

Periodically Correlated Space-Time Autoregressive Hilbertian Processes

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ABSTRACT

In this paper, we introduce periodically correlated space-time autoregressive processes with values in Hilbert spaces. The existence conditions and the strong law of large numbers are established. Moreover, we present an estimator for the autocorrelation parameter of such processes.

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1. INTRODUCTION

In time series analysis, periodically correlated (PC) processes, which can be categorized in the class of nonstationary harmonizable processes, have been widely used to characterize various real life phenomena, that exhibit some kind of seasonal behavior. Franses [1], Gardner [2], Gardner *et al.* [3] and Hurd and Miamee [4] and some other researchers remark the importance of PC processes theoretically and in applied fields, such as metrology, communication, economics, etc. Besides, Hilbertian PC processes of weak type were introduced and studied by Soltani and Shishehbor [5,6]. These processes have interesting time domain and spectral structures.

Among various models in the analysis of time series, autoregressive (AR) models are of great importance. Bosq [7] generalized the classical AR models to processes with values in Hilbert spaces by introducing autoregressive Hilbertian (ARH) models. In his fundamental work, Bosq [7] provides basic results on Hilbertian strongly second order AR and moving-average processes. The existence, covariance structure, parameter estimation, strong law of large numbers and central limit theorem are also covered in his book. These models attract the attention of various researchers, such as Mourid [8], Besse and Cardot [9], Pumo [10], Mas [11,12] and Horvath *et al.* [13], and are applied drastically in modeling functional time series.

The PC autoregressive Hilbertian process of order one (PCARH(1)) was introduced by Soltani and Hashemi [14]. They studied the structure and existence of PC ARH processes by embedding them into higher dimensions, and provided necessary and sufficient conditions for the existence of these processes. They considered the law of large numbers, the central limit theorem, and also suggested some methods for parameter estimation.

Space-time processes are of great importance in studying spatial processes. The space-time autoregressive moving average (STARMA) models were developed by Pfeifer and Deutsch [15–19]. Processes that can be modeled by the STARMA models are characterized by a single random variable observed at N fixed sites in space. The dependencies between the N time series are incorporated in the model through hierarchical $N \times N$ weighting matrices, specified prior to analyzing the data. These weighting matrices should incorporate the relevant physical characteristics of the system into the model. Each of the N time series are simultaneously modeled as linear combinations of past observations and disturbances, as well as weighted past observations and disturbances at neighboring sites.

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A STARMA(p, q, m) model is formulated as follows:

$$y_t = \sum_{k=1}^p \sum_{l=0}^{\lambda_k} \phi_{kl} W_l y_{t-k} - \sum_{k=1}^q \sum_{l=0}^{m_k} \theta_{kl} W_l \varepsilon_{t-k} + \varepsilon_t \quad (1)$$

where p, q are the temporal AR and MA lags, λ_k and m_k are the spatial lags, y_t is the $N \times 1$ vector of observations at time t at the N sites, W_l is the $N \times N$ matrix of weights for spatial order l and, finally, ε_t is the random disturbance at time t , which is normally distributed. The weighting matrices, in AR and moving-average parts, are the same as Pfeifer and Deutsch [17]. This model has found numerous applications ranging from environmental (Pfeifer and Deutsch [17]; Stoffer [20]) to epidemiological (Pfeifer and Deutsch [15]) and economical (Pfeifer and Bodily [21]) problems.

In the last decades, the technological advances in various fields, such as chemometrics, engineering, finance and medicine, makes it possible to observe samples as curves. In these cases, it is common to assume that the sample has been generated by a stochastic function. To analyze this type of data, it is convenient to use the tools provided by a recent area of statistics, known as functional data analysis (FDA).

Recently, FDA have been developed in the context of spatial statistics (Bosq [7]; Ruiz-Medina and Salmeron [22]). Ruiz-Medina [23,24] introduced and studied the structural properties of spatial autoregressive and moving-average Hilbertian processes, which are called SARH(1) and SMAH(1), respectively, in abbreviation.

