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A Characterization of a Local Vector Valued Bollobás Theorem

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Abstract. In this paper, we are interested in giving two characterizations for the so-called property $\mathbf{L}_{o,o}$, a local vector valued Bollobás type theorem. We say that (X, Y) has this property whenever given $\varepsilon > 0$ and an operador $T : X \to Y$, there is $\eta = \eta(\varepsilon, T)$ such that if x satisfies $\|T(x)\| > 1 - \eta$, then there exists $x_0 \in S_X$ such that $x_0 \approx x$ and T itself attains its norm at x_0. This can be seen as a strong (although local) Bollobás theorem for operators. We prove that the pair (X, Y) has the $\mathbf{L}_{o,o}$ for compact operators if and only if so does $(X, \mathbb{K})$ for linear functionals. This generalizes at once some results due to D. Sain and J. Talponen. Moreover, we present a complete characterization for when $(X \widehat{\otimes}_\pi Y, \mathbb{K})$ satisfies the $\mathbf{L}_{o,o}$ for linear functionals under strict convexity or Kadec–Klee property assumptions in one of the spaces. As a consequence, we generalize some results in the literature related to the strongly subdifferentiability of the projective tensor product and show that $(L_p(\mu) \times L_q(\nu); \mathbb{K})$ cannot satisfy the $\mathbf{L}_{o,o}$ for bilinear forms.

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1. Introduction

It has now been 60 years since Bishop and Phelps proved that every bounded linear functional can be approximated by norm-attaining ones [2]. Since then, several researchers have been working in norm-attaining theory in many different directions and it is out of doubt one of the most traditional topics in Functional Analysis nowadays. Bollobás [3] pushed further the Bishop–Phelps

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theorem by proving that if $\varepsilon > 0$, then there exists $\eta(\varepsilon) > 0$ such that whenever $x^* \in S_{X^*}$ and $x \in S_X$ satisfy $|x^*(x)| > 1 - \eta(\varepsilon)$, then there exist a new functional $x_0^* \in S_{X^*}$ and a new element $x_0 \in S_X$ such that

$$|x_0^*(x_0)| = 1, \qquad \|x_0 - x\| < \varepsilon, \qquad \text{and} \qquad \|x_0^* - x^*\| < \varepsilon \tag{1.1}$$

(our notation is standard and can be found in Sect. 1.1 at the end of this introduction). Let us notice that the Bishop–Phelps theorem plays an important role in non-reflexive spaces since otherwise *every* functional attains its norm. On the other hand, Bollobás theorem does make sense in the reflexive setting and, in this case, the functional x^* necessarily attains its norm; so it would be natural to wonder whether a version of Bollobás theorem *without* changing the initial functional x^* holds in general, that is, whether it is possible to take $x_0^* = x^*$ in (1.1). In a more general situation, we are wondering the following: given $\varepsilon > 0$, is it possible to find $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy $\|T(x)\| > 1 - \eta(\varepsilon)$, one can find a new element $x_0 \in S_X$ such that $\|T(x_0)\| = 1$ and $\|x_0 - x\| < \varepsilon$? It is easy to see that the pair $(X, \mathbb{K})$ satisfies it whenever X is a uniformly convex Banach space and it turns out that this is in fact a characterization for uniformly convex spaces (see [13, Theorem 2.1]). Nevertheless, there is no way of getting such a similar statement for linear operators: indeed, the authors in [7] proved that if X and Y are real Banach spaces of dimension greater than or equal to 2, then the pair (X, Y) always fails such a property. Therefore, *the only* hope for getting positive results in the context of operators would be by considering a weakening of the mentioned property and that was done in [4,8,9,16,17] (and more recently in [5,6] as a tool to get positive results on different norm-attainment notions). More specifically, we have the following property.

Definition 1.1. Let X, Y be Banach spaces. We say that the pair (X, Y) has the $\mathbf{L}_{o,o}$ for operators if given $\varepsilon > 0$ and $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$, there exists $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies $\|T(x)\| > 1 - \eta(\varepsilon, T)$, there exists $x_0 \in S_X$ such that

$$\|T(x_0)\| = 1 \qquad \text{and} \qquad \|x_0 - x\| < \varepsilon. \tag{1.2}$$

Notice that if the pair (X, Y) satisfies such a property, then, in particular, *every* operator has to be norm-attaining. Consequently, the Banach space X must be reflexive by James' theorem. By using a result due to Godefroy et al. [12] and a characterization by Franchetti and Payá [11], it turns out that the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for linear functionals if and only if X^* is strongly subdifferentiable (SSD, for short; see its definition below). On the other hand, at the same way that it happens in the classical norm-attaining theory (see, for instance, [14]), the $\mathbf{L}_{o,o}$ was studied for compact operators [16,17]: we say that (X, Y) has the $\mathbf{L}_{o,o}$ for compact operators if given $\varepsilon > 0$ and a norm-one compact operator T, there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies $\|T(x)\| > 1 - \eta(\varepsilon, T)$, there exists $x_0 \in S_X$ satisfying conditions (1.2). It is known that whenever X is strictly convex and Y is an arbitrary Banach space, the pair (X, Y) has the $\mathbf{L}_{o,o}$ for compact operators if and only if the dual X^* is Fréchet differentiable (see [17, Theorem 2.3]); and when X satisfies the Kadec–Klee property then (X, Y) has the $\mathbf{L}_{o,o}$ for compact operators for every Banach space Y (see [16, Theorem 2.12]).

