

REFLECTIONS IN TOPOLOGICAL ALGEBRAIC STRUCTURES

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ABSTRACT. Let \mathcal{C} be an epireflective category of **Top** and let $\Gamma_{\mathcal{C}}$ be the epireflective functor associated with \mathcal{C} . If \mathcal{A} denotes a (semi)topological algebraic subcategory of **Top**, we study when $\Gamma_{\mathcal{C}}(\mathcal{A})$ is an epireflective subcategory of \mathcal{A} . We prove that this is always the case for semi-topological structures and we find some sufficient conditions for topological algebraic structures. We also study when the epireflective functor preserves products, subspaces and other properties. In particular, we solve an open question about the coincidence of epireflections proposed by Echi and Lazaar in [6, Question 1.6] and repeated in [7, Question 1.9]. Finally, we apply our results in different specific topological algebraic structures.

1. INTRODUCTION

In this paper we deal with some applications of epireflective functors in the investigation of topological algebraic structures. In particular, we are interested in the following question: Let \mathcal{C} be an epireflective subcategory of **Top** (i.e., productive and hereditary, hence containing the 1-point space as the empty product), and let $r_{\mathcal{C}}$ be the epireflective functor associated with \mathcal{C} . If \mathcal{A} denotes a (semi)topological varietal subcategory of **Top** (that is, a subcategory that is closed under products, subalgebras and homomorphic images), we study when $\Gamma_{\mathcal{C}}(\mathcal{A})$ is an epireflective subcategory of \mathcal{A} . From a different viewpoint, this question has attracted the interest of many researchers recently (cf. [18, 21, 22, 24, 25, 26, 27, 30]) and, to some extent, our motivation for

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this research has been to give a unified approach to this topic. First, we recall some definitions and basic facts.

A full subcategory \mathcal{A} of a category \mathcal{B} is *reflective* if the canonical embedding of \mathcal{A} in \mathcal{B} has a left adjoint $r_{\mathcal{A}} : \mathcal{B} \rightarrow \mathcal{A}$ (called *reflection*). Thus for each \mathcal{B} -object B there exists an \mathcal{A} -object $r_{\mathcal{A}}B$ and a \mathcal{B} -morphism $r_{(B,\mathcal{A})} : B \rightarrow r_{\mathcal{A}}B$ such that for each \mathcal{B} -morphism $f : B \rightarrow A$ to an \mathcal{A} -object A , there exists a unique \mathcal{A} -morphism $\bar{f} : r_{\mathcal{A}}B \rightarrow A$ such that the following diagram commutes

$$\begin{array}{ccc} B & \xrightarrow{r_{(B,\mathcal{A})}} & r_{\mathcal{A}}B \\ & \searrow f & \swarrow \bar{f} \\ & & A \end{array}$$

The pair $(r_{\mathcal{A}}B, r_{(B,\mathcal{A})})$ is called the \mathcal{A} -*reflection* of B and the morphism $r_{(B,\mathcal{A})}$ is called \mathcal{A} -*reflection arrow*. If all \mathcal{A} -reflection arrows are epimorphisms, then the subcategory \mathcal{A} is said to be *epireflective*. The functor $r : \mathcal{B} \rightarrow \mathcal{A}$, which is called the *reflector*, assigns to each \mathcal{B} -morphism $f : X \rightarrow Y$, the \mathcal{A} -morphism $r_{\mathcal{A}}(f)$ that is determined by the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow r_{(X,\mathcal{A})} & & \downarrow r_{(Y,\mathcal{A})} \\ r_{\mathcal{A}}X & \xrightarrow{r_{\mathcal{A}}(f)} & r_{\mathcal{A}}Y \end{array}$$

2. BASIC FACTS

We collect in this section some known facts about epireflective categories that will be used along the paper. Here, we look at the category **Top** of topological spaces and continuous functions. Following Kennison [16], by a *topological property* \mathcal{P} , we mean a full subcategory of **Top** which is closed under the formation of isomorphic (= homeomorphic) objects. In general, such subcategories are called *isomorphism-closed*, or *replete*. When speaking about *subcategories*, we will always suppose that they are full and replete. A topological property \mathcal{P} is *hereditary* (*resp. divisible, productive, or coproductive*) if the objects of \mathcal{P} are closed under the formation of subspaces (*resp. quotient spaces, product spaces, or coproduct spaces.*) Here, the terms “product”, “coproduct”, “quotient space” and “subspace” are used in their topological sense. In particular, Kennison proved that a full subcategory \mathcal{P} of **Top** is epireflective if and only if \mathcal{P} is hereditary and productive (cf. [16]). Well known examples of reflective subcategories in **Top** are: the classes of all T_0 , T_1 , T_2 , and T_3 spaces, the class of all regular spaces, the completely regular spaces, the class of all totally disconnected spaces (cf. [16]). For $T_{3.5}$ spaces there exist the following reflections: the Stone-Čech compactification, the Hewitt realcompactification.

Let \mathcal{C} denote an epireflective subcategory of **Top**. That is, for each topological space X , there exists an associated topological space $r_{\mathcal{C}}X \in \mathcal{C}$ and a surjective continuous function $r_{(X,\mathcal{C})}: X \longrightarrow r_{\mathcal{C}}X$ such that for every continuous function $f: X \longrightarrow Y$, with $Y \in \mathcal{C}$, there exists a continuous function $\tilde{f}: r_{\mathcal{C}}X \longrightarrow Y$ (unique by surjectivity of $r_{(X,\mathcal{C})}$) such that the following diagram commutes

$$\begin{array}{ccc}
 X & \xrightarrow{\Gamma_{(X,\mathcal{C})}} & r_{\mathcal{C}}X \\
 & \searrow f & \swarrow \tilde{f} \\
 & & Y
 \end{array}$$

For a topological space X and a continuous map $f: X \rightarrow Y$ between topological spaces we write $r_{\mathcal{C}}X$ and $r_{\mathcal{C}}(f): r_{\mathcal{C}}X \rightarrow r_{\mathcal{C}}Y$ for the image of X, Y and f by the epireflection functor associated to \mathcal{C} . If the subcategory \mathcal{C} is clear from the context, we omit it from the index. Furthermore, the space $r_{\mathcal{C}}X$ is uniquely determined up to a homeomorphism, and $\Gamma_{(X,\mathcal{C})}$ is uniquely determined up to composition from the right with a homeomorphism. It is well known that for every subcategory \mathcal{A} of **Top** there exists a smallest epi-reflective subcategory $\mathcal{C}(\mathcal{A})$ in **Top** containing \mathcal{A} , namely spaces homeomorphic to subspaces of products of spaces from \mathcal{A} , (cf. [16]). It is said that \mathcal{A} *generates* $\mathcal{C}(\mathcal{A})$. In case there is a single space X with $\mathcal{C}(\{X\}) = \mathcal{C}$, we say that X *generates* \mathcal{C} and \mathcal{C} is called *simply generated* by X . (cf. [12, 13]). For example the class **Top**₀ of T_0 spaces is generated by the Sierpiński space. However, the class **Top**₁ of T_1 spaces is not simply generated. In fact, the class **Top**₁ is generated by the family of cofinite spaces. Furthermore, given an infinite cardinal κ , the space κ_{cof} , which is the set κ equipped with the cofinite topology, simply generates the epireflective subcategory of **Top** generated by all T_1 -spaces of cardinality at most κ (see [9, 12]).

For self-completeness, we recall below a realization of the epireflection associated to an epireflective subcategory $\mathcal{C}(\mathcal{A})$ that is generated by a subcategory \mathcal{A} of **Top**.

Let $\mathbf{F}(X, \mathcal{A})$ denote the class of all continuous functions of X onto spaces in \mathcal{A} . We set the following equivalence relation: for $f: X \rightarrow Y$ and $g: X \rightarrow Z$ in $\mathbf{F}(X, \mathcal{A})$, it is said that f and g are equivalent, $f \sim g$, if there is a homeomorphism $\psi: Y \rightarrow Z$ such

that $g = \psi \circ f$. Set $\kappa = |X|$. Since every continuous image of X can be considered as a subset of κ , the family of equivalence classes $\mathbf{E}(X, \mathcal{A}) = \mathbf{F}(X, \mathcal{A})/\sim$ defines a set. Let $E = E(X, \mathcal{A})$ be the set defined by selecting a fixed element in each equivalence class in $\mathbf{E}(X, \mathcal{A})$ and let $\varphi_{\mathcal{A}(X)} = \Delta_E X: X \rightarrow \prod_{f \in E} f(X)$ be the diagonal map of X into the product $\Pi_E X = \prod_{f \in E} f(X)$. We have that $\varphi_{\mathcal{A}(X)}$ is a continuous function from X into $\Pi_E X$ and, since $\Pi_E X \in \mathcal{E}(\mathcal{A})$, it follows that $\varphi_{\mathcal{A}(X)}(X) \in \mathcal{E}(\mathcal{A})$. It is easy to check that $(\mathcal{A}(X), \varphi_{\mathcal{A}(X)})$ satisfies the universal property of a reflection. Indeed, let $h: X \rightarrow Y$ be a continuous function from X into $Y \in \mathcal{A}$. Then, there exists $f \in \mathbf{F}(X, \mathcal{A})$, say $f: X \rightarrow Z$, such that $f \sim h$. Let $\psi: Z \rightarrow Y$ be a homeomorphism with $h = \psi \circ f$ and let π_f be the canonical projection of $\Pi_E X$ in $f(X)$. We have $f = \pi_f \circ \varphi_{\mathcal{A}(X)}$, which yields $h = \psi \circ (\pi_f \circ \varphi_{\mathcal{A}(X)}) = (\psi \circ \pi_f) \circ \varphi_{\mathcal{A}(X)}$. The general case, when $Y \in \mathcal{E}(\mathcal{A})$, follows easily observing that Y is a subspace of a product of spaces in \mathcal{A} .

