Received: 20 December 2019 | Revised: 18 June 2020 | Accepted: 22 June 2020

DOI: 10.1002/mana.201900554

MATHEMATISCHE NACHRICHTEN

ORIGINAL PAPER

Cosets of normal subgroups and powers of conjugacy classes

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Abstract

Let G be a finite group and let $K = x^G$ be the conjugacy class of an element x of G. In this paper, it is proved that if N is a normal subgroup of G such that the coset xN is the union of K and K^{-1} (the conjugacy class of the inverse of x), then N and the subgroup $\langle K \rangle$ are solvable. As an application, we prove that if there exists a natural number $n \geq 2$ such that $K^n = K \cup K^{-1}$, then $\langle K \rangle$ is solvable.

KEYWORDS

characters, conjugacy classes, cosets of normal subgroups, powers of conjugacy classes

MSC (2020)

20C15, 20D20, 20E45

1 | INTRODUCTION

Let G be a finite group, let N be a normal subgroup of G, and let $x \in G$. In [3] it was proved, by appealing to the Classification of Finite Simple Groups, that whenever all elements of the coset xN are conjugate (to x) in G then N is solvable. In fact, the result goes further and, for example, it is shown that if all elements in xN are p-elements for some odd prime p, then N is solvable, and if in addition they are conjugate, then N has normal p-complement. For $p = 2$, the fact that all elements in xN are 2-elements does not imply the solvability of N, however, the second assertion remains true. In this note, we investigate the case in which a coset xN is the union of the conjugacy class of x and that of its inverse, and our first objective is to prove the following.

Theorem A. *Let G be a finite group and let N be a normal subgroup of G. Let $K = x^G$ be the conjugacy class of an element $x \in G$. Suppose that $xN = K \cup K^{-1}$. Then $\langle K \rangle$ is solvable, and as a consequence so is N.*

We employ Theorem A to address a concrete problem on products of conjugacy classes. We recall that Arad and Fisman's conjecture asserts that the product of two non-trivial conjugacy classes cannot be a conjugacy class in a non-abelian finite simple group. Even though it remains unsolved, this subject is of keen interest for many authors, who have tried to find solvability conditions related to the product of conjugacy classes. For instance, a specific case of Arad and Fisman's conjecture is the following. If K is a conjugacy class, then the fact that K^2 is again a conjugacy class implies that $\langle K \rangle$ is solvable (Theorem A of [3]), and likewise, when K^n is a conjugacy class for some $n \geq 3$ (Theorem A of [1]). We study a particular case of the following conjecture, which was posed in [1].

Conjecture. *Let G be a group and let K be a conjugacy class of G. If $K^n = D \cup D^{-1}$ for some $n \geq 2$ where D a conjugacy class of G, then $\langle K \rangle$ is solvable.*

The hypotheses in the above conjecture are not unusual and it is not difficult, for instance with the help of [2], to find numerous examples (see Examples 3 and 4 of [1] for the case $n = 2$ and also see Section 3 for $n = 3$). It turns out that either

$|K| = |D|/2$ or $|K| = |D|$. The first case was already solved in [1], and our contribution here concerns the case $K = D$. We note that the case $n = 2$ with $K = D$ was already treated in Theorem D of [1].

Theorem B. *Let G be a finite group and let $K = x^G$ be a conjugacy class of G. If $K^n = K \cup K^{-1}$ for some $n \geq 2$, then $\langle K \rangle$ is solvable.*

2 | COSETS AND CHARACTERS

Throughout the paper, we will follow the notation of [5]. We start by stating two preliminary results on products of conjugacy classes, whose proofs are based on the Classification of Finite Simple Groups.

Lemma 2.1. *Let G be a group and let K, L and D be non-trivial conjugacy classes of G such that $KL = D$ with $|D| = |K|$. Then $\langle L \rangle$ is solvable.*

Proof. See Lemma 2 of [1]. □

The original statement of the above lemma includes some more result but just ensuring solvability needs the Classification of Finite Simple Groups. We also need the following extension of Theorem A of [3]. The original result also gives a characterization by means of characters, however just the following statement requires the Classification.

Theorem 2.2. *Let K be a conjugacy class of a group G. If there is $n \geq 2$ such that K^n is a conjugacy class, then $\langle K \rangle$ is solvable.*

Proof. See Theorem A of [1]. □

Next we study some properties of the character values for the conjugacy classes that we are dealing with.