Our first aim in the present paper is to generalize [16, Theorem 2.12] and [17, Theorem 2.3] at once. Indeed, we have the following theorem.

Theorem A. *Let X be a reflexive Banach space. The following are equivalent.*

(i) *The pair (X, Y) satisfies the $\mathbf{L}_{o,o}$ for compact operators for every Banach space Y.*

(ii) *The pair $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ (equivalently, X^* is SSD).*

Our second main result deals with a strengthening of the $\mathbf{L}_{o,o}$ in the context of bilinear forms (see [8]).

Definition 1.2. [8, Definition 2.1]. Let X, Y be Banach spaces. We say that $(X \times Y; \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for bilinear forms if given $\varepsilon > 0$ and $B \in \mathcal{B}(X \times Y; \mathbb{K})$ with $\|B\| = 1$, there exists $\eta(\varepsilon, B) > 0$ such that whenever $(x, y) \in S_X \times S_Y$ satisfies $|B(x, y)| > 1 - \eta(\varepsilon, B)$, there exists $(x_0, y_0) \in S_X \times S_Y$ such that

$$|B(x_0, y_0)| = 1, \qquad \|x_0 - x\| < \varepsilon, \qquad \text{and} \qquad \|y_0 - y\| < \varepsilon.$$

It is known that $(X \times Y; \mathbb{K})$ satisfies the $\mathbf{L}_{o,o}$ for bilinear forms whenever

(a) X, Y are finite dimensional;

(b) X is finite dimensional and Y is uniformly convex;

(c) $X = \ell_p$ and $Y = \ell_q$ if and only if $p > q'$, where q' is the conjugate index of q.

(see Proposition 2.2.(a), Lemma 2.6, and Theorem 2.7.(b) of [8], respectively). By observing items (a), (b), and (c) above, one might think that the reflexivity of the projective tensor product $X \widehat{\otimes}_\pi Y$ plays an important role here (notice that (c) gives the result for ℓ_p-spaces *exactly* when the projective tensor product $\ell_p \widehat{\otimes}_\pi \ell_q$ is reflexive (see [15, Corollary 4.24])). And this is indeed not a coincidence: we have the following result, which gives a complete characterization for the $\mathbf{L}_{o,o}$ in terms of the reflexivity of $X \widehat{\otimes}_\pi Y$ and also relates the $\mathbf{L}_{o,o}$ in different classes of functions under strict convexity or Kadec–Klee property assumptions on X. For the necessary terminology on approximation properties, we send the reader to the very end of Sect. 1.1.

Theorem B. *Let X be a strictly convex Banach space or a Banach space satisfying the Kadec–Klee property. Let Y be an arbitrary Banach space. Assume that either X or Y enjoys the AP, or that the pair (X, Y^*) satisfies the pointwise-BCAP. The following are equivalent.*

(a) *$\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$ and both $(X, \mathbb{K}), (Y, \mathbb{K})$ have the $\mathbf{L}_{o,o}$ for linear functionals.*

(b) *$X \widehat{\otimes}_\pi Y$ is reflexive and both $(X, \mathbb{K}), (Y, \mathbb{K})$ have the $\mathbf{L}_{o,o}$ for linear functionals.*

(c) $(X \times Y; \mathbb{K})$ *has the* $\mathbf{L}_{o,o}$ *for bilinear forms.*
(d) $(X \widehat{\otimes}_\pi Y, \mathbb{K})$ *has the* $\mathbf{L}_{o,o}$ *for linear functionals.*

As a consequence of Theorem B, we have that $(L_p(\mu) \times L_q(\nu); \mathbb{K})$ *cannot* satisfy the $\mathbf{L}_{o,o}$ for bilinear forms for every $1 < p, q < \infty$ and for not purely atomic measures μ, ν, since $L_p(\mu) \widehat{\otimes}_\pi L_q(\nu)$ is never reflexive (see [15, Theorem 4.21 and Corollary 4.22]). We conclude the paper with a discussion about the relation between the $\mathbf{L}_{o,o}$ in $\mathcal{B}(X \times Y, \mathbb{K})$ when we view $\mathcal{B}(X \times Y, \mathbb{K})$ as an space of operators or a dual space.