The following facts are easily verified.

Proposition 2.1. *Let \mathcal{C}_1 and \mathcal{C}_2 be epireflective subcategories in \mathbf{Top} such that $\mathcal{C}_2 \subseteq \mathcal{C}_1$. Then the pair*

$(r_{\mathcal{C}_2}(r_{\mathcal{C}_1} X), r_{(r_{\mathcal{C}_1} X, \mathcal{C}_2)})$ is a realization of the \mathcal{C}_2 -reflection of X in \mathbf{Top} .

Definition 2.2. A class \mathcal{C} in \mathbf{Top} is *closed under supertopologies* if whenever $(X, \tau) \in \mathcal{C}$ and ρ is a topology on X finer than τ , it follows that $(X, \rho) \in \mathcal{C}$.

The following result, whose proof is folklore, clarifies the action the epi-reflection functor for subcategories closed under supertopologies. (see [13] for the proof, which is straightforward anyway).

Theorem 2.3. *An epireflective subcategory \mathcal{C} in \mathbf{Top} is closed under supertopologies if and only if the reflection arrow $r_{(X,\mathcal{C})}$ is a quotient mapping.*

A topological space is called *functionally Hausdorff*, or *Urysohn*, if distinct points can be separated by a real-valued continuous function, or if any two distinct points have disjoint closed neighbourhoods, respectively.

Corollary 2.4. *The reflection arrow $r_{(X,\mathcal{C})}$ is a quotient mapping for each of the following subcategories of \mathbf{Top} defined by the separation axioms: T_0 , T_1 , T_2 , functionally Hausdorff, and Urysohn.*

The next result gives a general realization of the reflection functor for categories whose reflection arrows are quotients. We omit its easy proof here.

Proposition 2.5. *Let X be a topological space and let \mathcal{C} denote an epireflective subcategory of \mathbf{Top} whose reflection arrows are quotient maps. If $R_{\mathcal{C}}$ is the intersection of all equivalence relations $R \subseteq X^2$ on X such that $X/R \in \mathcal{C}$, then $r_{\mathcal{C}}X = X/R_{\mathcal{C}}$.*

By Theorem 2.3, the proposition above applies to the epireflections defined by the separation axioms: T_0 , T_1 , T_2 , functionally Hausdorff, and Urysohn. In particular, if \mathcal{C}_1 and \mathcal{C}_2 denote the subcategories defined by T_1 and T_2 , we have the following characterization, whose proof is folklore.

Proposition 2.6. *Let X be a topological space and let R be an equivalence relation on X . The following assertions are fulfilled:*

- (1) X/R is T_1 if and only if each equivalence class in R is closed in X .

(2) If the space X/R is Hausdorff then R is closed in $X \times X$. Conversely, if R is a closed subset of $X \times X$ and in addition the quotient map $\varphi: X \rightarrow X/R$ is open, then X/R is Hausdorff.

Corollary 2.7. *Let X be a topological space. Then*

- (1) $r_{e_1}X = X/R_{e_1}$ where R_{e_1} is the intersection of all equivalence relations whose equivalence classes are closed in X .
- (2) Let R_X be the smallest equivalence relation that is closed in $X \times X$. Then $r_{e_2}X$ is canonically homeomorphic to $r_{e_2}(X/R_X)$.

Proof. The verification of (1) is clear.

As for the proof of (2), by Proposition 2.5, we know that there is an equivalence relation R_2 on X such that $r_{e_2}X = X/R_2$. Furthermore, Proposition 2.6(2) implies that R_2 must be closed in $X \times X$. Hence $R_X \subseteq R_2$. Thus we have the following commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\pi_X} & X/R_X \\
 \Gamma_{(X,e_2)} \downarrow & & \downarrow \Gamma_{(X/R_X,e_2)} \\
 r_{e_2}X & \xrightarrow{\Gamma_{e_2}(\pi_X)} & r_{e_2}(X/R_X) \\
 & \xleftarrow{\varphi} &
 \end{array}$$

Here, the map φ is continuous because π_X and $\Gamma_{X/R_X,e_2}$ are both quotient morphisms.

Thus, $\Gamma_{e_2}(\pi_X)$ and φ are each other inverse, which completes the proof. \square

3. EPIREFLECTIVE CATEGORIES IN TOPOLOGICAL AND SEMITOPOLOGICAL ALGEBRAIC STRUCTURES

So far, only categories of topological spaces have been considered. However, our main interest lies on topological algebraic categories. Taking the terminology of Hart and Kunen [11], in what follows an *algebraic system* \mathcal{L} is a set (possibly empty or infinite) of symbols of constants, symbols of functions (every function symbol has a finite arity ≥ 1 , i.e., $\Phi: X^n \rightarrow X$) and a set of equations $\Sigma_{\mathcal{L}}$ that the elements, functions and constants must satisfy. (E.g., $\forall x \, ex = x$; this contains a constant sign, a binary operation sign and a variable.) An algebraic system is also called a *variety*, but further we will use the terminology *algebraic system*.

For a categorical generalization of algebraic, and varietal also called monadic functors, see [1, 14]

A *structure* \mathfrak{U} for \mathcal{L} is a set A (the domain) together with elements $c_{\mathfrak{U}}$ (of) and functions $\Phi_{\mathfrak{U}}: A^n \rightarrow A$, for $n \geq 1$ a natural number, the arity of the respective operation, corresponding to the constants and operations in \mathcal{L} (these we call the *specifications of the constants and operations from \mathcal{L} in \mathfrak{U}*), that satisfy the equations established in $\Sigma_{\mathcal{L}}$. E.g., when we talk about groups, it is understood that $\mathcal{L} = \{\cdot, i, 1, \Sigma_{\mathcal{L}}\}$ (symbols of the product, inverse element, identity and $\Sigma_{\mathcal{L}}$ denotes the equations that define a group). In general, groups (and other algebraic systems) are displayed as $\mathfrak{U} = (A; \cdot, i, 1)$, avoiding the use of the corresponding set of equations $\Sigma_{\mathcal{L}}$ for short. Here, only algebraic systems that are specified by a set of equations are considered (cf. [10]).

For two structures \mathfrak{U} and \mathfrak{V} for \mathcal{L} we say that $f: \mathfrak{U} \rightarrow \mathfrak{V}$ is *algebraically an \mathcal{L} -homomorphism* if $f(c_{\mathfrak{U}}) = c_{\mathfrak{V}}$ for each constant symbol c of \mathcal{L} , and $f(\Phi_{\mathfrak{U}}(x_1, \dots, x_n)) = \Phi_{\mathfrak{V}}(f(x_1), \dots, f(x_n))$ for each function symbol Φ of \mathcal{L} , of arity $n \geq 1$.

A *topological structure* (resp. *semitopological structure*) for \mathcal{L} is a pair (\mathfrak{U}, τ) where \mathfrak{U} is a structure for \mathcal{L} , and τ is a topology on A making all functions in \mathfrak{U} continuous (resp. separately continuous). We write \mathfrak{U} for (\mathfrak{U}, τ) if the topology is understood.

Let \mathfrak{U} and \mathfrak{V} be two (semi)topological structures of \mathcal{L} , and $f : \mathfrak{U} \rightarrow \mathfrak{V}$. The map f is an \mathcal{L} -*homomorphism* from \mathfrak{U} to \mathfrak{V} iff f is continuous and is algebraically an \mathcal{L} -homomorphism.

The class consisting of \mathcal{L} -topological (resp. \mathcal{L} -semitopological) structures and \mathcal{L} -morphisms defines a subcategory of **Top** that will be denoted by **Top** \mathcal{L} (resp. **STop** \mathcal{L}). For example, the category of topological groups **TopGrp** is specified by $\mathcal{L} = (\cdot, i, 1)$ with arities $(2, 1, 0)$.

We thank the referee for the information and references that follows next.

Remark 3.1. The notions we are dealing with here appeared at the end of the 19-th century in a text by Whitehead [29], but the theory began to develop well only after the theory of lattices was sufficiently well developed, in the thirties of the 20th century. This is the subject of *universal algebra*. Thus, what we have named an *algebraic system*, is also called a *variety*. As a general references to universal algebras, it is pertinent to mention here the volumes [5, 10, 17].

Definition 3.2. Let Φ be a n -ary function on X , i.e., $\Phi : X^n \rightarrow X$, where $n \geq 1$ is an integer. A Φ -congruence in X is an equivalence relation R in X such that if $x_i, y_i \in X$, $i = 1, \dots, n$ and $(x_i, y_i) \in R$ for $i = 1, \dots, n$, then $(\Phi(x_1, \dots, x_n), \Phi(y_1, \dots, y_n)) \in R$. (Observe that the analogue of this for constants c this is automatically satisfied, since $(c, c) \in R$).

