BLOCH FUNCTIONS ON THE UNIT BALL OF A BANACH SPACE

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Abstract. The space of Bloch functions on bounded symmetric domains is extended by considering Bloch functions f on the unit ball B_E of finite and infinite dimensional complex Banach spaces in two different ways: by extending the classical Bloch space considering the boundness of $(1-\|x\|^2)\|f'(x)\|$ on B_E and by preserving the invariance of the correspondiing seminorm when we compose with automorphisms φ of B_E. We study the connection between these spaces proving that they are different in general and prove that all bounded analytic functions on B_E are Bloch functions in both ways.

Introduction

The classical Bloch space $\mathcal{B}$ of analytic functions on the open unit disk $\mathbf{D}$ of $\mathbb{C}$ plays an important role in geometric function theory and it has been studied by many authors. K. T. Hahn and R. M. Timoney extended the notion of Bloch function by considering bounded homogeneous domains in $\mathbb{C}^n$, such as the unit ball B_n and the polydisk $\mathbf{D}^n$ (see [12, 18, 19]). O. Blasco, P. Galindo and A. Miralles extended the notion to the infinite dimensional setting by considering Bloch functions on the unit ball of an infinite dimensional Hilbert space (see [4, 5, 6]) and C. Chu, H. Hamada, T. Honda and G. Kohr considered Bloch functions on bounded symmetric domains which may be also infinite dimensional (see [8]).

In this article, we will deal with a finite or infinite dimensional complex Banach space E and we will consider two possible extensions of the classical Bloch space. The first one extends the classical Bloch space by considering *the natural Bloch space* $\mathcal{B}_{nat}(B_E)$ of holomorphic functions f on B_E such that $\|f\|_{nat} = \sup_{x\in B_E}(1-\|x\|^2)\|f'(x)\| < \infty$. The second one extends the space defined in [8] by considering *the invariant Bloch space* $\mathcal{B}_{inv}(B_E)$ of holomorphic functions f on the unit ball B_E of a complex Banach space E such that $\|f\|_{inv} = \sup_{\varphi\in Aut(B_E)} \|(f\circ\varphi)'(0)\| < \infty$. The only known case where $\|\cdot\|_{nat}$ and $\|\cdot\|_{inv}$ are equivalent seminorms and $\mathcal{B}_{nat}(B_E) = \mathcal{B}_{inv}(B_E)$ is when E is a finite or infinite dimensional Hilbert space (see [4, 18]). We will prove that there are spaces E satisfying $\mathcal{B}_{inv}(B_E) \subsetneq \mathcal{B}_{nat}(B_E)$ and other ones such that $\mathcal{B}_{nat}(B_E) \subsetneq \mathcal{B}_{inv}(B_E)$. Finally we will give a Schwarz-type lemma for complex Banach spaces and will prove that the space of bounded analytic functions on B_E given by $H^\infty(B_E)$ is strictly contained in both $\mathcal{B}_{inv}(B_E)$ and $\mathcal{B}_{nat}(B_E)$.

1. Background

1.1. **The classical Bloch space.** The classical *Bloch space* $\mathcal{B}$ (see [15]) is the space of analytic functions $f:\mathbf{D}\longrightarrow\mathbb{C}$ satisfying:

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbf{D}}(1-|z|^2)|f'(z)| < \infty$$

endowed with the norm:

$$\|f\|_{Bloch} = |f(0)| + \|f\|_{\mathcal{B}} < \infty$$

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so that $(\mathcal{B}, \|\cdot\|_{Bloch})$ becomes a Banach space. The seminorm $\|\cdot\|_{\mathcal{B}}$ is invariant by automorphisms of $\mathbf{D}$, that is, $\|f\circ\varphi\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$ for any $f\in\mathcal{B}$ and $\varphi\in Aut(\mathbf{D})$. Recall that:

$$H^\infty = \{f : \mathbf{D}\to\mathbb{C} : f \text{ is holomorphic and bounded }\}$$

is a Banach space endowed with the sup-norm $\|f\|_\infty = \sup_{z\in\mathbf{D}}|f(z)|$. It is well-known (see for instance [21]) that:

Proposition 1.1. *H^∞ is properly contained in $\mathcal{B}$ and $\|f\|_{\mathcal{B}} \le \|f\|_\infty$ for any $f\in H^\infty$.*

For further information and references about the classical Bloch space $\mathcal{B}$, the reader is referred to [3, 21].

1.2. **Holomorphic functions on B_E and the pseudohyperbolic distance.** We will denote by E, F complex Banach spaces. Given $x\in E$ and $r>0$, we will denote by $B(x,r)$ the ball given by $y\in E$ such that $\|y-x\|<r$. We will denote by B_E the open unit ball $B(0,1)$ of E. A function $f : B_E\to F$ is said to be holomorphic if it is Fréchet differentiable at every $x\in B_E$ or, equivalently, if $f(x)=\sum_{n=1}^\infty P_n(x)$ for all $x\in B_E$, where P_n is an $n-$homogeneous continuous polynomial, that is, the restriction to the diagonal of a continuous $n-$linear form on the n-fold space $E\times\cdots\times E$ into F. We will denote by $H(B_E,F)$ the space of holomorphic functions from B_E into F. If $F=\mathbb{C}$, we just denote the space by $H(B_E)$. For further information on holomorphic functions on complex Banach spaces, see [9] or [14].