1.1. Terminology and Background

Here will be working with Banach spaces over the real or complex field $\mathbb{K}$. The unit ball and unit sphere of a Banach space X are denoted by B_X and S_X, respectively. The symbols $\mathcal{L}(X, Y)$ and $\mathcal{B}(X \times Y; \mathbb{K})$ stand for the (bounded) linear operators and bilinear forms, respectively. When $Y = \mathbb{K}$, $\mathcal{L}(X, Y)$ becomes simply X^*, the topological dual space of X. We say that $T \in \mathcal{L}(X, Y)$ attains its norm if $\|T(x_0)\| = \|T\|$ for some $x_0 \in S_X$ and we say that $B \in \mathcal{B}(X \times Y; \mathbb{K})$ attains its norm if $|B(x_0, y_0)| = \|B\|$ for some $(x_0, y_0) \in S_X \times S_Y$.

The norm of X is said to be strongly subdifferentiable (SSD, for short) at the point $x \in X$ if the one-side limit

$$\lim_{t \to 0^+} \frac{\|x + th\| - \|x\|}{t}$$

exists uniformly for $h \in B_X$. Let us notice that the norm of X is Fréchet differentiable at x if and only if it is Gâteux differentiable and SSD at x. When we say that X is SSD we mean that the norm of X is SSD at every $x \in S_X$.

The projective tensor product of two Banach spaces X and Y is the completion of $X \otimes Y$ endowed with the norm given by

$$\|z\|_\pi = \inf \left\{ \sum_{n=1}^{\infty} \|x_n\| \|y_n\| : \sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty, z = \sum_{n=1}^{\infty} x_n \otimes y_n \right\}.$$

We denote the projective tensor product of X and Y endowed with the above norm by $X \widehat{\otimes}_\pi Y$. It is well-known (and we will be using these facts with no explicit mention throughout the paper) that $\|x \otimes y\| = \|x\| \|y\|$ for every $x \in X$ and $y \in Y$, and that the closed unit ball of $X \widehat{\otimes}_\pi Y$ is the closed convex hull of $B_X \otimes B_Y = \{x \otimes y : x \in B_X, y \in B_Y\}$. Moreover, we have that $(X \widehat{\otimes}_\pi Y)^* = \mathcal{B}(X \times Y; \mathbb{K})$ under the action of a bounded bilinear form B as a bounded linear functional on $X \widehat{\otimes}_\pi Y$ given by

$$\left\langle B, \sum_{n=1}^{\infty} x_n \otimes y_n \right\rangle = \sum_{n=1}^{\infty} B(x_n, y_n)$$

and $(X\widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*)$ under the action of a bounded linear operator T as a bounded linear functional on $X\widehat{\otimes}_\pi Y$ given by

$$\left\langle T, \sum_{n=1}^{\infty} x_n \otimes y_n \right\rangle = \sum_{n=1}^{\infty} \langle y_n, T(x_n)\rangle.$$

Analogously, we have that $(X\widehat{\otimes}_\pi Y)^* = \mathcal{L}(Y, X^*)$.

Recall that a Banach space is said to have the approximation property (AP, in short) if for every compact subset K of X and every $\varepsilon > 0$, there exists a finite-rank operator $T : X \longrightarrow X$ such that $\|T(x) - x\| \leq \varepsilon$ for every $x \in K$. We also make use of the so-called *pointwise bounded compact approximation property* (pointwise-BCAP, for short) defined recently in [10]: we say that a pair of Banach space (X, Y) has the *pointwise-BCAP* if for every operator $T \in \mathcal{L}(X, Y)$, there exists a constant $\lambda_T \geq 1$ such that $T \in \lambda_T \overline{B_{\mathcal{K}(X,Y)}}^{\tau_c}$, where τ_c denotes the topology of compact convergence in $\mathcal{L}(X, Y)$. We refer the reader to [15] for background on the beautiful tensor products of Banach spaces and approximation properties theories.

Finally, let us recall that a Banach space X satisfies the Kadec–Klee property if the weak and the norm topologies coincide in the unit sphere of X.

2. Proofs of Theorems A and B

We start this section by giving the proof of Theorem A.

Proof of Theorem A. (i)⇒(ii). Suppose that (X, Y) has the $\mathbf{L}_{o,o}$ for compact operators. Let $\varepsilon > 0$ and $x^* \in S_{X^*}$ be given, and let us prove that $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for linear functionals. Define $T : X \longrightarrow Y$ by $T(x) := x^*(x)y_0$ for some $y_0 \in S_Y$. Then, $\|T\| = \|x^*\| = 1$ and T is compact. By hypothesis, there is $\eta(\varepsilon, T) > 0$ witnessing the definition of the property $\mathbf{L}_{o,o}$. Let us set $\eta(\varepsilon, x^*) := \eta(\varepsilon, T) > 0$. Let $x_0 \in S_X$ be such that $|x^*(x_0)| > 1 - \eta(\varepsilon, T)$. Then, $\|T(x_0)\| = \|x^*(x_0)y_0\| = |x^*(x_0)| > 1 - \eta(\varepsilon, T)$ and by the assumption there is $x_1 \in S_X$ such that $\|T(x_1)\| = 1$ and $\|x_1 - x_0\| < \varepsilon$. Then, $|x^*(x_1)| = \|T(x_1)\| = 1$ and $\|x_1 - x_0\| < \varepsilon$, that is, $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for linear functionals.
(ii) ⇒ (i). Suppose that $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for linear functionals. By contradiction, suppose that there exist $\varepsilon_0 > 0$, $T \in \mathcal{K}(X, Y)$ with $\|T\| = 1$, and $(x_n) \subseteq S_X$ such that