Let \mathcal{L} be an algebraic system and let \mathfrak{U} be a structure for \mathcal{L} . If X is the domain of \mathfrak{U} and R is an equivalence relation on X that is a $\Phi_{\mathfrak{U}}$ -congruence for all function symbols $\Phi \in \mathcal{L}$, then we say that R is an \mathcal{L} -congruence.

The following proposition are well known, we include them for the reader's sake.

Proposition 3.3. [10, Lem. 2, p. 36] *Let Φ be a n -ary function on X and let R be a Φ -congruence. If $\pi: X \rightarrow X/R$ is the quotient map, then there is an n -ary map $\Phi_R: (X/R)^n \rightarrow X/R$ defined by $\Phi_R(\pi(x_1), \dots, \pi(x_n)) = \pi(\Phi(x_1, \dots, x_n))$.*

Corollary 3.4. [10, Th. 2, p. 58] *Let \mathcal{L} be an algebraic system and let \mathfrak{U} be a structure for \mathcal{L} . If X is the domain of \mathfrak{U} and R is an \mathcal{L} -congruence on X , then X/R is the domain for a structure $\mathfrak{V} = \mathfrak{U}/R$ for \mathcal{L} , with constant symbols c/R for constant symbols $c \in X$, and with function symbols $\Phi_{\mathfrak{V}}$ satisfying $\Phi_{\mathfrak{V}}(\pi(x_1), \dots, \pi(x_n)) = \pi(\Phi_{\mathfrak{U}}(x_1, \dots, x_n))$. Here $\pi: X \rightarrow X/R$ is the natural quotient map.*

Proposition 3.5. [10, Th. 1, p. 57] *Let Φ and Ψ be n -ary maps on X and Y , respectively. If $f: X \rightarrow Y$ is a map such that $f(\Phi(x_1, \dots, x_n)) = \Psi(f(x_1), \dots, f(x_n))$ for all $(x_1, \dots, x_n) \in X^n$, then there is a Φ -congruence R on X and an injective map $\tilde{f}: X/R \rightarrow Y$ that makes the following diagram commutative*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \pi & \nearrow \tilde{f} \\ & X/R & \end{array}$$

In particular $\tilde{f}(\Phi_R(\pi(x_1), \dots, \pi(x_n))) = \Psi(f(x_1), \dots, f(x_n))$.

The following result is a generalization of the first isomorphism theorem for arbitrary \mathcal{L} -structures.

Theorem 3.6. [10, Th. 1, p. 57] *Let $f: X \rightarrow Y$ be an \mathcal{L} -homomorphism from \mathfrak{A} to \mathfrak{B} . Then there is an \mathcal{L} -congruence R on X and an injective map $\tilde{f}: X/R \rightarrow Y$ that makes the following diagram commutative*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \pi & \nearrow \tilde{f} \\ & X/R & \end{array}$$

where \tilde{f} is an \mathcal{L} -homomorphism from \mathfrak{A}/R to \mathfrak{B} .

The next proposition is essential in many subsequent results. Here we present a stronger version than our initial result, for finitely many spaces, which has been kindly offered to us by the referee. Recall that given a family of topological spaces $\{X_i\}_{i \in I}$, a map $f: X_1 \times \dots \times X_n \rightarrow Y$ is named *separately continuous* when for every $i \in I$ and $\vec{x} \in \prod_{j \neq i} X_j$, the map $f_{\vec{x}}: X_i \rightarrow Y$, defined by $f_{\vec{x}}(x) \stackrel{\text{def}}{=} f(x; \vec{x})$, is continuous for every $\vec{x} \in \prod_{j \neq i} X_j$, where the symbol $;$ is used to mean that the variable x is placed at the coordinate i .

Proposition 3.7. *Let $\{X_i\}_{i \in I}$ a family of topological spaces and let $f: \prod_{i \in I} X_i \rightarrow Y$ be a separately continuous map into a topological space Y . If \mathcal{C} is an epireflective subcategory of \mathbf{Top} , then there is a (necessarily unique) separately continuous map $\bar{f}: \prod_{i \in I} r_{\mathcal{C}} X_i \rightarrow r_{\mathcal{C}} Y$.*

Proof. Since colimits are preserved by reflections, and actually by all functors having a left adjoint (cf. Adámek-Herrlich-Strecker, Prop. 18.10 (given there in dual formulation), or Herrlich-Strecker, Theorem 27.7), it will suffice to prove that the topology of separate continuity on a product (the so called *cross topology*) is a final structure, for certain maps from the factors of the product, hence is a colimit of a diagram having objects among the factors of the product.

Indeed, let X_i , for $i \in I$, be non-empty topological spaces (for some factor the empty space the product of the underlying sets is empty, so there is a unique topology on it.) Here for simplicity later we omit the index set I . We define on the underlying set of $\prod X_i$ the following topology. Let $\langle x_i \rangle \in \prod X_i$ have the following neighbourhood base. We take any open sets G_i of X_i containing x_i , and then the neighbourhood base of $\langle x_i \rangle$ will consist of the sets $\prod_{j \neq i} \{x_j\} \times G_i$ (a *cross*). This is the intersection of the topologies $X_i \times (\prod_{j \neq i} X_j)_{\text{discr}}$, where $i \in I$ is arbitrary, and the second factor has the discrete topology. Observe that this second factor can be given also in another way. We take the topological sum of $|\prod_{j \neq i} X_j|$ copies of X_i . Observe that the underlying set functor $U :: \mathbf{Top} \rightarrow \mathbf{Set}$ has both a left adjoint (discrete topologies on sets) and a right adjoint (indiscrete topologies on sets). Therefore U preserves all limits and colimits from \mathbf{Top} to \mathbf{Set} , in particular products.

In order to take the intersection of topologies, which is once more a colimit, we have to identify each set $\mathbf{r}_c X_i \times \prod_{j \neq i} \mathbf{r}_c X_j$ canonically to the set $\prod_{j \in I} \mathbf{r}_c X_j$. However, U preserves products, hence this makes the desired identification (by *iteration of products*).

In particular, the sum diagram, as a colimit, is preserved, therefore we have the sum of $|\prod_{j \neq i} X_j|$ many copies of $\mathbf{r}_c X_j$, with the canonical injections from the $\mathbf{r}_c X_j$ s,

where $j \neq i$. This is once more the product of $r_{\mathcal{C}}X_i$ and a discrete space of cardinality $|\prod_{j \neq i} X_j|$. We ought to reduce this cardinality to $|\prod_{j \neq i} r_{\mathcal{C}}X_j|$.

Observe that coequalizers, being colimits, are preserved by reflections. However, the image will be only a coequalizer, but not a cointersection of extremal epimorphisms. It will be a cointersection only if the domains of the morphisms (to be coequalized) are the same. This brings in once more that the cardinality $|\prod_{j \neq i} X_j|$ ought to be reduced to $|\prod_{j \neq i} r_{\mathcal{C}}X_j|$. Now recall that cointersections can be made iteratively. Namely, we divide the maps to be coequalized to (equivalence) classes, take the coequalizer of each class, which is an extremal epimorphism. Now we have a lot of extremal epimorphisms (as many as there are classes) and we have to take the cointersection of them, which is once more an extremal epimorphism. This gives the coequalizer of all considered morphisms.

Now the classes will be formed as follows. One class is formed by all $\langle x_j \mid j \neq i \rangle$, for which the images $\langle r_{\mathcal{C}, X_j}(x_j) \mid j \neq i \rangle$ are a fixed point of $\prod_{j \neq i} r_{\mathcal{C}}X_j$. \square

The next corollary follows from Proposition 3.7. However, we include the proof here for the reader's sake.

Corollary 3.8. *Let $\{X_1, \dots, X_n\}$ be finitely many topological spaces and let $f: X_1 \times \dots \times X_n \rightarrow Y$ be a separately continuous map into a topological space Y . If \mathcal{C} is an epireflective subcategory of \mathbf{Top} , then there is a (necessarily unique) separately continuous map $\bar{f}: r_{\mathcal{C}}X_1 \times \dots \times r_{\mathcal{C}}X_n \rightarrow r_{\mathcal{C}}Y$ such that $\bar{f}((r_{(X_1, \mathcal{C})}(x_1), \dots, r_{(X_n, \mathcal{C})}(x_n))) = r_{(Y, \mathcal{C})}(f((x_1, \dots, x_n)))$.*

Proof. In order to simplify the notation, we treat the case $n = 2$ only, as this is representative for the general case. The proof for $n > 2$ is obtained proceeding by induction.