The space $H^\infty(B_E)$ is given by $\{f : B_E\to\mathbb{C} : f \textit{ is holomorphic and bounded }\}$ and it becomes a Banach space when endowed with the sup-norm $\|f\|_\infty = \sup_{x\in B_E}|f(x)|$.

1.3. **The automorphisms on B_E.** We will denote by $Aut(B_E)$ all the automorphisms of B_E, that is, all the bijective biholomorphic maps $\varphi : B_E\to B_E$. It is well-known that if B_E is a bounded symmetric domain (including the unit ball of a Hilbert space and the finite or infinite dimensional polydisc) then B_E is homogeneous, that is, it acts transitively on B_E. Hence, if B_E is a bounded symmetric domain, then $\{\varphi(0) : \varphi\in \mathrm{Aut}(B_E)\} = B_E$. Kaup and Upmeier (see [13]) proved that:

Proposition 1.2. *Let E be a complex Banach space and let B_E be its open unit ball. Then there exists a closed subspace V of E such that $V\cap B_E$ is a bounded symmetric domain in V and $V\cap B_E = \{\varphi(0) : \varphi\in Aut(B_E)\}$.*

1.4. **The pseudohyperbolic and the hyperbolic distance.** If E is a complex Banach space, the pseudohyperbolic distance ρ_E for $x,y\in B_E$ is defined by:

$$\rho_E(x,y) = \sup\{|f(x)| : \ f\in H^\infty(B_E),\ \|f\|_\infty<1,\ f(y)=0\}. \tag{1.1}$$

For $z,w\in\mathbf{D}$, we will denote the pseudohyperbolic distance by ρ and is given by:

$$\rho(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right|.$$

We recall the following well-known results:

Proposition 1.3. *Let E be a complex Banach space and $x,y\in B_E$. Then:*

a) $\rho(f(x),f(y)) \le \rho_E(x,y)$ *for any $f\in H^\infty(B_E)$ such that $\|f\|_\infty<1$.*
b) $\rho_E(\varphi(x),\varphi(y)) \le \rho_E(x,y)$ *for all holomorphic mappings $\varphi : B_E\to B_E$. The equality is satisfied if and only if $\varphi\in Aut(B_E)$.*

The hyperbolic distance β_E for $x,y\in B_E$ is defined by:

$$\beta_E(x,y) = \frac{1}{2}\log\left(\frac{1+\rho_E(x,y)}{1-\rho_E(x,y)}\right).$$

When B_E is a bounded symmetric domain, it was proved that any $f \in \mathcal{B}_{inv}(B_E)$ is Lipschitz for the hyperbolic distance (see [5] and [8]), that is, there exists $M > 0$ such that for any $x, y \in B_E$:

$$|f(x) - f(y)| \leq M\beta_E(x, y).$$

1.5. **The space of Bloch functions on bounded symmetric domains.** The study of Bloch functions on bounded symmetric domains of $\mathbb{C}^n$ was extended by Hahn [12] and Timoney by using the Bergman metric (see [18, 19]). In particular, this study includes the unit euclidean ball B_n and the polydisc $\mathbf{D}^n$. The study of Bloch functions on bounded symmetric domains of infinite dimensional Banach spaces was introduced by Blasco, Galindo and Miralles for the Hilbert case (see [4]) and by Chu, Hamada, Honda and Kohr for general bounded symmetric domains by means of the Kobayashi metric (see [8]). If we consider these domains as the unit ball B_E of a JB^*-triple E, the corresponding Bloch space is the set of holomorphic functions on B_E which satisfy that $\sup_{x\in B_E} Q_f(z) < \infty$. In this case:

$$Q_f(z) = \sup_{x\in X\setminus\{0\}} \frac{|f'(z)(x)|}{k(z, x)}$$

and $k(z, x)$ is the infinitesimal Kobayashi metric for B_E (see [8] for more details). It was proved that:

$$\sup_{x\in B_E} Q_f(x) = \sup_{\varphi\in Aut(B_E)} \|(f \circ \varphi)'(0)\|$$

so the Bloch space on a bounded symmetric domain can be described in terms of the automorphisms of B_E. The authors also proved that $H^\infty(B_E)$ is strictly contained in the Bloch space.

2. The space of Bloch functions on B_E

2.1. **Two different definitions.** Let E be a complex Banach space and consider its open unit ball denoted by B_E. Bearing in mind the definition of the classical Bloch space taking the supremum of $(1 - |z|^2)|f'(z)|$ for $z \in \mathbb{C}, |z| < 1$, we can extend it for $f \in H(B_E)$ by defining what we call *the natural Bloch seminorm*:

$$\|f\|_{nat} = \sup_{x\in B_E} (1 - \|x\|^2)\|f'(x)\|$$

where $f'(x) \in E^*$ denotes the derivative of f at the point x. The space $\mathcal{B}_{nat}(B_E)$ is given by

$$\mathcal{B}_{nat}(B_E) = \{f \in H(B_E) : \|f\|_{nat} < \infty\}.$$

It is clear that $\|\cdot\|_{nat}$ is a seminorm for $\mathcal{B}_{nat}(B_E)$ and this space can be endowed with the norm $\|f\|_{nat-Bloch} = |f(0)| + \|f\|_{nat}$. Hence $(\mathcal{B}_{nat}(B_E), \|\cdot\|_{nat-Bloch})$ is a Banach space.

