_i are the coefficients of the vector field X_i that defines s on the open set V_i, as described before (1.1). The singular scheme of $\mathcal{F} = [s]$ is the singular scheme of any section in $[s]$.

We say that $\mathcal{F} = [s]$ *has isolated singularities* if $\dim Z_s = 0$.

In a series of papers [6–8], with the precedent of [24], Campillo and the third author have proved that, given a foliation

$$[s] \in \mathbb{P}H^0(\mathbb{P}^n, \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(-d), \Theta_{\mathbb{P}^n})) = \mathrm{Fol}(\mathcal{O}_{\mathbb{P}^n}(-d), \mathbb{P}^n)$$

with isolated singularities and $d > 1$, $[s]$ is the unique foliation in $\mathrm{Fol}(\mathcal{O}_{\mathbb{P}^n}(-d), \mathbb{P}^n)$ with singular scheme $Z = Z_s$. We summarize this statement saying that a foliation with isolated singularities of degree $d > 1$ in a projective space of dimension $n \geq 2$ is uniquely determined by its singular scheme.

In this paper, we study the extension of this result to foliations with isolated singularities on Hirzebruch surfaces S_δ, with $\delta \geq 0$.

Other results of this type, dealing with foliations (or distributions) of rank different from 1 in projective spaces, are given in [1, 13, 22].

To state our results, we recall from Section 2 below that every invertible sheaf $\mathcal{L}$ on S_δ has the form $\mathcal{O}_{S_\delta}(-d_1, -d_2)$ for some $d_1, d_2 \in \mathbb{Z}$. Hence, foliations on S_δ come from sections

$$s \in H^0(S_\delta, \mathrm{Hom}(\mathcal{O}_{S_\delta}(-d_1, -d_2), \Theta_{S_\delta})).$$

We work within the toric structure of S_δ. This point of view gives us a way to represent every section s by a bi-homogeneous polynomial affine 1-form Ω on $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$ (an affine 1-form, for short) in essentially the same way as a projective 1-form (say) in $\mathbb{P}^2$ is representable by a polynomial homogeneous 1-form in $(\mathbb{C}^3 \setminus \{0\})$ (see Proposition 3.2). This representation is one of our main tools. On the one hand, because it allows us to prove that if s has isolated singularities, then $d_1 \geq 0$ and $d_2 \geq 0$ if $\delta = 0$, and $d_1 \geq -1$ and $d_2 \geq 0$ if $\delta \geq 1$ (see Proposition 3.6, which is a refinement of [23, Proposition 2.2]). On the other hand, because the coefficients of Ω generate the ideal of the singular scheme Z of s (see Remark 3.8).

Global endomorphisms Φ of TS_δ play a central role. To start, Corollary 4.2 shows that, for $\delta \neq 1$ and Φ invertible, all foliations $[\Phi(s)]$ have the same singular scheme as $[s]$ does and most of them are different from $[s]$; therefore in this case, the singular scheme does not determine the foliation on the contrary to what happens in the projective case. However, we prove in Theorem 5.2 that for $d_2 \geq 1$ and $d_1 \geq 1$ (in case $\delta = 0$), and $d_1 \geq 2$ (in case $\delta \geq 2$), these foliations $[\Phi(s)]$ are the unique ones that share their singular scheme with $[s]$. We prove moreover that, in case $\delta = 1$, the foliation $[s]$ is uniquely determined by its singular scheme if $d_2 \geq 1$ and $d_1 \geq 0$.

Some preliminaries on Hirzebruch surfaces that will be used throughout the paper are given in Section 2. In Section 3, we give the aforementioned representation of the sections $s \in H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^*)$ in terms of poly-

nomial bi-homogeneous affine 1-forms, as well as one in terms of vector fields. Theorem 4.1 on the structure of the global endomorphisms of the tangent bundle of Hirzebruch surfaces is the main content of Section 4 and we see it as one of our leading results. Theorem 5.2, the main result of the paper, is proved in Section 5.

Notation. Throughout the paper, the structure sheaf $\mathcal{O}_{S_\delta}$ of S_δ will be denoted by $\mathcal{O}$. The sheaves of sections of the tangent TS_δ and cotangent T^*S_δ bundles will be denoted respectively by Θ_{S_δ} and $\Omega^1_{S_\delta}$. For a line bundle L, its associated invertible sheaf will be denoted by $\mathcal{O}(L)$. If it has the form $\mathcal{L} = \mathcal{O}(d_1, d_2)$, then the sheaves of sections of the bundles $TS_\delta \otimes L$ and $T^*S_\delta \otimes L$ will be denoted respectively by $\Theta_{S_\delta}(d_1, d_2)$ and $\Omega^1_{S_\delta}(d_1, d_2)$.

2 Preliminaries on Hirzebruch surfaces

For an integer $\delta \geq 0$, the Hirzebruch surface S_δ is the ruled surface

$$\psi : S_\delta \to \mathbb{P}^1 \tag{2.1}$$

associated to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-\delta))$, where $\mathbb{P}^1$ is the complex projective line [25, Chapter V, Corollary 2.13].

The surjective map ψ gives S_δ the structure of a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, and its fibres constitute *the ruling* of S_δ.

Let F and M be two generators of the divisor class group $\mathrm{Cl}(S_\delta)$ such that $F^2 = 0$, $M^2 = \delta$ and $F \cdot M = 1$. If $\delta > 0$, let M_0 denote the class of the $(-\delta)$-curve of S_δ, that is, the unique irreducible curve of S_δ with negative self-intersection (if $\delta = 0$, take M for M_0). For simplicity, for each $E \in \mathrm{Cl}(S_\delta)$, E will also denote the image of E in $\mathrm{Cl}(S_\delta) \otimes \mathbb{Q}$. For every $d_1, d_2 \in \mathbb{Z}$, the invertible sheaf $\mathcal{O}(d_1, d_2)$ corresponds to the class $d_1 F + d_2 M$.

The *cone of curves* $NE(S_\delta)$ of S_δ is the convex cone of $\mathrm{Cl}(S_\delta) \otimes \mathbb{Q}$ generated by the images of the effective classes. Its dual cone $NE(S_\delta)^\vee$ (with respect to the intersection form) is called the *nef cone* and is denoted by $P(S_\delta)$. Specifically,

$$P(S_\delta) := NE(S_\delta)^\vee = \{E \in \mathrm{Cl}(S_\delta) \otimes \mathbb{Q} \mid E \cdot C \geq 0 \text{ for any effective divisor } C \text{ on } S_\delta\}.$$

The *ample cone* $\mathrm{Amp}(S_\delta)$ of S_δ is the convex cone of $\mathrm{Cl}(S_\delta) \otimes \mathbb{Q}$ whose elements are the ample classes. These classes are described in the following proposition (whose proof can be deduced from [25, Chapter V, Corollary 2.18]).

Proposition 2.1. *A class $d_1 F + d_2 M$ of $\mathrm{Cl}(S_\delta)$ is ample if and only if $d_1, d_2 > 0$.*

Since $\mathrm{Amp}(S_\delta)$ is the topological interior of $P(S_\delta)$ (see [27]), it follows from Proposition 2.1 that $P(S_\delta)$ is the convex cone spanned by F and M. Moreover, the topological closure of $NE(S_\delta)$ is equal to $P(S_\delta)^\vee$ and, therefore, it is the convex cone spanned by F and M_0; since both generators are effective, one has that $NE(S_\delta)$ is closed and it is spanned by the classes F and M_0. From these facts, the following result is clear.

Proposition 2.2. *A class $d_1 F + d_2 M$ in $\mathrm{Cl}(S_\delta)$ is effective if and only if $d_1 + \delta d_2 \geq 0$ and $d_2 \geq 0$.*

The Hirzebruch surface S_δ also has the structure of a toric variety, that is, it can be regarded as the quotient of $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$ by an action of the algebraic torus $(\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$. Indeed, by considering coordinates (X_0, X_1, Y_0, Y_1) in $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$, the action is given by

$$(\lambda, \mu) \cdot (X_0, X_1, Y_0, Y_1) := (\lambda X_0, \lambda X_1, \mu Y_0, \lambda^{-\delta} \mu Y_1)$$

for all $(\lambda, \mu) \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$ (see [12], where S_δ appears as $\mathbb{F}(0, \delta)$). Thus, we have a natural quotient map

$$\pi : (\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\}) \to S_\delta. \tag{2.2}$$

For integers d_1 and d_2, a polynomial $H(X_0, X_1, Y_0, Y_1) \in \mathbb{C}[X_0, X_1, Y_0, Y_1]$ is said to be *bi-homogeneous of bi-degree* (d_1, d_2) if every monomial $X_0^\alpha X_1^\beta Y_0^\gamma Y_1^\mu$ appearing in H with non-zero coefficient satisfies that $\alpha + \beta - \delta\mu = d_1$ and $\gamma + \mu = d_2$. For any effective divisor $d_1 F + d_2 M$ in S_δ, the non-zero global sections of $\mathcal{O}(d_1, d_2)$ correspond to bi-homogeneous polynomials of bi-degree (d_1, d_2).