In this article, we introduce PC space-time autoregressive Hilbertian (PCSTARH) processes as an extension of Pfeifer and Deutsch's model to an infinite dimensional Hilbert space, and provide a theorem which demonstrates their existence, the strong law of large numbers and parameter estimation. Our methodology is to embed every PCSTARH(1, 1) with period T into a subclass of PCARH^k(1) processes, and then applied the results of Soltani and Hashemi [14].

This article is organized as follows. In Section 2, we provide the required definitions and present conditions which guarantees the existence of the model. The strong law of large number of this model is presented in Section 3. In Section 4, we find estimators for autocorrelation operators and prove their consistency and the last section is devoted to some conclusions.

2. PC SPACE-TIME PROCESS

Let \mathcal{H} stands for a real separable Hilbert space equipped with the scalar product $\langle \cdot, \cdot \rangle$, the norm $\| \cdot \|$ and the Borel σ -field \mathcal{B} . Besides, let $\mathcal{L}(\mathcal{H})$ denotes the Hilbert space of bounded linear operators on \mathcal{H} . Consider (Ω, \mathcal{F}, P) stands for a probability space. A random variable with values in \mathcal{H} is an \mathcal{F}/\mathcal{B} measurable mapping from Ω into \mathcal{H} . A random variable X is called strongly second order if $E\|X\|^2 < \infty$. For the sake of simplicity, we refer to strongly second order random variables with values in \mathcal{H} as \mathcal{H} -valued random variables throughout the paper.

For \mathcal{H} -valued random variables X and Y , the covariance and the cross-covariance operators are defined in terms of tensorial product, as follows:

$$C_X(x) := \mathbb{E}[(X \otimes X)x] = \mathbb{E}\langle X, x \rangle X, \quad (2)$$

$$C_{X,Y}(x) := \mathbb{E}[(X \otimes Y)x] = \mathbb{E}\langle X, x \rangle Y, \quad (3)$$

respectively. Note that, throughout this work, X_{it} will denote an \mathcal{H} -valued random variable at time t and site i .

In the first step, let us define PC and PC Hilbertian white noise processes, along with a sequence of T -periodic bounded linear operators.

Definition 2.1. An \mathcal{H} -valued stochastic process $\{X_{it}, i = 1, \dots, k, t \in \mathbb{Z}\}$ is said to be PC space-time with period T , PCS in abbreviation, if

$$C_{X_{in}, X_{im}}(x) = C_{X_{i(n+T)}, X_{i(m+T)}}(x), \quad (4)$$

for each $n, m \in \mathbb{Z}, x \in \mathcal{H}$ and some integer $T > 0$.

The smallest such T is called the period of the process. If $T = 1$, the process is called space-time stationary.

Definition 2.2. A PC space-time \mathcal{H} -valued process $\{\varepsilon_{it}, i = 1, \dots, k, t \in \mathbb{Z}\}$ is called white noise (PCHWN) if it satisfies the following properties:

- (i) $\mathbb{E}(\varepsilon_{it}) = 0$, $0 < E\|\varepsilon_{it}\|^2 = \sigma_{it}^2 < \infty$ for every $i = 1, \dots, k, t \in \mathbb{Z}$, where \mathbb{E} stands for the expected value based on the Bochner integral.
- (ii) $C_{\varepsilon_{it}}(x) = C_{\varepsilon_{i(t+T)}}(x)$ for every $i = 1, \dots, k, t \in \mathbb{Z}$, when $x \in \mathcal{H}$.

(iii) $C_{\varepsilon_{it}, \varepsilon_{jl}}(\mathbf{x}) = 0$ for all $t \neq l$ or $i \neq j$ when $\mathbf{x} \in \mathcal{H}$.

Definition 2.3. A sequence $\{\rho_t, t \in \mathbb{Z}\}$ in $\mathcal{L}(\mathcal{H})$ is called T -periodic if $\rho_t = \rho_{t+T}$.

For a bounded linear operator, A , and an $n \times p$ matrix, B , we define the multiplication of A and B , $C=AB$, as an $n \times p$ matrix, whose elements are $C_{ij} = AB_{ij}$. Here and subsequently, X_{it} denotes an \mathcal{H} -valued random variable at time t and site i .