On the other hand, bearing in mind the definition of the Bloch space in [8] and to preserve the invariance of the corresponding seminorm when composing with an automorphism, we define for $f \in H(B_E)$, the *invariant Bloch semi-norm* by:

$$\|f\|_{inv} = \sup_{\varphi\in Aut(B_E)} \|(f \circ \varphi)'(0)\|$$

and the space $\hat{\mathcal{B}}_{inv}(B_E)$ will be given by:

$$\hat{\mathcal{B}}_{inv}(B_E) = \{f \in H(B_E) : \|f\|_{inv} < \infty\}.$$

Let $\|f\|_{inv-Bloch} = |f(0)| + \|f\|_{inv}$. If B_E is a bounded symmetric domain, then $\|\cdot\|_{inv-Bloch}$ is a norm for $\hat{\mathcal{B}}_{inv}(B_E)$. Nevertheless, if we deal with general Banach spaces E, $\|f\|_{inv-Bloch}$ fails to be a norm since Proposition 1.2 does not assure that the closed subspace V of E

satisfies $V \cap B_E = B_E$ if B_E is not a bounded symmetric domain. We consider the quotient space:

$$\mathcal{B}_{inv}(B_E) = \hat{\mathcal{B}}_{inv}(B_E)/\sim$$

where $f \sim g$ in $\hat{\mathcal{B}}_{inv}(B_E)$ if and only if $\|f\|_{inv} = \|g\|_{inv}$ and $f(0) = g(0)$. Hence $\mathcal{B}_{inv}(B_E)$ is a vector space and we can endow it with the norm given by $\|[f]\|_{inv-Bloch} = \|f\|_{inv-Bloch}$ for any $[f] \in \mathcal{B}_{inv}(B_E)$. Then $\mathcal{B}_{inv}(B_E)$ becomes a Banach space. We will write f to refer to $[f]$ if there can be no possible confusion. If we deal with bounded symmetric domains B_E, the space of Bloch functions $\mathcal{B}_{inv}(B_E)$ coincides with $\hat{\mathcal{B}}_{inv}(B_E)$ defined in [8].

As we have mentioned, the seminorms $\|\cdot\|_{nat}$ and $\|\cdot\|_{inv}$ are equivalent if E is a finite or infinite dimensional Hilbert space, so we have that $\mathcal{B}_{nat}(B_E) = \mathcal{B}_{inv}(B_E)$ in this case. Dealing with bounded symmetric domains, it was proved (see Corollary 3.5 in [8]):

Corollary 2.1. *Let B_E be a bounded symmetric domain and $f \in \mathcal{B}_{inv}(B_E)$. Then for any $x \in B_E$ we have:*

$$\|f'(x)\| \leq \frac{\|f\|_{inv}}{1 - \|x\|^2}.$$

Hence, it is clear that $\|f\|_{nat} = \sup_{x \in B_E} \|f'(x)\|(1 - \|x\|^2) \leq \|f\|_{inv}$. So we obtain:

Proposition 2.2. *Let E be a complex Banach space such that B_E is a bounded symmetric domain. Then, $\mathcal{B}_{inv}(B_E) \subseteq \mathcal{B}_{nat}(B_E)$ and $\|f\|_{nat} \leq \|f\|_{inv}$.*

In this section, we will give examples where these spaces are different even for some bounded symmetric domains. Indeed, for general Banach spaces E we will show that it is not true that $\hat{\mathcal{B}}_{inv}(B_E) \subseteq \mathcal{B}_{nat}(B_E)$ or $\mathcal{B}_{nat}(B_E) \subseteq \hat{\mathcal{B}}_{inv}(B_E)$.

2.2. **The case $E = (\mathbb{C}^n, \|\cdot\|_\infty)$ and $E = c_0$.** Let $E = (\mathbb{C}^n, \|\cdot\|_\infty)$ or $E = c_0$, whose open unit ball is a bounded symmetric domain so-called (finite or infinite dimensional) polydisc and which is usually denoted by $\mathbf{D}^n$ and B_{c_0} respectively. Bloch functions on the finite or infinite polydisc were studied in [18] and [8] respectively. We prove that the Bloch spaces defined with each of the seminorms $\|\cdot\|_{nat}$ and $\|\cdot\|_{inv}$ are different even if B_E is the bidisc $\mathbf{D}^2$.

For any $f \in H(B_E)$ and $x \in B_E$ we have that $f'(x)$ belongs to ℓ_1^n or ℓ_1 respectively, so we can identify $f'(x)$ by $\left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \cdots\right)$ and $\|f'(x)\| = \sum_{k=1}^n \left|\frac{\partial f}{\partial x_k}(x)\right|$, where n can be finite or infinite.

2.2.1. *Automorphisms on $E = (\mathbb{C}^n, \|\cdot\|_\infty)$ and $E = c_0$.* Consider for any $z \in \mathbf{D}$ the automorphism $\varphi_z : \mathbf{D} \to \mathbb{C}$ given by

$$\varphi_z(w) = \frac{w - z}{1 - \bar{z}w},$$

which satisfies that $\varphi_z'(0) = -(1 - |z|^2)$. It is well-known that any $\varphi \in Aut(\mathbf{D})$ is given by $\varphi(w) = e^{i\alpha}\varphi_z(w)$ for some $z \in \mathbf{D}$ and $\alpha \in [0, 2\pi[$.