Given a line bundle L on S_δ, the Chern class $c(L) = af + bh \in H^2(S_\delta, \mathbb{Z})$ (considered below [23, Definition 1.1]) is expressed in bi-degree form by setting $e = \delta, f = F$ and $f' = M$ (so that $[B_0]^* = M_0$). Then it holds that

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\delta}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.3}$$

For instance, the Chern class

$$c(K_{S_\delta}) = (2g - 2)f - 2h = -2f - 2h$$

of the canonical bundle K_{S_δ} (see [23, Lemma 1.3]) corresponds to the *canonical sheaf* $\mathcal{K}_{S_\delta} = \mathcal{O}(\delta - 2, -2)$.

The surface S_δ is covered by the four affine open sets $U_{ij}, i, j \in \{0, 1\}$, given by

$$U_{ij} := \{\pi(X_0, X_1, Y_0, Y_1) \in S_\delta \mid X_i \neq 0 \text{ and } Y_j \neq 0\}, \tag{2.4}$$

where π is the quotient map (2.2). Since $\pi(X_0, X_1, Y_0, Y_1) = \pi(1, X_1/X_0, 1, X_0^\delta Y_1/Y_0)$ in U_{00}, the open set U_{00} is identified with $\mathbb{C}^2$ by means of the isomorphism

$$\pi(X_0, X_1, Y_0, Y_1) \mapsto (x_{00}, y_{00}),$$

where $x_{00} := X_1/X_0$ and $y_{00} := X_0^\delta Y_1/Y_0$. Similarly, U_{10} is identified with $\mathbb{C}^2$ by means of the isomorphism $\pi(X_0, X_1, Y_0, Y_1) \mapsto (x_{10}, y_{10})$, where $x_{10} := X_0/X_1$ and $y_{10} := X_1^\delta Y_1/Y_0$. The change of coordinates map in the overlap of U_{00} and U_{10} is given by

$$\varphi_{00}^{10} : U_{00} \cap U_{10} \subseteq U_{00} \to U_{00} \cap U_{10} \subseteq U_{10}, \quad (x_{00}, y_{00}) \mapsto (1/x_{00}, x_{00}^\delta y_{00}) = (x_{10}, y_{10}).$$

If C is the curve on S_δ defined by the zero locus of a bi-homogeneous polynomial $H(X_0, X_1, Y_0, Y_1)$, then the intersection $C \cap U_{00}$ is the zero locus of the polynomial in the affine coordinates x_{00} and y_{00} given by

$$\tilde{H}^{00}(x_{00}, y_{00}) := H(1, x_{00}, 1, y_{00}). \tag{2.5}$$

Analogously, for each $i, j \in \{0, 1\}$, we can obtain affine coordinates (x_{ij}, y_{ij}) for every affine open set U_{ij}, change of coordinates maps

$$\varphi_{ij}^{i'j'} : U_{ij} \cap U_{i'j'} \subseteq U_{ij} \to U_{ij} \cap U_{i'j'} \subseteq U_{i'j'}, \tag{2.6}$$

and an equation $\tilde{H}^{ij} = 0$ for the intersection of C with U_{ij}.

3 Representation of foliations on S_δ by affine vector fields and 1-forms

Recall from Section 1 that a foliation $\mathcal{F}$ on the Hirzebruch surface S_δ is given by the class $[s]$ of a global section $s \in H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^*)$. Let $\mathcal{L} = \mathcal{O}(-d_1, -d_2)$ be the tangent sheaf of $\mathcal{F}$. We follow [12, Section 3.1] to obtain the vector field representation of s. To that end, define

$$\mathcal{H} := \mathcal{O}(1, 0)^{\oplus 2} \oplus \mathcal{O}(0, 1) \oplus \mathcal{O}(-\delta, 1),$$

and consider the *Euler* exact sequence given by

$$0 \to \mathcal{O}^{\oplus 2} \xrightarrow{j} \mathcal{H} \xrightarrow{d\pi} \Theta_{S_\delta} \to 0. \tag{3.1}$$

Taking the tensor product with $\mathcal{L}^* = \mathcal{O}(d_1, d_2)$ in the sequence (3.1), we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}(d_1, d_2)^{\oplus 2} \xrightarrow{j \otimes 1} \mathcal{H}(d_1, d_2) \xrightarrow{d\pi \otimes 1} \Theta_{S_\delta}(d_1, d_2) \longrightarrow 0. \tag{3.2}$$

The long exact sequence associated to (3.2) reads

$$0 \longrightarrow H^0(S_\delta, \mathcal{O}(d_1, d_2))^{\oplus 2} \xrightarrow{j \otimes 1^0} H^0(S_\delta, \mathcal{H}(d_1, d_2)) \xrightarrow{d\pi \otimes 1^0} H^0(S_\delta, \Theta_{S_\delta}(d_1, d_1)) \xrightarrow{\delta^0}$$

$$\xrightarrow{\delta^0} H^1(S_\delta, \mathcal{O}(d_1, d_2))^{\oplus 2} \xrightarrow{j \otimes 1^1} H^1(S_\delta, \mathcal{H}(d_1, d_2)) \xrightarrow{d\pi \otimes 1^1} H^1(S_\delta, \Theta_{S_\delta}(d_1, d_2)) \xrightarrow{\delta^1} \cdots, \tag{3.3}$$

where

$$H^q(S_\delta, \mathcal{H}(d_1, d_2)) = H^q(S_\delta, \mathcal{O}(d_1 + 1, d_2))^{\oplus 2} \oplus H^q(S_\delta, \mathcal{O}(d_1, d_2 + 1)) \oplus H^q(S_\delta, \mathcal{O}(d_1 - \delta, d_2 + 1)) \quad (3.4)$$

for

$$q = 0, 1, 2 \quad \text{and} \quad (j \otimes 1^0)(H_1, H_2) = (X_0 H_1, X_1 H_1, Y_0 H_2, -\delta Y_1 H_1 + Y_1 H_2).$$

The sequence (3.3) has the following interpretation: Any section s *in the image of* $d\pi \otimes 1^0$ is uniquely determined by a vector field

$$X = V_0 \frac{\partial}{\partial X_0} + V_1 \frac{\partial}{\partial X_1} + W_0 \frac{\partial}{\partial Y_0} + W_1 \frac{\partial}{\partial Y_1}, \quad (3.5)$$

where

$$V_0, V_1 \in H^0(S_\delta, \mathcal{O}(d_1 + 1, d_2)),$$
$$W_0 \in H^0(S_\delta, \mathcal{O}(d_1, d_2 + 1)),$$
$$W_1 \in H^0(S_\delta, \mathcal{O}(d_1 - \delta, d_2 + 1)),$$

up to the addition of multiples of the *radial* vector fields

$$R_1 := X_0 \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1} - \delta Y_1 \frac{\partial}{\partial Y_1} \quad \text{and} \quad R_2 := Y_0 \frac{\partial}{\partial Y_0} + Y_1 \frac{\partial}{\partial Y_1}.$$

Remark 3.1. We say for brevity that a section $s \in H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^*)$ is *representable by an affine vector field X* if s lies in the image of the map $d\pi \otimes 1^0$ from (3.3). If this is the case, vector fields X_{ij} that define s in the covering (2.4) may be computed by writing the product $d\pi \cdot X$ in the coordinates (x_{ij}, y_{ij}) described above.

Of course, *every* section s is representable by an affine vector field if and only if the map $d\pi \otimes 1^0$ is surjective, and this is the case *if* (but not *only if*) $h^1(S_\delta, \mathcal{L}^*) = 0$ (see Remark 3.3 below).

Foliations on S_δ may be also defined in terms of 1-forms. Indeed, considering the covering $\{V_i\}$ of S_δ and vector fields X_i associated to a global section $s \in H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^*)$ as in (1.1), the 1-form associated to s is given by a collection of 1-forms Ω_i on V_i such that $\Omega_i(X_i) = 0$. These 1-forms glue together into a global section (*the annihilator of s*) in $H^0(S_\delta, \Omega^1_{S_\delta}(d_1 + 2 - \delta, d_2 + 2))$.

Now we proceed with this construction. Let $\mathcal{L} = \mathcal{O}(-d_1, -d_2)$, with $d_1, d_2 \in \mathbb{Z}$.