For each fixed point $c \in X_2$, the map $f_c: X_1 \rightarrow Y$, defined by $f_c(x_1) := f(x_1, c)$, is continuous. Accordingly, there exists a continuous function $\mathbf{r}_e(f_c): \mathbf{r}_e X_1 \rightarrow \mathbf{r}_e Y$ that makes commutative the following digram

$$\begin{array}{ccc} X & \xrightarrow{f_c} & Y \\ \mathbf{r}_{(X,e)} \downarrow & & \downarrow \mathbf{r}_{(Y,e)} \\ \mathbf{r}_e X & \xrightarrow{\mathbf{r}_e(f_c)} & \mathbf{r}_e Y \end{array}$$

Thus, the map $\mathbf{r}_e(f_c)$ is defined, for each $\mathbf{r}_e X_1 \ni x'_1 = \mathbf{r}_{(X_1,e)}(x_1)$, by

$$\mathbf{r}_e(f_c)(x'_1) = \mathbf{r}_e(f_c)(\mathbf{r}_{(X_1,e)}(x_1)) = \mathbf{r}_{(Y,e)}(f_c(x_1)) = \mathbf{r}_{(Y,e)}(f(x_1, c)).$$

Furthermore, we have that if $\mathbf{r}_{(X,e)}(x_1) = \mathbf{r}_{(X,e)}(u_1)$ then $\mathbf{r}_e(f)(x_1, c) = \mathbf{r}_e(f)(u_1, c)$.

This crucial fact will be widely applied in what follows.

The equality above implies that the map $\tilde{f}: \mathbf{r}_e X_1 \times X_2 \rightarrow Y$, defined by

$$\tilde{f}((\mathbf{r}_{(X_1,e)}(x_1), x_2)) := \mathbf{r}_{(Y,e)}(f(x_1, x_2)),$$

is well defined and separately continuous. Repeating the argument for $\tilde{f}^d: X_2 \rightarrow Y$, defined by $\tilde{f}^d(x_2) := \tilde{f}(d, x_2)$ with $d \in \mathbf{r}_e X_1$, for each $d \in \mathbf{r}_e X_1$ there exists a continuous map $\mathbf{r}_e(\tilde{f}^d): \mathbf{r}_e X_2 \rightarrow \mathbf{r}_e Y$, satisfying

$$\mathbf{r}_e(\tilde{f}^d)(\mathbf{r}_{(X_2,e)}(x_2)) = \mathbf{r}_{(Y,e)}(\tilde{f}^d(x_2)).$$

It is now clear that the map $\overline{f}: \mathbf{r}_e X_1 \times \mathbf{r}_e X_2 \rightarrow \mathbf{r}_e Y$ defined by

$$\overline{f}(\mathbf{r}_{(X_1,e)}(x_1), \mathbf{r}_{(X_2,e)}(x_2)) = \mathbf{r}_{(Y,e)}(f((x_1, x_2)))$$

is well defined and separately continuous. □

As a consequence of the previous result, it follows that epireflections respect semitopological structures in the best possible fashion.

Proposition 3.9. *Let $\mathbf{STop}\mathcal{L}$ be a category of semitopological structures. If \mathcal{C} is an epireflective subcategory of \mathbf{Top} , then $r_{\mathcal{C}}(\mathbf{STop}\mathcal{L}) \subseteq \mathbf{STop}\mathcal{L}$. That is, for each \mathcal{L} -structure $(\mathfrak{U}, X) \in \mathbf{STop}\mathcal{L}$, where X is the domain of \mathfrak{U} , we have that $r_{\mathcal{C}}X$ is the domain of an \mathcal{L} -structure \mathfrak{V} such that $(\mathfrak{V}, r_{\mathcal{C}}X) \in \mathbf{STop}\mathcal{L}$. Furthermore, the reflection arrow $r_{(X, \mathcal{C})}: X \rightarrow r_{\mathcal{C}}X$ is an \mathcal{L} -homomorphism in $\mathbf{STop}\mathcal{L}$ for each $(\mathfrak{U}, X) \in \mathbf{STop}\mathcal{L}$.*

Proof. Let (\mathfrak{U}, X) be a semitopological structure for $\mathbf{STop}\mathcal{L}$. We equip $r_{\mathcal{C}}X$ with the algebraic structure \mathfrak{V} built by taking the constants $c_{\mathfrak{V}} := r_{(X, \mathcal{C})}(c_{\mathfrak{U}})$ for all constants $c \in \mathcal{L}$ and, if $\Phi \in \mathcal{L}$ is a separately continuous n -ary function symbol, then we apply Corollary 3.8 in order to define, for $x'_i \in r_{\mathcal{C}}X$, with $x'_i = r_{(X, \mathcal{C})}(x_i)$, for $x_i \in X$, as

$$\Phi_{\mathfrak{V}}((r_{(X, \mathcal{C})}(x_1), \dots, r_{(X, \mathcal{C})}(x_n))) = r_{(X, \mathcal{C})}(\Phi_{\mathfrak{U}}((x_1, \dots, x_n))).$$

This definition implies that $\Phi_{\mathfrak{V}}: (r_{\mathcal{C}}X)^n \rightarrow r_{\mathcal{C}}X$ is a well defined separately continuous n -ary function symbol for all $\Phi \in \mathcal{L}$. Thus $r_{\mathcal{C}}X$ is equipped with a semitopological \mathcal{L} -structure canonically inherited from the \mathcal{L} -structure in X . Furthermore, it is also clear that $r_{(X, \mathcal{C})}$ is a continuous \mathcal{L} -homomorphism. \square

We are now in position of establishing the main result in this section.

Theorem 3.10. *Let $\mathbf{STop}\mathcal{L}$ be a category of semitopological structures. If \mathcal{C} is an epireflective subcategory of \mathbf{Top} , then $r_{\mathcal{C}}(\mathbf{STop}\mathcal{L})$ is an epireflective subcategory of $\mathbf{STop}\mathcal{L}$.*

Proof. First, observe that, since $r_{(X, \mathcal{C})}: X \rightarrow r_{\mathcal{C}}X$ is an \mathcal{L} -homomorphism for all \mathcal{L} -structure $(\mathfrak{U}, X) \in \mathbf{STop}\mathcal{L}$, it follows that the equivalence relation $\{(x_1, x_2) \in X^2 :$

$\mathbf{r}_{(X,e)}(x_1) = \mathbf{r}_{(X,e)}(x_2)$ is an \mathcal{L} -congruence in X for all $(\mathfrak{U}, X) \in \mathbf{STop}\mathcal{L}$. By Proposition 3.9, we know that $\mathbf{r}_e(\mathbf{STop}\mathcal{L})$ is equipped with a semicontinuous \mathcal{L} -structure, where the reflection arrows $\mathbf{r}_{(X,e)}$ are epimorphisms in \mathbf{Top} . Thus it will suffice to show that \mathbf{r}_e preserves \mathcal{L} -morphisms.

Let (\mathfrak{U}, X) and (\mathfrak{U}', Y) be two semitopological structures in $\mathbf{STop}\mathcal{L}$ and let $(\mathfrak{V}, \mathbf{r}_e X)$ and $(\mathfrak{V}', \mathbf{r}_e Y)$ the structures in $\mathbf{STop}\mathcal{L}$, with domains $\mathbf{r}_e X$ and $\mathbf{r}_e Y$ canonically associated to the former by Proposition 3.9.

Let $f: X \rightarrow Y$ be a continuous \mathcal{L} -morphism. If $\Phi \in \mathcal{L}$ is a separately continuous n -ary function symbol, by the commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \mathbf{r}_{(X,e)} \downarrow & & \downarrow \mathbf{r}_{(Y,e)} \\ \mathbf{r}_e X & \xrightarrow{\mathbf{r}_e(f)} & \mathbf{r}_e Y \end{array}$$

it follows

$$\begin{aligned} (1) \quad \mathbf{r}_e(f)(\Phi_{\mathfrak{V}}(\mathbf{r}_{(X,e)}(x_1) \cdots, \mathbf{r}_{(X,e)}(x_n))) &= \mathbf{r}_e(f)(\mathbf{r}_{(X,e)}(\Phi_{\mathfrak{U}}((x_1, \cdots, x_n)))) \\ &= \mathbf{r}_{(Y,e)}(f(\Phi_{\mathfrak{U}}((x_1, \cdots, x_n)))) \\ &= \mathbf{r}_{(Y,e)}(\Phi_{\mathfrak{U}'}((f(x_1) \cdots, f(x_n)))) \\ &= \Phi_{\mathfrak{V}'}((\mathbf{r}_{(Y,e)}(f(x_1)), \cdots, \mathbf{r}_{(Y,e)}(f(x_n)))) \\ &= \Phi_{\mathfrak{V}'}((\mathbf{r}_e(f)(\mathbf{r}_{(X,e)}(x_1)) \cdots, \mathbf{r}_e(f)(\mathbf{r}_{(X,e)}(x_n)))). \end{aligned}$$

□

Theorem 3.10 allows us to obtain a neat realization of epireflections whose reflection arrows are quotient maps. Remark that we have shown that the epireflection functor in \mathbf{Top} coincides with the epireflection functor in $\mathbf{STop}\mathcal{L}$.