Now let $x = (x_1, x_2, \cdots) \in B_E$ and consider the automorphism $\varphi_x : B_E \to B_E$ given by

$$\varphi_x(y) = \left(\frac{x_1 - y_1}{1 - \bar{x_1}y_1}, \frac{x_2 - y_2}{1 - \bar{x_2}y_2}, \cdots\right).$$

It is well-known that any $\varphi \in Aut(B_E)$ is given by $\varphi = (\varphi_1, \varphi_2, \cdots)$ where $\varphi_k \in Aut(\mathbf{D})$ (see [16] and [11] for the finite and infinite dimensional case respectively).

We are interested in the calculation of $\|(f \circ \varphi)'(0)\|$ in order to calculate $\|f\|_{inv}$, where $f : B_E \to \mathbb{C}$ is a holomorphic function and φ is an automorphism of B_E, so we can consider, without loss of generality, that $\varphi = \varphi_x$ since $|e^{i\alpha}z| = |z|$ for $z \in \mathbf{D}$, $\alpha \in [0, 2\pi[$ and $(e^{i\alpha_1}z_1, e^{i\alpha_2}z_2, \cdots) \in B_E$ for any $(z_1, z_2, \cdots) \in B_E$ and $\alpha_1, \alpha_2, \cdots \in [0, 2\pi[$.

Proposition 2.3. *Let $f \in H(B_E)$ for $E = (\mathbb{C}^n, \|\cdot\|_\infty)$ or $E = c_0$. If $x \in B_E$, then:*

$$\|(f\circ\varphi_x)'(0)\| = \sum_{k=1}^{n}(1-|x_k|^2)\left|\frac{\partial f}{\partial x_k}(x)\right|,$$

where n can be finite or infinite.

Proof. It is clear that:

$$\varphi_x'(0) = \begin{pmatrix} -(1-|x_1|^2) & 0 & \cdots \\ 0 & -(1-|x_2|^2) & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

Since $f\circ\varphi_x$ is well-defined on B_E and bearing in mind that $\varphi_x(0) = x$, we have:

$$f\circ\varphi_x)'(0) = f'(\varphi_x(0))\circ\varphi_x'(0) =$$

$$\begin{pmatrix} \frac{\partial f}{\partial x_1}(x) & \frac{\partial f}{\partial x_2}(x) & \cdots \end{pmatrix} \circ \begin{pmatrix} -(1-|x_1|^2) & 0 & \cdots \\ 0 & -(1-|x_2|^2) & \cdots \\ 0 & 0 & \cdots \end{pmatrix}$$

so:

$$\|(f\circ\varphi_x)'(0)\| = \sum_{k=1}^{n}(1-|x_k|^2)\left|\frac{\partial f}{\partial x_k}(x)\right|,$$

where n can be finite or infinite. □

Hence:

Corollary 2.4. *For any $f \in \mathcal{B}_{inv}(B_E)$, we have that:*

$$\|f\|_{inv} = \sup_{x\in B_E}\|(f\circ\varphi_x)'(0)\| = \sup_{x\in B_E}\sum_{k=1}^{n}(1-|x_k|^2)\left|\frac{\partial f}{\partial x_k}(x)\right|$$

where n can be finite or infinite.

Proposition 2.5. *Let $E = (\mathbb{C}^2, \|\cdot\|_\infty)$ and consider the bidisc $B_E = \mathbf{D}^2$. Then:*

$$\mathcal{B}_{inv}(\mathbf{D}^2)) \subsetneq \mathcal{B}_{nat}(\mathbf{D}^2).$$

Proof. It is clear that $\mathcal{B}_{inv}(\mathbf{D}^2) \subseteq \mathcal{B}_{nat}(\mathbf{D}^2)$ by Proposition 2.2. To prove that these spaces are different, consider $f(z,w) = (w+1)\log(z-1)$. Then, $\frac{\partial f}{\partial z} = \frac{w+1}{z-1}$ and $\frac{\partial f}{\partial w} = \log(z-1)$. Notice that:

$$|f\|_{nat} = \sup_{(z,w)\in\mathbf{D}^2}(1-\sup\{|z|,|w|\}^2)\left(\left|\frac{\partial f}{\partial z}(z,w)\right| + \left|\frac{\partial f}{\partial w}(z,w)\right|\right)$$

so:

$$\|f\|_{nat} \le \sup_{|z|<1}(1-|z|^2)\left(\left|\frac{w+1}{z-1}\right| + |\log(z-1)|\right) \le 2\sup_{|z|<1}(1-|z|)\left(\frac{2}{1-|z|} + |\log(z-1)|\right) \le$$

$$4 + \sup_{|z|<1}|z-1||\log(z-1)|$$

It is clear that $w\log w$ is bounded on the set of complex numbers w such that $|w| \le 2$ since $|w\log w| \le |w||\log|w| + i\arg w| \le |w|(\log|w| + 2\pi)$ and $t\log t \to 0$ when $t \to 0$ so there exists a constant $M > 0$ such that $\|f\|_{nat} \le 4 + M < \infty$ and conclude that $f \in \mathcal{B}_{nat}(\mathbf{D}^2)$.