First, we see from [25, Section II, Exercise 5.16 (b)] applied to $\mathcal{F} = \Theta_{S_\delta}$ that the evaluation map

$$b : \Theta_{S_\delta} \times \wedge^2 \Omega^1_{S_\delta} \to \Omega^1_{S_\delta}, \quad b(X, \omega) = \omega(X),$$

induces an isomorphism

$$\Theta_{S_\delta} \otimes \mathcal{K}_{S_\delta} = \Theta_{S_\delta} \otimes \wedge^2 \Omega^1_{S_\delta} \xrightarrow{\tilde{b}} \Omega^1_{S_\delta},$$

which gives in turn an isomorphism $\Theta_{S_\delta} \cong \Omega^1_{S_\delta} \otimes \mathcal{K}^*_{S_\delta} = \Omega^1_{S_\delta}(2 - \delta, 2)$. Hence,

$$\Theta_{S_\delta} \otimes \mathcal{L}^* = \Theta_{S_\delta}(d_1, d_2) \cong \Omega^1_{S_\delta}(d_1 + 2 - \delta, d_2 + 2),$$

and we obtain that

$$H^q(S_\delta, \Theta_{S_\delta}(d_1, d_2)) \cong H^q(S_\delta, \Omega^1_{S_\delta}(d_1 + 2 - \delta, d_2 + 2)) \quad \text{for } q = 0, 1, 2. \quad (3.6)$$

Now we seek for bi-homogeneous polynomial affine 1-forms that represent the sections of

$$H^0(S_\delta, \Omega^1_{S_\delta}(d_1 + 2 - \delta, d_2 + 2)).$$

To that end, first we dualize (3.1):

$$0 \to \Omega^1_{S_\delta} \xrightarrow{d\pi^*} \mathcal{H}^* \xrightarrow{j^*} \mathcal{O}^{\oplus 2} \to 0. \quad (3.7)$$

Then we twist (3.7) by $\mathcal{O}(d_1 - \delta + 2, d_2 + 2)$:

$$0 \longrightarrow \Omega^1_{S_\delta}(d_1 - \delta + 2, d_2 + 2) \xrightarrow{d\pi^* \otimes 1} \mathcal{H}^*(d_1 - \delta + 2, d_2 + 2) \xrightarrow{j^* \otimes 1} \mathcal{O}(d_1 - \delta + 2, d_2 + 2)^{\oplus 2} \longrightarrow 0.$$

And then we consider the long exact sequence associated to the exact sequence above, part of which reads as follows:

$$0 \longrightarrow H^0(S_\delta, \Omega^1_{S_\delta}(d_1 - \delta + 2, d_2 + 2)) \xrightarrow{d\pi^* \otimes 1^0} H^0(S_\delta, \mathcal{H}^*(d_1 - \delta + 2, d_2 + 2)) \xrightarrow{j^* \otimes 1^0}$$

$$\xrightarrow{j^* \otimes 1^0} H^0(S_\delta, \mathcal{O}(d_1 - \delta + 2, d_2 + 2))^{\oplus 2} \xrightarrow{\delta^0} H^1(S_\delta, \Omega^1_{S_\delta}(d_1 - \delta + 2, d_2 + 2)) \xrightarrow{d\pi^* \otimes 1^1} \cdots,$$

where

$$H^q(S_\delta, \mathcal{H}^*(d_1 - \delta + 2, d_2 + 2))$$
$$= H^q(S_\delta, \mathcal{O}(d_1 - \delta + 1, d_2 + 2))^{\oplus 2} \oplus H^q(S_\delta, \mathcal{O}(d_1 - \delta + 2, d_2 + 1)) \oplus H^q(S_\delta, \mathcal{O}(d_1 + 2, d_2 + 1))$$

for $q = 0, 1, 2$, and

$$(j^* \otimes 1^0)(A_0, A_1, B_0, B_1) = (X_0 A_0 + X_1 A_1 - \delta Y_1 B_1, Y_0 B_0 + Y_1 B_1).$$

As a consequence, we deduce that, given an invertible sheaf $\mathcal{L} = \mathcal{O}(-d_1, -d_2)$ on S_δ, a foliation

$$\mathcal{F} = [s] \in \mathbb{P} H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^*)$$

may not be representable by a polynomial affine vector field, but it is always representable by some bi-homogeneous differential 1-form.

Proposition 3.2. *Let $\mathcal{L} = \mathcal{O}(-d_1, -d_2)$, with $d_1, d_2 \in \mathbb{Z}$. Then any foliation $\mathcal{F}$ in $\mathrm{Fol}(\mathcal{L}, S_\delta)$ is uniquely determined (up to multiplication by a non-zero scalar) by a differential 1-form*

$$\Omega = A_0 \, dX_0 + A_1 \, dX_1 + B_0 \, dY_0 + B_1 \, dY_1, \tag{3.8}$$

where

$$A_0, A_1 \in H^0(S_\delta, \mathcal{O}(d_1 - \delta + 1, d_2 + 2)),$$
$$B_0 \in H^0(S_\delta, \mathcal{O}(d_1 - \delta + 2, d_2 + 1)),$$
$$B_1 \in H^0(S_\delta, \mathcal{O}(d_1 + 2, d_2 + 1))$$

are bi-homogeneous polynomials (not all of them equal to 0) that satisfy the following two conditions:

$$\begin{cases} \Omega(R_1) = X_0 A_0 + X_1 A_1 - \delta Y_1 B_1 = 0, \\ \Omega(R_2) = Y_0 B_0 + Y_1 B_1 = 0. \end{cases} \tag{3.9}$$

Moreover, if $i, j \in \{0, 1\}$, a differential 1-form defining $\mathcal{F}$ in the affine open set U_{ij} is given by

$$\Omega_{ij} = \tilde{A}^{ij}_{i'} dx_{ij} + \tilde{B}^{ij}_{j'} dy_{ij},$$

where $\{i'\} := \{0, 1\} \setminus \{i\}$, $\{j'\} := \{0, 1\} \setminus \{j\}$ and the correspondence $H \mapsto \tilde{H}$ is given by equation (2.5) and its ilk just below it.

Let τ be the kernel of the Jacobian of ψ in (2.1). It is a sub-line bundle of TS_δ and induces an exact sequence

$$0 \to \tau \to TS_\delta \xrightarrow{d\psi} N \to 0,$$

where N is the normal bundle to the ruling (see [23, (1.2)]). We see from [23, Lemma 1.4] and (2.3) that $\mathcal{O}(\tau) = \mathcal{O}(-\delta, 2)$ and $\mathcal{O}(N) = \mathcal{O}(2, 0)$, so that the sequence above corresponds to

$$0 \to \mathcal{O}(-\delta, 2) \to \Theta_{S_\delta} \xrightarrow{d\psi} \mathcal{O}(2, 0) \to 0. \tag{3.10}$$

The spaces of foliations with tangent bundles τ and N will play a role in the results that follow (especially in Proposition 3.6 below). For this reason, our next three remarks gather information about them.

Remark 3.3. We study the representation by affine vector fields of sections in $H^0(S_\delta, \Theta_{S_\delta}(\delta, -2))$ and in $H^0(S_\delta, \Theta_{S_\delta}(-2, 0))$, in the context of (3.3).

Claim: $h^1(S_\delta, \mathcal{O}(\tau^*)) = h^1(S_\delta, \mathcal{O}(\delta, -2))$ and $h^1(S_\delta, \mathcal{O}(N^*)) = h^1(S_\delta, \mathcal{O}(-2, 0))$ are equal to $1 \neq 0$. Indeed, let $D = \delta F - 2M$ and recall that $K_{S_\delta} - D = -2F$, so that $h^q(S_\delta, \mathcal{O}(\delta, -2)) = h^{2-q}(S_\delta, \mathcal{O}(-2, 0))$ for $q = 0, 1, 2$, by Serre duality. Moreover,

$$h^0(S_\delta, \mathcal{O}(\delta, -2)) = 0 \quad \text{because } -2 < 0,$$

and $h^0(S_\delta, \mathcal{O}(-2, 0)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$. Hence, the Euler characteristic $\chi(\mathcal{O}(\delta, -2))$ is equal, on the one hand, to $-h^1(S_\delta, \mathcal{O}(\delta, -2))$ and on the other hand, to $\frac{1}{2}D \cdot (D - K_{S_\delta}) + \chi(\mathcal{O}) = \frac{1}{2}(-4) + 1 = -1$, by the Riemann–Roch theorem.

Having proved the claim, it follows from (3.3) that *no section $s \in H^0(S_\delta, \Theta_{S_\delta}(\delta, -2))$ is representable by an affine vector field*. Indeed, we see from (3.4) that

$$h^0(S_\delta, \mathcal{H}(\delta, -2)) = h^0(S_\delta, \mathcal{O}(1 + \delta, -2)) + h^0(S_\delta, \mathcal{O}(\delta, -1)) + h^0(S_\delta, \mathcal{O}(0, -1)) = 0,$$

because each summand is equal to 0 (by the argument in the displayed equation above). The best we can say from this computations on the value of $h^0(S_\delta, \Theta_{S_\delta}(\delta, -2))$ is that it is greater than or equal to 1 (from (3.10)) and that it is less than or equal to 2 (from (3.3)); see Remark 3.4 below for the actual value.

