McShane-Whitney extensions for fuzzy Lipschitz maps

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Abstract

We present a McShane-Whitney extension theorem for real-valued fuzzy Lipschitz maps defined between fuzzy metric spaces. Motivated by the potential applications of the obtained results, we generalize the mathematical theory of extensions of Lipschitz maps to the fuzzy context. We develop the problem in its full generality, explaining the similarities and differences with the classical case of extensions on metric spaces.

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1. Introduction and basic definitions

The so called McShane-Whitney Theorem is a classical result which establishes that, given a real valued function on a subspace of a metric space, it can always be extended to the whole space preserving the Lipschitz constant. This theoretical result can be easily proved using a constructive procedure: the extension is given by concrete known formulas involving a supremum—the McShane formula— or an infimum—the Whitney formula—. The aim of the present paper is to obtain a similar result for the case of fuzzy Lipschitz maps on fuzzy metric spaces as a contribution to the general theory of these spaces, that could find applications in different fields.

In the case of finite spaces, the formulas cited above provide effective computational tools for getting these Lipschitz extensions, and allow to develop useful applications. Among them, we are particularly interested in the construction of algorithms for artificial intelligence. A classic and simple model for some machine learning developments is based on controlled extensions of real valued functions that act in subspaces of metric spaces. For example, suppose we have a metric model for a given problem —i.e., the individuals in the model are elements of a subspace (S_0, d) of a metric space (X, d)—, and assume that the learning tool is a real function I. A basic reinforcement learning scheme is then provided by an increasing subset of metric subspaces $S_0 \subseteq S_1 \subseteq S_2 \cdots \subseteq X$ and the corresponding extensions of the index I_0, I_1, $I_2, \cdots$, which improve as the process progresses, since we incorporate new information about the system at each step. Similar arguments provide standard tools in distance learning (see for example [8]). This and other issues on the relation among machine learning and Lipschitz functions are of current interest; the reader can find concrete information about in [1, 8, 9, 19, 32, 24] and the references therein. Concretely, the authors of the present paper are involved in the development of algorithms for reinforcement learning based on the basic idea explained above. Although this is still an open project, some results on this research have been already published and can be found in [5].

On the other hand, nowadays it seems to be an objective of the scientific community to enrich the set of mathematical tools for machine learning by introducing increasingly sophisticated fuzzy methods. The nature of the topic itself and the multiple applications of the different areas of the machine learning motivated an early introduction of techniques of fuzzy mathematics in these areas, both to justify theoretical developments and for concrete applications (see for example [18, 36]).

Motivated in part by this situation, the second objective of this work is to merge the theoretical contexts of Lipschitz maps and fuzzy metrics to facilitate the construction of new tools in machine learning. In particular, our aim is to adapt a central theorem of the theory of Lipschitz functions —the McShane-Whitney extension theorem for real valued Lipschitz functions— to the framework of fuzzy metric spaces (see [2, 4, 31, 34]).

Let us recall some fundamental definitions. A map $f : (X, d) \to (Y, q)$ between metric spaces is said to be Lipschitz if there is a constant $K > 0$ such that

$$q(f(x), f(y)) \leq K d(x, y), \quad x, y \in X.$$

The Lipschitz constant of f is the infimum of all the constants K satisfying the inequality. It is well known that Lipschitz real functions acting in metric subspaces can always be extended to the whole metric space preserving the Lipschitz constant. Indeed, the so called McShane-Whitney Theorem establishes that if S is a subspace of a metric space (X, d) and $f : S \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant K, there exists an extension to X with the same Lipschitz constant. In fact, a lot of extensions are available. For example, the following function

$$f^M(x) := \sup_{s \in S}\{f(s) - K\, d(s, x)\}, \quad x \in X,$$

that is called the McShane extension of f, gives one of them, and also the Whitney extension defined as

$$f^W(x) := \inf_{s \in S}\{f(s) + K\, d(s, x)\}, \quad x \in X.$$

The aim of the present paper is to extend this result to the setting of real-valued fuzzy Lipschitz functions $f : (X, M, *) \to (\mathbb{R}, N, \circledast)$ between fuzzy metric spaces as introduced in [37]: f is fuzzy Lipschitz if for every $t > 0$, the supremum

$$\sup_{x \neq y} \frac{1 - N(f(x), f(y), t)}{1 - M(x, y, t)}$$

is finite. However, in order to extend the result in a coherent way, we will first consider functions $f : X \times (0, \infty) \to \mathbb{R}$, that is, we will consider explicitly the dependence of the original map on the parameter t. Our main theoretical result —Corollary 22— gives an extension theorem for fuzzy Lipschitz maps.

If $(X, M, *)$ is a fuzzy metric space and $(\mathbb{R}, N, \circledast)$ is what we call a Euclidean fuzzy metric space, suppose $S \subseteq X$ and $f : (S, M, *) \times (0, \infty) \to (\mathbb{R}, N, \circledast)$ is a fuzzy Lipschitz map. Then, under some requirements on M, f can be extended to a fuzzy Lipschitz map $f : X \times (0, \infty) \to \mathbb{R}$.

Although this result seems to be the most adequate generalization after understanding the nature of fuzzy Lipschitz maps, from the point of view of the applications seems to be relevant the case of extending fuzzy Lipschitz functions non-depending on t acting in S to fuzzy Lipschitz functions non-depending on t acting in X. We will show that this is also possible using our construction.

We intend in this way to provide new tools for opening the door to a new methodological approach to machine learning, including fuzzy notions in the design of algorithms of artificial intelligence that use Lipschitz extensions. Following the principles underlying the fuzzy philosophy, our technique allows to modulate, using the "t" parameter, the level of uncertainty under which the function fulfills the Lipschitz inequality. This idea is automatically translated into the context of machine learning, thus allowing to build in future research work a technique to introduce probabilistic interpretations into forecasting methods based on reinforcement learning such as those that can be found in [5].

The structure of the paper is as follows. We begin recalling in Section 2 some necessary concepts about fuzzy metric spaces. Our technique for extending a fuzzy Lipschitz map is based on a method for constructing a family of metrics from a fuzzy metric. The problem of obtaining a compatible metric from a fuzzy metric has already treated in the literature [6, 12, 26, 27]. We summarize some results in Section 3 and provide a new method in Lemma 7. Section 4 is devoted to introduce the concept of fuzzy Lipschitz function as considered in [37]. For completeness, we provide in this section some results about the relationship of this concept with other notions of Lipschitzness in the fuzzy framework coming from fuzzy contractiveness notions. Finally, in Section 5 we obtain a McShane-Whitney extension theorem for fuzzy Lipschitz maps taking values in what we call a Euclidean fuzzy metric space.