$$1 \geq \|T(x_n)\| \geq 1 - \frac{1}{n} \tag{2.1}$$

but satisfying that $\operatorname{dist}(x_n, \mathrm{NA}(T)) \geq \varepsilon_0$, where $\mathrm{NA}(T) = \{x \in S_X : \|T(x)\| = \|T\|\}$. Since X is reflexive and $(x_n)_{n=1}^{\infty}$ is bounded, we may (and we do) assume that $x_n \xrightarrow{w} x_0$ for some $x_0 \in B_X$. Since T is a compact operator, we have that $T(x_n) \xrightarrow{\|\cdot\|} T(x_0)$. By (2.1), we have that $\|T(x_0)\| = 1$ and, in particular, $x_0 \in S_X$. Let us take $y_0^* \in S_{Y^*}$ to be such that $y^*(T(x_0)) = \|T(x_0)\| = 1$. Consider $x_0^* := T^* y_0^* \in S_{X^*}$. Then $x_0^*(x_0) = T^* y_0^*(x_0) = y^*(T(x_0)) = 1$. Since

$x_n \xrightarrow{w} x_0$, we have that $x_0^*(x_n) \longrightarrow x_0^*(x_0) = 1$ as $n \to \infty$. Since $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for linear functionals, there is $(x_n') \subseteq S_X$ such that $x_0^*(x_n') = 1$ for every $n \in \mathbb{N}$ and $\|x_n' - x_n\| \to 0$. This shows that

$$1 = x_0^*(x_n') = T^* y_0^*(x_n') = y_0^*(T(x_n')),$$

that is, $1 = y_0^*(T(x_n')) \leq \|T(x_n')\| \leq \|T\| = 1$, so, $\|T(x_n')\| = 1$ and then $x_n' \in \mathrm{NA}(B)$. The convergence $\|x_n' - x_n\| \to 0$ yields the desired contradiction.

Remark 2.1. [17, Theorem 2.3] says that X is strictly convex and the pair (X, Y) has the $\mathbf{L}_{o,o}$ for compact operators if and only if X^* is Fréchet differentiable. Let us notice that X^* is Fréchet differentiable if and only if X is strictly convex and $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ (see [9, Theorem 2.5]). Therefore, Theorem A generalizes [17, Theorem 2.3] as we no longer need strict convexity on X. On the other hand, [16, Theorem 2.12] says that if X is a reflexive space which satisfies the Kadec–Klee property, then (X, Y) has the $\mathbf{L}_{o,o}$ for compact operators for every Y. This is also covered by our Theorem A since whenever X is a reflexive space satisfying the Kadec–Klee property, the pair $(X, \mathbb{K})$ satisfies the $\mathbf{L}_{o,o}$ (see [9, Propositions 2.2 and 2.6]).

We now present the proof of Theorem B.

Proof of Theorem B. (a) $\Rightarrow$ (b). If we assume (a), then we have that X and Y are both reflexive, and that every operator from X into Y^* is compact. So, $\mathcal{L}(X, Y^*) = (X \widehat{\otimes}_\pi Y)^*$ is reflexive by Ryan [15, Theorem 4.19]. Then, $X \widehat{\otimes}_\pi Y$ is reflexive.
(b) $\Rightarrow$ (a). If $X \widehat{\otimes}_\pi Y$ is reflexive, then so is $\mathcal{L}(X, Y^*) = (X \widehat{\otimes}_\pi Y)^*$. Since (X, Y^*) has the pointwise-BCAP (or either X or Y has the AP), we have that $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$ (see [10, the diagram on pg.4] for the pointwise-BCAP assumption and [15, Theorem 4.21] for the AP assumption).
(a) $\Rightarrow$ (c). Suppose that $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$ and assume that both $(X, \mathbb{K})$ and $(Y, \mathbb{K})$ satisfy the $\mathbf{L}_{o,o}$ for linear functionals. By contradiction, let us assume that $(X \times Y; \mathbb{K})$ fails to have the $\mathbf{L}_{o,o}$ for bilinear forms. Then, there exist $\varepsilon_0 > 0$, $B \in \mathcal{B}(X \times Y; \mathbb{K})$ with $\|B\| = 1$, and $(x_n, y_n)_{n=1}^\infty \subseteq S_X \times S_Y$ such that

$$1 \geq B(x_n, y_n) \geq 1 - \frac{1}{n} \tag{2.2}$$

and whenever $(u, v) \subset S_X \times S_Y$ is such that $B(u, v) = 1$, we have that

$$\|u - x_n\| \geq \varepsilon_0 \quad \text{or} \quad \|v - y_n\| \geq \varepsilon_0. \tag{2.3}$$