Now we calculate $\|f\|_{inv}$. Notice that:

$$\|f\|_{inv} = \sup_{(z,w)\in\mathbf{D}^2} (1-|z|^2)\left|\frac{\partial f}{\partial z}(z,w)\right| + (1-|w|^2)\left|\frac{\partial f}{\partial w}(z,w)\right| \geq$$

$$\sup_{(z,w)\in\mathbf{D}^2} (1-|z|)\left|\frac{w+1}{z-1}\right| + (1-|w|)\left|\log(z-1)\right|.$$

Evaluating at $w=0$:

$$\|f\|_{inv} \geq \sup_{|z|<1}\left\{(1-|z|)\frac{1}{|z-1|} + |\log(z-1)|\right\}$$

and taking $z_n = 1-1/n$ we have:

$$\|f\|_{inv} \geq \sup_{n\in\mathbb{N}}\left\{(1-|z_n|)\frac{1}{|z_n-1|} + |\log(z_n-1)|\right\} = \sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\frac{1}{\frac{1}{n}} + |\log\left(\frac{-1}{n}\right)|\right\} \geq$$

$$\sup_{n\in\mathbb{N}}\left\{1+\left|\log\left(\frac{1}{n}\right)\right|\right\}$$

and since $1+\left|\log\left(\frac{1}{n}\right)\right| \to \infty$ when $n\to\infty$ we have that $\|f\|_{inv}=\infty$ so $f\notin \mathcal{B}_{inv}(\mathbf{D}^2)$. □

Hence, we have:

Corollary 2.6. *Let E be $(\mathbb{C}^n, \|\cdot\|_\infty)$ for $n\geq 2$ or c_0. Then, $\mathcal{B}_{inv}(B_E)\subsetneq \mathcal{B}_{nat}(B_E)$.*

Proof. It is clear that $\mathcal{B}_{inv}(B_E)\subseteq\mathcal{B}_{nat}(B_E)$ because of Proposition 2.2. Consider $f\in H(\mathbf{D}^2)$ given in Proposition 2.5. The function $g(x_1,x_2,x_3,\cdots)=f(x_1,x_2)$ belongs to $\mathcal{B}_{nat}(B_E)$ but $g\notin\mathcal{B}_{inv}(B_E)$. □

2.3. **The case $E = L_p(\Omega,\mu)$.** L. L. Stachó and E. Vesentini (see [17] and [20]) proved that for measure spaces $E=L_p(\Omega,\mu)$, $1\leq p<\infty$, $p\neq 2$ and $\mu(\Omega)<\infty$, we have:

$$Aut(B_E)=\{U|_{B_E} : U \text{ is a surjective linear isometry of } E\}.$$

Hence $\varphi'(0)=\varphi$ and $\varphi(0)=0$ for any $\varphi\in Aut(B_E)$. We will prove that the behaviour of the unit ball B_E of these spaces is completely different to bounded symmetric domains when we deal with spaces of Bloch functions on B_E.

Proposition 2.7. *Let $E=L_p(\Omega,\mu)$, $1\leq p<\infty$, $p\neq 2$ and $\mu(\Omega)<\infty$. Then:*

$$\mathcal{B}_{nat}(B_E)\subsetneq\hat{\mathcal{B}}_{inv}(B_E).$$

Proof. Let $f\in\mathcal{B}_{nat}(B_E)$ and $\varphi\in Aut(B_E)$. Since φ is the restriction of a surjective linear isometry to B_E, we have:

$$\|(f\circ\varphi)'(0)\| = \|f'(\varphi(0))\circ\varphi'(0)\| = \|f'(0)\circ\varphi'(0)\| = \|f'(0)\| \leq \sup_{x\in B_E}(1-\|x\|^2)\|f'(x)\|$$

so $\mathcal{B}_{nat}(B_E)\subseteq\hat{\mathcal{B}}_{inv}(B_E)$. However, J. M. Ansemil, R. Aron and S. Ponte (see [1] and [2]) proved that given any two disjoint balls in an infinite dimensional complex Banach space E, there exists an entire function on E which is bounded on one and unbounded on the other. We consider the balls $B_1=\frac{1}{2}B_E=\{x\in E:\|x\|<\frac{1}{2}\}$ and $B_2=B(x_0,\frac{1}{5}):=\{x\in E: \|x-x_0\|<\frac{1}{5}\}$ for a fixed $x_0\in E$ such that $\|x_0\|=\frac{3}{4}$. Then, there exists an entire function f on E such that $f|_{B_1}$ is bounded and $f|_{B_2}$ is unbounded, so there exists $(x_n)\subset B_2$ such that $|f(x_n)|\to\infty$ when $n\to\infty$. By the Mean Value Theorem (see Theorem 13.8 in [14]) we have that:

$$|f(x_n)-f(0)|\leq\|x_n\|\sup_{0\leq\lambda\leq1}\|f'(\lambda x_n)\|,$$

so:
$$\frac{|f(x_n)| - |f(0)|}{\|x_n\|} \leq \frac{|f(x_n) - f(0)|}{\|x_n\|} \leq \sup_{0\leq\lambda\leq 1} \|f'(\lambda x_n)\|$$
and since $\|x_n\| \leq \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$, we have that:
$$\frac{|f(x_n)| - |f(0)|}{\|x_n\|} \geq \frac{20}{19}\left(|f(x_n)| - |f(0)|\right) \to \infty$$
when $n \to \infty$, so we can take a sequence $(\lambda_n) \subset [0,1]$ such that $\|f'(\lambda_n x_n)\| \to \infty$ when $n \to \infty$. Hence:
$$(1 - \|\lambda_n x_n\|^2)\|f'(\lambda x_n)\| \geq (1 - \|\lambda_n x_n\|)\|f'(\lambda_n x_n)\| \geq (1 - \|x_n\|)\|f'(\lambda_n x_n)\| \geq \frac{1}{20}\|f'(\lambda_n x_n)\| \to \infty$$
when $n \to \infty$. Since $B_1, B_2 \subset B_E$, we conclude that $\|f\|_{nat} = \infty$ but, as we have observed above, $\|f\|_{inv} = \|f'(0)\| < \infty$ so $\mathcal{B}_{nat}(B_E) \subsetneq \hat{\mathcal{B}}_{inv}(B_E)$. □

2.4. **Bloch functions and Lipschitz functions for the hyperbolic metric.** As we have mentioned in Subsection 1.4, any $f \in \mathcal{B}_{inv}(B_E)$ is Lipschitz for the corresponding hyperbolic distance β_E on B_E for any Banach space E such that B_E is a bounded symmetric domain (see [8]). We will prove that this is no longer true if we deal with B_E which are not bounded symmetric domains. Consider the spaces L_p from Subsection 2.3. Then:

Proposition 2.8. *Let $E = L_p(\Omega, \mu)$, $1 \leq p < \infty$, $p \neq 2$ and $\mu(\Omega) < \infty$. Then there exists $f \in \hat{\mathcal{B}}_{inv}(B_E)$ which is not Lipschitz for the corresponding hyperbolic distance β_E on B_E.*

Proof. Look at the proof of Proposition 2.7. Take the balls B_1, B_2, f the function which is defined there and the sequence $(x_n) \subset B_E$ such that $|f(x_n)| \to \infty$ when $n \to \infty$. So $|f(x_n) - f(0)| \geq |f(x_n)| - |f(0| \to \infty$ when $n \to \infty$ but:
$$\beta_E(x_n, 0) = \frac{1}{2}\log\left(\frac{1 + \|x_n\|}{1 - \|x_n\|}\right)$$
which is bounded on (x_n) since $1 - \|x_n\| \geq 1 - \frac{19}{20} = \frac{1}{20}$. □

3. Bounded functions on B_E are Bloch functions

In [4] and [8] it was proved that $H^\infty(B_E) \subsetneq \mathcal{B}_{inv}(B_E)$ when E is a Hilbert space or B_E is a bounded symmetric domain respectively. In this section, we will prove that this result remains true if we deal with any complex Banach space E and any Bloch space, that is, and $H^\infty(B_E) \subsetneq \mathcal{B}_{nat}(B_E)$ and $H^\infty(B_E) \subsetneq \hat{\mathcal{B}}_{inv}(B_E)$.

First we recall the following result which is an application of the Schwarz lemma (see page 641 in [7]).

Theorem 3.1. *If $g : B(x_0, r) \to \mathbb{C}$ is analytic and $g(x_0) = 0$, then:*
$$|g(y)| \leq ||g||_\infty \frac{\|y - x_0\|}{r} \text{ if } \|y - x_0\| < r,$$
where $\|g\|_\infty$ denotes the sup-norm of g on $B(x_0, r)$.

As consequence, it is clear by the definition of the pseudohyperbolic distance on B_E given by (1.1) that:

Corollary 3.2. *Let E be a complex Banach space and $x, y \in B_E$. Then:*
$$\rho_E(y, x) \leq \frac{\|y - x\|}{r} \text{ for all } y \in B(x, r).$$

Proposition 3.3. *Let $f \in H^\infty(B_E)$ such that $\|f\|_\infty \le 1$. Then, for any $x_0 \in B_E$, we have that:*

$$(1-\|x_0\|)\|f'(x_0)\| \le 1 - |f(x_0)|^2.$$

Proof. First we consider that $\|f\|_\infty < 1$ and let $x_0 \in B_E$. Applying Corollary 3.2, for $r > 0$ such that $\|x_0\| + r < 1$, we have that:

$$\rho_E(y, x_0) \le \frac{\|y - x_0\|}{r} \qquad \text{for all } y \in B(x_0, r).$$

Taking limits when $r \to (1-\|x_0\|)^-$ we get:

$$\rho_E(y, x_0) \le \frac{\|y - x_0\|}{1-\|x_0\|} \text{ for all } y \in B(x_0, 1-\|x_0\|).$$

Notice that for any $x_0 \in B_E$, $f'(x_0)$ is the functional on E satisfying:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{\|x - x_0\|} = 0,$$

so given $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\left| \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{\|x - x_0\|} \right| < \varepsilon$$

if $\|x - x_0\| < \delta$. Without loss of generality, we can choose δ such that $x_0 + B(x_0, \delta) \subset B_E$. For x such that $\|x - x_0\| < \delta$ we have that:

$$\left| \frac{f'(x_0)(x - x_0)}{\|x - x_0\|} \right| < \varepsilon + \frac{|f(x) - f(x_0)|}{\|x - x_0\|}.$$