2. Fuzzy metric spaces

The origins of fuzzy metric spaces are due mainly to Menger [21] (see also [29]) who introduced the concept of probabilistic metric space which gives a probabilistic interpretation of the distance between two points by assigning a distribution function with every pair of elements. This concept has evolved in the last decades to various concepts of fuzzy metrics. One of the most widespread notion is that due to George and Veeramani [10] (see also [11]) and this will be the notion that we will work with. Let us recall its definition.

Definition 1 ([10])**.** *A triple* $(X, M, *)$ *is called a* fuzzy metric space *if* X *is a nonempty set,* $*$ *is a continuous* t*-norm, and* M *is a fuzzy set on* $X^2 \times (0, \infty)$ *such that for each* $x, y, z \in X$ *and* $t, s > 0$,

(1) $M(x, y, t) > 0$,

(2) $M(x, y, t) = 1$ *if and only if* $x = y$,

(3) $M(x, y, t) = M(y, x, t)$,

(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$, *and*

(5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$, *is continuous.*

The pair $(M, *)$ *is said to be a fuzzy metric on* X.

Example 2 ([10])**.** *Given a metric space* (X, d)*, let* M_d *be the fuzzy set on* $X^2 \times (0, \infty)$ *defined by*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

For every continuous t-norm $*$, $(M_d, *)$ *is a fuzzy metric on* X *which is called the standard fuzzy metric induced by* d.

The class of stationary fuzzy metric spaces was introduced in [15] when the authors were studying completions of fuzzy metric spaces. We extend this definition as follows.

Definition 3 (see [15])**.** *A fuzzy metric space* $(X, M, *)$ *is said to be* eventually stationary *(or* $(M, *)$ *is a eventually stationary fuzzy metric on* X*) if we can find* $t_0 > 0$ *such that the function* $M(x, y, \cdot) : [t_0, +\infty) \to [0, 1]$ *is constant for every* $x, y \in X$.

$(M, *)$ *is said to be* stationary *if* $M(x, y, \cdot)$ *is constant for every* $x, y \in X$.

Another class of fuzzy metric spaces which includes the stationary fuzzy metric spaces are the so-called strong fuzzy metric spaces introduced and studied in [12].

Definition 4 ([12])**.** *A fuzzy metric space* $(X, M, *)$ *is said to be* strong *if*

$$M(x, y, t) * M(y, z, t) \leq M(x, z, t)$$

for all $x, y, z \in X$ *and all* $t > 0$.

It is obvious that every stationary fuzzy metric space is strong. Furthermore, $(M, *)$ is strong if and only if $\{(M_t, *) : t > 0\}$ is a family of stationary fuzzy metrics on X associated to M, where $M_t : X \times X \times (0, +\infty) \to [0, 1]$ is given by $M_t(x, y, s) = M(x, y, t)$, for all $x, y \in X$ and all $s > 0$.

3. Metrics from fuzzy metrics

George and Veeramani [10] showed that every fuzzy metric $(M, *)$ on a nonempty set X generates a Hausdorff topology $\tau(M)$ on X. Moreover, this topology is metrizable [14]. This naturally leads to the problem of constructing in an easy way a metric d on X compatible with the topology $\tau(M)$ such that we can infer results for the fuzzy metric $(M, *)$ from classic results about metrics. In this way Radu constructed in [26] such a metric d which has been successfully applied to prove fixed point theorems for complete fuzzy metric spaces from classic results in the context of metric spaces. This construction was improved and modified in [6, 27] which allowed the authors to prove several fixed point theorems for different types of contractions in the context of fuzzy metric spaces. Their main tool is the construction of a metric from a fuzzy metric such that it preserves a certain contraction notion. In a different way, Gregori, Morillas and Sapena [12] developed a method for constructing a metric from a strong fuzzy metric as follows.

Proposition 5 ([12])**.** *Let* $(X, M, *)$ *be a strong fuzzy metric space such that* $* \geq *_L$*, where* $*_L$ *is the Łukasiewicz t-norm. Let* $\{M_t : t > 0\}$ *be the family of stationary fuzzy metrics associated to* $(M, *)$*. Then*

(i) $\{d_t^M : t > 0\}$ *is a family of metrics on* X *where* $d_t^M(x, y) = 1 - M(x, y, t)$ *for all* $x, y \in X$ *and all* $t > 0$.

(ii) $d = \sup_{t>0} d_t^M$ *is a metric on* X *such that* $\tau(M_t) \subseteq \tau(d)$.

We notice that part (i) of Proposition 5 provides the following characterization of strong fuzzy metrics with respect to the Łukasiewicz t-norm $*_{\text{L}}$.

Proposition 6. *A fuzzy metric space $(X, M, *_L)$ is strong if and only if $\{d_t^M : t > 0\}$ is a family of metrics on X where $d_t^M(x,y) = 1 - M(x,y,t)$ for all $x, y \in X$ and all $t > 0$.*

Proof. Necessity follows from Proposition 5.

For the converse, fix $t > 0$. Then given $x, y, z \in X$ we have that

$$\begin{aligned} d_t^M(x,y) + d_t^M(y,z) &\geq d_t^M(x,z) \\ 1 - M(x,y,t) + 1 - M(y,z,t) &\geq 1 - M(x,z,t) \\ 1 - M(x,y,t) - M(y,z,t) &\geq -M(x,z,t) \\ M(x,y,t) + M(y,z,t) - 1 &\leq M(x,z,t) \\ \max\{M(x,y,t) + M(y,z,t) - 1, 0\} &\leq M(x,z,t) \\ M(x,y,t) *_{\mathrm{L}} M(y,z,t) &\leq M(x,z,t). \end{aligned}$$

Since t is arbitrary then $(M, *_{\mathrm{L}})$ is strong. ☐

Observe that the method provided by Proposition 5 of constructing a metric d from a fuzzy metric $(M, *)$ is only valid when $(M, *)$ is strong. Moreover d does not preserve important properties of M since, for example, $\tau(d) \neq \tau(M)$ in general.

Next we present a new method for constructing a family of metrics from a fuzzy metric which is better behaved for our purposes. This method is based on an standard "convexification" process inspired by classical metrization theorems [35, Theorem 23.4] and the obtention of a subadditive function by means of the inf-convolution [7] (it is a particular case of the metrics d_ε considered in [20] when $\varepsilon = \infty$).