Since X and Y are reflexive and both $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are bounded, we may assume (and we do) that $x_n \xrightarrow{w} x_0$ and $y_n \xrightarrow{w} y_0$ for some $x_0 \in B_X$ and $y_0 \in B_Y$. Let $T \in \mathcal{L}(X, Y^*) = (X \widehat{\otimes}_\pi Y)^*$ be arbitrary. By assumption, we have that $T \in \mathcal{K}(X, Y^*)$. Since $x_n \xrightarrow{w} x_0$ and T is compact, we have that $T(x_n) \xrightarrow{\|\cdot\|} T(x_0)$ and then since

$$|T(x_n)(y_n) - T(x_0)(y_0)| \leq \|T(x_n) - T(x_0)\| \|y_n\| + |T(x_0)(y_n) - T(x_0)(y_0)|,$$

we have that

$$T(x_n)(y_n) \longrightarrow T(x_0)(y_0) \tag{2.4}$$

as $n \to \infty$ for every $T \in (X\widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$. This means that $x_n \otimes y_n \xrightarrow{w} x_0 \otimes y_0$. In particular, since $B \in \mathcal{B}(X \times Y; \mathbb{R}) = (X\widehat{\otimes}_\pi Y)^*$, we have that $B(x_n, y_n) \longrightarrow B(x_0, y_0)$ and by (2.2), $B(x_0, y_0) = 1$. In particular, $x_0 \in S_X$ and $y_0 \in S_Y$.

Let us consider $T_B \in \mathcal{L}(X, Y^*)$ and $S_B \in \mathcal{L}(Y, X^*)$ to be the associated linear operators to the bilinear form B. We have that

$$T_B^*(y_0)(x_0) = T_B(x_0)(y_0) = B(x_0, y_0) = 1,$$

which shows that $T_B^*(y_0) \in S_{X^*}$. Analogously, $S_B^*(x_0)(y_0) = 1$ and $S_B^*(x_0) \in S_{Y^*}$.

Claim: We have that

($\star$) $T_B^*(y_0)(x_n) \longrightarrow 1$ as $n \to \infty$.
($\star\star$) $S_B^*(x_0)(y_n) \longrightarrow 1$ as $n \to \infty$.

We only prove ($\star$) since ($\star\star$) is analogous. As T_B^* is a compact operator and $y_n \xrightarrow{w} y_0$, we have that $T_B^*(y_n) \xrightarrow{\|\cdot\|} T_B^*(y_0)$. At the same time, by (2.4) we have that

$$T_B^*(y_n)(x_n) = T_B(x_n)(y_n) \longrightarrow T_B(x_0)(y_0) = 1$$

as $n \to \infty$. Therefore,

$$\begin{aligned} |T_B^*(y_n)(x_n) - T_B^*(y_0)(x_n)| &= |(T_B^*(y_n) - T_B^*(y_0)(x_n))| \\ &\leq \|T_B^*(y_n) - T_B^*(y_0)\| \longrightarrow 0 \end{aligned}$$

and so $T_B^*(y_0)(x_n) \longrightarrow 1$ as $n \to \infty$.

Let us prove that $\|x_n - x\| \longrightarrow \infty$ as $n \to \infty$. Assume first that X satisfies the Kadec–Klee property. Since $x_0 \in S_X$ and $x_n \xrightarrow{w} x_0$, we have that $\|x_n - x_0\| \to 0$ as $n \to \infty$. We prove that the same holds if X is taken to be strictly convex. Indeed, by using ($\star$), we have that $T_B^*(y_0)(x_n) \longrightarrow 1$ as $n \to \infty$. Since $(X, \mathbb{K})$ satisfies the $\mathbf{L}_{o,o}$ and X is strictly convex, we have that X^* is Fréchet differentiable (see [9, Theorem 2.5.(b)] and then, by the Šmulyan lemma, we have that $\|x_n - x_0\| \longrightarrow 0$ as $n \to \infty$ as desired.

To conclude the proof of this implication, let us set $y_0^* := S_B^*(x_0) \in S_{Y^*}$. Then, $y_0^*(y_0) = 1$ and $y_0^*(y_n) \longrightarrow 1$ as $n \to \infty$ by ($\star\star$). Since $(Y, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for linear functionals, there is $(y_n') \subseteq S_Y$ such that $y_0^*(y_n') = 1$ and $\|y_n' - y_n\| \longrightarrow 0$ as $n \to \infty$. This means that

$$1 = y_0^*(y_n') = S_B^*(x_0)(y_n') = B(x_0, y_n').$$

Since $\|x_n - x_0\| \longrightarrow 0$ and $\|y_n' - y_n\| \longrightarrow 0$ as $n \to \infty$, we get a contradiction with (2.3).