For the case of sections $s \in H^0(S_\delta, \Theta_{S_\delta}(-2, 0))$, we have already seen (a couple of lines above) that $h^0(S_\delta, \mathcal{O}(-2, 0)) = 0$. Similar computations to the ones above show that

$$h^0(S_\delta, \mathcal{H}(-2, 0)) = h^0(S_\delta, \mathcal{O}(-1, 0)) + h^0(S_\delta, \mathcal{O}(-2, 1)) + h^0(S_\delta, \mathcal{O}(-(\delta + 2), 1)) = h^0(S_\delta, \mathcal{O}(-2, 1)),$$

which is equal to 0 for $\delta = 0, 1$, and it is equal to $\delta - 1$ for $\delta \geq 2$. *The conclusion is that no such section s is representable by an affine vector field for $\delta = 0, 1$, and that $h^0(S_\delta, \Theta_{S_\delta}(-2, 0)) \geq \delta - 1$, for $\delta \geq 2$.*

We conclude this remark by saying that in Remark 3.5 below we will show however that

$$h^0(S_\delta, \Theta_{S_\delta}(-2, 0)) = \begin{cases} 1 & \text{if } \delta = 0, \\ 0 & \text{if } \delta = 1, \\ \delta - 1 & \text{if } \delta \geq 2. \end{cases}$$

Together with Remark 3.1, this computation shows that *for $\delta \geq 2$, in the corresponding exact sequence (3.3), the map $d\pi \otimes 1^0$ is not only injective but also surjective, and hence *the map δ^0 is the zero map*.

Remark 3.4. The *unique foliation* in $\mathrm{Fol}(\mathcal{O}(-\delta, 2), S_\delta)$ is the one given by the ruling $S_\delta \to \mathbb{P}^1$. Indeed, by Proposition 3.2, any foliation $\mathcal{F}$ in $\mathrm{Fol}(\mathcal{O}(-\delta, 2), S_\delta)$ is representable by an affine differential 1-form Ω as in (3.8), where

$$A_0, A_1 \in H^0(S_\delta, \mathcal{O}(1, 0)),$$
$$B_0 \in H^0(S_\delta, \mathcal{O}(2, -1)),$$
$$B_1 \in H^0(S_\delta, \mathcal{O}(\delta + 2, -1))$$

are bi-homogeneous polynomials that satisfy the conditions in (3.9). Since the last two sheaves have no non-zero global section, it follows that $B_0 = B_1 = 0$. Hence, $\mathcal{F}$ is defined by any non-zero scalar multiple of the differential form $\Omega_\tau = X_1 \, dX_0 - X_0 \, dX_1$, which corresponds to the ruling. Finally, we see from the local expressions at the end of Lemma 3.4 that $\mathcal{F}$ *has no singularities.*

Remark 3.5. Now consider the case of foliations with tangent sheaf $\mathcal{O}(N) = \mathcal{O}(2, 0)$. On the one hand, recall from [23, Proposition 2.4] that any $\mathcal{F}$ in $\mathrm{Fol}(\mathcal{O}(2, 0), S_\delta)$ *with isolated singularities* is actually smooth (that is, it has no singularities at all). On the other hand, Brunella in [4] (as quoted in [30]) states that a rational surface Z carries a smooth holomorphic foliation $\mathcal{G}$ if and only if Z is a Hirzebruch surface and $\mathcal{G}$ a rational fibration. With these facts in mind, it should be clear that $\mathcal{F}$ is a rational fibration only if $\delta = 0$. Now we prove it: It follows from Proposition 3.2 that $\mathcal{F}$ in $\mathrm{Fol}(\mathcal{O}(2, 0), S_\delta)$ is representable by an affine differential 1-form Ω

as in (3.8), where

$$A_0, A_1 \in H^0(S_\delta, \mathcal{O}(-(\delta + 1), 2)),$$
$$B_0 \in H^0(S_\delta, \mathcal{O}(-\delta, 1)),$$
$$B_1 \in H^0(S_\delta, \mathcal{O}(0, 1))$$

are bi-homogeneous polynomials that satisfy (3.9). We distinguish three cases.

Case 1: $\delta = 0$. As in Remark 3.4, one can prove that the *unique foliation* $\mathcal{F}$ in $\mathrm{Fol}(\mathcal{O}(2, 0), S_0)$ is the one defined by $\Omega = Y_1 dY_0 - Y_0 dY_1$: the ruling of $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with respect to the projection onto the second factor. Hence, $\mathcal{F}$ is smooth.

Case 2: $\delta = 1$. The complex vector spaces $H^0(S_1, \mathcal{O}(-2, 2))$ and $H^0(S_1, \mathcal{O}(-1, 1))$ consist of the scalar multiples of Y_1^2 and Y_1, respectively, and $H^0(S_1, \mathcal{O}(0, 1))$ contains only linear forms in Y_0 and Y_1. Hence, the affine 1-forms (3.8) have the following shape:

$$\Omega = a Y_1^2 dX_0 + b Y_1^2 dX_1 + a_1 Y_1 dY_0 + Y_1(c_0 X_0 + c_1 X_1) dY_1,$$

where $a, b, a_1, c_0, c_1 \in \mathbb{C}$. Thus it follows easily that *the unique Ω that satisfies conditions* (3.9) *is* $\Omega = 0$.

Case 3: $\delta \geq 2$. *We claim that no foliation $\mathcal{F}$ in* $\mathrm{Fol}(\mathcal{O}(2, 0), S_\delta)$ *has isolated singularities.* Indeed, consider a 1-form Ω as in (3.8) that represents such a foliation. Let D denote the divisor $-(\delta + 1)F + 2M$. Then $D \cdot M_0 < 0$ and $(D - M_0) \cdot M_0 < 0$, which means that M_0 is a double fixed component of the complete linear system $|D|$ and, therefore, Y_1^2 divides both A_0 and A_1. Now, from the first equation in (3.9) one gets that Y_1 divides B_1 and, by the second equation therein, that Y_1^2 divides B_0. This is a contradiction unless $B_0 = B_1 = 0$, because $B_0 \in H^0(S_\delta, \mathcal{O}(-\delta, 1))$ and the latter consists of the scalar multiples of Y_1. Thus, we conclude that $\Omega = Y_1^2(A_0' dX_0 + A_1' dX_1)$ for some $A_j' \in H^0(S_\delta, \mathcal{O}(\delta - 1, 0)) = H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(\delta - 1))$, and, from the first equation in (3.9), that

$$\Omega = \Omega_N = Y_1^2 A_{\delta-2}(X_0, X_1)(X_1 dX_0 - X_0 dX_1) = Y_1^2 A_{\delta-2}(X_0, X_1)\Omega_\tau \tag{3.11}$$

for some $A_{\delta-2} \in H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(\delta - 2))$ (see Remark 3.4 above for Ω_τ). We conclude from (3.11) that $\mathcal{F}$ has no isolated singularities and, moreover, from (3.6), that

$$h^0(S_\delta, \Theta_{S_\delta}(-2, 0)) = h^0(S_\delta, \Omega^1_{S_\delta}(-\delta, 2)) = h^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(\delta - 2)) = \delta - 1.$$

Our next result is a refinement of [23, Proposition 2.2]: It computes those tangent sheaves $\mathcal{L}$ for which a foliation $\mathcal{F} \in \mathrm{Fol}(\mathcal{L}, S_\delta)$ may have isolated singularities.

Proposition 3.6. *Let $d_1, d_2 \in \mathbb{Z}$ and let $\mathcal{L} = \mathcal{O}(-d_1, -d_2)$ be an invertible sheaf on S_δ such that there exists a foliation $\mathcal{F} \in \mathrm{Fol}(\mathcal{L}, S_\delta)$ with isolated singularities. If $\delta = 0$ (resp. $\delta \geq 1$), then either $\mathcal{L} \cong \mathcal{O}(\tau)$ or $\mathcal{L} \cong \mathcal{O}(N)$, or $d_1 \geq 0$ and $d_2 \geq 0$ (resp. either $\mathcal{L} \cong \mathcal{O}(\tau)$, or $d_1 \geq -1$ and $d_2 \geq 0$).*

Proof. Under the correspondence (2.3) between the expressions $d_1 F + d_2 M$ and $af + bh$ of a class in $\mathrm{Cl}(S_\delta)$, [23, Proposition 2.2] states that if there exists a foliation

$$\mathcal{F} \in \mathrm{Fol}(\mathcal{L}, S_\delta)$$

with isolated singularities, then either $\mathcal{L} \cong \mathcal{O}(\tau)$, or $\mathcal{L} \cong \mathcal{O}(N)$, or $(2 + d_1)F + d_2 M$ belongs to the closure of $\mathrm{Amp}(S_\delta)$ (which coincides with the nef cone $P(S_\delta)$). By Proposition 2.1, the latter condition is equivalent to the system of inequalities $d_1 \geq -2$, $d_2 \geq 0$. In the case $\delta = 0$, the double ruling of S_0 shows that the mentioned system is equivalent to $d_1 \geq 0$ and $d_2 \geq 0$, and the statement for $\delta = 0$ has been proved.