Lemma 7. *Let $(X, M, *)$ be a fuzzy metric space, and let $\varphi : [0,1) \to [0,1)$ be an increasing function such that $\varphi^{-1}(0) = \{0\}$. Fix $t > 0$ and consider the function $p_t : X \times X \to \mathbb{R}^+$ defined by*

$$p_t(x,y) := \inf\left\{ \sum_{i=1}^{n} \varphi\big(1 - M(x_i, x_{i+1}, t)\big) \,:\, x_1 = x,\, x_{n+1} = y,\ x_i \in X \right\}$$

for every $x, y \in X$. Then

(i) p_t is a pseudo metric on X, and

(ii) if $(x,y) \mapsto \varphi\big(1 - M(x,y,t)\big)$ *satisfies the triangular inequality, then*

$$p_t(\cdot,\cdot) = \varphi\big(1 - M(\cdot,\cdot,t)\big),$$

and it defines a metric.

Proof. (i) The function is clearly symmetric, due to the symmetry of M. On the other hand, a direct calculation using the properties of the infimum gives the triangular inequality.

(ii) Note that, if $\varphi\big(1 - M(\cdot,\cdot,t)\big)$ satisfies the triangular inequality then it is a pseudo metric. Moreover given $x, y \in X$ we have that for every $x_1, \ldots, x_{n+1} \in X$ such that $x_1 = x$, $x_{n+1} = y$ then

$$\varphi\big(1 - M(x,y,t)\big) \leq \sum_{i=1}^{n} \varphi\big(1 - M(x_i, x_{i+1}, t)\big),$$

and so the infimum $p_t(x,y)$ coincides with $\varphi\big(1 - M(x,y,t)\big)$. On the other hand, we have that $\varphi\big(1 - M(x,y,t)\big) = 0$ if and only if $1 - M(x,y,t) = 0$, and by the definition of fuzzy metric this happens if and only if $x = y$. This gives (ii). □

In the literature we can find a lot of examples of fuzzy metrics but many of them are constructed starting from a classic metric [13, 28]. We next show that the above Lemma allows to recover the metric from the fuzzy metric in some cases.

Example 8 (cf. [12, Example 25])**.** *Let* (X,d) *be a metric space and let us consider, following [13, Example 5], the fuzzy metric* $(M,\cdot)$ *on* X *given by*

$$M(x,y,t) := e^{(-d(x,y)/g(t))}, \quad x, y \in X, \;\; t > 0,$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ *is an increasing continuous function.*

Let us consider $\varphi : [0,1) \to [0,1)$ *given by* $\varphi(x) = -\log(1-x)$*. It is obvious that* $\varphi^{-1}(0) = \{0\}$ *and that* φ *is subadditive since it is concave. Following Lemma 7, given* $t > 0$ *we can construct the following metric* p_t *on* X *:*

$$p_t(x,y) := \varphi(1 - M(x,y,t)) = -\log(1 - 1 + e^{-d(x,y)/g(t)}) = \frac{d(x,y)}{g(t)}.$$

Observe that if $(X, M, *)$ is a strong fuzzy metric space, then by Proposition 5 $\{d_t^M : t > 0\}$ is a family of metrics on X. Thus, if φ is a metric preserving function [7] then $\varphi(1 - M(\cdot, \cdot, t))$ is also a metric on X for all $t > 0$ so we obtain (ii) of the previous result.

We next present an example borrowed from [17], where Lemma 7 can be applied but Proposition 5 cannot.

Example 9 ([17, Example 2]). *Let $X = \{a, b, c\}$ and $M : X \times X \times (0, +\infty) \to [0, 1]$ given by*

$$M(x, y, t) = M(y, x, t) = \begin{cases} 1 & \text{if } x = y \\ \dfrac{2t+1}{2t+2} & \text{if } x = a \text{ or } x = b \text{ and } y = c \\ \dfrac{t}{t+2} & \text{if } x = a, y = b \end{cases}.$$

It was proved in [17] that $(M, *_L)$ is a fuzzy metric on X which is not strong. Notice that by Proposition 6, there has to be $t > 0$ such that d_t^M is not a metric. In fact, d_t^M is not a metric for every $t > 0$, since

$$d_t^M(a, b) = 1 - M(a, b, t) = \frac{2}{t+2} \nleq d_t^M(a, b) + d_t^M(b, c)$$
$$= 1 - M(a, b, t) + 1 - M(b, c, t) = \frac{2}{2t+2}.$$

Hence, we cannot apply Proposition 5 to construct a metric from $(M, *_L)$.

Nevertheless, Lemma 7 allows this construction. If we consider the function $\varphi : [0, 1) \to [0, 1)$ given by $\varphi(x) = x$ then this lemma provides a family of metrics $\{p_t : t > 0\}$ where

$$p_t(x, y) = \begin{cases} \dfrac{2}{2t+2} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

for every $t > 0$.

4. Fuzzy Lipschitz maps

Since our aim is to obtain a McShane-Whitney extension theorem in the context of fuzzy metric spaces, we must look for an appropriate notion of Lipschitz function in this context. Hence it is natural to analyze the different notions of fuzzy Lipschitz maps that have been proposed in the literature. We collect some of them in the following definition.

Definition 10. *Let* $(X, M, *)$ *and* $(Y, N, \circledast)$ *be two fuzzy metric spaces. A map* $f : (X, M, *) \to (Y, N, \circledast)$ *is said to be*

(1) fuzzy Lipschitz *[37] if given* $t > 0$ *there exists* $k > 0$ *such that*

$$1 - N(f(x), f(y), t) \leq k(1 - M(x, y, t))$$

for every $x, y \in X$*. In this case, given* $t > 0$ *the* t-dilation *of* f *is*

$$dil(f, t) := \sup_{x \neq y} \frac{1 - N(f(x), f(y), t)}{1 - M(x, y, t)} < \infty.$$

Moreover f *is said to be* stationary fuzzy Lipschitz *if*

$$dil(f) = \sup_{t>0} dil(f, t) < \infty.$$

(2) GS-fuzzy Lipschitz *[16] with constant* $k > 0$ *if*

$$\frac{1}{N(f(x), f(y), t)} - 1 \leq k \left(\frac{1}{M(x, y, t)} - 1 \right)$$

for each $x, y \in X$ *and* $t > 0$*.*

(3) SBR-fuzzy Lipschitz *[30] with constant* $k > 0$ *if*

$$N(f(x), f(y), kt) \geq M(x, y, t)$$

for each $x, y \in X$ *and* $t > 0$*.*

If f *is fuzzy Lipschitz and* $dil(f, t) < 1$ *for every* $t > 0$ *then we say that* f *is* fuzzy contractive*. The corresponding notions fuzzy expansive and fuzzy nonexpansive maps are defined when* $dil(f, t) > 1$ *and* $dil(f, t) = 1$ *for every* $t > 0$*, respectively.*

Similar notions can be considered for the other two concepts of fuzzy Lipschitz functions.