In the following, we provide the definition of PC space-time autoregressive Hilbertian process of order one (PCSTARH(1, 1)).

Definition 2.4. A Hilbertian process $\{X_{it}, i = 1, \dots, k, t \in \mathbb{Z}\}$ is called a PCSTARH(1) with period T , if it is PC and satisfies

$$X_{it} = \phi_t X_{i(t-1)} + \psi_t \sum_{j=1}^k w_{ij} X_{j(t-1)} + \varepsilon_{it}, \tag{5}$$

where $\{\phi_t, t \in \mathbb{Z}\}$ and $\{\psi_t, t \in \mathbb{Z}\}$ are T -periodic sequences in $\mathcal{L}(\mathcal{H})$ as the AR parameters of lag time $t, \{w_{ij}, i = 1, \dots, k, j = 1, \dots, k\}$ are coefficients that satisfy the following properties:

$$\begin{aligned} w_{ij} &\geq 0, \quad \forall i, j = 1, \dots, k \\ w_{ii} &= 0, \quad \forall i = 1, \dots, k \\ \sum_{j=1}^k w_{ij} &= 1, \end{aligned} \tag{6}$$

and $\{\varepsilon_{it}, i = 1, \dots, k, t \in \mathbb{Z}\}$ is a PCHWN.

Note that, if we define $\mathbf{X}_t := (X_{1t}, X_{2t}, \dots, X_{kt})'$ and $\varepsilon_t := (\varepsilon_{1t}, \dots, \varepsilon_{kt})'$, as \mathcal{H}^k -valued random variables, and $W = (w_{ij})$ as a $k \times k$ weight matrix, then we can rewrite (5) as:

$$\mathbf{X}_t = (\phi_t I + \psi_t W) \mathbf{X}_{(t-1)} + \varepsilon_t \tag{7}$$

In the sequel, we prove that the class of PCSTARH(1, 1) processes can be embedded into the class of PCARH^k(1) processes. Let us present the following lemma that is crucial in our approach.

Lemma 2.1. The Hilbertian process $\mathbf{X}_t = (\phi_t I + \psi_t W) \mathbf{X}_{(t-1)} + \varepsilon_t$ is also a PCARH^k(1) process, where $\{\phi_t, t \in \mathbb{Z}\}$ and $\{\psi_t, t \in \mathbb{Z}\}$ are T -periodic sequences in $\mathcal{L}(\mathcal{H})$ and W is a weight matrix.

Proof. We first show that $\rho_t = \phi_t I + \psi_t W$ is a T -periodic sequence in $\mathcal{L}(\mathcal{H}^k)$. Since ϕ_t and ψ_t are T -periodic bounded linear operators, we have

$$\rho_{t+T} = \phi_{t+T} I + \psi_{t+T} W = \phi_t I + \psi_t W.$$

It can be shown that ε_t is PCHWN, since

- (1) $\mathbb{E}(\varepsilon_t) = \mathbb{E}(\varepsilon_{1t}, \varepsilon_{2t}, \dots, \varepsilon_{kt})' = 0$,
- (2) $C_{\varepsilon_{n+T}}(\mathbf{x}) = C_{\varepsilon_n}(\mathbf{x})$, because,

$$\begin{aligned} C_{\varepsilon_{n+T}}(\mathbf{x}) &= \mathbb{E} \langle \varepsilon_{n+T}, \mathbf{x} \rangle \varepsilon_{n+T} \\ &= \mathbb{E} \left(\sum_{i=1}^k \langle \varepsilon_{i(n+T)}, x_i \rangle \varepsilon_{1(n+T)}, \dots, \sum_{i=1}^k \langle \varepsilon_{i(n+T)}, x_i \rangle \varepsilon_{k(n+T)} \right)' \\ &= \mathbb{E} \left(\sum_{i=1}^k \langle \varepsilon_{in}, x_i \rangle \varepsilon_{1n}, \dots, \sum_{i=1}^k \langle \varepsilon_{in}, x_i \rangle \varepsilon_{kn} \right)' = C_{\varepsilon_n}(\mathbf{x}) \end{aligned}$$

- (3) $C_{\varepsilon_n, \varepsilon_m}(\mathbf{x}) = 0$ for each $n \neq m, \mathbf{x} \in \mathcal{H}^k$.