Now assume that $\delta \geq 1$. First, $\mathcal{L}$ cannot be isomorphic to $\mathcal{O}(N)$ by Remark 3.5. Finally, assume that $d_1 = -2$ and $d_2 \geq 0$. We will show that every $\mathcal{F} \in \mathrm{Fol}(\mathcal{O}(-2, -d_2), S_\delta)$ has no isolated singularities. Indeed, by Proposition 3.2, $\mathcal{F}$ is representable by an affine differential 1-form

$$\Omega = A_0 \, dX_0 + A_1 \, dX_1 + B_0 \, dY_0 + B_1 \, dY_1,$$

where

$$A_0, A_1 \in H^0(S_\delta, \mathcal{O}(-(1+\delta), d_2+2)),$$
$$B_0 \in H^0(S_\delta, \mathcal{O}(-\delta, d_2+1)),$$
$$B_1 \in H^0(S_\delta, \mathcal{O}(0, d_2+1))$$

satisfy the conditions in (3.9). If D denotes the divisor $-(1+\delta)F + (d_2+2)M$, it holds that

$$D \cdot M_0 = -(1+\delta) < 0 \quad \text{and} \quad (D - M_0) \cdot M_0 = -1 < 0.$$

Therefore, the complete linear system $|D|$ has M_0 as double fixed component. This shows that Y_1^2 divides A_0 and A_1. It follows from (3.9) that Y_1 divides B_1 and B_0 as well, and hence that Y_1 is a factor of all the coefficients of Ω, which shows that $\mathcal{F}$ has no isolated singularities. This finishes the proof.   □

Proposition 3.7. *Let* $\mathcal{L} = \mathcal{O}(-d_1, -d_2)$ *be an invertible sheaf on* S_δ *such that* $d_2 \geq 0$ *and, either* $\delta = 0$ *and* $d_1 \geq 0$, *or* $\delta \geq 1$ *and* $d_1 \geq -1$. *Then* $h^1(S_\delta, \mathcal{L}^*) = 0$.

Proof. If $d_1, d_2 \geq 0$, the result follows from [28, Proposition 2.3]. So let us assume $\delta \geq 1$, $d_1 = -1$ and $d_2 \geq 0$. Since $(-F + d_2M) \cdot M_0 = -1$, it holds that M_0 is a fixed component of the complete linear system $|-F + d_2M|$, and therefore

$$h^0(S_\delta, \mathcal{O}(-1, d_2)) = h^0(S_\delta, \mathcal{O}(\delta-1, d_2-1)) = \frac{\delta}{2}d_2(d_2+1) = \chi(\mathcal{O}(-1, d_2)),$$

where the second equality comes again from [28, Proposition 2.3] and the third from the Riemann–Roch theorem. The result follows (by Serre duality) from

$$h^2(S_\delta, \mathcal{O}(-1, d_2)) = h^0(S_\delta, \mathcal{O}(\delta-1, -(d_2+2))) = 0. \qquad □$$

Remark 3.8. Under the conditions of Proposition 3.7, every foliation in $\mathrm{Fol}(\mathcal{L}, S_\delta)$ has two equivalent descriptions: through affine vector fields (3.5) – in view of Remark 3.1 – and through some affine differential 1-form (3.8). Moreover, Proposition 3.6 shows that this double description includes all foliations on S_δ with isolated singularities, except the one with tangent bundle τ and, in the case $\delta = 0$, also the one with tangent bundle N (see Remark 3.3 above). However, the foliations associated to these exceptional cases are actually smooth (by Remarks 3.4 and 3.3, respectively).

Assume now that a section $s \in H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^*)$ is representable by an affine vector field X as in (3.5). Then the affine 1-form Ω in (3.8) that corresponds to s is given by

$$\Omega = \begin{vmatrix} dX_0 & dX_1 & dY_0 & dY_1 \\ X_0 & X_1 & 0 & -\delta Y_1 \\ 0 & 0 & Y_0 & Y_1 \\ V_0 & V_1 & W_0 & W_1 \end{vmatrix}$$

$$= \begin{vmatrix} X_1 & 0 & -\delta Y_1 \\ 0 & Y_0 & Y_1 \\ V_1 & W_0 & W_1 \end{vmatrix} dX_0 - \begin{vmatrix} X_0 & 0 & -\delta Y_1 \\ 0 & Y_0 & Y_1 \\ V_0 & W_0 & W_1 \end{vmatrix} dX_1 + \begin{vmatrix} X_0 & X_1 \\ V_0 & V_1 \end{vmatrix} (-Y_1 dY_0 + Y_0 dY_1)$$

$$= A_0\, dX_0 + A_1\, dX_1 + B_0\, dY_0 + B_1\, dY_1. \qquad (3.12)$$

This follows because $\Omega(R_1) = \Omega(R_2) = \Omega(X) = 0$. A further conclusion is that the sheaf of ideals I_Z of the singular scheme $Z = Z_s$ of the section s is the ideal $I_Z = (A_0, A_1, B_0, B_1) \subset \mathcal{O}$ generated by the coefficients of Ω: this can be deduced from the local expressions for Ω at the end of Proposition 3.2 together with the just proven fact that $\Omega(X) = 0$.

4 Global endomorphisms of TS_δ

In this section, we compute the space of global endomorphisms of the tangent bundle of a Hirzebruch surface. This computation will be essential to establish the main result of this paper.

Theorem 4.1. *Consider the Hirzebruch Surface S_δ, with $\delta \geq 0$. The space of global endomorphisms of its tangent bundle TS_δ has dimension*

$$h^0(S_\delta, \mathrm{Hom}_{\mathcal{O}}(\Theta_{S_\delta}, \Theta_{S_\delta})) = \begin{cases} 2 & \text{if } \delta = 0, \\ \delta & \text{if } \delta \geq 1. \end{cases}$$

Moreover, every such global endomorphism Φ is uniquely determined by a matrix A which is equal to

$$A(a, d) = a \cdot \mathbf{1}_{2\times 2} \oplus d \cdot \mathbf{1}_{2\times 2} \qquad\qquad \text{if } \delta = 0,$$
$$A(a) = a \cdot \mathbf{1}_{4\times 4} \qquad\qquad \text{if } \delta = 1,$$

$$A(a, C) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ X_1 Y_1 C(X) & -X_0 Y_1 C(X) & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad \text{if } \delta \geq 2,$$

where $a, d \in \mathbb{C}$, $\mathbf{1}_{n\times n}$ denotes the $n \times n$ identity matrix ($n \in \mathbb{N}$) and $C(X) = C(X_0, X_1)$ is a homogeneous polynomial of degree $\delta - 2$.

Moreover, the following properties are satisfied:

(i) *Φ is invertible if and only if $ad \neq 0$ when $\delta = 0$, and $a \neq 0$ when $\delta \geq 1$.*

(ii) *For any invertible sheaf $\mathcal{L}$ on S_δ, if $\Phi \in \mathrm{H}^0(S_\delta, \mathrm{Hom}_{\mathcal{O}}(\Theta_{S_\delta}, \Theta_{S_\delta}))$ and $s \in \mathrm{H}^0(S_\delta, \mathrm{Hom}_{\mathcal{O}}(\mathcal{L}, \Theta_{S_\delta}))$, then*

$$\Phi \circ s = \Phi(s) \in \mathrm{H}^0(S_\delta, \mathrm{Hom}_{\mathcal{O}}(\mathcal{L}, \Theta_{S_\delta})).$$

In particular, if s is representable by the affine vector field

$$X = V_0 \frac{\partial}{\partial X_0} + V_1 \frac{\partial}{\partial X_1} + W_0 \frac{\partial}{\partial Y_0} + W_1 \frac{\partial}{\partial Y_1},$$

then $\Phi(s)$ is representable by the affine vector field

$$X' := V_0' \frac{\partial}{\partial X_0} + V_1' \frac{\partial}{\partial X_1} + W_0' \frac{\partial}{\partial Y_0} + W_1' \frac{\partial}{\partial Y_1},$$

where[1] $(V_0', V_1', W_0', W_1')^t = A \cdot (V_0, V_1, W_0, W_1)^t$ and the matrix A represents Φ.

(iii) *Under the hypothesis of representability in (ii), let*

$$\Omega = A_0\, dX_0 + A_1\, dX_1 + B_0\, dY_0 + B_1\, dY_1 \quad \text{and} \quad \Omega' = A_0'\, dX_0 + A_1'\, dX_1 + B_0'\, dY_0 + B_1'\, dY_1$$

be the affine 1-forms that represent the sections s and $\Phi(s)$, respectively, through (3.12) and let A represent Φ. Then

$$(A_0', A_1', B_0', B_1') = (A_0, A_1, B_0, B_1) \cdot A.$$

The hypothesis of representability in (ii) and (iii) holds, in particular, for any section s where the corresponding invertible sheaf $\mathcal{L} = \mathcal{O}(-d_1, -d_2)$ satisfies the conditions of Proposition 3.7.