Yun, Hwang and Chang proved in [37] that a fuzzy Lipschitz map is always continuous, and they studied the relationship of fuzzy Lipschitz maps with GS-fuzzy contractive self-maps. For example, they proved [37, Corollary 18] that every GS-fuzzy contractive self-map is fuzzy nonexpansive or fuzzy contractive. For completeness, we study more in deep the relationship between the different concepts of fuzzy Lipschitz function considered in Definition 10.

Proposition 11. *Let $f : (X, M, *) \to (Y, N, \circledast)$ be a map between two fuzzy metric spaces.*

(i) If f is GS-fuzzy contractive then it is SBR-fuzzy nonexpansive.

(ii) If f is SBR-fuzzy contractive then it is GS-fuzzy nonexpansive.

(iii) f is GS-fuzzy nonexpansive if and only if f is SBR-fuzzy nonexpansive.

(iv) f is SBR-fuzzy nonexpansive (equivalently, GS-fuzzy nonexpansive) if and only if it is fuzzy contractive or fuzzy nonexpansive.

Proof. (i) Let $0 < k < 1$ such that

$$N(f(x), f(y), t) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))}$$

for each $x, y \in X$ and $t > 0$. Given any $k' \geq 1$ we have that

$$\begin{aligned} N(f(x), f(y), k't) \geq N(f(x), f(y), t) &\geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))} \\ &> \frac{M(x, y, t)}{M(x, y, t) + (1 - M(x, y, t))} = M(x, y, t), \end{aligned}$$

so f is SBR-fuzzy nonexpansive.

(ii) Let $0 < k < 1$ such that $N(f(x), f(y), kt) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$. Given any $k' \geq 1$ we have that

$$\begin{aligned} N(f(x), f(y), t) \geq N(f(x), f(y), kt) &\geq M(x, y, t) \\ &\geq \frac{M(x, y, t)}{M(x, y, t) + k'(1 - M(x, y, t))}. \end{aligned}$$

(iii) This is obvious since if $k = 1$ we have that

$$N(f(x), f(y), kt) = N(f(x), f(y), t)$$

and

$$M(x, y, t) = \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))}.$$

(iv) The statement follows from the next equivalences.

$$\begin{aligned} N(f(x),f(y),t) &\geq M(x,y,t) && \text{for all } x,y\in X \text{ and } t>0, \Leftrightarrow \\ 1-N(f(x),f(y),t) &\leq 1-M(x,y,t) && \text{for all } x,y\in X \text{ and } t>0, \Leftrightarrow \\ \frac{1-N(f(x),f(y),t)}{1-M(x,y,t)} &\leq 1 && \text{for all distinct } x,y\in X \text{ and } t>0, \end{aligned}$$

which finally gives $dil(f,t)\leq 1$ for all $t>0$. □

Remark 12. *Notice that by the above proof, if f is GS-fuzzy contractive then f is an Edelstein map [23], i.e.*

$$N(f(x),f(y),t) > M(x,y,t)$$

for all $x,y\in X$ and all $t>0$ (see [37, Lemma 15]). We also observe that GS-fuzzy contractiveness of a self-map implies fuzzy contractiveness; this follows from [37, Lemma 16, Theorem 17].

Corollary 13. *Let $f:(X,M,*)\to(Y,N,\circledast)$ be a map between two fuzzy metric spaces such that $(N,\circledast)$ is stationary. The following statements are equivalent.*

(i) *f is SBR-fuzzy Lipschitz;*
(ii) *f is SBR-fuzzy nonexpansive;*
(iii) *f is SBR-fuzzy contractive;*
(iv) *f is GS-fuzzy contractive;*
(v) *f is fuzzy contractive or fuzzy nonexpansive.*

Proof. Equivalence between (i), (ii) and (iii) is obvious from the definitions. (iii) implies (iv) is also clear since for any $0<k<1$ we have that

$$\begin{aligned} N(f(x),f(y),t) = N(f(x),f(y),kt) &\geq M(x,y,t) \\ &\geq \frac{M(x,y,t)}{M(x,y,t)+k(1-M(x,y,t))}. \end{aligned}$$

The other implications are deduced from Proposition 11. □

Remark 14. *Notice that if f is fuzzy Lipschitz but not fuzzy contractive then f is not necessarily SBR-fuzzy Lipschitz. In fact, let us consider $X=[0,1]$ and the fuzzy metrics $(M,*_L)$, $(N,\cdot)$ on X given by*

$$\begin{aligned} M(x,y,t) &= e^{-\frac{|x-y|}{t}} \\ N(x,y,t) &= 1-|x-y| \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then the map $f : (X, M, *_L) \to (X, N, \cdot)$ given by $f(x) = x$ is fuzzy Lipschitz. In fact, we have that

$$dil(f,t) = \sup_{x \neq y} \frac{1 - N(f(x), f(y), t)}{1 - M(x,y,t)} = \sup_{x \neq y} \frac{|x-y|}{1 - e^{-\frac{|x-y|}{t}}} = \frac{1}{1 - e^{-\frac{1}{t}}} < \infty.$$

Hence f is fuzzy Lipschitz but not fuzzy contractive.

Nevertheless, f is not SBR-fuzzy Lipschitz. Otherwise we would have

$$N(f(x), f(y), t) = 1 - |x-y| \geq M(x,y,t) = e^{-\frac{|x-y|}{t}}$$

for all $x, y \in X = [0,1]$ and all $t > 0$. Hence, the function $g(x,t) : [0,1] \times (0, +\infty) \to \mathbb{R}$ given by $g(x,t) = 1 - x - e^{-\frac{x}{t}}$ should be greater than 0 for every $t > 0$ and every $x \in [0,1]$. However, for example, $g'(x,1) = -1 + e^{-x} \leq 0$ for every $x \in X$. Since $g(0,1) = 0$ then $g(x,1) < 0$ for every $x \in]0,1]$, which is a contradiction.

We also notice that f is not GS-fuzzy Lipschitz.

We next show a particular case where the concepts of SBR-fuzzy Lipschitz function and GS-fuzzy Lipschitz function coincide.

Proposition 15. *Let (X, d) and (Y, q) be two metric spaces. The following statements are equivalent:*

(1) $f : (X, d) \to (Y, q)$ is Lipschitz with constant $k > 0$;

(2) $f : (X, M_d, *) \to (Y, M_q, \circledast)$ is SBR-fuzzy Lipschitz with $k > 0$;

(3) $f : (X, M_d, *) \to (Y, M_q, \circledast)$ is GS-fuzzy Lipschitz with constant $k > 0$.