Corollary 3.11. *Let $\mathbf{STop}\mathcal{L}$ be a category of semitopological structures. If \mathcal{C} is an epireflective subcategory of \mathbf{Top} whose reflection arrows $r_{(X,\mathcal{C})}$ are quotient maps then $r_{\mathcal{C}}X = X/R_{\mathcal{C}}$ for all $(\mathfrak{A}, X) \in \mathbf{STop}\mathcal{L}$, where $R_{\mathcal{C}}$ coincides with the intersection of all \mathcal{L} -congruences R such that $X/R \in \mathcal{C}$.*

Proof. It suffices to apply Proposition 2.5 and Theorem 3.10. □

On the other hand, using Corollary 2.7, we obtain

Corollary 3.12. *Let $\mathbf{STop}\mathcal{L}$ be a category of semitopological structures and let X be a space in $\mathbf{STop}\mathcal{L}$. Then $r_{e_1}X = X/R_{e_1}$ where R_{e_1} is the intersection of all equivalence relations such that equivalence classes are closed in X , furthermore R_{e_1} is an \mathcal{L} -congruence in X .*

The following result was established by Tkachenko [26, Theorem 3.4] for semitopological groups. Here, we obtain a variant of his proof as an application of our results.

Corollary 3.13. *Let \mathcal{L} denote the algebraic system defined by groups and let $\mathbf{STop}\mathcal{L}$ be the corresponding category semitopological groups (group operation is separately continuous). Then $r_{e_1}G = G/H$ for all $G \in \mathbf{STop}\mathcal{L}$, where H is the intersection of all closed subgroups of G .*

Proof. First, remark that if R is an \mathcal{L} -congruence defined on G , then the R -equivalence classes are cosets of the normal subgroup $H_R \stackrel{\text{def}}{=} \{x \in G : (x, e_G) \in R\}$. Thus, using

Corollary 3.12, we obtain that $r_{e_1}G = G/H$, where H is a normal closed subgroup of G . Let us see that H the smallest closed subgroup of G . Indeed if K is a closed subgroup of G , then G/K is a T_1 space, therefore if $\pi: G \rightarrow G/K$ is the respective quotient mapping, we have a group homomorphism $r_{e_1}(\pi): G/H \rightarrow G/K$ that makes the following diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/K \\ & \searrow \mathbf{r}_{(G, e_1)} & \nearrow \mathbf{r}_{e_1}(\pi) \\ & & G/H \end{array}$$

commutative. This obviously means that $H \subseteq K$. \square

4. PRODUCTS

In this section we deal with the epireflections that preserve products, which is a crucial fact in order to study the preservation of topological structures.

Let \mathcal{C} be an epireflective category in **Top** and let $\{X_i\}$ a set (resp. finite set) of topological spaces. Then we have the following commutative diagram

$$\begin{array}{ccc} \prod X_i & \xrightarrow{\prod \mathbf{r}_{(X_i, e)}} & \prod r_e X_i \\ \mathbf{r}_{(\prod X_i, e)} \downarrow & & \downarrow \text{id} \\ r_e \prod X_i & \xrightarrow{\mathbf{r}_e(\prod \mathbf{r}_{(X_i, e)})} & \prod r_e X_i \end{array}$$

Set $\mu_e \stackrel{\text{def}}{=} r_e(\prod \mathbf{r}_{(X_i, e)})$. We notice that μ_e is defined uniquely by the condition that the following diagram commutes for every j

$$\begin{array}{ccc}
r_{\mathcal{C}} \prod X_i & \xrightarrow{\mu_{\mathcal{C}}} & \prod r_{\mathcal{C}} X_i \\
& \searrow r_{\mathcal{C}}(\pi_{X_j}) & \swarrow \pi_{r_{\mathcal{C}} X_j} \\
& & r_{\mathcal{C}} X_j
\end{array}$$

here $\pi_{r_{\mathcal{C}} X_j}$ denotes the respective canonical projection. If $\mu_{\mathcal{C}}$ is a homeomorphism onto $\prod r_{\mathcal{C}} X_i$ for every family of topological spaces (resp. finite family of topological spaces) then we say that \mathcal{C} *preserves products* (resp. \mathcal{C} *preserves finite products*).

For example, from [20, Prop. 6.2.1], there follows the following sufficient condition for the preservation of finite products.

Proposition 4.1. *Let \mathcal{C} be an epireflective category in \mathbf{Top} and let $\{X_i : i \in I\}$ be a set of topological spaces such that $r_{(X,e)}$ is open for all $X \in \mathbf{Top}$. Then \mathcal{C} preserves the product $\prod_{i \in I} X_i$.*

Proof. Clearly the map $\mu_{\mathcal{C}}$ defined above is onto. To prove that $\mu_{\mathcal{C}}$ is injective, it will suffice to repeat *mutatis mutandis* the argument used by T. Ishii in [20, Prop. 6.2.1]. Therefore, we have shown that $\mu_{\mathcal{C}}$ is a bijection. Moreover, since every reflection arrow $r_{(X,e)}$ is open, it follows that the reflection arrow $\prod_{i \in I} r_{(X_i,e)}$ is also open. Therefore, we have the commutative diagram

$$\begin{array}{ccc}
& \prod X_i & \\
r_{(\prod X_i, e)} \swarrow & & \searrow \prod r_{(X_i, e)} \\
r_{\mathcal{C}} \prod X_i & \xrightarrow{\mu_{\mathcal{C}}} & \prod r_{\mathcal{C}} X_i
\end{array}$$

Since both $r_{(\prod X_i, e)}$ and $\prod r_{(X_i, e)}$ are open and $\mu_{\mathcal{C}}$ is continuous, it follows that $\mu_{\mathcal{C}}$ is a homeomorphism. \square

It is a well-known fact that the T_0 -reflection in **Top** preserves all products but the T_1 , T_2 and T_3 reflections do not preserve all finite products (cf. [15, §1]).

The following result improves Proposition 3.8 for epireflections that preserve finite products. Again, we thank the referee for letting us notice that our results can be extended to infinite products.

Proposition 4.2. *Let $\{X_i : i \in I\}$ be a set of topological spaces and let $f: \prod_{i \in I} X_i \longrightarrow Y$ be a continuous map into a topological space Y . If \mathcal{C} is an epireflective subcategory of **Top** such that the epireflection preserves the product $\prod_{i \in I} X_i$, then there is a (necessarily unique) continuous map $\bar{f}: \prod_{i \in I} r_{\mathcal{C}} X_i \longrightarrow r_{\mathcal{C}} Y$ such that*

$$\bar{f}(\langle r_{(X_i, \mathcal{C})}(x_i) \rangle) = r_{(Y, \mathcal{C})}(f(\langle x_i \rangle)).$$

Proof. By hypothesis, the canonical homeomorphism

$$\mu_{\mathcal{C}}: r_{\mathcal{C}} \prod_{i \in I} X_i \rightarrow \prod_{i \in I} r_{\mathcal{C}} X_i$$

satisfies that the inverse mapping defined by $\mu_{\mathcal{C}}^{-1}(\langle r_{(X_i, \mathcal{C})}(x_i) \rangle) = r_{(\prod_{i \in I} X_i, \mathcal{C})}(\langle x_i \rangle)$ is well defined (i.e., it depends only on $r_{(X_i, \mathcal{C})}(x_i)$ and not on the choice of x_i), and is continuous. Therefore, we have the following commutative diagram:

$$\begin{array}{ccc}
 r_{\mathcal{C}} \prod_{i \in I} X_i & \begin{array}{c} \xleftarrow{\mu_{\mathcal{C}}} \\ \xrightarrow{\mu_{\mathcal{C}}^{-1}} \end{array} & \prod_{i \in I} r_{\mathcal{C}} X_i \\
 \downarrow r_{\mathcal{C}}(f) & \searrow & \swarrow \bar{f} \\
 & & r_{\mathcal{C}} Y \\
 \uparrow r_{(\prod_{i \in I} X_i, \mathcal{C})} & & \swarrow r_{(Y, \mathcal{C})} \\
 \prod_{i \in I} X_i & \xrightarrow{f} & Y
 \end{array}$$

Hence $\bar{f} = r_c(f) \circ \mu_c^{-1}$ is a continuous map. The equality stated in Prop. 4.2 follows from the commutativity of the lower square in the diagram. \square

The following result improves Theorem 3.10 when the reflection functor preserves products.

Theorem 4.3. *Let $\mathbf{Top}\mathcal{L}$ be a category of topological structures. If \mathcal{C} is an epireflective subcategory of \mathbf{Top} such that the epireflection preserves products, then $r_c(\mathbf{Top}\mathcal{L})$ is an epireflective subcategory of $\mathbf{Top}\mathcal{L}$.*

Proof. It suffices to observe that $\Phi_{r_c X}$ is continuous for every n -ary function symbol $\Phi \in \mathcal{L}$ as a consequence of Proposition 4.2. \square

Corollary 4.4. *Let $\mathbf{Top}\mathcal{L}$ be a category of topological structures and let \mathcal{C}_0 be the class of T_0 , then $r_{c_0}(\mathbf{Top}\mathcal{L})$ is an epireflective subcategory of $\mathbf{Top}\mathcal{L}$.*

Proof. The classes of T_0 spaces is both epireflective and preserve products in \mathbf{Top} . \square

Remark 4.5. Let \mathbf{TopGrp} denote the category of topological groups and let \mathcal{C}_0 denote the epireflective subcategory of T_0 spaces in \mathbf{Top} . Then $r_{c_0}(\mathbf{TopGrp})$ is an epireflective subcategory of \mathbf{TopGrp} . Furthermore, since every T_0 topological group is $T_{3.5}$, it follows that every member in $r_{c_0}(\mathbf{TopGrp})$ is $T_{3.5}$. In other words, for topological groups, the T_0 reflection and the $T_{3.5}$ reflection coincide.