Proof. We start with the computation of the matrix A associated to a global endomorphism Φ. Consider the open covering $\{U_{ij}\}_{0\leq i,j\leq 1}$ of S_δ from (2.4), where $TS_\delta|_{U_{ij}} \simeq U_{ij} \times \mathbb{C}^2$ for all $i, j \in \{0, 1\}$. A global endomorphism Φ of TS_δ is given by a collection of 2×2 matrices $\{M_{ij} \simeq \Phi|_{U_{ij}}\}_{0\leq i,j\leq 1}$ such that the entries of each matrix M_{ij} are regular functions on U_{ij} and, for every point p belonging to an overlap $U_{ij} \cap U_{i'j'}$, we have

$$M_{i'j'}|_p = J_{ij}^{i'j'}|_p \cdot M_{ij}|_p \cdot (J_{ij}^{i'j'}|_p)^{-1}, \tag{4.1}$$

where $J_{ij}^{i'j'}$ denotes the Jacobian matrix of the change of coordinates map $\varphi_{ij}^{i'j'}$ from (2.6) and $B|_p$ denotes the matrix obtained by evaluating the entries of B at p. Write

$$M_{00} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.2}$$

---

1 The superscript t denotes the transpose of the vector and the dot $\cdot$ denotes matrix multiplication.

where a, b, c and d are regular functions on U_{00} (that is, they are given by polynomials in $\mathbb{C}[x_{00}, y_{00}]$). By considering an arbitrary point $p = (x_{00}, y_{00}) = (x_{01}, y_{01}) \in U_{00} \cap U_{01}$ (given in coordinates in both open subsets), equation (4.1) becomes

$$M_{01}|_p = \begin{pmatrix} 1 & 0 \\ 0 & -y_{01}^2 \end{pmatrix} \begin{pmatrix} a(x_{01}, y_{01}^{-1}) & b(x_{01}, y_{01}^{-1}) \\ c(x_{01}, y_{01}^{-1}) & d(x_{01}, y_{01}^{-1}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -y_{01}^{-2} \end{pmatrix} \tag{4.3}$$

$$= \begin{pmatrix} a(x_{01}, y_{01}^{-1}) & -b(x_{01}, y_{01}^{-1}) \cdot y_{01}^{-2} \\ -c(x_{01}, y_{01}^{-1}) \cdot y_{01}^2 & d(x_{01}, y_{01}^{-1}) \end{pmatrix}. \tag{4.4}$$

Since the entries of M_{01} must be regular functions on U_{01} (polynomials in $\mathbb{C}[x_{01}, y_{01}]$) and $p \in U_{00} \cap U_{01}$ is arbitrary, we have that, necessarily, $b = 0$, $a(x_{00}, y_{00}) = a(x_{00})$, $d(x_{00}, y_{00}) = d(x_{00})$ (that is, a and d depend only on x_{00}) and the degree of c in y_{00} is less than or equal to 2.

Now we plug these conditions into (4.2) and compute (4.1) with $(i, j) = (0, 0)$, $(i', j') = (1, 1)$ and $p = (x_{00}, y_{00}) = (x_{11}, y_{11})$ being an arbitrary point in $U_{00} \cap U_{11}$. We obtain that

$$M_{11}|_p = \begin{pmatrix} a(x_{11}^{-1}) & 0 \\ \delta y_{11} x_{11}^{-1}[a(x_{11}^{-1}) - d(x_{11}^{-1})] + y_{11}^2 x_{11}^{(\delta-2)} c(x_{11}^{-1}, y_{11}^{-1} x_{11}^{\delta}) & d(x_{11}^{-1}) \end{pmatrix}. \tag{4.5}$$

Hence, by reasoning as above, since M_{11} is defined by regular entries, this is the case only if the functions a and d are constant and these constants must be equal if $\delta \geq 1$ (this follows from the lower-left entry of the matrix above). Moreover, since the polynomial c expressed in coordinates u and v must have the shape

$$c(u, v) = c_0(u) + c_1(u)v + c_2(u)v^2$$

for some univariate polynomials $c_i(u)$, we see that the term $y_{11}^2 x_{11}^{-(\delta+2)} c(x_{11}^{-1}, y_{11}^{-1} x_{11}^{\delta})$ comes from a regular function in U_{11} if and only if $c = 0$ (resp. $c_0 = c_1 = 0$ and $c_2(u)$ has degree less than or equal to $\delta - 2$) if $\delta \in \{0, 1\}$ (resp. if $\delta \geq 2$).

We have shown, so far, that the restriction of a global endomorphism Φ to the affine subset U_{00} must be given by a matrix of the type

$$M_{00} = \begin{cases} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} & \text{if } \delta = 0, \\[2mm] \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} & \text{if } \delta = 1, \\[2mm] \begin{pmatrix} a & 0 \\ c(x_{00})y_{00}^2 & a \end{pmatrix} & \text{if } \delta \geq 2, \end{cases} \tag{4.6}$$

where $a, d \in \mathbb{C}$ and c is a polynomial in one variable of degree less than or equal to $\delta - 2$. Using (4.4) and (4.5), we deduce that, for $\delta \in \{0, 1\}$, the matrices M_{01} and M_{11} coincide with M_{00} and that, for $\delta \geq 2$, we have

$$M_{01} = \begin{pmatrix} a & 0 \\ -c(x_{01})y_{01}^2 & a \end{pmatrix} \quad \text{and} \quad M_{11} = \begin{pmatrix} a & 0 \\ c(x_{11}^{-1})x_{11}^{\delta-2}y_{11}^2 & a \end{pmatrix}.$$

We deduce similarly that M_{10} also coincides with M_{00} for $\delta \in \{0, 1\}$ and that, for $\delta \geq 2$, we have

$$M_{10} = \begin{pmatrix} a & 0 \\ -c(x_{10}^{-1})x_{10}^{\delta-2}y_{10}^2 & a \end{pmatrix}.$$

Noticing that every collection of four matrices as before (that is, with a, d and c satisfying the given conditions) also satisfies the remaining conditions from (4.1), we conclude the part of the statement concerning the dimension of the space of global endomorphisms of T_{S_δ} and (i).

The matrix representation A of Φ also follows from these computations: according to the different values of δ, we obtain the entries of the matrix A (say from equation (4.6)): $C(X)$ is the homogeneous form associated to the polynomial $c(x_{00})$. Then one verifies that, according to the different values of δ, the restrictions of A to the open sets U_{00}, U_{01}, U_{11} and U_{10} coincide, respectively, with the matrices $M_{00}, M_{01}, M_{11}, M_{10}$ described above.

Now we prove (ii). Let Φ be as before and consider a foliation $\mathcal{F} = [s]$ in $\mathrm{Fol}(\mathcal{L}, S_\delta)$ such that s is representable by an affine vector field X (as in the statement). The restriction $s|_{U_{00}}$ of the section s to this open set is the vector field $(d\pi \cdot X)|_{U_{00}}$, where π comes from (2.2), and it is given by

$$s|_{U_{00}} = (-x_{00}\tilde{V}_0^{00} + \tilde{V}_1^{00})\frac{\partial}{\partial x_{00}} + (\delta y_{00}\tilde{V}_0^{00} - y_{00}\tilde{W}_0^{00} + \tilde{W}_1^{00})\frac{\partial}{\partial y_{00}}.$$

Then the restriction of $\Phi(s)$ to U_{00} can be computed by using the matrix M_{00} as follows:

$$\Phi(s)|_{U_{00}} = \begin{cases} a(-x_{00}\tilde{V}_0^{00} + \tilde{V}_1^{00})\dfrac{\partial}{\partial x_{00}} + d(\delta y_{00}\tilde{V}_0^{00} - y_{00}\tilde{W}_0^{00} + \tilde{W}_1^{00})\dfrac{\partial}{\partial y_{00}} & \text{if } \delta = 0, \\[2ex] a(-x_{00}\tilde{V}_0^{00} + \tilde{V}_1^{00})\dfrac{\partial}{\partial x_{00}} + a(\delta y_{00}\tilde{V}_0^{00} - y_{00}\tilde{W}_0^{00} + \tilde{W}_1^{00})\dfrac{\partial}{\partial y_{00}} & \text{if } \delta = 1, \\[2ex] a(-x_{00}\tilde{V}_0^{00} + \tilde{V}_1^{00})\dfrac{\partial}{\partial x_{00}} \\[1ex] \quad + [c(x_{00})y_{00}^2(-x_{00}\tilde{V}_0^{00} + \tilde{V}_1^{00}) + a(\delta y_{00}\tilde{V}_0^{00} - y_{00}\tilde{W}_0^{00} + \tilde{W}_1^{00})]\dfrac{\partial}{\partial y_{00}} & \text{if } \delta \geq 2. \end{cases}$$

This concludes the proof of (ii) because the vector field $(d\pi \cdot X')|_{U_{00}}$ coincides, in each case, with the above ones, and the same happens when considering the remaining open sets U_{10}, U_{01} and U_{11}.