Proof. (1) $\Leftrightarrow$ (2) Fix $k > 0$. The equivalence is a consequence of the following:

$$\begin{aligned} M_q(f(x), f(y), kt) \geq M_d(x,y,t) &\Leftrightarrow \frac{kt}{kt + q(f(x), f(y))} \geq \frac{t}{t + d(x,y)} \\ &\Leftrightarrow \frac{kt}{t} \geq \frac{kt + q(f(x), f(y))}{t + d(x,y)} \\ &\Leftrightarrow kt + kd(x,y) \geq kt + q(f(x), f(y)) \\ &\Leftrightarrow kd(x,y) \geq q(f(x), f(y)). \end{aligned}$$

(1) $\Leftrightarrow$ (3) This equivalence was proved in [16] in case of GS-fuzzy contractive self-maps but the proof is similar in this case. $\square$

It is natural to wonder whether in the above proposition we can add an equivalent statement involving fuzzy Lipschitz functions. Nevertheless, this is not possible since although it was proved in [37, Lemma 8] that if a map $f : (X, d) \to (Y, q)$ between two metric spaces is Lipschitz then $f : (X, M_d, *) \to (Y, M_q, \circledast)$ is fuzzy Lipschitz, the converse is not true in general as the next example shows (notice that by Propositions 11 and 15, the converse is true if $f : (X, M_d, *) \to (Y, M_q, \circledast)$ is fuzzy contractive or fuzzy nonexpansive).

Example 16. *Let us consider the real line $\mathbb{R}$, and the discrete metric d and the Euclidean metric e on $\mathbb{R}$. It is obvious that the map $f : (\mathbb{R}, d) \to (\mathbb{R}, e)$ given by $f(x) = x^2$ is not Lipschitz since $\{|x^2 - y^2| : x, y \in \mathbb{R}\}$ is not bounded. Nevertheless $f : (\mathbb{R}, M_d, \cdot) \to (\mathbb{R}, M_e, \cdot)$ is fuzzy Lipschitz since given $t > 0$ we have that*

$$
\begin{aligned}
dil(f,t) &= \sup_{x \neq y} \frac{1 - M_e(f(x), f(y), t)}{1 - M_d(x, y, t)} = \sup_{x \neq y} \frac{\frac{e(f(x),f(y))}{t+e(f(x),f(y))}}{\frac{d(x,y)}{t+d(x,y)}} \\
&= \sup_{x \neq y} \frac{e(f(x), f(y))}{d(x,y)} \frac{t + d(x,y)}{t + e(f(x), f(y))} = \sup_{x \neq y} \frac{|x^2 - y^2|(t+1)}{t + |x^2 - y^2|} \leq t + 1.
\end{aligned}
$$

As a consequence of the previous results, we can infer that there is a close relationship between the different concepts of fuzzy Lipschitz maps mainly when the Lipschitz constant is less than or equal to 1. But due to its similarity with the crisp concept of Lipschitz map, we consider the definition of fuzzy Lipschitz map due to Yun, Hwang and Chang [37] more suitable for the aim of this paper. Nevertheless, for reasons that will become clear in the next section, it is convenient to use a broader definition of fuzzy Lipschitz map including the dependence of the parameter t.

Definition 17. *Given two fuzzy metric spaces $(X, M, *), (Y, N, \circledast)$, we will say that a map*

$$f : X \times (0, \infty) \to Y$$

is a fuzzy Lipschitz map if the "extended dilation"

$$dil(f,t) := \sup_{x \neq y} \frac{1 - N(f(x,t), f(y,t), t)}{1 - M(x, y, t)}$$

is finite for every $t > 0$.

We will show that the extension results can be applied to this class of maps: if the original fuzzy Lipschitz map f depends on t, then the extension depends on t too. But we will also prove that, under reasonable requirements, if f does not depend on t —despite being a fuzzy Lipschitz map—, then we can choose an extension which also does not.

We can write the definition of our Lipschitz-type maps in the usual terms, that is, involving inclusions of sets and inequalities as in the classical presentation of Lipschitz maps. In this way, we can say that f is fuzzy Lipschitz if and only if for each $x \in X$ and $t > 0$ there is a constant $K(t)$ such that

$$f(B_\varepsilon^M(x,t),t) \subseteq K(t)\, B_\varepsilon^N(f(x,t),t),$$

which gives an inequality as the next one for each $t > 0$,

$$1 - N(f(x,t), f(y,t), t) \leq K(t)\Big(1 - M(x,y,t)\Big), \quad x, y \in X.$$

Notice that we can take $K(t) = dil(f,t)$.

5. The McShane-Whitney extension theorem for fuzzy Lipschitz maps

In what follows we show our main result, the McShane-Whitney extension theorem in the realm of fuzzy metric spaces. To achieve this, we will consider a special type of fuzzy metrics on $\mathbb{R}$.

Definition 18. *A fuzzy metric $(M,*)$ on $\mathbb{R}$ is said to be a* Euclidean fuzzy metric *if there are functions $\phi, g : \mathbb{R}^+ \to \mathbb{R}^+$ such that ϕ is increasing and*

$$M(x,y,t) = 1 - \phi(|x-y|)g(t).$$

In this case we will denote M by $M_{\phi,g}$ and we will say that $(\mathbb{R}, M_{\phi,g}, *)$ is a Euclidean fuzzy metric space.

Example 19. *Let us consider $X = [0,1]$ and $M : X^2 \times (0,+\infty) \to [0,1]$ given by $M(x,y,t) = 1-|x-y|$ for all $x, y \in X$ and all $t > 0$. It is easy to see that $(M, *_L)$ is a Euclidean stationary fuzzy metric induced by the functions $\phi, g : \mathbb{R}^+ \to \mathbb{R}^+$ given by*

$$\phi(x) = x$$
$$g(x) = 1$$

for all $x \in X$.