There are more general structures than groups where the epireflective subcategories that are closed under finer topologies preserve products. An example of this are Mal'tsev spaces.

Definition 4.6. A *Mal'tsev operation* on a topological space X is a map $\Phi: X^3 \rightarrow X$ satisfying the identity $\Phi(x, x, y) = \Phi(y, x, x) = y$ for all $x, y \in X$. A space is a topological *Mal'tsev space* if it admits a continuous Mal'tsev operation.

For example, if G is a topological group, then the map $(x, y, z) \mapsto xy^{-1}z$ is a Mal'tsev operation on G . Hence every topological group is a Mal'tsev space. In like manner, a *semitopological Mal'tsev space* is a space that admits a separately continuous Mal'tsev operation.

The classes **TopMlt** (resp. **STopMlt**) of topological (resp. semitopological) Mal'tsev spaces are algebraic systems in **Top** with the continuous maps that respect these algebraic structures as arrows.

According to Gartside, Reznichenko and Sipacheva [8] topological Mal'tsev spaces were introduced by Uspenskij in [28] and have subsequently been studied by several authors. In this section we deal with these spaces and our main motivation is to transfer much of the behavior of topological groups to Mal'tsev spaces. We will see that most epi-reflective functors that preserve the topological group structure also respect the topological Mal'tsev operation.

The following result is attributed to Mal'tsev [19] by Reznichenko and Uspenskij [23, 4.11]. Our formulation is somewhat more general.

Lemma 4.7. *Let (X, Φ) be a semitopological Mal'tsev space and let R be a Φ -congruence in X . Then the quotient map $\pi: X \rightarrow X/R$ is open.*

Proposition 4.8. *Let (X, Φ) be a semitopological Mal'tsev space and let \mathcal{C} be an epi-reflective class in **Top** closed under supertopologies. Then the reflection arrow $r_{(X, \mathcal{C})}$ is an open map.*

Proof. By Theorem 3.10, we know that $(r_e X, r_e(\Phi))$ is a semitopological Mal'tsev space. Furthermore, the reflection arrow $r_{(X,e)}$ is a Φ -homomorphism. By Theorem 3.6, there is a Φ -congruence R such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{r_{(X,e)}} & r_e X \\ & \searrow \pi & \nearrow \tilde{r}_{(X,e)} \\ & X/R & \end{array}$$

By Theorem 2.3, $\tilde{r}_{(X,e)}$ is a quotient Φ -homomorphism. Since π is open by Lemma 4.7 and $\tilde{r}_{(X,e)}$ is one-to-one, it follows that $r_{(X,e)}$ is open. \square

From Proposition 4.1 and Proposition 4.8, we obtain.

Theorem 4.9. *Let \mathcal{C} be an epireflective category in \mathbf{Top} that is closed under finer topologies. If $\{X_i\}$ is a family of semitopological Mal'tsev spaces, then r_e preserves the product of $\{X_i\}$.*

Corollary 4.10. *If \mathcal{C} denotes the epireflective category of \mathbf{Top} defined by any of the following separation axioms: T_0 , T_1 , T_2 and functionally Hausdorff, then r_e preserves arbitrary products of semitopological Mal'tsev spaces.*

The next result is an application of the techniques developed in this paper. It shows that the modification of a Mal'tsev space by most separation axioms is a Mal'tsev space.

Theorem 4.11. *If \mathcal{C} denotes the epireflective category of \mathbf{Top} defined by any of the following separation axioms: T_0 , T_1 , T_2 and functionally Hausdorff, and \mathbf{TopMlt} is*

the subcategory of topological Mal'tsev spaces. Then $r_e(\mathbf{TopMlt})$ is an epireflective subcategory of \mathbf{TopMlt} .

Proof. Use Corollary 4.10 and Theorem 4.3. □

5. SUBSPACES

We say that an epireflection *preserves* subspaces if the following holds. For $i: A \hookrightarrow X$ an inclusion of a subspace we have that $r_e(i): r_e A \rightarrow r_e X$ is an inclusion of a subspace as well. More exactly: there is an inclusion of a subspace $i: B \hookrightarrow r_e X$ and a homeomorphism h such that $j = r_e(i) \circ h$. In this section, we study when epireflection functors preserve subspaces in topological algebraic structures.

Definition 5.1. Let X be a topological space. Two subsets A and B of X are said to be *completely separated* in X if there exists a continuous real valued map f such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

The space X is said to be *functionally Hausdorff* if any two different points can be completely separated by a real valued continuous function.

The space X is said to be *completely regular* if every closed set F of X is completely separated from any point $x \notin F$. A completely regular T_1 -space is called a $T_{3,5}$ -space.

A topological space X is said to be *Urysohn* if any two distinct points have disjoint closed neighbourhoods.

It is easy to show that epireflection functors do not preserve subspaces in general. In order to see this, consider the epireflective subcategories defined by the separation axioms T_0 , T_1 , T_2 , *Urysohn*, T_3 , *functionally Hausdorff*, *regular*, *completely regular* and $T_{3,5}$, that we denote by \mathcal{C}_0 , \mathcal{C}_1 , \mathcal{C}_2 , \mathcal{C}_u , \mathcal{C}_3 , \mathcal{C}_{fh} , \mathcal{C}_r , \mathcal{C}_{cr} and $\mathcal{C}_{3,5}$ respectively.

Take a set X of arbitrary infinite cardinality and an ideal point $p \notin X$ and set $X^* \stackrel{\text{def}}{=} X \cup \{p\}$ and consider the following two topologies on X^*

$$\tau_1 = \{U \subseteq X^* : p \notin U\} \cup \{X^*\}$$

$$\tau_2 = \{U \subseteq X^* : p \in U\} \cup \{\emptyset\}.$$

Set $X_i \stackrel{\text{def}}{=} (X^*, \tau_i)$, $1 \leq i \leq 2$ and pick a point x in X_1 . Since every neighbourhood of p in X_1 contains x , we have $r_{(X_1, \mathcal{C}_i)}(x) = r_{(X_1, \mathcal{C}_i)}(p)$ for each separation axiom in $\{T_1, T_2, \text{Urysohn}, \text{functionally Hausdorff}, \text{regular}, \text{completely regular and } T_{3.5}\}$. That is to say $r_{(X_1, \mathcal{C}_i)}$ is a single-valued map. Now, take X , which is a discrete, dense, open subset of X_1 . We have that $r_{\mathcal{C}_i}X = X \neq r_{(X_1, \mathcal{C}_i)}(X) = r_{(X_1, \mathcal{C}_i)}(p)$.

As for the \mathcal{C}_r (resp. \mathcal{C}_{cr}) reflection, remark that no nonempty closed subset of X_1 is contained in a proper open subset. Therefore $r_{\mathcal{C}_r}X_1 = r_{\mathcal{C}_{cr}}X_1$ is the indiscrete space and again $r_{\mathcal{C}_r}X = r_{\mathcal{C}_{cr}}X = X$, which yields $r_{(X_1, \mathcal{C}_r)}(X) \neq r_{\mathcal{C}_r}$ and $r_{(X_1, \mathcal{C}_{cr})}(X) \neq r_{\mathcal{C}_{cr}}$. This completes the proof for open subsets. For closed subsets, it suffices to take the space X_2 .

Definition 5.2. Let X be a topological space and let \mathcal{A} be a subset of **Top**. A subset A of X is said \mathcal{A} -*oset* if there is a space $Y \in \mathcal{A}$ and a continuous map $f: X \rightarrow Y$ such that $A = f^{-1}(U)$ for some open subset U of Y . It is clear that the family of all \mathcal{A} -osets forms a subbase in X for the initial topology $\tau_{\mathcal{A}}$, with respect to all continuous maps $X \rightarrow Y \in \mathcal{A}$. The subsets $G \in \tau_{\mathcal{A}}$ are called \mathcal{A} -*open*. A subset F of X is said \mathcal{A} -*closed*, if $X \setminus F$ is \mathcal{A} -open.

In case $\mathcal{A} = \mathcal{C}_i$ for some $i \in \{T_1, T_2, u, r, fh, cr, T_{3.5}\}$, we will use the symbolism \mathbf{T}_i -*open* for short.