Lemma 2.3. *Let G be a finite group and let N be a normal subgroup of G. Let $K = x^G$ be the conjugacy class of an element $x \in G$. Suppose that $xN \subseteq K \cup K^{-1}$. If $\chi \in \mathrm{Irr}(G)$ does not contain N in its kernel, then $\chi(x)$ is a purely imaginary number.*

Proof. The proof is reminiscent of the proof of Lemma 3.1 of [3]. Let $\mathfrak{X}$ be a representation of G that affords χ. We know that $\mathfrak{X}$ can be linearly extended to $\mathbb{C}[G]$ and for every conjugacy class of G, say T, we denote by $\widehat{T}$ the sum of all elements in T in the group algebra $\mathbb{C}[G]$. Since N is a disjoint union of conjugacy classes of G, the sum

$$\widehat{N} = \sum_{n \in N} n \in \mathbf{Z}(\mathbb{C}[G])$$

and, by Schur's lemma, $\mathfrak{X}(\widehat{N})$ is a scalar matrix. The trace of $\mathfrak{X}(\widehat{N})$ is

$$\sum_{n \in N} \chi(n) = |N|[\chi_N, 1_N] = 0,$$

so $\mathfrak{X}(\widehat{N}) = O$, where O denotes the zero matrix, and $\mathfrak{X}(\widehat{zN}) = \mathfrak{X}(z)\mathfrak{X}(\widehat{N}) = O$, for every $z \in G$.

Observe that, by hypothesis, $KN \subseteq K \cup K^{-1}$, and since KN is a union of conjugacy classes then $KN = K$ or $KN = K \cup K^{-1}$. Also, we know that $\mathfrak{X}(\widehat{K}\widehat{N}) = \sum_{x \in K} \mathfrak{X}(\widehat{xN}) = O$ and, analogously, $\mathfrak{X}(\widehat{K^{-1}}\widehat{N}) = O$. On the other hand, we write $\widehat{K}\widehat{N} = m_1\widehat{K} + m_2\widehat{K^{-1}}$ with $m_1, m_2 \in \mathbb{N}$, where we allow $m_2 = 0$ for the case $KN = K$. By taking inverses, we have $\widehat{K^{-1}}\widehat{N} = m_2\widehat{K} + m_1\widehat{K^{-1}}$. By taking traces, we have

$$\begin{aligned}\mathrm{Trace}\left(\mathfrak{X}\left(\widehat{K}\widehat{N} + \widehat{K^{-1}}\widehat{N}\right)\right) &= (m_1 + m_2)|K|\chi(x) + (m_1 + m_2)|K|\chi\left(x^{-1}\right) \\ &= (m_1 + m_2)|K|\left(\chi(x) + \chi\left(x^{-1}\right)\right) = 0.\end{aligned}$$

Thus, $\chi(x) = -\chi\left(x^{-1}\right) = -\overline{\chi(x)}$, so the lemma is proved. □

We are ready to prove Theorem A, which we state again.

Theorem 2.4. *Let G be a finite group and let N be a normal subgroup of G. Let $K = x^G$ be the conjugacy class of an element $x \in G$. Suppose that $xN = K \cup K^{-1}$. Then $\langle K \rangle$ is solvable, and as a consequence so is N.*

Proof. The proof is reminiscent of the proof of Theorem 3.2(c) of [3]. The case in which K is a real class is just a particular case of the mentioned theorem although we will include this case in our proof.

It is clear that $xN = x^{-1}N$ and then $x^2N = N = \left(K \cup K^{-1}\right)\left(K \cup K^{-1}\right)$, so in particular $K^2 \subseteq N$. Consequently, $\left\langle K^2 \right\rangle \leq N$. Moreover, we have $K^3 \subseteq NK = K \cup K^{-1}$. If $K^3 = K$ or $K^3 = K^{-1}$, it follows that $\langle K \rangle$ is solvable by Theorem 2.2 and then N is solvable as well because obviously $N \leq \langle K \rangle$. Thus we can assume that $K^3 = K \cup K^{-1}$. Further, it can also be assumed that $K^{2n+1} = K^3$ for every $n \geq 1$. Indeed, $\left(K^2\right)^n \subseteq N$ for every $n \geq 1$, so $K^{2n+1} \subseteq KN = K \cup K^{-1}$, and in fact, the equality can be assumed to hold again by Theorem 2.2, as wanted. As a consequence, x must be a 2-element. Let $C = \mathbf{C}_G(x)$ be the centralizer of x in G. We have

$$|G : C| = |G : NC||NC : C| = |K|.$$

Moreover, we have $|K| = |N|$ or $|N|/2$, depending on whether $K = K^{-1}$ or not. Also, if $n \in \mathbf{C}_N(x)$, then as xn has the same order as x, we have that n must be a 2-element too. Accordingly, $\mathbf{C}_N(x)$ is a 2-group and it follows that either $|G : NC| = |\mathbf{C}_N(x)|/2$ or $|\mathbf{C}_N(x)|$ is a 2-power. As a consequence, we obtain $G = NPC$, for every Sylow 2-subgroup P of G. Moreover, if we choose P such that $x \in P$, then we can replace G by the subgroup $H = NP$, since the conjugacy class of x in H coincides with K. Therefore, we can assume that the index $|G : N|$ is a power of 2.