(c) $\Rightarrow$ (d). Suppose that $(X \times Y; \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for bilinear forms. To prove that $(X\widehat{\otimes}_\pi Y, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for linear functionals, let us fix $B \in (X\widehat{\otimes}_\pi Y)^* =$

$\mathcal{B}(X \times Y; \mathbb{K})$ with $\|B\| = 1$. By hypothesis, given $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon, B) > 0$. We use similar arguments from [6, Proposition 4.3].

Let $z \in S_{X \widehat{\otimes}_\pi Y}$ be such that

$$\operatorname{Re}\langle B, z\rangle > 1 - \frac{\eta(\varepsilon, B)^2}{2}.$$

By Ryan [15, Proposition 2.8] we can find $(x_n)_{n=1}^\infty \subseteq S_X$, $(y_n)_{n=1}^\infty \subseteq S_Y$, and $(\lambda_n)_{n=1}^\infty \subseteq \mathbb{R}^+$ such that $z = \sum_{n=1}^\infty \lambda_n x_n \otimes y_n$ and such that

$$\sum_{n=1}^\infty \lambda_n < 1 + \frac{\eta(\varepsilon, B)^2}{2}.$$

Consider the sets

$$I := \{n \in \mathbb{N} : \operatorname{Re} B(x_n, y_n) > 1 - \eta(\varepsilon, B)\}$$

and $J := I^c$. Hence, we have that

$$\begin{aligned} 1 - \frac{\eta(\varepsilon, B)^2}{2} < \operatorname{Re}\langle B, z\rangle &= \sum_{n=1}^\infty \lambda_n \operatorname{Re} B(x_n, y_n) \\ &= \sum_{n\in I} \lambda_n \operatorname{Re} B(x_n, y_n) + \sum_{n\in J} \lambda_n \operatorname{Re} B(x_n, y_n) \\ &\leq \sum_{n\in I} \lambda_n + (1 - \eta(\varepsilon, B)) \sum_{n\in J} \lambda_n \\ &= \sum_{n=1}^\infty \lambda_n - \eta(\varepsilon, B) \sum_{n\in J} \lambda_n \\ &< 1 + \frac{\eta(\varepsilon, B)^2}{2} - \eta(\varepsilon, B) \sum_{n\in J} \lambda_n, \end{aligned}$$

that is, $\eta(\varepsilon, B) \sum_{n\in J} \lambda_n < \eta(\varepsilon, B)^2$ and then

$$\sum_{n\in J} \lambda_n < \eta(\varepsilon, B). \tag{2.5}$$

Notice that for each $n \in I$, we have that $\operatorname{Re} B(x_n, y_n) > 1 - \eta(\varepsilon, B)$. Then, by hypothesis, there exists $(x'_n, y'_n)_{n\in I} \subseteq S_X \times S_Y$ such that

$$|B(x'_n, y'_n)| = 1, \quad \|x'_n - x_n\| < \varepsilon, \quad \text{and} \quad \|y'_n - y_n\| < \varepsilon.$$

Let us write $B(x'_n, y'_n) = e^{i\theta_n}$ with some $\theta_n \in \mathbb{R}$ for every $n \in I$. Let us notice that, for each $n \in I$, we have

$$\begin{aligned}
1 - \eta(\varepsilon, B) < \operatorname{Re} B(x_n, y_n) &= \operatorname{Re} B(x_n - x'_n + x'_n, y_n) \\
&= \operatorname{Re} B(x_n - x'_n, y_n) + \operatorname{Re} B(x'_n, y_n) \\
&\leq \|x_n - x'_n\| + \operatorname{Re} B(x'_n, y_n - y'_n + y'_n) \\
&= \|x_n - x_n\|' + \operatorname{Re} B(x'_n, y_n - y'_n) + \operatorname{Re} B(x'_n, y'_n) \\
&\leq \|x_n - x'_n\| + \|y_n - y'_n\| + \operatorname{Re} B(x'_n, y'_n) \\
&< 2\varepsilon + \operatorname{Re} B(x'_n, y'_n)
\end{aligned}$$

that is, $1 - \operatorname{Re} B(x'_n, y'_n) < 2\varepsilon + \eta(\varepsilon, B)$ for each $n \in I$. Now, since $1 = |B(x'_n, y'_n)|^2 = \operatorname{Re} B(x'_n, y'_n)^2 + \operatorname{Im} B(x'_n, y'_n)^2$, we have that, for every $n \in I$,

$$1 > 1 - 2\varepsilon - \eta(\varepsilon, B) + \operatorname{Im} B(x'_n, y'_n)^2$$

which implies that $\operatorname{Im} B(x'_n, y'_n)^2 < 2\varepsilon + \eta(\varepsilon, B)$. Thus, if $n \in I$, we have that

$$\begin{aligned}
|1 - e^{i\theta}| &= |1 - B(x'_n, y'_n)| \\
&= \sqrt{(1 - \operatorname{Re} B(x'_n, y'_n))^2 + \operatorname{Im} B(x'_n, y'_n)^2} \\
&< \sqrt{(2\varepsilon + \eta(\varepsilon, B))^2 + 2\varepsilon + \eta(\varepsilon, B)} \\
&< \sqrt{4\varepsilon + 2\eta(\varepsilon, B)}.
\end{aligned}$$