Lemma 5.3. *Let X be a topological space and let $\mathcal{C} = \mathcal{C}(\mathcal{A})$ be an epireflective subcategory of \mathbf{Top} that is generated by a family of spaces $\mathcal{A} \subseteq \mathbf{Top}$. Given a subset A of X , the following assertions are equivalent:*

- (a) A is \mathcal{C} -open (res. \mathcal{C} -closed).
- (b) $A = r_{(X,\mathcal{C})}^{-1}(U)$, for some open (resp. closed) subset U of $r_{\mathcal{C}}X$.
- (c) A is \mathcal{A} -open (res. \mathcal{A} -closed)

Proof. (a) \Rightarrow (b). Let A be a \mathcal{C} -open subset of X . Then there are $Y \in \mathcal{C}$, V open in Y and a continuous map $f: X \rightarrow Y$ such that $f^{-1}(V) = A$. By the functorial definition of epireflections, there exists a continuous map $g: r_{\mathcal{C}}X \rightarrow Y$ such that $g \circ r_{(X,\mathcal{C})} = f$. Therefore $A = f^{-1}(V) = r_{(X,\mathcal{C})}^{-1}(g^{-1}(V))$ and it suffices to take $U = g^{-1}(V)$. Now if $A \subseteq X$ is \mathcal{C} -open, then $A = \bigcup_{i \in I} (\bigcap_{j \in J_i} A_{ij})$, where each J_i is finite and $A_{ij} = r_{(X,\mathcal{C})}^{-1}(U_{ij})$ for some open subset U_{ij} of $r_{\mathcal{C}}X$. Then

$$A = \bigcup_{i \in I} \left(\bigcap_{j \in J_i} r_{(X,\mathcal{C})}^{-1}(U_{ij}) \right) = r_{(X,\mathcal{C})}^{-1} \left(\bigcup_{i \in I} \left(\bigcap_{j \in J_i} U_{ij} \right) \right)$$

and we may choose

$$U \stackrel{\text{def}}{=} \bigcup_{i \in I} \left(\bigcap_{j \in J_i} U_{ij} \right).$$

(b) \Rightarrow (c). We have seen in Section 2 that the space $r_{\mathcal{C}}X$ can be realized as the diagonal of a product $\prod_{f \in E} Y_f$ where $Y_f \in \mathcal{A}$ for all $f \in E$ and f stands for a surjective continuous map $f: X \rightarrow Y_f$. Thus, the family $\{\pi_f^{-1}(W) \cap r_{\mathcal{C}}X : W \text{ open in } Y_f, f \in E\}$ form an open subbase in $r_{\mathcal{C}}X$ and, as a consequence, the topologies $\tau_{\mathcal{C}}$ and $\tau_{\mathcal{A}}$ coincide. Therefore, if $A = r_{(X,\mathcal{C})}^{-1}(U)$ for some open subset U of $r_{\mathcal{C}}X$, it follows that $A \in \tau_{\mathcal{A}}$.

(c) \Rightarrow (a) is obvious. □

The following result characterizes when an epireflection functor preserves subspaces.

Proposition 5.4. *Let \mathcal{C} be an epireflective subcategory of \mathbf{Top} and let X be a topological space. If A is a subspace of X we have that the epireflection \mathcal{C} preserves the subspace $A \hookrightarrow X$ if and only if the following two properties are satisfied:*

- (1) *For all a_1, a_2 in A such that $r_{(A,\mathcal{C})}(a_1) \neq r_{(A,\mathcal{C})}(a_2)$, we have $r_{(X,\mathcal{C})}(a_1) \neq r_{(X,\mathcal{C})}(a_2)$;*
- (2) *For every \mathcal{C} -closed (resp. \mathcal{C} -open) subset F of A , there is a \mathcal{C} -closed (resp. \mathcal{C} -open) subset E of X such that $E \cap A = F$.*

Proof. Suppose that \mathcal{C} preserves the subspace $A \hookrightarrow X$. For simplicity, we assume that the \mathcal{C} -image of the subspace inclusion $A \hookrightarrow X$ is the subspace inclusion $r_e A \hookrightarrow r_e X$ (that is $r_e A = r_{(X,\mathcal{C})}(A)$). Then (1) is obviously satisfied. As for (2) let U be a \mathcal{C} -open subset of A . By Lemma 5.3, there is an open subset V in $r_e A$ such that $U = r_{(A,\mathcal{C})}^{-1}(V)$. Furthermore, by hypothesis, we may assume that there is an open set W in $r_e X$ such that $V = r_e A \cap W$. Thus

$$U = r_{(A,\mathcal{C})}^{-1}(V) = r_{(A,\mathcal{C})}^{-1}(r_e A \cap W) \subseteq r_{(X,\mathcal{C})}^{-1}(r_e A) \cap r_{(A,\mathcal{C})}^{-1}(W) = r_{(X,\mathcal{C})}^{-1}(r_e A) \cap r_{(X,\mathcal{C})}^{-1}(W).$$

Since also $U \subseteq A$, therefore we have

$$U \subseteq A \cap r_{(X,\mathcal{C})}^{-1}(r_e A) \cap r_{(X,\mathcal{C})}^{-1}(W),$$

and, conversely,

$$U = r_{(A,\mathcal{C})}^{-1}(V) = r_{(A,\mathcal{C})}^{-1}(r_e A \cap W) \supseteq A \cap r_{(X,\mathcal{C})}^{-1}(W),$$

since $a \in A$ and $r_{(X,\mathcal{C})}(a) = r_{(A,\mathcal{C})}(a) \in W$ imply $r_{(A,\mathcal{C})}(a) \in r_e A \cap W$. This proves that $U = r_{(X,\mathcal{C})}^{-1}(W) \cap A$.

Conversely, suppose that (1) and (2) are satisfied. Let $j : A \hookrightarrow X$ be the inclusion of the subspace A to X and let $g : r_e A \longrightarrow r_e X$ be a continuous mapping that makes the following diagram

$$\begin{array}{ccc}
A & \xrightarrow{j} & X \\
\mathbf{r}_{(A,\mathcal{C})} \downarrow & & \downarrow \mathbf{r}_{(X,\mathcal{C})} \\
\mathbf{r}_{\mathcal{C}}A & \xrightarrow{g} & \mathbf{r}_{\mathcal{C}}X
\end{array}$$

commutative

It is clear, by (1), that g is injective. We will show that g is also open.

Let V be an open subset in $\mathbf{r}_{\mathcal{C}}A$. Then $\mathbf{r}_{(A,\mathcal{C})}^{-1}(V)$ is \mathcal{C} -open in A . By (2), there is a \mathcal{C} -open subset U of X such that $\mathbf{r}_{(A,\mathcal{C})}^{-1}(V) = j^{-1}(U \cap j(A))$. By the commutativity of the diagram, surjectivity of $\mathbf{r}_{(A,\mathcal{C})}$ and $\mathbf{r}_{(X,\mathcal{C})}$, and taking into account that U is the inverse image of an open set in $\mathbf{r}_{\mathcal{C}}X$, it follows that $g(V) = \mathbf{r}_{(X,\mathcal{C})}(U) \cap \mathbf{r}_{(X,\mathcal{C})}(A)$, which is open in $\mathbf{r}_{\mathcal{C}}X$. This completes the proof. \square

Definition 5.5. Let X be a topological space and let \mathcal{A} denote a class of topological spaces, we say that a subset A of X is \mathcal{A} -*embedded* when for every continuous map $f: A \rightarrow Z$, with $Z \in \mathcal{A}$, there exists $Y \in \mathcal{A}$, with $Z \subseteq Y$, a continuous map $\bar{f}: X \rightarrow Y$ such that $\bar{f}|_A = f$. In other words, every continuous map on A taking values in a space in \mathcal{A} can be extended to a continuous map on X taking values in a possibly different larger space in \mathcal{A} .

Corollary 5.6. *Let X be a topological space and let $\mathcal{C} = \mathcal{C}(\mathcal{A})$ be the epireflective subcategory of \mathbf{Top} that is generated by \mathcal{A} . If A is an \mathcal{A} -embedded subspace of X , then the epireflection functor associated to \mathcal{C} preserves the subspace $A \hookrightarrow X$.*

Proof. We apply Proposition 5.4 for the proof. In order to verify assertion (1), suppose that a_1, a_2 are two points in A such that $\mathbf{r}_{(A,\mathcal{C})}(a_1) \neq \mathbf{r}_{(A,\mathcal{C})}(a_2)$. We have seen in Section

2 that the space $r_{\mathcal{C}}A$ can be realized as the diagonal of a product $\prod_E X = \prod_{f \in E} Y_f$ where $Y_f \in \mathcal{A}$ for all $f \in E$ and f stands for a surjective continuous map $f: A \rightarrow Y_f$. Thus, there is a map $f: A \rightarrow Z \in \mathcal{A}$ such that $f(a_1) \neq f(a_2)$. Then there exists $Y \in \mathcal{A}$, with $Z \subseteq Y$ and a continuous map $\bar{f}: X \rightarrow Y$ that makes following diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & Z \\ \downarrow j_{(A,X)} & & \downarrow j_{(Z,Y)} \\ X & \xrightarrow{\bar{f}} & Y \end{array}$$

commutative. Thus, we have that $\bar{f}(a_1) \neq \bar{f}(a_2)$, which implies that $r_{(X,\mathcal{C})}(a_1) \neq r_{(X,\mathcal{C})}(a_2)$.

As for assertion (2), it suffices to observe that the collection of \mathcal{A} -osets forms an open subbase for the topology $\tau_{\mathcal{A}} = \tau_{\mathcal{C}}$. \square

In [25], M. Tkachenko proves that the \mathcal{C}_0 -reflection respects arbitrary subgroups. The next corollary improves this result.

Corollary 5.7. *The \mathcal{C}_0 -reflection preserves subspaces.*

Even though the \mathcal{C}_1 -reflection does not preserve subspaces, our methods provide a neat characterization of this property. First we need the following lemma.