On the other hand, as x is a 2-element, it is well-known that

$$\chi(x)^{|x|} \equiv \chi(1) \mod 2$$

(in the ring of algebraic integers), for every $\chi \in \mathrm{Irr}(G)$. Now, since G/N is a 2-group, the degree of every non-linear irreducible character of G containing N in its kernel is a power of 2. Also, if $\chi \in \mathrm{Irr}(G)$ does not contain N in its kernel and it is real-valued, necessarily $\chi(x) = 0$ by Lemma 2.3, and hence, by the above congruence, $\chi(1)$ is even. Therefore, we conclude that all non-linear real-valued irreducible characters of G have even degree. By Theorem A of [6], this is equivalent to the fact that G has normal 2-complement, so in particular, G is solvable. Then N and $\langle K \rangle$ are solvable as well. □

We wonder whether Theorem A will still be true when the hypothesis $xN = K \cup K^{-1}$ is weakened to $xN \subseteq K \cup K^{-1}$ as is the case with Lemma 2.3, however, we have not been able to prove it by using similar methods.

3 | PROOF OF THEOREM B

We employ the results of Section 2 to solve a specific case of the conjecture of the Introduction. For that purpose, we need to work with the complex group algebra $\mathbb{C}[G]$. Let $L_1, \dots, L_k$ be the conjugacy classes of a finite group G and let S be a G-invariant set of G, then we can write the sum $\widehat{S} = \sum_{i=1}^{k} n_i \widehat{L_i}$ with $n_i \in \mathbb{N}$ for $1 \leq i \leq k$. We write $\left(\widehat{S}, \widehat{L_i}\right) = n_i$. We will use the following well-known properties.

Lemma 3.1. *If L_1, L_2 and L_3 are conjugacy classes of a finite group G, then*

1. $\left(\widehat{L_1}\widehat{L_2}, \widehat{L_3}\right) = \left(\widehat{L_1^{-1}}\widehat{L_2^{-1}}, \widehat{L_3^{-1}}\right)$;
2. $\left(\widehat{L_1}\widehat{L_2}, \widehat{L_3}\right) = |L_2||L_3|^{-1}\left(\widehat{L_1}\widehat{L_3^{-1}}, \widehat{L_2^{-1}}\right)$.

Proof. This easily follows, for instance, from Theorem 4.6 of [4]. □

We are ready to give the proof of Theorem B.

Proof of Theorem B. We can assume that K is non-real, since the real case is a particular case of Theorem 2.2. Also, as we pointed out in the Introduction, the case $n = 2$ is already proved in Theorem D of [1]. Henceforth, we will assume that $n \geq 3$ and argue by induction on $|G|$. Since K^{n-1} is a G-invariant set, we write $K^{n-1} = L_1 \cup \cdots \cup L_s$ where L_i are distinct conjugacy classes of G (the trivial class may be included) for every $1 \leq i \leq s$. Then

$$K^n = KK^{n-1} = K(L_1 \cup \cdots \cup L_s) = K \cup K^{-1}.$$

Suppose that L_i is a non-trivial conjugacy class. If either $KL_i = K$ or $KL_i = K^{-1}$, by Lemma 2.1, we know that $\langle L_i \rangle$ is solvable. Consider now $\overline{G} = G/\langle L_i \rangle$ and observe from the hypothesis that $\overline{K^n} = \overline{K} \cup \overline{K^{-1}}$. Then $\left\langle \overline{K} \right\rangle$ is solvable by induction. Notice that if $\overline{K} = \overline{K^{-1}}$, then $\overline{K^n} = \overline{K}$ and $\left\langle \overline{K} \right\rangle$ is solvable again by Theorem 2.2. Consequently, $\langle K \rangle$ is solvable.