The proof of (iii) is a straightforward computation (whose details we omit). Notice that Ω' is obtained from (3.12) by replacing the last row (V_0, V_1, W_0, W_1) in the determinant therein by the row (V_0', V_1', W_0', W_1') where $(V_0', V_1', W_0', W_1')^t = A \cdot (V_0, V_1, W_0, W_1)^t$. □

Corollary 4.2. *Let $\mathcal{L} = \mathcal{O}(-d_1, -d_2)$ be an invertible sheaf on S_δ such that every section $s \in H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^*)$ is representable by an affine vector field X as in (3.5). Fix a section s and let Z be its singular scheme with sheaf of ideals I_Z. Let $0 \neq \Phi \in H^0(S_\delta, \mathrm{Hom}_{\mathcal{O}}(\Theta_{S_\delta}, \Theta_{S_\delta}))$ be a global endomorphism and let $s' = \Phi(s)$ have singular scheme Z'. Then the following assertions hold:*

(i) *$Z \subseteq Z'$ and $Z = Z'$ if Φ is invertible.*

(ii) *Let A be the matrix associated to Φ by Theorem 4.1. If $\delta = 1$, then Φ is invertible and $\mathcal{F} = [s] = [s']$. If $\delta \neq 1$ and Φ is invertible, then the condition $\mathcal{F} = [s] = [s']$ is equivalent to the condition $a = d$, for $\delta = 0$, and it is equivalent to the conditions $C = 0$ or $C \neq 0$ and $X_0 V_1 - X_1 V_0 = 0$, for $\delta \geq 2$.*

(iii) *If Φ is not invertible and $s' \neq 0$, then Z' is one-dimensional.*

Proof. Let

$$\Omega = A_0\, dX_0 + A_1\, dX_1 + B_0\, dY_0 + B_1\, dY_1 \quad \text{and} \quad \Omega' = A_0'\, dX_0 + A_1'\, dX_1 + B_0'\, dY_0 + B_1'\, dY_1$$

be the affine 1-forms that represent the sections s and s', respectively (see (3.12)).

For every $\delta \geq 0$, Theorem 4.1 (iii) states, in matrix notation, that

$$(A_0', A_1', B_0', B_1') = (A_0, A_1, B_0, B_1) \cdot A.$$

By Remark 3.8, this equality implies that $I_{Z'} \subseteq I_Z$, which shows that $Z \subseteq Z'$, and the first part in (i) is proved.

Now we divide the proof of the second part in (i) and of the remaining statements into three cases:

Case 1: $\delta = 0$. By (3.12),

$$\Omega = (Y_0 W_1 - Y_1 W_0)(X_1 dX_0 - X_0 dX_1) + (X_0 V_1 - X_1 V_0)(-Y_1 dY_0 + Y_0 dY_1),$$

and $A = A(a, d)$ by Theorem 4.1. Then it follows from (iii) therein that

$$\Omega' = a \cdot (Y_0 W_1 - Y_1 W_0)(X_1 dX_0 - X_0 dX_1) + d \cdot (X_0 V_1 - X_1 V_0)(-Y_1 dY_0 + Y_0 dY_1).$$

Since $ad \neq 0$, it follows from Remark 3.8 that $I_Z = I_{Z'}$ (and hence that $Z = Z'$). This finishes the proof of (i) in this case 1. With respect to (ii), it is obvious that $[s] = [s']$ if and only if $a = d$. Finally, if (say) $a = 0$, then $\Omega' = d \cdot (X_0 V_1 - X_1 V_0)(-Y_1 dY_0 + Y_0 dY_1) \neq 0$, and we see from the final statement of Proposition 3.2 that the restriction of Ω' to the open set U_{00} is given by

$$\Omega'_{00} = d \cdot \tilde{B}_1^{00}(x_{00}, y_{00})\, dy_{00} = d \cdot y_{00}(\tilde{V}_1^{00} - x_{00}\tilde{V}_0^{00})\, dy_{00},$$

so that $\{(x_{00}, 0)\} \subset Z' \cap U_{00}$ and (iii) follows.

Case 2: $\delta = 1$. This case follows at once from the fact (see Theorem 4.1) that $A = A(a) = a \cdot \mathbf{1}_{4\times4}$.

Case 3: $\delta \geq 2$. By recalling the expression for Ω from (3.12) and the fact that $A = A(a, C)$, it follows from (iii) in Theorem 4.1 that

$$\begin{aligned}\Omega' &= a \cdot \Omega - Y_1 C B_0 \cdot (X_1 dX_0 - X_0 dX_1) \\ &= a \cdot \Omega + (X_0 V_1 - X_1 V_0) Y_1^2 C \cdot (X_1 dX_0 - X_0 dX_1),\end{aligned} \tag{4.7}$$

so that $A_0' = a \cdot A_0 - X_1 Y_1 C B_0$, $A_1' = a \cdot A_1 + X_0 Y_1 C B_0$ and $B_j' = a \cdot B_j$ for $j = 0, 1$.

If $a \neq 0$, then $A_0 - \frac{1}{a} A_0' \in (B_0) = (B_0') \subset I_{Z'}$, so that $A_0 \in I_{Z'}$. Similar arguments show that $A_1 \in I_{Z'}$ and then $I_Z \subset I_{Z'}$, which, together with the above, shows that $Z = Z'$ and the proof of (i) is complete.

The second equality in (4.7) proves (ii).

Finally, (iii) follows from the observation that the 1-form in the second equality in (4.7) coincides with $a \cdot \Omega + (X_0 V_1 - X_1 V_0) \cdot \Omega_N$, where Ω_N comes from (3.11) within Remark 3.5. There it was shown that Ω_N does not have isolated singularities and hence Ω' either does not, whenever $a = 0$. □

5 Foliations with isolated singularities on S_δ that share singular scheme

If $\mathcal{F} = [s] \in \mathbb{P}H^0(S_\delta, \Theta_{S_\delta}(d_1, d_2))$ is a foliation with isolated singularities on S_δ, then for every invertible endomorphism Φ of TS_δ all foliations $[\Phi(s)]$ share their singular scheme with $[s]$, by Corollary 4.2. Our main result, Theorem 5.2, states that these are the only ones whenever $d_2 \geq 1$ and $d_1 \geq 1$ for $\delta = 0$, $d_1 \geq 2$ for $\delta \geq 2$, and $d_1 \geq 0$ for $\delta = 1$. In this latter case, we see that $[s]$ is uniquely determined by its singular scheme. Notice that this result holds for all foliations with ample cotangent bundle, with the exception of the cases $\delta \geq 2$ and $d_1 = 1$. We devote this final section to the proof of Theorem 5.2.

Lemma 5.1. *Consider the families of invertible sheaves on S_δ given by*

$$\mathcal{L} = \mathcal{O}(-d_1, -d_2) \quad and \quad \mathcal{E} = \mathcal{L} \otimes \mathcal{K}_{S_\delta} = \mathcal{O}(\delta - (d_1 + 2), -(d_2 + 2)),$$

where $\mathcal{K}_{S_\delta} = \mathcal{O}(\delta - 2, -2)$ is the canonical sheaf of S_δ, $d_2 \geq 1$ and $d_1 \geq 1$ for $\delta = 0$, $d_1 \geq 0$ for $\delta = 1$, and $d_1 \geq 2$ for $\delta \geq 2$.

Then

$$h^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{E}) = 0 = h^1(S_\delta, \Theta_{S_\delta} \otimes \mathcal{E}).$$

Proof. From the exact sequence (3.10), we obtain the exact sequence

$$0 \to \mathcal{O}(-(d_1 + 2), -d_2) \to \Theta_{S_\delta} \otimes \mathcal{E} \xrightarrow{d\psi \otimes 1} \mathcal{O}(\delta - d_1, -(d_2 + 2)) \to 0.$$

Considering its associated long exact sequence, we deduce that it suffices to prove

$$h^0(S_\delta, \mathcal{O}(-(d_1 + 2), -d_2)) = 0 = h^0(S_\delta, \mathcal{O}(\delta - d_1, -(d_2 + 2))), \tag{5.1}$$

$$h^1(S_\delta, \mathcal{O}(-(d_1 + 2), -d_2)) = 0 = h^1(S_\delta, \mathcal{O}(\delta - d_1, -(d_2 + 2))), \tag{5.2}$$

to get the desired equalities in the statement. For a start, both equations in (5.1) hold for every $\delta \geq 0$ *and for every* $d_1 \in \mathbb{Z}$ because $-(d_2 + 2) < -d_2 < 0$.

In order to prove the equalities in (5.2), we recall from [28, Proposition 2.3] that $h^1(S_\delta, \mathcal{O}(a, b)) = 0$ for integer numbers $a \geq 0$, $b \geq -1$ and $\delta \geq 0$. Therefore, by Serre duality, we have

$$h^1(S_\delta, \mathcal{O}(\delta - (a + 2), -(b + 2))) = 0 \quad \text{for } a \geq 0, b \geq -1 \text{ and } \delta \geq 0. \tag{5.3}$$

Substitute the value $b = d_2 - 2 \geq -1$ in (5.3) to get

$$h^1(S_\delta, \mathcal{O}(\delta - (a + 2), -d_2)) = 0 \quad \text{for } a \geq 0, d_2 \geq 1 \text{ and } \delta \geq 0. \tag{5.4}$$

Then (5.4) proves the first equality in (5.2). In fact, it holds, for $\delta = 0$ for any value of $a = d_1 \geq 0$ for $\delta = 1$ for any value of $a = d_1 + 1 \geq 0$ (that is, for any value of $d_1 \geq -1$), and finally for $\delta \geq 2$ for any value of $a = d_1 + \delta \geq 0$ (that is, for any value of $d_1 \geq -\delta$).