Consequently, the proof is completed.

Assumption A₁: There are integers $k_0, k_1, \dots, k_{T-1} \in [1, \infty)$, such that $\sum_{i=0}^{T-1} \|\rho_i\|^{k_i} < 1$, where $\rho_i = \phi_i I + \psi_i W$.

Corollary 2.1. If $\sum_{i=0}^{T-1} 2^{k_i} (\|\phi_i\|^{k_i} + \|\psi_i\|^{k_i} \|W\|^{k_i}) < 1$, then assumption \mathbf{A}_1 holds.

Proof. It is enough to apply the known inequality

$$(x + y)^p \leq 2^p(x^p + y^p),$$

where x, y and p are positive.

We now state a theorem, concerning existence and uniqueness of the PCSTARH(1,1) process.

Theorem 2.1. Under the assumption \mathbf{A}_1 , the equation $\mathbf{X}_t = (\phi_t I + \psi_t W)\mathbf{X}_{(t-1)} + \varepsilon_t$ has a unique solution given by

$$\mathbf{X}_{nT+i} = \sum_{j=0}^{\infty} A_{j,nT+i} \varepsilon_{nT+i-j} = \sum_{k=0}^{\infty} \sum_{l=0}^{T-1} [A_{T,nT+i}]^k A_{l,nT+i} \varepsilon_{(n-k)T+i-l}, \tag{8}$$

where $A_{0,i} = I, A_{1,i} = \rho_i, A_{2,i} = \rho_i \rho_{i-1}, \dots, A_{k,i} = \rho_i \rho_{i-1} \dots \rho_{i-k+1}$ and $\rho_i = \phi_i I + \psi_i W$.

Proof. Based on Lemma 2.1, \mathbf{X}_t is a PCARH^k(1) process that can be written as $\mathbf{X}_t = \rho_t \mathbf{X}_{(t-1)} + \varepsilon_t$ with $\rho_t = \phi_t I + \psi_t W$. Then we can apply Theorem 2.1 of Soltani and Hashemi [14] to prove that under the assumption \mathbf{A}_1 , a PCARH(1) process has a unique solution as in Equation (8) and the proof is then completed.

2.1. Strong Law of Large Number

In this section, we prove the strong law of large numbers for PCSTARH(1, 1) processes.

Definition 2.5. A PCSTARH(1, 1), $\{X_{it}, i = 1, \dots, k, t \in \mathbb{Z}\}$, is said to be standard if assumption \mathbf{A}_1 holds.

Theorem 2.2. Let $\{X_{it}, i = 1, \dots, k, t \in \mathbb{Z}\}$ be a standard PCSTARH(1,1) and $X_{i,0}, X_{i,1}, \dots, X_{i,NT-1}$ be a finite segment of this model. Then, as $N \rightarrow \infty$,

$$\frac{n^{\frac{1}{4}}}{(\log n)^\beta} \frac{S_{i,n}(X)}{n} \xrightarrow{a.s} 0, \quad \beta > \frac{1}{2},$$

where $S_{i,n}(X) = \sum_{t=0}^{n-1} X_{it}$ and $n = NT$.

Proof. By defining $\mathbf{X}_t = (X_{1t}, X_{2t}, \dots, X_{kt})'$, \mathbf{X}_t is a PCARH^k(1) process and, using Theorem 2.2 of Soltani and Hashemi [14], we have

$$\frac{n^{\frac{1}{4}}}{(\log n)^\beta} \frac{S_n(X)}{n} \xrightarrow{a.s} 0, \quad \beta > \frac{1}{2}.$$

and the proof is completed.

3. ESTIMATION OF THE AUTOCORRELATION PARAMETERS

Parameter estimation is an important feature of model identification. In this section, the parameters of PCSTARH(1, 1) model are estimated using the method of moment.