Now, let us define

$$z' := \sum_{n \in I} \lambda_n e^{-i\theta_n} x'_n \otimes y'_n \in X \widehat{\otimes}_\pi Y.$$

We have that

$$\langle B, z' \rangle = \sum_{n \in I} \lambda_n e^{-i\theta_n} B(x'_n, y'_n) = \sum_{n \in I} \lambda_n = \|z'\|_\pi.$$

On the other hand, since

$$\begin{aligned}
\|x'_n \otimes y'_n - x_n \otimes y_n\| &\leq \|x'_n \otimes y'_n - x'_n \otimes y_n\| + \|x'_n \otimes y_n - x_n \otimes y_n\| \\
&\leq \|y'_n - y_n\| + \|x'_n - x_n\| \\
&< 2\varepsilon
\end{aligned}$$

we have that (here we use (2.5) and the fact that $|1-e^{i\theta}| < \sqrt{4\varepsilon + 2\eta(\varepsilon, B)}$)

$$\begin{aligned}
\|z'-z\|_\pi &= \left\| \sum_{n\in I} \lambda_n e^{-i\theta_n} x_n' \otimes y_n' - \sum_{n\in I} \lambda_n x_n \otimes y_n - \sum_{n\in J} \lambda_n x_n \otimes y_n \right\| \\
&\leq \left\| \sum_{n\in I} \lambda_n (e^{-i\theta_n} x_n' \otimes y_n' - x_n \otimes y_n) \right\| + \sum_{n\in J} \lambda_n \\
&\leq \sum_{n\in I} \lambda_n |e^{-i\theta_n} - 1| + \left\| \sum_{n\in I} \lambda_n (x_n' \otimes y_n' - x_n \otimes y_n) \right\| + \sum_{n\in J} \lambda_n \\
&< \left(\sqrt{4\varepsilon + 2\eta(\varepsilon, B)} \right) \sum_{n\in I} \lambda_n + 2\varepsilon \sum_{n\in I} \lambda_n + \sum_{n\in J} \lambda_n \\
&< \left(\sqrt{4\varepsilon + 2\eta(\varepsilon, B)} \right) \left(1 + \frac{\eta(\varepsilon, B)^2}{2} \right) + 2\varepsilon \left(1 + \frac{\eta(\varepsilon, B)^2}{2} \right) + \eta(\varepsilon, B) \\
&= \left(\sqrt{4\varepsilon + 2\eta(\varepsilon, B)} + 2\varepsilon \right) \left(1 + \frac{\eta(\varepsilon, B)^2}{2} \right) + \eta(\varepsilon, B).
\end{aligned}$$

In particular, we have that $\|z'\| > 0$ and we may define $z'' := \frac{z'}{\|z'\|} \in S_{X\widehat{\otimes}_\pi Y}$. Notice that, since

$$\|z''-z'\| = \left\| \frac{z'}{\|z'\|} - z' \right\| = |1 - \|z'\|| \leq \|z - z'\|$$

we have that

$$\begin{aligned}
\|z''-z\| &\leq \|z''-z'\| + \|z'-z\| \\
&< 2\|z'-z\| \\
&< 2\left(\sqrt{4\varepsilon + 2\eta(\varepsilon, B)} + 2\varepsilon \right) \left(1 + \frac{\eta(\varepsilon, B)^2}{2} \right) + 2\eta(\varepsilon, B).
\end{aligned}$$

Finally, notice that

$$\langle B, z'' \rangle = \left\langle B, \frac{z'}{\|z'\|} \right\rangle = 1.$$

This shows that $(X\widehat{\otimes}_\pi Y, \mathbb{R})$ satisfies the $\mathbf{L}_{o,o}$ for linear functionals.
(d) ⇒ (b). Suppose that $(X\widehat{\otimes}_\pi Y, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for linear functionals. By Dantas et al. [9, Theorem 2.3], $X\widehat{\otimes}_\pi Y$ is reflexive and $(X\widehat{\otimes}_\pi Y)^*$ is SSD. Since X^*, Y^* are closed subspaces of $(X\widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*) = \mathcal{L}(Y, X^*)$, we have that both X^*, Y^* are SSD [11]. Therefore, both $(X, \mathbb{K})$ and $(Y, \mathbb{K})$ satisfy the $\mathbf{L}_{o,o}$ for linear functionals.