Choose a sequence (ε_n) of positive numbers such that $\varepsilon_n \to 0$ when $n \to \infty$ and consider their corresponding (δ_n) sufficiently small to satisfy that $x_0 + B(x_0, \delta_n) \subset B_E$ and $\sup_{n\in\mathbb{N}} \{\|x_0\| + \delta_n\} < 1$. Since:

$$\|f'(x_0)\| = \sup_{y \in B_E} |f'(x_0)(y)|,$$

we choose vectors $(y_n) \subset B_E$, $\|y_n\| \to 1$ such that $\|f'(x_0)\| = \lim_{n\to\infty} |f'(x_0)(y_n)|$ and define $x_n \in B_E$ by:

$$x_n = x_0 + \delta_n \frac{y_n}{\|y_n\|}.$$

It is clear that $x_n \in B_E$ since $x_0 + B(x_0, \delta_n) \subset B_E$ and $\sup_{n\in\mathbb{N}} \{\|x_0\| + \delta_n\} < 1$. We have:

$$\left\| \frac{x_n - x_0}{\|x_n - x_0\|} - y_n \right\| = \left\| \frac{y_n}{\|y_n\|} - y_n \right\| \to 0$$

when $n \to \infty$ since $\|y_n\| \to 1$. Hence:

$$\|f'(x_0)\| = \lim_{n\to\infty} |f'(x_0)(y_n)| = \lim_{n\to\infty} \frac{|f'(x_0)(x_n - x_0)|}{\|x_n - x_0\|}.$$

Notice that:

$$\frac{|f'(x_0)(x_n - x_0)|}{\|x_n - x_0\|} < \varepsilon_n + \frac{|f(x_n) - f(x_0)|}{\|x_n - x_0\|} \le \varepsilon_n + \left| \frac{f(x_n) - f(x_0)}{1 - \overline{f(x_0)} f(x_n)} \right| \frac{\left|1 - \overline{f(x_0)} f(x_n)\right|}{\|x_n - x_0\|}.$$

Since the pseudohyperbolic distance is contractive for f, we have that:

$$\frac{|f'(x_0)(x_n - x_0)|}{\|x_n - x_0\|} < \varepsilon_n + \rho_E(x_n, x_0) \frac{\left|1 - \overline{f(x_0)} f(x_n)\right|}{\|x_n - x_0\|}.$$

Next observe that there is no loss of generality in assuming $\delta_n \to 0$, so we can consider that $\delta_n \leq 1 - \|x_0\|$. Hence by Corollary 3.2:

$$\rho_E(x_n, x_0) \leq \frac{\|x_n - x_0\|}{1 - \|x_0\|}.$$

so:

$$\left|\frac{f'(x_0)(x_n - x_0)}{\|x_n - x_0\|}\right| < \varepsilon_n + \frac{\|x_n - x_0\|}{1 - \|x_0\|}\frac{|1 - \overline{f(x_0)}f(x_n)|}{\|x_n - x_0\|} = \varepsilon_n + \frac{|1 - \overline{f(x_0)}f(x_n)|}{1 - \|x_0\|}.$$

Taking limits when $n \to \infty$, we have:

$$\|f'(x_0)\| \leq \frac{1}{1 - \|x_0\|}(1 - |f(x_0)|^2)$$

and we are done. Suppose now that $\|f\|_\infty = 1$. Then, there exists a sequence of functions $(f_n) \subset H^\infty(B_E)$ (for instance, $f_n(x) = (1 - 1/n)f(x)$) such that f_n converges uniformly to f on B_E. We apply the inequality to the functions (f_n) and taking limits when $n \to \infty$, we are done. □

Proposition 3.4. *Let E be a complex Banach space. Then $H^\infty(B_E) \subseteq \mathcal{B}_{nat}(B_E)$ and the map $Id : (H^\infty(B_E), \|\cdot\|_\infty) \to (\mathcal{B}_{nat}(B_E), \|\cdot\|_{nat-Bloch})$ is continuous.*

Proof. Let $f \in H^\infty(B_E)$. Then $f/\|f\|_\infty$ has sup-norm 1 and we can apply Proposition 3.3. Then, for any $x \in B_E$ we have:

$$\frac{\|f'(x)\|}{\|f\|_\infty} \leq \frac{1}{1 - \|x\|}\left(1 - \frac{|f(x)|^2}{\|f\|_\infty^2}\right) \leq \frac{1}{1 - \|x\|},$$

so $(1 - \|x\|^2)\|f'(x)\| = (1 + \|x\|)(1 - \|x\|)\|f'(x)\| \leq 2\|f\|_\infty$ and we obtain that $H^\infty(B_E) \subset \mathcal{B}_{nat}(B_E)$. Adding up $|f(0)|$ to the left term we obtain that $\|Id(f)\|_{nat-Bloch} \leq 3\|f\|_\infty$, so bounded functions are Bloch functions and the inclusion is continuous. □

Now we prove that the same result remains true if we deal with $\hat{\mathcal{B}}_{inv}(B_E)$ instead of $\mathcal{B}_{nat}(B_E)$.