Let $\mathbf{X}_0, \dots, \mathbf{X}_{n-1}$ be a finite segment from $\mathbf{X}_t = (\phi_t I + \psi_t W)\mathbf{X}_{(t-1)} + \varepsilon_t$, where n is a multiple of T , $n = NT$, and $\mathbf{X}_t = (X_{1t}, X_{2t}, \dots, X_{kt})'$. To estimate the parameters, ϕ_t and ψ_t , we first estimate $\rho_t = \phi_t I + \psi_t W$.

The classical method of moments provides the following normal equations:

$$D_{l-1} = \rho_l C_{l-1}, \quad l = 1, \dots, T, \tag{9}$$

where

$$\rho_l = \phi_l I + \psi_l W, \tag{10}$$

$$\begin{aligned} C_{l-1} &= \mathbb{E}(\mathbf{X}_{l-1} \otimes \mathbf{X}_{l-1}), \\ D_{l-1} &= \mathbb{E}(\mathbf{X}_l \otimes \mathbf{X}_{l-1}). \end{aligned} \tag{11}$$

Since C_{l-1} is a compact operator, it has the following spectral decomposition

$$C_{l-1} = \sum_{m \in \mathbb{N}} \lambda_{m,l-1} (\mathbf{e}_{m,l-1} \otimes \mathbf{e}_{m,l-1}), \quad \sum \lambda_{m,l-1} \leq \infty, \tag{12}$$

where $(\lambda_{m,l-1})_{m \geq 1}$ is a sequence of the positive eigenvalues of C_{l-1} and $(\mathbf{e}_{m,l-1})_{m \geq 1}$ is a complete orthonormal system in \mathcal{H}^k . We define $\pi_{m,l-1}$ as the associated sequence of projections, hence, $\pi_{m,l-1} = \mathbf{e}_{m,l-1} \otimes \mathbf{e}_{m,l-1}$ and $\Pi_{k_n,l-1} = \sum_{j=1}^{k_n} \pi_{j,l-1}$.

First, it is crucial to note that, since C_{l-1}^{-1} is not necessarily invertible, we can not deduce from (9) that $\rho_l = D_{l-1} C_{l-1}^{-1}$, $l = 1, \dots, T$. A necessary and sufficient condition for C_{l-1}^{-1} to be defined is that $\text{Ker}(C_{l-1}) = 0$, i.e. $C_{l-1}(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

From Equation (9), we have

$$D_{l-1}(\mathbf{e}_{j,l-1}) = \rho_l C_{l-1}(\mathbf{e}_{j,l-1}) = \lambda_{j,l-1} \rho_l(\mathbf{e}_{j,l-1})$$

Then, for any $\mathbf{x} \in \mathcal{H}^k$, the derived equation leads to the representation

$$\rho_l(\mathbf{x}) = \rho_l \left(\sum_{j=1}^{\infty} \langle \mathbf{x}, \mathbf{e}_{j,l-1} \rangle \mathbf{e}_{j,l-1} \right) = \sum_{j=1}^{\infty} \frac{D_{l-1}(\mathbf{e}_{j,l-1})}{\lambda_{j,l-1}} \langle \mathbf{x}, \mathbf{e}_{j,l-1} \rangle. \tag{13}$$

Equation (13) gives a core idea for the estimation of ρ_l . Therefore, we estimate D_{l-1} , $\lambda_{j,l-1}$ and $\mathbf{e}_{j,l-1}$ empirically and substitute them in Equation (13). For this purpose, the estimated eigen elements $(\hat{\lambda}_{j,l-1}, \hat{\mathbf{e}}_{j,l-1})_{1 \leq j \leq n}$ will be obtained using the empirical covariance operator $\hat{C}_{l-1} = \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{X}_{l-1+kT} \otimes \mathbf{X}_{l-1+kT}$. Besides, $\hat{\pi}_{j,l-1}$ is the empirical counterpart of $\pi_{j,l-1}$ and $\hat{\Pi}_{k_n,l-1} = \sum_{j=1}^{k_n} \hat{\pi}_{j,l-1}$ is the projector on the space spanned by the k_n first eigenvectors of \hat{C}_{l-1} . Note that by the finite sample, the entire sequence $(\lambda_{j,l-1}, \mathbf{e}_{j,l-1})$ cannot be estimated and just a truncated version can be obtained, which leads to

$$\hat{\rho}_l(\mathbf{x}) = \sum_{j=1}^{k_n} \frac{\hat{D}_{l-1}(\hat{\mathbf{e}}_{j,l-1})}{\hat{\lambda}_{j,l-1}} \langle \mathbf{x}, \hat{\mathbf{e}}_{j,l-1} \rangle. \tag{14}$$