Example 20 (cf. [13, Example 6]). *Let us consider two functions $\phi, g : \mathbb{R}^+ \to \mathbb{R}^+$ such that ϕ is strictly increasing, subadditive, bounded by $k > 0$ and $\phi^{-1}(0) = \{0\}$, meanwhile g is decreasing, bounded by $\frac{1}{k}$ and greater than 0. Then it is easy to see that $(M, *_L)$ is a Euclidean fuzzy metric on $\mathbb{R}$ where*

$$M(x, y, t) = 1 - \phi(|x - y|)g(t).$$

We only check the triangle inequality. Let $x, y, z \in \mathbb{R}$ and $t, s > 0$. Since ϕ is subadditive and strictly increasing then $\phi(|x-y|) \leq \phi(|x-y|+|y-z|) \leq \phi(|x-z|) + \phi(|z-y|)$. Using that g is decreasing we have that

$$\begin{aligned}\phi(|x-y|)g(t+s) &\leq \phi(|x-z|)g(t+s) + \phi(|z-y|)g(t+s) \\ &\leq \phi(|x-z|)g(t) + \phi(|z-y|)g(s).\end{aligned}$$

That is,

$$1 - \phi(|x-y|)g(t+s) \geq 1 - \phi(|x-z|)g(t) - \phi(|z-y|)g(s),$$

what gives

$$M(x, y, t+s) \geq M(x, z, t) *_L M(z, y, s).$$

Let $(X, M, *)$ be a fuzzy metric space and consider a Euclidean fuzzy metric space $(\mathbb{R}, N_{\phi,g}, \circledast)$. If $f : (X, M, *) \times (0, \infty) \to (\mathbb{R}, N_{\phi,g}, \circledast)$ is a fuzzy Lipschitz map then for each $t > 0$ there is a positive real number $K(t)$ such that

$$\begin{aligned}1 - N_{\phi,g}(f(x,t), f(y,t), t) &\leq K(t)\Big(1 - M(x,y,t)\Big) \\ \phi(|f(x,t) - f(y,t)|)g(t) &\leq K(t)\Big(1 - M(x,y,t)\Big)\end{aligned}$$

for all $x, y \in X$. By composing with the increasing function ϕ^{-1} we obtain

$$|f(x,t) - f(y,t)| \leq \phi^{-1}\Big(\frac{K(t)}{g(t)}\big(1 - M(x,y,t)\big)\Big), \quad x, y \in X.$$

Due to the fact that $|\cdot|$ is a norm on $\mathbb{R}$, we automatically get from the inequality above that for each $t > 0$, the inequality

$$|f(x,t) - f(y,t)|$$

$$\leq \inf_n \Big\{ \sum_{i=1}^n \phi^{-1}\Big(\frac{K(t)}{g(t)}\big(1-M(x_i,x_{i+1},t)\big)\Big), x_1=x, x_2, ..., x_n, x_{n+1}=y,\ x_i \in X\Big\},$$

where $x, y \in S$. We will write $d_{t,f}(x,y)$ for the second term of this inequality, that is clearly a pseudometric on X for each $t > 0$. Therefore, we obtain in this way a typical Lipschitz inequality for the function $f(\cdot,t) : X \to \mathbb{R}$, where the spaces X and $\mathbb{R}$ are assumed to be pseudometric spaces with the explained pseudometrics.

This construction gives the proof of the following extension theorem. Note that we have to take the elements x_i in all the set X in the definition of $d_{t,f}$ and not only the ones of S.

Theorem 21. *Let $(X, M, *)$ be a fuzzy metric space and $(\mathbb{R}, N_{\phi,g}, \circledast)$ be a Euclidean fuzzy metric space. Let $S \subseteq X$ and suppose that $f : (S, M, *) \times (0,\infty) \to (\mathbb{R}, N_{\phi,g}, \circledast)$ is a fuzzy Lipschitz map. Then the following statements are equivalent.*

(i) $f(\cdot,t) : (S, d_{t,f}) \to \mathbb{R}$ is nonexpansive for all $t > 0$.

(ii) The function f can be extended as a fuzzy Lipschitz map to $X \times (0,\infty)$.

Proof. (i) $\Rightarrow$ (ii) Fix $t > 0$. By assumption, the function $f(\cdot,t) : (X, d_{t,f}) \to \mathbb{R}$ is Lipschitz so, for example, the McShane formula provides an extension $\hat{f}(\cdot,t)$ of $f(\cdot,t)$ to all the set X. Therefore, we have in particular that for each $x, y \in X$,

$$|\hat{f}(x,t) - \hat{f}(y,t)| \leq d_{t,f}(x,y) \leq \phi^{-1}\left(\frac{K(t)}{g(t)}\big(1 - M(x,y,t)\big)\right),$$

and so

$$1 - N_{\phi,g}(\hat{f}(x,t), \hat{f}(y,t), t) = \phi(|f(x,t) - f(y,t)|)g(t) \leq K(t)\big(1 - M(x,y,t)\big).$$

Since $t > 0$ is arbitrary, $\hat{f}$ is a fuzzy Lipschitz map.

(ii) $\Rightarrow$ (i) Suppose that f allows an extension $\hat{f}$ to the whole X which is fuzzy Lipschitz. Then given $t > 0$ we have that

$$\begin{aligned} 1 - N_{\phi,g}(\hat{f}(x,t), \hat{f}(y,t), t) &\leq K(t)(1 - M(x,y,t)) \\ \phi(|\hat{f}(x,t) - \hat{f}(y,t)|)g(t) &\leq K(t)\big(1 - M(x,y,t)\big) \\ |\hat{f}(x,t) - \hat{f}(y,t)| &\leq \phi^{-1}\left(\frac{K(t)}{g(t)}\big(1 - M(x,y,t)\big)\right) \qquad (1) \end{aligned}$$

for all $x, y \in X$. Fix now $x, y \in S$. Given $n \in \mathbb{N}$ and $\{x_1, \dots, x_{n+1}\} \subseteq X$ such that $x_1 = x$ and $x_{n+1} = y$ we can use inequality (1) to obtain

$$|f(x,t) - f(y,t)| = |\hat{f}(x,t) - \hat{f}(y,t)|$$

$$\leq \sum_{i=1}^{n} |\hat{f}(x_i,t) - \hat{f}(x_{i+1},t)| \leq \sum_{i=1}^{n} \phi^{-1}\Big(\frac{K(t)}{g(t)}\big(1 - M(x_i, x_{i+1}, t)\big)\Big).$$

Since this holds for all $n \in \mathbb{R}$ and for all $x_i\,'s$, we obtain that for $x, y \in S$,

$$|f(x,t) - f(y,t)| \leq d_{t,f}(x,y).$$

Therefore, (i) holds. □

The following result will be the main tool for working in concrete applications.

Corollary 22. *Let $(X, M, *)$ be a fuzzy metric space and $(\mathbb{R}, N_{\phi,g}, \circledast)$ be a Euclidean fuzzy metric space. Let $S \subseteq X$ and suppose that $f : (S, M, *) \times (0, \infty) \to (\mathbb{R}, N_{\phi,g}, \circledast)$ is a fuzzy Lipschitz map with extended dilation $K(t)$. Assume also that for every $t > 0$ the map $\rho_t : X \times X \to \mathbb{R}^+$ given by*

$$\rho_{t,f}(x,y) := \phi^{-1}\left(\frac{K(t)}{g(t)}\big(1 - M(x,y,t)\big)\right), \quad x, y \in X,$$

is a metric on X. Then the function f can be extended as a fuzzy Lipschitz map to $X \times (0, \infty)$.