Remark 2.2. In Theorem B, the assumptions that X or Y has the AP or that the pair (X, Y^*) satisfies the pointwise-BCAP are only used to get that (b) ⇒ (a); on the other hand, implications (a) ⇒ (c) ⇒ (d) ⇒ (b) are valid without these assumptions (notice that (c) ⇒ (d) ⇒ (b) are valid for general Banach spaces while in (a) ⇒ (c) we assume that X is strictly convex or that satisfies the Kadec–Klee property).

Remark 2.3. The approximation properties in Theorem B are technical assumptions we need in order to guarantee that the equality $\mathcal{L}(X,Y^*) = \mathcal{K}(X,Y^*)$ holds true. In fact, this is what we use to get (b) $\Rightarrow$ (a). Let us notice also that, as far as we know, it is an open problem whether the fact $X\widehat{\otimes}_\pi Y$ is reflexive implies that $\mathcal{L}(X,Y^*) = \mathcal{K}(X,Y^*)$ is true for general reflexive spaces X and Y. A positive answer for this problem would provide us a general characterization for Theorem B.

As an immediate consequence of Theorem B, we have the following corollary. Notice that item (a) below was proved also in [8, Theorem 2.7.(b)].

Corollary 2.1. *Let $1 < p, q < \infty$ and let q' be the conjugate index of q.*

(a) $(\ell_p \times \ell_q; \mathbb{K})$ *satisfies the* $\mathbf{L}_{o,o}$ *for bilinear forms if and only if* $p > q'$.

(b) $(L_p(\mu), L_q(\nu); \mathbb{K})$ *fails the* $\mathbf{L}_{o,o}$ *for bilinear forms for not purely atomic measures* μ, ν.

Proof. Under the assumption of (a), we have that the projective tensor product $\ell_p\widehat{\otimes}_\pi\ell_q$ is reflexive (see [15, Corollary 4.24]). For (b), since $L_p(\mu)\widehat{\otimes}_\pi L_q(\nu)$ contains complemented isomorphic copies of ℓ_1 for every p, q, it is never reflexive (see [15, Theorem 4.21 and Corollary 4.22]). Therefore, both items follow immediately by applying Theorem B. $\square$

Let us conclude the paper by commenting on the $\mathbf{L}_{o,o}$ for different classes of functions. Let X and Y be Banach spaces. In $\mathcal{B}(X \times Y; \mathbb{K})$, as we have seen in Theorem B, one can consider:

(A) the $\mathbf{L}_{o,o}$ for linear functionals seeing $\mathcal{B}(X \times Y; \mathbb{K})$ as $(X\widehat{\otimes}_\pi Y)^*$,
(B) the $\mathbf{L}_{o,o}$ for operators seeing $\mathcal{B}(X \times Y; \mathbb{K})$ as $\mathcal{L}(X, Y^*)$, and, of course,
(C) the $\mathbf{L}_{o,o}$ for bilinear forms.

We have the following relation between properties (A), (B), and (C):

- *General implications.* Clearly, we have that (C) $\Rightarrow$ (B) by considering the associated bilinear for $B_T \in \mathcal{B}(X \times Y; \mathbb{K})$ of a given operator $T \in \mathcal{L}(X, Y^*)$. Also, by our Theorem B (implication (c) $\Rightarrow$ (d)) and noticing that, for this implication, we do not need any assumption on X besides reflexivity (not even approximation property assumptions), we also have that (C) $\Rightarrow$ (A).
- *Not true implications.* (B) does not imply (A) or (C) in general. Indeed, by [1, Theorem 2.4.10], for every $1 < p < \infty$, we have that $\mathcal{L}(c_0, \ell_p) = \mathcal{K}(c_0, \ell_p) = \mathcal{L}(\ell_{p'}, \ell_1)$, where p' is the conjugate index of p. We have that $(\ell_{p'}, \ell_1)$ has the $\mathbf{L}_{o,o}$ for operators by Theorem A (since $(\ell_{p'}, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for linear functionals) but neither $(\ell_{p'} \times c_0; \mathbb{K})$ nor $(\ell_{p'}\widehat{\otimes}_\pi c_0; \mathbb{K})$ can have the $\mathbf{L}_{o,o}$ for bilinear forms and for linear functionals, respectively, since c_0 is not reflexive.
- *Implications with extra assumptions.* Assume that either X or Y has the AP, or that the pair (X, Y^*) has the pointwise-BCAP. In this case, implication (A) $\Rightarrow$ (B) holds. Indeed, if $(X\widehat{\otimes}_\pi Y; \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for linear

functionals, then $X\widehat{\otimes}_\pi Y$ must be reflexive and, by one of our assumptions, every operator from X into Y^* is compact and by Theorem B, the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for linear functionals. By Theorem A, the pair (X, Y^*) has the $\mathbf{L}_{o,o}$ for operators. Finally, if X or Y has the AP (or (X, Y^*) has the pointwise-BCAP) and X or Y is stricly convex, then (A)⇒(C).

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Declaration
Conflict of interests All authors declare that they have no conflict of interest.

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