If k_n grows to infinity by the sample size, the estimator $\hat{\rho}_l$ will be consistent. On the other hand, we know that $\lambda_{j,l-1} \rightarrow 0$. Hence, it will be a delicate issue to control the behavior of $\frac{1}{\lambda_{j,l-1}}$. In fact, a small error in the estimation of $\lambda_{j,l-1}$ can have an enormous impact on (14).

We now turn to estimate ϕ_l and ψ_l . Let $B = (1, 0, \dots, 0)'$ and W be an invertible matrix, then

$$\phi_l = B' \rho_l B \tag{15}$$

$$\psi_l = W^{-1}(\rho_l - \phi_l I). \tag{16}$$

To study consistency of estimators, we begin by proving the consistency of $\hat{\rho}$. In order to study consistency of the estimators, we need the following assumptions:

Assumption B₁: $\mathcal{X} = \{\mathbf{X}_n; n \in \mathbb{Z}\}$ is a standard PCARH(1) such that $E\|\mathbf{X}_n\|^4 \leq \infty$ for all $n \in \mathbb{Z}$.

Assumption B₂: $\lambda_{1,l-1} \geq \lambda_{2,l-1} \geq \dots \geq 0$.

Assumption B₃: $\hat{\lambda}_{k_n,l-1} \geq 0$, a.s.

Following Hashemi and Soltani [25], the next theorem can be proved.

Theorem 3.1. Suppose that **B₁**, **B₂** and **B₃** hold and $\rho_l, l = 1, \dots, T$, are Hilbert Schmidt operators. Then, if for some $\beta \geq 1$,

$$\lambda_{k_n,l-1}^{-1} \sum_{j=1}^{\infty} a_{j,l-1} = O(n^{\frac{1}{4}} (\log n)^\beta), \tag{17}$$

we obtain

$$\|\hat{\rho}_l - \rho_l\| \xrightarrow{a.s} 0, \quad l = 1, \dots, T, \tag{18}$$

where $a_{j,l-1} = 2\sqrt{2} \max \left[(\lambda_{j-1,l-1} - \lambda_{j,l-1})^{-1}, (\lambda_{j,l-1} - \lambda_{j+1,l-1})^{-1} \right]$ if $j \geq 2$, and $a_{1,l-1} = 2\sqrt{2}(\lambda_{1,l-1} - \lambda_{2,l-1})^{-1}$.

Proof. Consider the decomposition

$$\begin{aligned} (\hat{\rho}_l - \rho_l)(\mathbf{x}) &= [\hat{\rho}_l(\mathbf{x}) - \rho_l \Pi_{k_n, \ell-1}(\mathbf{x})] \\ &\quad + [\rho_l \Pi_{k_n, \ell-1}(\mathbf{x}) - \rho_l \tilde{\Pi}_{k_n, \ell-1}(\mathbf{x})] \\ &\quad + [\rho_l \tilde{\Pi}_{k_n, \ell-1}(\mathbf{x}) - \rho_l(\mathbf{x})] \\ &:= \hat{\alpha}_l(\mathbf{x}) + \hat{b}_l(\mathbf{x}) + \hat{c}_l(\mathbf{x}), \end{aligned} \quad (19)$$

and put

$$\hat{\alpha}_l = \sup_{\|\mathbf{x}\| \leq 1} \|\hat{\alpha}_l(\mathbf{x})\|, \quad \hat{\beta}_l = \sup_{\|\mathbf{x}\| \leq 1} \|\hat{b}_l(\mathbf{x})\|, \quad \hat{\gamma}_l = \sup_{\|\mathbf{x}\| \leq 1} \|\hat{c}_l(\mathbf{x})\|.$$

It is easy to see that

$$\begin{aligned} \hat{\alpha}_l &\xrightarrow{a.s.} 0, \\ \hat{\beta}_l &\xrightarrow{a.s.} 0, \\ \hat{\gamma}_l &\xrightarrow{a.s.} 0, \end{aligned} \quad (20)$$

and the proof is completed. For more details, see Theorem (3.3) of Hashemi and Soltani [25].