Proof. If the above defined map is a metric on X, then we clearly have that it coincides with $d_{t,f}$. Then we directly get (i) in Theorem 21, and the result holds. □

Remark 23. *Observe that under the hypotheses of the previous corollary, we can provide directly an extension of f by using:*

- *the following modified McShane formula*

$$f^M(x,t) = \sup_{s \in S}\left\{f(s) - \phi^{-1}\left(\frac{K(t)}{g(t)}(1 - M(x,s,t))\right)\right\}$$

  for all $x \in X$ and all $t > 0$.

- *the following modified Whitney formula*

$$f^W(x,t) = \inf_{s\in S}\left\{ f(s) + \phi^{-1}\left(\frac{K(t)}{g(t)}(1 - M(x,s,t))\right)\right\}$$

for all $x \in X$ and all $t > 0$.

Remark 24. *The main problem to define the extension of a fuzzy Lipschitz function is that the Lipschitz extension obtained by our method depends on the parameter t. However, under some assumptions, it can be proved that we can obtain an extension not depending on t. For example, let $(X, M, *), (\mathbb{R}, N_{\phi,g}, \circledast)$ be two stationary fuzzy metric spaces such that $(N_{\phi,g}, \circledast)$ is a Euclidean fuzzy metric. Given $S \subseteq X$ and a fuzzy Lipschitz function $f : (S, M, *) \to (Y, N_{\phi,g}, \circledast)$, then f is a stationary fuzzy Lipschitz function. Furthermore, if ϕ^{-1} is subadditive then it is easy to see [7] that the function ρ_t considered in the previous corollary is a metric on X which does not depend on t, so the function f can be extended to the whole X without depending on t (see also [3]).*

Nevertheless, we can improve a little bit this result as follows.

Proposition 25. *Let $(X, M, *)$ be a eventually stationary fuzzy metric space with $* \geq *_L$ and $(\mathbb{R}, N_{\phi,g}, \circledast)$ be a Euclidean stationary fuzzy metric space such that ϕ is strictly increasing and ϕ^{-1} is subadditive. Let $S \subseteq X$ and suppose that $f : (S, M, *) \to (\mathbb{R}, N_{\phi,g}, \circledast)$ is fuzzy Lipschitz. Then f can be extended as a fuzzy Lipschitz map to X.*

Proof. Although we could use Corollary 22 to prove this proposition, for completeness, we present a direct proof by using a suitable McShane formula.

Since $(N_{\phi,g}, \circledast)$ is stationary then g must be constant. Otherwise, if $g(t_1) \neq g(t_2)$ for some $t_1, t_2 > 0$ then $N_{\phi,g}(0, 1, t_1) = 1 - \phi(|1 - 0|)g(t_1) \neq 1 - \phi(|1 - 0|)g(t_2) = N_{\phi,g}(0, 1, t_2)$ which contradicts stationarity of $(N_{\phi,g}, \circledast)$.

Furthermore, since $(X, M, *)$ is eventually stationary we can find $t_0 > 0$ such that $M(x, y, t) = M(x, y, s)$ for all $x, y \in X$ and all $t, s \geq t_0$. By assumption we can find $K(t_0) > 0$ such that

$$1 - N_{\phi,g}(f(s), f(s'), t_0) \leq K(t_0)(1 - M(s, s', t_0))$$

for all $s, s' \in S$. Observe that $d_{t_0} : X \times X \to [0, +\infty)$ given by $d_{t_0}(x, y) = \frac{K(t_0)}{g(t_0)}(1 - M(x, y, t_0))$ is a metric on X.

Consider $\hat{f} : X \to \mathbb{R}$ defined by the following modified McShane formula:

$$\hat{f}(x) = \sup_{s \in S} \left\{ f(s) - \phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1 - M(x, s, t_0)) \right) \right\}$$

for all $x \in X$. Let us check that $\hat{f}$ is fuzzy Lipschitz. Given $x, y \in X$ we have that

$$\begin{aligned}
1 - N_{\phi,g}(\hat{f}(x), \hat{f}(y), t) &= \phi(|\hat{f}(x) - \hat{f}(y)|) g(t_0) \\
&= \phi\Big(\Big| \sup_{s \in S} \left\{ f(s) - \phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1 - M(x, s, t_0)) \right) \right\} \\
&- \sup_{s \in S} \left\{ f(s) - \phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1 - M(y, s, t_0)) \right) \right\} \Big| \Big) g(t_0) \\
&\leq \phi\Big(\Big| \sup_{s \in S} \Big\{ f(s) - \phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1 - M(x, s, t_0)) \right) \\
&- f(s) + \phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1 - M(y, s, t_0)) \right) \Big\} \Big| \Big) g(t_0)
\end{aligned}$$

$$\begin{aligned}
&= \phi\left(\left| \sup_{s \in S} \left\{ \phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1 - M(y, s, t_0)) \right) - \phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1 - M(x, s, t_0)) \right) \right\} \right| \right) \\
&= \phi(\left(\left| \sup_{s \in S} \left\{ \phi^{-1}(d_{t_0}(y, s)) - \phi^{-1}(d_{t_0}(x, s)) \right\} \right| \right) g(t_0) \\
&\leq \phi\left(\phi^{-1}(d_{t_0}(x, y)) \right) g(t_0) = K(t_0)(1 - M(x, y, t_0)) \\
&\leq K(t_0)(1 - M(x, y, t))
\end{aligned}$$

where in the last inequality we have used that $M(x, y, t) = M(x, y, t_0)$ whenever $t \geq t_0$ and $M(x, y, t) \leq M(x, y, t_0)$ whenever $t < t_0$. Consequently, $\hat{f}$ is fuzzy Lipschitz.

Moreover, $\hat{f}$ extends f to X. In fact, if $s_0 \in S$ then

$$f(s_0) = f(s_0) - \phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1 - M(s_0, s_0, t_0)) \right) \leq \hat{f}(s_0).$$

On the other hand, since f is fuzzy Lipschitz for every $s \in S$ we have that

$$\begin{aligned}
1 - N_{\phi,g}(f(s), f(s_0), t_0) &\leq K(t_0)(1 - M(s, s_0, t_0)), \\
\phi(|f(s) - f(s_0)|) g(t_0) &\leq K(t_0)(1 - M(s, s_0, t_0)), \\
|f(s) - f(s_0)| &\leq \phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1 - M(s, s_0, t_0)) \right),
\end{aligned}$$

and then

$$f(s)-\phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1-M(s,s_0,t_0))\right)\leq f(s_0).$$

Hence $\hat{f}(s_0)\leq f(s_0)$ so $\hat{f}(s_0)=f(s_0)$. □

Remark 26. *Notice that under the conditions of the previous proposition, we can provide directly an extension of f not depending on t by using*

- *the following modified McShane formula (as used in the proof)*

$$f^M(x)=\sup_{s\in S}\left\{f(s)-\phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1-M(x,s,t_0))\right)\right\}$$

  for all $x\in X$, or

- *the following modified Whitney formula*

$$f^W(x)=\inf_{s\in S}\left\{f(s)+\phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1-M(x,s,t_0))\right)\right\}$$

  for all $x\in X$.