Therefore, we can assume that $KL_i = K \cup K^{-1}$ for every non-trivial class L_i. By Lemma 3.1(2) and (1), we know that

$$0 \neq \left(\widehat{K}\widehat{L_i}, \widehat{K^{-1}}\right) = \frac{|L_i|}{|K|}\left(\widehat{K^2}, \widehat{L_i^{-1}}\right) = \frac{|L_i|}{|K|}\left(\widehat{K^{-2}}, \widehat{L_i}\right),$$

and thus $L_i \subseteq K^{-2}$. We deduce that $K^{n-1} \subseteq K^{-2} \cup \{1\}$. On the other hand, $\left|K^2\right| = \left|K^{-2}\right| \leq \left|K^{n-1}\right| \leq \left|K^2\right| + 1$ (in the first inequality we are using that $n \geq 3$), thus either $K^{n-1} = K^{-2}$ or $K^{n-1} = K^{-2} \cup \{1\}$. In both cases, $K^{-2} \subseteq K^{n-1}$. Moreover, if L_i is a non-trivial conjugacy class, again by applying Lemma 3.1(2) and (1), we have

$$0 \neq \left(\widehat{K}\widehat{L_i}, \widehat{K}\right) = \frac{|L_i|}{|K|}\left(\widehat{K}\widehat{K^{-1}}, \widehat{L_i^{-1}}\right) = \frac{|L_i|}{|K|}\left(\widehat{K}\widehat{K^{-1}}, \widehat{L_i}\right).$$

Hence $L_i \subseteq KK^{-1}$. In addition, $K^{-1}L_i \subseteq K^{-1}K^{-1}K \subseteq K^{n-1}K = K^n$ and consequently, $K^{-1}L_i \subseteq K \cup K^{-1}$. If $K^{-1}L_i = K$ or $K^{-1}L_i = K^{-1}$, then $\langle K \rangle$ is solvable by arguing as before. We can assume then that $K^{-1}L_i = K \cup K^{-1}$ for every non-trivial L_i. In particular, by applying Lemma 3.1(1) and (2)

$$0 \neq \left(\widehat{K^{-1}}\widehat{L_i}, \widehat{K}\right) = \left(\widehat{K}\widehat{L_i^{-1}}, \widehat{K^{-1}}\right) = \frac{|L_i|}{|K|}\left(\widehat{K^2}, \widehat{L_i}\right),$$

which means that $L_i \subseteq K^2$. Therefore, $K^{n-1} \subseteq K^2 \cup \{1\}$. Analogously as above, taking cardinalities we obtain that either $K^{n-1} = K^2$ or $K^{n-1} = K^2 \cup \{1\}$. In both cases, $K^2 \subseteq K^{n-1}$. Hence $K^3 \subseteq K^n = K \cup K^{-1}$. By applying Theorem 2.2, it can be assumed that $K^3 = K \cup K^{-1}$. Taking into account that $K^2 = K^{-2}$, we obtain

$$K^5 = K^2K^3 = K^2\left(K \cup K^{-1}\right) = K^3 \cup K^{-2}K^{-1} = K^3 \cup K^{-3} = K \cup K^{-1}.$$

Inductively, we easily get $K^{2k+1} = K \cup K^{-1}$ for every $k \geq 1$, and as a consequence, $K\left\langle K^2 \right\rangle = K \cup K^{-1}$. The fact that $K \cup K^{-1}$ is a union of cosets of the normal subgroup $\langle K^2 \rangle$ shows that $\left|\left\langle K^2 \right\rangle\right|$ divides $2|K|$. Now, note that $|K| \leq \left|K^2\right| < 1 + \left|K^2\right| \leq \left|\left\langle K^2 \right\rangle\right|$, so we conclude that $\left|\left\langle K^2 \right\rangle\right| = 2|K|$. By cardinalities, it follows that $x\left\langle K^2 \right\rangle = K \cup K^{-1}$, and then, we apply Theorem A to get that $\left\langle K^2 \right\rangle$ is solvable. Now, notice that if $x, y \in K$, then $x\left\langle K^2 \right\rangle = y\left\langle K^2 \right\rangle$. This implies that $\langle K \rangle/\left\langle K^2 \right\rangle$ is cyclic of order 2, so $\langle K \rangle$ is solvable as well. □

Example 3.2. We give an example of a group satisfying the hypotheses of Theorem B with $n = 3$, in which the order of the elements in K is not a prime. Let $G = \left\langle a, x \mid a^8 = x^2 = 1, a^x = a^3 \right\rangle$ the semidihedral group of order 16 and $K = a^G$, which satisfies $K^3 = K \cup K^{-1}$.

ACKNOWLEDGMENT

The authors are grateful to the referee for careful reading and many helpful comments and improvements on the paper. This research is partially supported by the Spanish Government, Proyecto PGC2018-096872-B-I00. The first named author is also supported by Proyecto UJI-B2019-03.

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