Next corollary states the consistency of ϕ and ψ estimators, define in (15) and (16).

Corollary 3.1. Under the assumptions of Theorem 3.1, $\hat{\phi}_l$ and $\hat{\psi}_l$ are consistent estimator.

4. CONCLUSION

Our objective in this paper is to introduce a new model for PC space-time data. We first introduce PC space-time autoregressive processes in an Hilbert space and provide conditions for their existence.

Our main aim is to show that there exists a relation between PCSTARH(1, 1) and PCARH(1) and so we can show some properties for PCSTARH(1, 1) models, such as the strong law of large numbers. PCSTARH models have sophisticated statistical structures that open up promising new resources for data modeling strategies. We have focused only on the theoretical setup and leave applied approaches for future research.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

AUTHORS' CONTRIBUTIONS

The authors had the same contribution in designing the model, developing the theory and writing the manuscript

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REFERENCES

1. P.H. Franses, Periodicity and Stochastic Trends in Economic Time Series, Oxford University Press, New York, NY, USA, 1996.
2. W.A. Gardner, Cyclostationarity in Communications and Signal Processing, IEEE Press, New York, NY, USA, 1994.
3. W.A. Gardner, A. Napolitano, L. Paura, Signal Process. 86 (2006), 639–697.

4. H.L. Hurd, A. Miamee, *Periodically Correlated Random Sequences: Spectral Theory and Practice*, Wiley, Hoboken, NJ, USA, 2007.
5. A.R. Soltani, Z. Shishehbor, *Acta Math. Hung.* 80 (1998), 265–270.
6. A.R. Soltani, Z. Shishehbor, *J. Georgian Math.* 6 (1999), 91–98.
7. D. Bosq, *Linear Processes in Function Spaces: Theory and Applications*, Berlin, Germany, Springer, 2000.
8. T. Mourid, *Acad. Sci.* 317 (1993), 1167–1172.
9. P. Besse, H. Cardot, *Canad. J. Stat.* 24 (1996), 467–487.
10. B. Pumo, *Stat. Inference Stoch. Process.* 3 (1999), 1–13.
11. A. Mas, *Stochastic Process Appl.* 99 (2002), 117–135.
12. A. Mas, *J. Multivariate Anal.* 98 (2007), 1231–1261.
13. L. Horvath, M. Huskova, P. Kokoszka, *J. Multivariate Anal.* 101 (2010), 352–367.
14. A.R. Soltani, M. Hashemi, *Stat. Inference Stochastic Process.* 14 (2011), 177–188.
15. P. Pfeifer, S. Deutsch, *Technometrics.* 22 (1980), 35–47.
16. P. Pfeifer, S. Deutsch, *Technometrics.* 22 (1980), 397–408.
17. P. Pfeifer, S. Deutsch, *SIAM J. Appl. Math. Ser. A.* 40 (1981), 133–136.
18. P. Pfeifer, S. Deutsch, *Geograph. Anal.* 13 (1981), 117–133.
19. P. Pfeifer, S. Deutsch, *Technometrics.* 23 (1981), 401–409.
20. D. Stoffer, *J. Am. Stat. Assoc.* 81 (1986), 762–772.
21. P. Pfeifer, S. Bodily, *J. Forecast.* 8 (1990), 255–272.
22. M.D. Ruiz-Medina, R. Salmero, *Stoch. Environ. Res. Risk Assess.* 24 (2010), 131–146.
23. M.D. Ruiz-Medina, *J. Multivar. Anal.* 102 (2011), 292–305.
24. M.D. Ruiz-Medina, *Environmetrics.* 23 (2012), 119–128.
25. M. Hashemi, A.R. Soltani, *Indian J. Sci. Technol.* 9 (2016), 1–6.