6. Applications: the extension formulas for two fuzzy metric spaces

To finish the paper, let us explain how to obtain explicit formulas providing parameterized families of extensions. These families can be used to build machine learning tools as the ones in [5], but integrating new fuzzy elements in them. This would provide more flexible algorithms, in the sense of not depending so heavily on the Lipschitz constant, which could easily increase a lot as a result of outliers among the data, producing as a consequence imprecise extensions. Although the same construction that we present here can be done for a broader class of fuzzy metric spaces, we will center our attention in two standard cases for the aim of clarity, and also because these particular cases can be easily implemented in the context of [5].

Let us consider a strong fuzzy metric space $(X,M,*)$ with $*\geq *_{\mathrm{L}}$ and the Euclidean fuzzy metric space $(\mathbb{R},N_E,*_{\mathrm{L}})$ (see Example 20), given by

$$N_E(x,y,t)=1-\min\{|x-y|,1\}g(t),\quad x,y\in\mathbb{R},$$

where $g:[0,+\infty)\to(0,1]$ is a decreasing function.

Let $S \subseteq X$ and $I : (S, M, *) \to (\mathbb{R}, N_E, *_{\mathrm{L}})$ be a fuzzy Lipschitz map. Since $(M, *)$ is a strong fuzzy metric, by using Proposition 5 we can obtain that the hypotheses of Corollary 22 are satisfied. Thus, we know (see Remark 23) that there are two canonical extensions of I, —provided by the McShane and Whitney formulas—, $I^M, I^W : X \times (0, +\infty) \to \mathbb{R}$. Parameter dependent interpolations of these functions can be considered as optimal extensions of I, and would be given by

$$I_\alpha(x,t) := \alpha(t)\, I^M(x,t) + \big(1 - \alpha(t)\big)\, I^W(x,t), \quad x \in X,\ t > 0.$$

Here, $\alpha : (0, +\infty) \to [0, 1]$.

- *The metric model depending on a parameter.* Take a metric space (X, d) and construct the associated strong fuzzy metric space $(X, M_k, *_{\mathrm{L}})$ (cf. [13, Example 6]) defined by

  $$M_k(x,y,t) = 1 - \frac{\min\{d(x,y), k\}}{h(t)}, \quad x, y \in X,$$

  where $k > 0$ and $h : (0, +\infty) \to (k, +\infty)$ is an increasing continuous function. Let $S \subseteq X$. Suppose that the function $I : (S, M_k, *_{\mathrm{L}}) \to (\mathbb{R}, N_E, *_{\mathrm{L}})$ is a fuzzy Lipschitz map. Its corresponding Lipschitz inequality is

  $$1 - N_E(I(x), I(y), t) \le K(t)\big(1 - M_k(x,y,t)\big), \quad x, y \in S,$$

  for all $x, y \in X$ and all $t > 0$, which can be rewritten as

  $$\min\{|I(x) - I(y)|, 1\} \le \frac{K(t)}{g(t)h(t)} \min\{d(x,y), k\}.$$

  That is, there is a function $Q : \mathbb{R}^+ \to \mathbb{R}^+$ such that

  $$\min\{|I(x) - I(y)|, 1\} \le Q(t) \min\{d(x,y), k\}$$

  for all $x, y \in S$ and all $t > 0$. Then the McShane and Whitney extensions of I to $X \times (0, \infty)$ are given by

  $$I^M(x,t) := \sup_{s \in S}\{I(s) - Q(t)\, \min\{d(s,x), k\}\}, \quad x \in X,$$

and

$$I^W(x,t) := \inf_{s\in S}\{I(s) + Q(t)\,\min\{d(s,x),k\}\}, \quad x\in X.$$

Thus, a possible family of extensions would be given by functions as

$$\begin{aligned} I_{\alpha(t)}(x,t) =& \alpha(t)\, I^M(x,t) + (1-\alpha(t))\, I^W(x,t) \\ =& \alpha(t)\, \sup_{s\in S}\{I(s) - Q(t)\,\min\{d(s,x),k\}\} \\ &+ (1-\alpha(t))\, \inf_{s\in S}\{I(s) + Q(t)\,\min\{d(s,x),k\}\}, \end{aligned}$$

$x\in X$, $t>0$. An adequate function $\alpha(t)$ could be given for example by an optimization procedure, in order to define a machine learning method incorporating fuzzy elements.

- *The exponential fuzzy model.* In this case, we consider the stationary fuzzy metric $(M_1,\cdot)$ given by [13, Example 5],

$$M_1(x,y,t) = e^{-d(x,y)}, \quad x,y\in X,$$

where (X,d) is a metric space. As above, let $S\subseteq X$ and $I:(S,M_1,\cdot)\to(\mathbb{R},N_E,*_{\mathrm{L}})$ be a fuzzy Lipschitz function. Then, for each $t>0$ we can find $K(t)>0$ such that

$$1 - N_E(I(x),I(y),t) \le K(t)\big(1 - M_1(x,y,t)\big), \quad x,y\in S,$$

for all $x,y\in X$ and all $t>0$, which can be rewritten as

$$\min\{|I(x)-I(y)|,1\} \le \frac{K(t)}{g(t)}(1-e^{-d(x,y)}).$$

Notice that since $\cdot \ge *_{\mathrm{L}}$ we have by Proposition 5 that $1-e^{-d(x,y)}$ is a metric on X.

Using the same arguments than in the previous example, we obtain that the family of extensions would be given by functions as

$$I_{\alpha(t)}(x,t) = \alpha(t)\, I^M(x,t) + (1-\alpha(t))\, I^W(x,t), \quad x\in X,\ t>0,$$

where

$$I^M(x,t) = \sup_{s\in S}\left\{I(s) - \frac{K(t)}{g(t)}\left(1-e^{-d(s,x)}\right)\right\}, \quad x\in X,\ t>0,$$

and

$$I^W(x,t) = \inf_{s\in S} \left\{ I(s) + \frac{K(t)}{g(t)} \left(1 - e^{-d(s,x)}\right) \right\}, \quad x \in X,\ t > 0.$$

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