Proposition 3.5. *Let E be a complex Banach space and B_E its open unit ball. Then $H^\infty(B_E) \subseteq \hat{\mathcal{B}}_{inv}(B_E)$ and the map $Id : (H^\infty(B_E), \|\cdot\|_\infty) \to (\mathcal{B}_{inv}(B_E), \|\cdot\|_{Bloch})$ is continuous.*

Proof. Let $f \in H^\infty(B_E)$. For any $\varphi \in Aut(B_E)$ it is clear that $f\circ\varphi \in H^\infty(B_E)$ and $\|f\circ\varphi\|_\infty = \|f\|_\infty$ since $\varphi(B_E) = B_E$. So by the proof of Proposition 3.4 we have $\|(f \circ \varphi)'(0)\| \leq 2\|f\|_\infty$ and hence $\|f\|_{inv} \leq 2\|f\|_\infty$. We conclude that $H^\infty(B_E) \subseteq \hat{\mathcal{B}}_{inv}(B_E)$ and in addition:

$$\|f\|_{inv-Bloch} = |f(0)| + \|f\|_{inv} \leq 3\|f\|_\infty$$

so Id is also continuous. □

Finally we prove that $H^\infty(B_E)$ is strictly contained in $\mathcal{B}_{nat}(B_E)$ and $\hat{\mathcal{B}}_{inv}(B_E)$.

Proposition 3.6. *For any inifinite dimensional complex Banach space E, we have that $H^\infty(B_E) \subsetneq \mathcal{B}_{nat}(B_E)$.*

Proof. We proved that $H^\infty(B_E) \subseteq \mathcal{B}_{nat}(B_E)$ in Proposition 3.4. Let $x_0 \in E$ and $\|x_0\| = 1$. By the Hahn-Banach Theorem, there exists $L \in E^*$ such that $\|L\| = L(x_0) = 1$. The function $f(x) = \log(1 - L(x))$ satisfies that $f \in \mathcal{B}_{nat}(B_E) \setminus H^\infty(B_E)$ since:

$$(1 - \|x\|^2)\|f'(x)\| = (1 - \|x\|^2)\frac{\|L\|}{|1 - L(x)|} \leq (1 - \|x\|^2)\frac{1}{1 - \|x\|} \leq 2$$

but there exists $(x_n) \subset B_E$ such that $x_n \to x_0$ and $\lim_{n\to} |f(x_n)| = |\log(1-L(x_n))| = \infty$, so $f \notin H^\infty(B_E)$. □

Proposition 3.7. *For any inifinite dimensional complex Banach space E, we have that $H^\infty(B_E) \subsetneq \hat{\mathcal{B}}_{inv}(B_E)$.*

Proof. We proved that $H^\infty(B_E) \subseteq \hat{\mathcal{B}}_{inv}(B_E)$ in Proposition 3.5. Let V be the closed subspace of E given by Proposition 1.2 satisfying that:

$$B_V = V \cap B_E = \{\varphi(0) : \varphi \in Aut(B_E)\}.$$

If $V = \{0\}$, then any automorphism φ would satisfy $\varphi(0) = 0$ and hence φ is the restriction of a linear isometry of E (see Proposition 1 in [10]). Then for any $f \in H(B_E)$ we have that $\|f\|_{inv} = \|f'(0)\| < \infty$ so any $f \in H(B_E)$ belongs to $\hat{\mathcal{B}}_{inv}(B_E)$ but it is well-known that there are unbounded holomorphic functions on B_E and we are done. If $V \neq \{0\}$, there exists a continuous linear map $L : V \to \mathbb{C}$ and $x \in V$ such that $\|L\| = 1$, $\|x\| = 1$ and $L(x) = \|L\|$. By the Hahn-Banach Theorem, there exists a continuous linear map L_1 on E such that $L_1|_V = L$ and $\|L_1\| = \|L\|$. We consider the map $f : B_E \to \mathbb{C}$ given by $f(x) = \log(1-L_1(x))$. It is clear that f is unbounded on B_E since $B_V \subset B_E$. Now we prove that $f \in \hat{\mathcal{B}}_{inv}(B_E)$. Notice that for any $\varphi \in Aut(B_E)$ we have that $L \circ \varphi \in H^\infty(B_E)$ since $\varphi(B_E) \subset B_E$ and $\|L \circ \varphi\|_\infty \leq \|L\|$=1. Let $h : \mathbf{D} \to \mathbb{C}$ given by $h(z) = \log(1-z)$, so $f(x) = (h \circ L)(x)$. For any $\varphi \in Aut(B_E)$ we have that:

$$\|(f\circ\varphi)'(0)\| = \|(h\circ L\circ\varphi)'(0)\| \leq \|h'(L(\varphi(0))\|\|(L\circ\varphi)'(0)\| \leq \frac{\|h\|_B}{1-|(L\circ\varphi)(0)|^2}\|(L\circ\varphi)'(0)\|$$

where $\|h\|_B$ denotes the Bloch seminorm for the classical Bloch space $\mathcal{B}$ and it is clear that $h \in \mathcal{B}$. Apply Proposition 3.3 to $L \circ \varphi$ at $x_0 = 0$ and conclude that $\|(f \circ \varphi)'(0)\| \leq \|h\|_B$ for any $\varphi \in Aut(B_E)$ so $\|f\|_{inv} < \infty$. □

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