For the proof of the second equality in (5.2), we replace the value $b = d_2 \geq 1$ in (5.3) to get

$$h^1(S_\delta, \mathcal{O}(\delta - (a + 2), -(d_2 + 2))) = 0 \quad \text{for } a \geq 0, d_2 \geq 1 \text{ and } \delta \geq 0. \tag{5.5}$$

According to the restrictions on the values of d_1, we see that the substitution $a = d_1 - 2 \geq 0$ in (5.5) proves the second equality in (5.2), *except* for the following cases:

$$h^1(S_0, \mathcal{O}(-1, -(d_2 + 2))) = 0, \quad d_2 \geq 1, \tag{5.6}$$

$$h^1(S_1, \mathcal{O}(d_1, -(d_2 + 2))) = 0, \quad d_2 \geq 1, d_1 = 0, 1. \tag{5.7}$$

The proof of these equalities follows from [23, Proposition 5.3]. Indeed, the invertible sheaves in (5.6) and (5.7) lie in the regions given by Proposition 3.6 ($d_1 \geq 0$ and $d_2 \geq 0$ for $\delta = 0$, and $d_1 \geq -1$ and $d_2 \geq 0$ for $\delta \geq 1$) where foliations with tangent sheaf $\mathcal{L} = \mathcal{O}(-d_1, -d_2)$ may have isolated singularities. They neither correspond to the ruling $\mathcal{O}(\tau) = \mathcal{O}(-\delta, 2)$ nor to the sheaf $\mathcal{O}(d, 0)$ associated to a Riccati foliation. Moreover, none of the invertible sheaves from (5.6) belong to the exceptional cases described in [23, Proposition 5.3 (1)] which correspond to $\delta = 0$ and $\mathcal{O}(-a, -b) = \mathcal{O}(0, -b)$. This finishes the proof of (5.6).

Finally, using (2.3), we see that the exceptional cases in [23, Proposition 5.3 (2)] correspond to $\delta = 1$ and

$$\mathcal{O}(-d_1, -d_2) = \mathcal{O}\left(n\frac{n-1}{2} + 1, -n\right) \quad \text{with } n \geq 2. \tag{5.8}$$

Then it is clear that none of the invertible sheaves in (5.7) has the form (5.8). This proves (5.7) and, in conclusion, the lemma. □

Let $[s] \in \mathbb{P}H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^*)$ be a foliation with isolated singularities and singular scheme Z with sheaf of ideals I_Z. In view of Corollary 4.2, there exist other foliations $[s'] \in \text{Fol}(\mathcal{L}, S_\delta) \simeq \mathbb{P}H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^*)$ with the same singular scheme. We seek for them through the following construction. Consider the bundle $E = TS_\delta \otimes L^*$. Then for its dual hold

$$E^* = (TS_\delta \otimes L^*)^* \simeq T^*S_\delta \otimes L \quad \text{and} \quad \bigwedge^2 E^* = \bigwedge^2 (T^*S_\delta \otimes L) \simeq \bigwedge^2 (T^*S_\delta) \otimes L^{\otimes 2} \simeq \mathcal{K}_{S_\delta} \otimes L^{\otimes 2}.$$

Hence, the Koszul resolution of Z (see [7]) may be written as

$$0 \to \bigwedge^2 \Omega^1_{S_\delta} \otimes \mathcal{L}^{\otimes 2} \xrightarrow{\iota_s} \Omega^1_{S_\delta} \otimes \mathcal{L} \xrightarrow{\iota_s} I_Z \to 0, \tag{5.9}$$

where the maps ι_s are contraction-by (or evaluation-at) s. The tensor product of (5.9) with $\Theta_{S_\delta} \otimes \mathcal{L}^*$ gives the exact sequence

$$0 \to \left(\bigwedge^2 \Omega^1_{S_\delta}\right) \otimes \Theta_{S_\delta} \otimes \mathcal{L}^* \xrightarrow{\iota_s \otimes 1} \Omega^1_{S_\delta} \otimes \Theta_{S_\delta} \xrightarrow{\iota_s \otimes 1} \Theta_{S_\delta} \otimes \mathcal{L}^* \otimes I_Z \to 0,$$

where $\Omega^1_{S_\delta} \otimes \Theta_{S_\delta} \simeq \text{Hom}_\mathcal{O}(\Theta_{S_\delta}, \Theta_{S_\delta})$. Letting $\mathcal{E} = \mathcal{L} \otimes \mathcal{K}_{S_\delta}$, the sequence above may be rewritten as

$$0 \to \Theta_{S_\delta} \otimes \mathcal{E} \xrightarrow{\iota_s \otimes 1} \text{Hom}_\mathcal{O}(\Theta_{S_\delta}, \Theta_{S_\delta}) \xrightarrow{\iota_s \otimes 1} \Theta_{S_\delta} \otimes \mathcal{L}^* \otimes I_Z \to 0,$$

with associated long exact sequence given by

$$0 \longrightarrow H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{E}) \xrightarrow{\iota_s \otimes 1^0} H^0(S_\delta, \text{Hom}_\mathcal{O}(\Theta_{S_\delta}, \Theta_{S_\delta})) \xrightarrow{\iota_s \otimes 1^0} H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^* \otimes I_Z) \xrightarrow{\delta^0}$$

$$\xrightarrow{\delta^0} H^1(S_\delta, \Theta_{S_\delta} \otimes \mathcal{E}) \xrightarrow{\iota_s \otimes 1^1} H^1(S_\delta, \text{Hom}_\mathcal{O}(\Theta_{S_\delta}, \Theta_{S_\delta})) \xrightarrow{\iota_s \otimes 1^1} H^1(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^* \otimes I_Z) \xrightarrow{\delta^1} \cdots . \tag{5.10}$$

Finally, notice that $H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^* \otimes I_Z)$ consists of those global sections in $H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^*)$ that vanish at Z and notice that the effect of the map $\iota_s \otimes 1^0$ in (5.10) on a global endomorphism Φ is $\iota_s \otimes 1^0(\Phi) = \Phi(s)$. Then (5.10) shows that *every section s' that vanishes on Z is of the form $s' = \Phi(s)$ for some endomorphism Φ if and only if the map $\iota_s \otimes 1^0$ is surjective, and this is the case if $h^1(S_\delta, \Theta_{S_\delta} \otimes \mathcal{E}) = 0$.* This conclusion, together with Lemma 5.1, gives our main result.

Theorem 5.2. *Let $\mathcal{F} = [s] \in \mathrm{Fol}(\mathcal{L}, S_\delta) \simeq \mathbb{P}H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^*)$ be a foliation on S_δ, where $\mathcal{L} = \mathcal{O}(-d_1, -d_2)$ satisfies that $d_2 \geq 1$, and $d_1 \geq 1$ if $\delta = 0$, $d_1 \geq 0$ if $\delta = 1$, and $d_1 \geq 2$ if $\delta \geq 2$.*

Assume that $[s]$ has isolated singularities, let Z be its singular scheme and consider any other section $s' \in H^0(S_\delta, \Theta_{S_\delta} \otimes \mathcal{L}^*)$ with the same singular scheme Z as $[s]$. Then there exists a global invertible endomorphism Φ of TS_δ such that $s' = \Phi(s)$. Moreover, if the affine 1-form

$$\Omega = A_0 \, dX_0 + A_1 \, dX_1 + B_0 \, dY_0 + B_1 \, dY_1$$

represents the section s, then any section $s' = \Phi(s)$ is represented by an affine 1-form Ω' where

$$\Omega' = \begin{cases} a \cdot (A_0 \, dX_0 + A_1 \, dX_1) + d \cdot (B_0 \, dY_0 + B_1 \, dY_1), & a, d \in \mathbb{C}^*, \text{ if } \delta = 0, \\ a \cdot \Omega, & a \in \mathbb{C}^*, \text{ if } \delta = 1, \\ a \cdot \Omega - Y_1 C(X_1, X_2) B_0 \cdot (X_1 dX_0 - X_0 dX_1), & a \in \mathbb{C}^*, \ C \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\delta - 2)), \text{ if } \delta \geq 2. \end{cases} \tag{5.11}$$

It follows in particular that if $\delta = 1$, then $\mathcal{F} = [s]$ is uniquely determined by Z, in the sense that $\mathcal{F}$ is the unique foliation with singular scheme Z.

Proof. The first statement follows from Lemma 5.1 and the construction described after its proof. The second statement (the one containing (5.11)) follows from Corollary 4.2, and the last one from the second line in (5.11). □

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