Lemma 5.8. *Let A be a subset of a topological space X . Then A is T_1 -closed if and only if there is a continuous mapping $f: X \rightarrow Y_{\text{cof}}$, where Y_{cof} is a set equipped with the cofinite topology, such that $A = f^{-1}(p)$ for a singleton $p \in Y$.*

Proof. Suppose that A is T_1 -closed. Then there is a T_1 -space Z and a continuous mapping $g: X \rightarrow Z$ such that $A = g^{-1}(B)$ for a closed subset B in Z . If we identify B with a singleton, say p_B , and define $Y \stackrel{\text{def}}{=} (Z \setminus B) \cup \{p_B\}$. Then the map $f: X \rightarrow Y_{\text{cof}}$, defined by $f(x) = g(x)$ if $x \notin g^{-1}(B)$ and $f(x) = p_B$ if $x \in g^{-1}(B)$, is continuous and $A = f^{-1}(p_B)$. The converse implication is obvious. \square

Since the category \mathbf{Top}_1 of T_1 -spaces is generated by the spaces equipped with the cofinite topology, we see that a subset A of a topological space X is T_1 -embedded if and only if for every continuous map $f: A \rightarrow Z_{\text{cof}}$, where Z is a set equipped with the cofinite topology, there exists a set Y , with $Z \subseteq Y$, a continuous map $\bar{f}: X \rightarrow Y_{\text{cof}}$ such that $\bar{f}|_A = f$.

Lemma 5.9. *Every T_1 -closed subspace A of a topological space X is T_1 -embedded.*

Proof. Let $f: A \rightarrow Z_{\text{cof}}^1$ be a continuous map defined on A . By Lemma 5.8, there is a continuous map $g: X \rightarrow Z_{\text{cof}}^2$ and a point $p \in Z^2$ such that $A = g^{-1}(p)$. Set $Y \stackrel{\text{def}}{=} Z^1 \sqcup Z^2$, the disjoint union of Z^1 and Z^2 , and define $\tilde{f}: X \rightarrow Y_{\text{cof}}$ by $\tilde{f}(x) = f(x)$ if $x \in A$ and $\tilde{f}(x) = g(x)$ if $x \notin A$. It is clear that inverse image by \tilde{f} of every singleton in Y is closed in X , which yields the continuity of the map. \square

Theorem 5.10. *If A is a T_1 -closed subspace of a topological space X , then $r_e A = r_{(X,e)}(A)$.*

Proof. Apply Corollary 5.6 and Lemma 5.9. \square

In [25, Lemma 3.7] M. Tkachenko proved that the T_1 -reflection preserves closed subgroups in the category of semitopological groups. The next Corollary is a variant of Tkachenko's result. Again, our formulation is somewhat more general. First, we

recall that a *left topological group* (resp. *right topological group*) is a group G equipped with a topology such that the left translations $x \mapsto ax$ are continuous (resp. the right translations $x \mapsto xa$ are continuous).

Corollary 5.11. *Let G be a left (resp. right) topological group and let H be a closed subgroup of G . Then $r_{\mathcal{C}}H = r_{(G,\mathcal{C})}(H)$.*

Proof. There is no loss of generality in assuming that G is a left topological group. Let $X \stackrel{\text{def}}{=} (G/H)$ be the quotient space G/H . It is easily seen that X is a T_1 -space. Furthermore, if $\pi: G \rightarrow X$ denotes the canonical quotient map, we have that $H = \pi^{-1}(\pi(H))$, which implies that H is T_1 -closed in G . Thus it suffices apply Theorem 5.10. □

6. COINCIDENCE OF EPIREFLECTIONS

The following general question is dealt with in this section: Let \mathcal{C} and \mathcal{E} be two epireflective subcategories of \mathbf{Top} such that $\mathcal{C} \supseteq \mathcal{E}$. Characterize the spaces X such that $r_{\mathcal{C}}X = \mathcal{E}(X)$. This topic has been studied in [6, 7] where it is left as a specific open question to characterize the spaces X for which $r_{\mathcal{C}_1}X = r_{\mathcal{C}_{3.5}}X$, where $r_{\mathcal{C}_1}$ and $r_{\mathcal{C}_{3.5}}$ are the epireflection functors associated to \mathcal{C}_1 and $\mathcal{C}_{3.5}$, the subcategories of T_1 -spaces and $T_{3.5}$ spaces, respectively.

Our approach is based on the notion of \mathcal{C} -open subset that has been introduced previously. First, we recall that in the category \mathbf{Top}_0 the epimorphisms are not the surjections, like they are for \mathbf{Top} . Hence also epireflective has another meaning (cf. [4]).

Theorem 6.1. *Let \mathcal{C} and \mathcal{E} be two epireflective subcategories of \mathbf{Top} such that $\mathbf{Top}_0 \supseteq \mathcal{C} \supseteq \mathcal{E}$ and let X be a topological space. Then $r_{\mathcal{C}}X = r_{\mathcal{E}}X$ if and only if every \mathcal{C} -open subset of X is \mathcal{E} -open.*

Proof. The “only if” part is obvious. As for the “if” part, consider the following commutative diagram

$$\begin{array}{ccc} & X & \\ \mathbf{r}_{(X,\mathcal{C})} \swarrow & & \searrow \mathbf{r}_{(X,\mathcal{E})} \\ r_{\mathcal{C}}X & \xrightarrow{\mathbf{r}_{(r_{\mathcal{C}}X,\mathcal{E})}} & r_{\mathcal{E}}X \end{array}$$

where $\mathbf{r}_{(r_{\mathcal{C}}X,\mathcal{E})}$ is the unique continuous map canonically defined since $\mathcal{C} \supseteq \mathcal{E}$. It will suffice to verify that $\mathbf{r}_{(r_{\mathcal{C}}X,\mathcal{E})}$ is 1-to-1 and open. Suppose first that x, y belong to X and $\mathbf{r}_{(X,\mathcal{C})}(x) \neq \mathbf{r}_{(X,\mathcal{C})}(y)$. By our initial assumption \mathcal{C} is included in \mathbf{Top}_0 . Thus there is an open subset W in $r_{\mathcal{C}}X$ that contains exactly one of these points. Assume wlog that $\mathbf{r}_{(X,\mathcal{C})}(x) \in W \not\ni \mathbf{r}_{(X,\mathcal{C})}(y)$, which yields $x \in \mathbf{r}_{(X,\mathcal{C})}^{-1}(W) \not\ni y$. By hypothesis and Lemma 5.3, there must be an open subset V of $r_{\mathcal{E}}X$ such that $\mathbf{r}_{(X,\mathcal{C})}^{-1}(W) = \mathbf{r}_{(X,\mathcal{E})}^{-1}(V)$. Thus $\mathbf{r}_{(X,\mathcal{E})}(x) \neq \mathbf{r}_{(X,\mathcal{E})}(y)$, which proves the injectivity of $\mathbf{r}_{(r_{\mathcal{C}}X,\mathcal{E})}$.

Now, let W be an arbitrary open subset of $r_{\mathcal{C}}X$. Again, there must be an open subset V of $r_{\mathcal{E}}X$ such that $\mathbf{r}_{(X,\mathcal{C})}^{-1}(W) = \mathbf{r}_{(X,\mathcal{E})}^{-1}(V)$. Furthermore, the commutativity of the diagram above implies that $\mathbf{r}_{(X,\mathcal{C})}^{-1}(W) = \mathbf{r}_{(X,\mathcal{C})}^{-1}(\mathbf{r}_{(r_{\mathcal{C}}X,\mathcal{E})}^{-1}(V))$. This implies that $W = \mathbf{r}_{(r_{\mathcal{C}}X,\mathcal{E})}^{-1}(V)$ and, as a consequence that $\mathbf{r}_{(r_{\mathcal{C}}X,\mathcal{E})}(W) = V$. This completes the proof. \square

The following result answers Question 1.6 in [6], repeated in [7, Question 1.9].

Corollary 6.2. *Let X be a topological space. Then $r_{\mathcal{C}_1}X = r_{\mathcal{C}_{3.5}}X$ if and only if every T_1 -closed subset F of X is completely separated from any point $x \notin F$.*

Proof. Necessity: Suppose that $r_{\mathcal{C}_1}X = r_{\mathcal{C}_{3.5}}X$. By Theorem 6.1, every T_1 -closed subset F of X is $\mathcal{C}_{3.5}$ -closed. Therefore, by Lemma 5.3, there is a closed subset $E \subseteq r_{\mathcal{C}_{3.5}}X$ such that $F = r_{(X,\mathcal{C})}^{-1}(E)$. Thus, $r_{(X,\mathcal{C})}(x) \notin E$ for all $x \notin F$. Since $r_{\mathcal{C}_{3.5}}X$ a $T_{3.5}$ space, this implies that F is completely separated from any point $x \notin F$.

Sufficiency: Let U be a T_1 -open subset of X , we must verify that U is $\mathcal{C}_{3.5}$ -open in order to apply Theorem 6.1. By hypothesis $X \setminus U$ is completely separated from any point $x \in U$. Hence $X \setminus U = r_{(X,\mathcal{C}_{3.5})}^{-1}(r_{(X,\mathcal{C}_{3.5})}(X \setminus U))$, which implies that $X \setminus U$ is $\mathcal{C}_{3.5}$ -closed and therefore U must be a $\mathcal{C}_{3.5}$ -open subset of X , which completes the proof. \square

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